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Applications of Game Theory to Distributed Routing and Delay Tolerant Networking

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ABSTRACT

This thesis focuses on the issues related to the selfish behavior of the agents in the communication networks. We are particularly interested in two situations in which these issues arise and we address game-theoretical framework to study them.

The first situation relates to communication networks using a distributed routing based on autonomous agents. Compared to a centralized routing, this type of routing offers significant advantages in terms of scalability, ease of deployment or robustness to failures and environmental disturbances. We investigate the convergence properties of the sequential best-response dynamics in a routing game over parallel links. The game involves a finite number of routing agents each of which decides how much flow to route on each of the links with the objective of minimizing its own costs. For some particular cases (e.g., two players), the convergence of the best-response dynamics can be proved by showing that this game has a potential function. For other cases, a potential function has remained elusive. We propose the use of non-linear spectral radius of the Jacobian of the best-response dynamics as an alternative approach to proving its convergence.

The second situation occurs in Delay Tolerant Networks (DTNs) that have been the subject of intensive research over the past decade. DTN has an idea to support communication in environments where connectivity is intermittent and where communication delays can be very long. We focus on game-theoretic models for DTNs. First, we propose an incentive mechanism to persuade selfish mobile nodes to participate in relaying messages, and investigate the influence of the information given by the source (number of existing copies of the message, age of these copies) to the relays on the rewards proposed. For static information polices, that is the same type of information given to all the relays, it is shown that the expected reward paid by the source is independent of the policy. However, the source can reduce the reward by dynamically adapting the type of information based on the meeting times with the relays. For the particular cases, we give some structural results of the optimal adaptive policy. Next, we consider the model where the source proposes a fixed reward. The mobile relays can decide to accept or not the packet and then to drop the packet in the future. This game can be modelled as a partially-observable stochastic game. For two relays, we have shown that the optimal policies for the relays relates to the threshold type.

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RÉSUMÉ ÉTENDU

Cette thèse porte sur les problématiques liées au comportement égoïste des agents dans les réseaux de communication. Nous étudions plus particulièrement deux situations dans lesquelles ces problématiques apparaissent. La première concerne les réseaux de communication utilisant un routage décentralisé basé sur des agents autonomes. Par rapport à un routage centralisé, ce type de routage offre des avantages significatifs en termes d'évolutivité, de facilité de déploiement ou encore de robustesse aux pannes et aux perturbations de l'environnement. La seconde situation apparaît dans les réseaux tolérants aux délais (en anglais: Delay Tolerant Network - DTN) qui ont fait l'objet d'intenses recherches ces dix dernières années pour permettre la communication dans des environnements où la connectivité n'est qu'intermittente et où les délais de communication peuvent être très long.

Dans ces deux situations, on est confronté au comportement égoïste des participants. En effet, dans un système de routage décentralisé non-coopératif, les agents autonomes sont en concurrence pour les ressources du réseau, chacun cherchant à optimiser les performances de son propre trafic. De même, dans les DTNs les nœuds mobiles qui sont censés servir de relai pour la communication entre les autres nœuds, peuvent ne pas être disposés à coopérer en raison de leurs objectifs individuels.

La théorie des jeux fournit un cadre théorique naturel pour ces environnements compétitifs dans lesquels des agents égoïstes sont en interaction. Elle donne différents concepts d'équilibre et peut être utilisée pour concevoir des mécanismes d'incitation conduisant à des résultats globaux efficaces et souhaitables en dépit du comportement égoïste des participants.

1 Théorie des jeux non-coopératifs

La théorie des jeux formalise une situation interactive comme un *jeu*. L'étude de la thèse concerne les jeux non-coopératifs, lorsque les participants ne sont pas autorisés à conclure une entente pour former des coalitions.

Un modèle de jeu contient les éléments suivants :

Joueurs.

- Un ensemble de *joueurs*, $\mathcal{N} = (1, 2, \cdots, n)$.

Stratégies.

- Pour chaque joueur i, un *ensemble de stratégies* S_i. Une *stratégie* $s_i \in S_i$ est une action que le joueur i peut prendre. Des stratégies, une pour chaque joueur dans le jeu, forment un *profil des stratégies* $\mathbf{s} = (s_1, s_2, \cdots, s_n)$.

Rétributions.

- Chaque profil possible de stratégies conduit à une issue bien définie du jeu à laquelle on peut associer une *rétribution* (ou gain) pour chaque joueur. La rétribution du joueur i est représentée par la fonction d'utilité, $u_i = u_i(s_1, s_2, \cdots, s_n)$. La valeur d'utilité indique comment le joueur apprécie le résultat.

1.1 Équilibre de Nash

Un concept clef de la théorie des jeux est l'équilibre de Nash. Pour des joueurs égoïstes qui agissent en maximisant leur propre rétribution, l'équilibre de Nash reflète un état stable à partir duquel aucun joueur ne peut améliorer son gain par une déviation unilatérale . Formellement, un vecteur des stratégies $\mathbf{s} \in \bigotimes_{k=1}^{n} S_k$ est un *équilibre de Nash* si pour tous les joueurs i et tout autre stratégie $s'_i \in \mathcal{S}_i$,

Équilibre de Nash en stratégies pures.

$$u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s'_i, \mathbf{s}_{-i}).$$

Cet équilibre est appelé un équilibre de Nash en stratégies pures (PNE). L'équilibre de Nash est stable en ce sens qu'une fois que les joueurs jouent une telle solution, il est dans l'intérêt de tous les joueurs de garder la même stratégie (Nisan et al., 2007).

1.2 Trouver des équilibres via une dynamique de meilleure réponse

L'équilibre de Nash n'est intéressant que si les joueurs peuvent apprendre à jouer un équilibre en interagisseant à plusieurs reprises. Un équilibre de Nash résulte alors de l'adaptation rationnelle des joueurs dans le jeu. La faon sans doute la plus naturelle de jouer à un jeu est de jouer sa *meilleure réponse*. Plus précisément, étant donné un profil de stratégies $\mathbf{s}$ qui n'est pas un équilibre de Nash pur, considérons un joueur arbitraire i. Son utilité sous le profil de stratégies $\mathbf{s}$ est $u_i(\mathbf{s})$. En supposant que tous les autres joueurs respectent leurs stratégies dans s_{-i}, le joueur i peut avantageusement changer son utilité en déviant unilatéralement de sa stratégie s_i vers une autre stratégie $s'_i \in S_i$. Un déviation de la stratégie s_i pour s_i' est dit être un *meilleure réponse* si s_i' maximise l'utilité du joueur i, $\max_{s_i' \in S_i} u_i(s_i', s_{-i})$.

Prouver que cette dynamique naturelle converge rapidement vers un équilibre permet de valider l'existence et la faisabilité de l'équilibre de Nash. Dans certains jeux, la dynamique de meilleure réponse conduit les joueurs à un équilibre de Nash en quelques étapes. Il existe quelques jeux pour lesquels les joueurs ne seront pas certains d'atteindre un équilibre en un nombre fini d'étapes, mais le vecteur des stratégies convergera vers cet équilibre. Si une dynamique de meilleure réponse atteint un état stable, cet état est clairement PNE. Cette dynamique est cyclique dans un jeu sans PNE. Elle peut également être cyclique et ne pas converger dans des jeux qui ont un PNE.

2 Convergence de la dynamique de meilleure réponse dans les jeux de routage sur des liens parallèles

Dans cette thèse, nous étudions des jeux de routage non coopératifs dans lesquels chaque flux origine-destination est contrôlé par un agent autonome qui décide comment son propre trafic est routé dans le réseau. Nous étudions la convergence des agents de routage autonomes vers un équilibre de Nash.

2.1 Jeux de routage non coopératif sur des liens parallèles

Nous étudions un jeu de routage non-coopératif dans un réseau de liens parallèles partagés par un nombre fini d'agents de routage, vus comme les joueurs du jeu. Nous considérons un ensemble $\mathcal{C} = \{1, \ldots, K\}$ d'agents de routage et un ensemble $\mathcal{S} = \{1, \ldots, S\}$ de liens. Chaque agent $i \in \mathcal{C}$ contrôle une partie non négligeable λ_i du trafic total, et cherche à partager son flux sur les liens afin de minimiser ses frais. Ce jeu de routage est représenté sur la Figure 1.

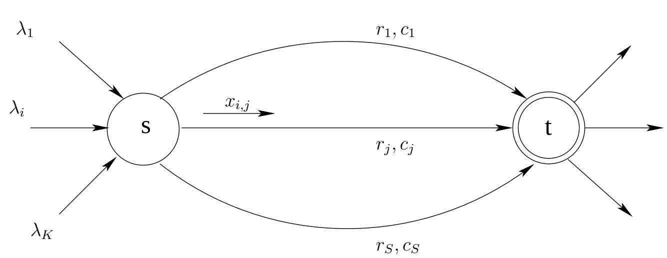

Figure 1: Un routage via des liens parallèles.

Le lien $j \in \mathcal{S}$ a la capacité r_j et un coût de traitement c_j par unité de temps est à payer pour chaque paquet envoyé sur ce lien.

$\mathbf{x}_i = (x_{i,j})_{j \in \mathcal{S}} \in \mathcal{X}_i$ est la stratégie de routage du joueur i, où $x_{i,j}$ est la quantité de trafic qu'il envoie sur la lien j, avec $0 \leq x_{i,j} < r_j$ pour toutes $j \in \mathcal{S}$, et $\sum_{j \in \mathcal{S}} x_{i,j} = \lambda_i$. $\mathcal{X}_i$ désigne l'ensemble des stratégies de routage pour le joueur i. Un profil de stratégies est un vecteur $\mathbf{x} = (\mathbf{x}_i)_{i \in \mathcal{C}}$ de l'espace produit des stratégies $\mathcal{X} = \bigotimes_{i \in \mathcal{C}} \mathcal{X}_i$ tel que $\sum_{i \in \mathcal{C}} x_{i,j} < r_j$ pour tout $j \in \mathcal{S}$. Il est supposé que $\sum_{i \in \mathcal{C}} \lambda_i < \sum_{j \in \mathcal{S}} r_j$. Le vecteur $\mathbf{x}_{-i}$ donne les stratégies de tous les joueurs sauf i.

Compte tenu des stratégies des autres, l'agent i vise à minimiser son coût total sous des contraintes de conservation du flux et de capacité. Le problème d'optimisation résolu par l'agent i dépend des décisions de routage des autres agents et est formulé comme suit :

$$\text{minimize } T_i(\mathbf{z}_i, \mathbf{x}_{-i}) = \sum_{j \in \mathcal{S}} \frac{c_j}{r_j} z_{i,j} \, \phi(\rho_j) \tag{BR-i}$$

soumis à

$$\mathbf{z}_i \in \mathcal{X}_i, \tag{1}$$

$$y_j = z_{i,j} + \sum_{k \neq i} x_{k,j}, \quad \forall j \in \mathcal{S}, \tag{2}$$

$$\rho_j = y_j / r_j, \quad \forall j \in \mathcal{S}, \tag{3}$$

$$\rho_j < 1, \quad \forall j \in \mathcal{S}, \tag{4}$$

où $\phi(\rho)$ est le coût d'un lien dont le taux d'utilisatin est ρ. Dans les réseaux de transport ou de communication, ϕ modélise le délai sur une route ou sur un lien.

Nous faisons les postulats suivants sur la fonction ϕ:

(A_1) $\phi : [0, 1) \to [0, \infty)$,

(A_2) $\lim_{\rho \to 1^-} \phi(\rho) = +\infty$,

(A_3) ϕ est continue, strictement croissante, convexe, et est deux fois continûment différentiable.

L'équilibre de Nash

$\mathbf{x}^* \in \mathcal{X}$ est un point d'équilibre de Nash (NEP) si $\mathbf{x}_i^*$ est une solution optimale du problème (BR-i) pour tous les joueurs $i \in \mathcal{C}$, si :

$$\mathbf{x}_i^* = \arg\min_{\mathbf{z} \in \mathcal{X}_i} T_i(\mathbf{z}, \mathbf{x}_{-i}^*), \quad \forall i \in \mathcal{C},$$

où $\mathbf{x}_{-i}^*$ est le vecteur des stratégies à l'équilibre de tous les joueurs autres que le joueur i.

Il résulte de nos hypothèses sur la fonction ϕ, que les fonctions de coût des liens sont un cas particulier des fonctions de type-B, telles que définies dans (Orda et al., 1993).

Comme montré dans le Théorème 2.1 de cette référence, cela implique l'existence d'un équilibre de Nash unique pour notre jeu de routage.

2.2 Dynamique de meilleure réponse

La meilleure réponse du joueur est définie comme sa stratégie optimale étant donnée la stratégie des autres. Soit $x^{(u)} : \mathcal{X} \to \mathcal{X}$, définie comme

$$x^{(u)}(\mathbf{x}) = \left(\arg\min_{\mathbf{z} \in \mathcal{X}_u} T_u(\mathbf{z}, \mathbf{x}_{-u}), \mathbf{x}_{-u} \right), \tag{5}$$

la meilleure réponse du joueur u à la stratégie $\mathbf{x}_{-u}$ des autres joueurs.

La *dynamique de meilleure réponse* consiste ensuite à ce que les joueurs jouent dans un certain ordre en optimisant leur propre stratégie en réponse à la stratégie des autres la plus récente connue.

Définissons arbitrairement un ordre pour le premier round, 1, 2, …, K, et supposons qu'il soit le même dans chaque round suivant. Définissons $\hat{x}^{(1)} : \mathcal{X} \to \mathcal{X}$ comme

$$\hat{x}^{(1)}(\mathbf{x}) = x^{(K)} \circ x^{(K-1)} \circ \ldots \circ x^{(1)}(\mathbf{x}), \tag{6}$$

le point atteint à partir du point $\mathbf{x}$ après un tour du jeu. Ensuite récursivement

$$\hat{x}^{(n)}(\mathbf{x}) = \hat{x}^{(1)} \circ \hat{x}^{(n-1)}(\mathbf{x}), \tag{7}$$

est le point atteint après n rounds.

La dynamique de meilleure réponse est représentée par la séquence $\{\hat{x}^{(n)}(\mathbf{x}_0)\}_{n \geq 1}$, où $\mathbf{x}_0$ est le profil de stratégies initial. Un équilibre de Nash a la propriété que la stratégie de chaque joueur est une meilleure réponse aux stratégies des autres joueurs. Par conséquent, si $\mathbf{x}_0$ est un équilibre alors la séquence restera au point $\mathbf{x}_0$. La principale question à laquelle nous cherchons à répondre est la suivante: est-ce que la dynamique de meilleure réponse pour le jeu de routage converge depuis n'importe quel point initial ?

2.3 Résultats de convergence connexes

Pour le jeu asymétrique (lorsque les volumes de trafic contrôlés par les agents sont différents), les résultats de convergence disponibles sont très peu nombreux. Dans (Orda et al., 1993), pour le jeu de routage à deux joueurs sur deux liens parallèles, la convergence vers l'équilibre de Nash unique a été prouvée en se basant sur la propriété de monotonie du flot d'un joueur sur chaque lien. Les auteurs soulignent eux-mêmes que ce type de preuve ne peut être utilisé pour des cas plus généraux. Altman *et al.* étudient également le cas de deux liens (Altman et al., 2001). En supposant que les fonctions de latence pour les liens sont linéaires, ils prouvent la convergence de la dynamique de meilleure réponse séquentielle pour un nombre arbitraire de joueurs en utilisant la condition de contraction. Plus récemment Mertzios, 2009 a prouvé la convergence pour le jeu de routage à deux joueurs et pour la grande classe des fonctions de latence de liens introduites dans (Orda et al., 1993). La preuve se fait sur un argument de type potentiel, à savoir, montrer que la quantité de flux qui est réalloué dans le réseau dans chaque étape est strictement décroissante. Cependant, cet argument ne paraît pas s'étendre facilement à plus de deux joueurs.

2.4 L'approche basée sur le rayon spectral non-linéaire

Nous proposons une approche différente pour l'étude de la convergence de la dynamique de meilleure réponse. L'idée clé pour prouver la convergence est d'étudier les matrices jacobiennes des fonctions de meilleure réponse, et d'analyser la faon dont les produits longs de ces matrices grandissent en fonction du nombre de mises à jour de meilleur réponse.

Une méthode habituelle pour prouver la convergence des itérations de l'opérateur $\hat{x}^{(1)} : \mathcal{X} \to \mathcal{X}$ est de montrer que cet opérateur est une contraction. La condition de contraction nécessite de trouver une norme appropriée dans laquelle la distance entre les itérations de la fonction à partir de deux points différents diminue avec chaque itération. Trouver une telle norme peut être très complexe. Notre idée, pour la fonction de meilleure réponse, consiste à observer qu'il suffit de trouver une norme dans laquelle la distance diminue asymptotiquement et non pas avec chaque itération. Cette condition plus faible peut être formalisée en se basant sur la notion de *rayon spectral non-linéaire*.

Pour une fonction $f : \mathcal{X} \to \mathcal{X}$, définissons l'ensemble des matrices jacobiennes

$$\mathcal{J}(f) = \{Df(x) : f \text{ est différentiable en } x\} \tag{8}$$

Définition 1. *Le rayon spectral non-linéaire d'une fonction $f : \mathcal{X} \to \mathcal{X}$ est défini comme (Mak et al., 2007):*

$$\bar{\rho}(f) = \limsup_{n\to\infty} \sup_{A_i \in \mathcal{J}(f)} \left\| \prod_{i=1}^{n} A_i \right\|^{1/n}.$$

Le rayon spectral non-linéaire correspond au rayon spectral joint $\hat{\rho}(\mathcal{M})$ de l'ensemble $\mathcal{M}$ de matrices, où $\mathcal{M} = \mathcal{J}(f)$.

Pour les opérateurs non-linéaires, le critère de convergence suivant a été formulé.

Théorème 1 (Mak et al., 2007, Théorème 1)**.** *Si $f : \mathcal{X} \to \mathcal{X}$ est Lipschitz-continue et a un rayon spectral non-linéaire strictement inférieur à 1, alors les itérations de f sont globalement asymptotiquement stables. De plus, la rapport de décroissance exponentielle, r, satisfait $0 < r \leq -\log(\bar{\rho}(f))$.*

Ainsi, au lieu d'exiger que l'opérateur de meilleure réponse soit une contraction, on peut montrer la convergence de la dynamique de meilleure réponse en montrant que:

1. $\hat{x}^{(1)}$ est Lipschitz-continu; et

2. $\bar{\rho}(\hat{x}^{(1)}) < 1$.

Nous avons montré que la fonction de meilleure réponse $\hat{x}^{(1)}$ satisfait à la première condition, à savoir être Lipschitz continue. Pour la deuxième condition, nous avons établi la structure des matrices jacobiennes de la fonction $\hat{x}^{(1)}$.

Structure des matrices jacobiennes

La matrice jacobienne de $\hat{x}^{(1)}$ est le produit de matrices jacobiennes des meilleures réponses des joueurs individuels. Pour un joueur u et un point $\mathbf{x} \in \mathcal{X}$ en lequel $x^{(u)}$ est différentiable, la matrice jacobienne de cette fonction est la matrice bloc

$$\mathrm{D}x^{(u)}(\mathbf{x}) = \begin{pmatrix} \frac{\partial x_1^{(u)}}{\partial \mathbf{x}_1}(\mathbf{x}) & \dots & \frac{\partial x_1^{(u)}}{\partial \mathbf{x}_K}(\mathbf{x}) \\ \vdots & & \vdots \\ \frac{\partial x_K^{(u)}}{\partial \mathbf{x}_1}(\mathbf{x}) & \dots & \frac{\partial x_K^{(u)}}{\partial \mathbf{x}_K}(\mathbf{x}) \end{pmatrix},$$

où (i,j)-bloc $\frac{\partial x_i^{(u)}}{\partial \mathbf{x}_j}(\mathbf{x})$ mesure la sensibilité de la stratégie du joueur i obtenue après la meilleure réponse du joueur u par rapport à un changement dans la stratégie du joueur j.

Définissons $\mathcal{S}_u(\mathbf{x}) = \{j \in \mathcal{S} : x_{u,j}^{(u)}(\mathbf{x}) > 0\}$ comme l'ensemble des liens utilisés par le joueur u dans sa meilleure réponse aux stratégies $\mathbf{x}_{-u}$ des autres joueurs. Basée sur la forme particulière de la fonction de coût T_u dans (BR-i), la contrainte de la conservation du flux, les conditions d'optimalité KKT, et les hypothèses sur la fonction $\phi(\cdot)$, nous avons établi la structure spécifique de la matrice jacobienne de la fonction $x^{(u)}$.

Théorème 2. *La matrice jacobienne de la fonction de meilleure réponse $x^{(u)}$ du joueur $u \in \mathcal{C}$ a la forme suivante*

$$\mathrm{D}x^{(u)}(\mathbf{x}) = \begin{pmatrix} I & \dots & 0 & \dots & 0 \\ \vdots & \ddots & & & \vdots \\ M_u(\mathbf{x}) & \dots & 0 & \dots & M_u(\mathbf{x}) \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & 0 & \dots & I \end{pmatrix},$$

et $M_u(\mathbf{x}) = \Psi_u(\Gamma_u B - I)\Theta_u$, où

- *B est la matrice $S \times S$ avec 1 dans chaque entrée, à savoir, $B = \mathbf{1}^T\mathbf{1}$,*

- $\Gamma_u = diag(\boldsymbol{\gamma}^{(u)})$ *et* $\Theta_u = diag(\boldsymbol{\theta}^{(u)})$ *où les vecteurs* $\boldsymbol{\gamma}^{(u)} = (\gamma_1^{(u)}, \cdots, \gamma_S^{(u)})$ *et* $\boldsymbol{\theta}^{(u)} = (\theta_1^{(u)}, \cdots, \theta_S^{(u)})$ *sont tels que*
  $0 < \gamma_i^{(u)} < 1, \forall i \in \mathcal{S}$, *et* $\sum_{i=1}^{S} \gamma_i^{(u)} = 1$, *et*
  $\frac{1}{2} \leq \theta_i^{(u)} < 1$,
- Ψ_u *une matrice diagonale positive telle que* $\Psi_{i,i}^{(u)} = 1$ *if* $i \in \mathcal{S}_u(\mathbf{x})$, *et* $\Psi_{i,i}^{(u)} = 0$ *sinon.*

Corollaire 1. *La matrice jacobienne de* $\hat{x}^{(1)}$ *a la forme*

$$\mathrm{D}\hat{x}^{(1)}(\mathbf{x}) = \prod_{u=K}^{1} \mathrm{D}x^{(u)}(\mathbf{x}).$$

2.5 Convergence de la dynamique de meilleure réponse

Nous avons formulé la conjecture suivante sur le rayon spectral non-linéaire de $\hat{x}^{(1)}$.

Conjecture 1. *Pour* K *and* S *fixés, tout ensemble* $\hat{\mathcal{J}}$ *de matrices ayant la forme donnée dans le Corollaire 1 a un rayon spectral joint strictement inférieure à* 1.

Sur les nombreuses expériences numériques que nous avons menées, la conjecture ci-dessus semble effectivement vraie.

Le principal résultat de cette étude est:

Théorème 3. *Si la conjecture 1 est vraie, alors la dynamique de meilleure réponse* (7) *pour le jeu de routage* (BR-i) *converge vers l'équilibre de Nash unique du jeu.*

On a ainsi obtenu une condition suffisante purement structurelle qui permet de réduire l'analyse de la convergence de la dynamique séquentielle de meilleure réponse à l'analyse du rayon spectral joint de certaines matrices. Nous avons pu montrer que la conjecture est valide et donc que le théorème s'applique dans deux cas non triviaux:
(a) les jeux de routage avec deux joueurs et pour un nombre arbitraire de liens;
(b) les jeux de routage avec K joueurs et un nombre arbitraire de liens pour des fonctions linéaires de latence de liens, ϕ.

3 Incitations à la collaboration des nœuds mobiles dans les DTN

Un autre champ d'application dans lequel on a des interactions concurrentielles est celui des réseaux tolérants aux délais (DTN). Les DTN ont été proposés pour permettre la communication dans des environnements où le chemin de bout-en-bout entre une source et une destination n'est disponible que sporadiquement. Les DTN utilisent l'approche *store-carry-forward*, dans laquelle les paquets sont transmis par la source aux noeuds mobiles qu'elle rencontre. Ces derniers servent de relais pour la transmission du message. Ils peuvent stocker le message et le transporter jusqu'à ce qu'une opportunité de communication avec la destination ou un autre relai apparaîsse.

Afin de minimiser le temps de livraison du message dans les réseaux mobiles, les algorithmes de routage utilisés dans les DTN impliquent généralement un routage multi-copies, dans lesquel le message est délivré si l'un des nœuds relais possédant une copie rencontre la destination. Dans cette thèse, nous nous focalisons sur le schéma de routage à deux sauts, qui est connu pour fournir un bon compromis entre le temps de livraison du message et la consommation de ressources (Al-Hanbali et al., 2008). Dans un tel schéma, la transmission d'une copie du message est autorisée en au plus deux étapes : un nœud relais qui a reu le message de la source ne peut pas le transmettre à un autre nœud relais, mais doit le délivrer directement à la destination s'il la rencontre.

Dans la pratique, les DTN sont composés de dispositifs mobiles comme des smartphones, des tablettes ou d'autres dispositifs mobiles disposant de multiples interfaces sans fil. Lorsqu'un nœud mobile doit économiser son énergie, ou en raison d'autres objectifs individuels, il peut ne pas être disposé à servir de relais pour la transmission de données entre d'autres nœuds.

Le comportement égoïste des nœuds d'un DTN et la nature décentralisée de leur prise de décision nécessite des mécanismes d'incitation appropriés pour que les nœuds acceptent de servir de relais, au bénéfice du réseau dans son ensemble.

3.1 Travaux connexes sur les mécanismes d'incitation pour DTNs

Dans la littérature sur les DTN (El-Azouzi et al., 2012; Zhang et al., 2007), plusieurs mécanismes d'incitation ont été récemment proposés. Shevade et al., 2008 utilise la technique Tit-for-Tat (TFT) pour concevoir un protocole de routage incitatif qui permet aux nœuds égoïstes du DTN de maximiser leurs utilités individuelles tout en se conformant aux contraintes du TFT. Mobicent (Chen et al., 2010) est un système d'incitation basé sur le crédit, qui intègre des crédits et des techniques cryptographiques pour résoudre les problèmes d'insertion d'arêtes et d'attaques cachées d'arêtes parmi des nœuds. PI (Lu et al., 2010) attache une incitation sur le paquet envoyé pour stimuler les nœuds égoïstes à coopérer dans la livraison du message. SMART (Zhu et al., 2009) est un système d'incitation sécurisé multicouches à base de crédits pour DTNs. Dans SMART, des monnaies en couches sont utilisées pour fournir des incitations aux nœuds égoïstes du DTN à transmettre un paquet. MobiGame (Wei et al., 2011) est un système d'incitation pour DTN qui est centré utilisateur et basé sur la réputation. En outre, Li et al., 2010 propose un routage égoïste dans les DTN, où un nœud exploite la volonté sociale pour déterminer si oui ou non il doit relayer des paquets pour les autres. Ning et al., 2011 formule la communication entre nœuds comme un jeu coopératif de deux personnes dans un système d'incitations basé sur le crédit pour promouvoir la collaboration. RELICS (Uddin et al., 2010) est un autre mécanisme d'incitation coopératif basé sur la réduction de la consommation d'énergie pour les DTN égoïstes, dans lequel une mesure de classement a été définie pour caractériser le comportement de transit d'un nœud. Dans (Wang et al., 2012), les auteurs ont proposé un système de diffusion qui encourage les nœuds à coopérer, et qui choisit les chemins de livraison pouvant atteindre un maximum de nœuds avec un minimum de transmissions. Un aspect fondamental qui est généralement ignoré dans la littérature sur les DTN est la difficulté à acquitter la réception du message, l'envoi du message d'acquittement pouvant nécessiter un délai important. En fait, l'échange de récompenses entre les nœuds mobiles ne devrait pas exiger de messages de retour. Le mécanisme que nous proposons prend en compte cette difficulté à acquitter la réception du message.

Nous étudions le routage à deux sauts dans les DTN et nous introduisons un mécanisme de récompense qui incite les nœuds à servir de relais. Dans ce mécanisme, un relais reoit une récompense si et seulement s'il est le premier à livrer le message à la destination. Dans notre mécanisme, nous évitons l'utilisation d'un message de retour avertissant la source que le message a été livré avec succès.

3.2 Modèle du système

Nous considérons un réseau sans fil avec un nœud source, un nœud destination et N nœuds relais. Nous supposons que la source et la destination sont fixes et ne sont pas à portée radio l'une de l'autre, et que d'autres nœuds sont en mouvement selon un modèle de mobilité donné.

Les mouvements des nœuds relais sont caractérisés par les processus de contact avec la source et la destination. Notre principale hypoyhèse est que les temps inter-contacts entre un relais et la source (resp. la destination) sont des variables aléatoires (*i.i.d.*) de premier et second moments finis. Soit T_s (resp. T_d) le temps aléatoire entre deux contacts consécutifs entre un relais et la source (resp. destination). Nous supposons que T_s et T_d sont indépendants. Plutôt que les temps inter-contacts eux-mêmes nous considérons des temps inter-contacts résiduels. Ainsi, le temp inter-contact résiduel $\tilde{T}_s$ est le temps aléatoire entre l'instant auquel le message est généré et l'instant auquel le relais rencontrera la source. $\tilde{T}_d$ est le temps aléatoire entre l'instant auquel un relais donné reoit le message de la source et l'instant auquel le relais rejoindra la destination.

A l'instant 0, la source génère un message pour la destination. La source veut que ce message soit livré à la destination aussi rapidement que possible via des nœuds relais. La source propose à chaque relais rencontré une récompense pour livrer le message[1]. Une

[1] Notons que puisque la source n'est pas informée quand le message atteint la destination, elle peut encore proposer le message à un relais même si le message a déjà été livré par un autre relais.

hypothèse importante que nous faisons est que les relais ne cherchent pas à faire de profit : un relais accepte le message à condition que la récompense promise par la source compense l'espérance du coût pour délivrer le message à la destination, tel qu'*estimé par le relais* lorsqu'il rencontre la source.

L'espérance du coût a plusieurs composantes. Un relais qui accepte le message de la source encourt toujours un *coût de réception* C_r. C'est un coût d'énergie fixe pour recevoir le message de la source. Il y a également un *coût de stockage* C_s par unité de temps relatif au stockage du message dans le buffer du relais. Une fois que le relais rencontre la destination, il peut livrer le message. Ceci génère un *coût de transmission* C_d qui est un coût d'énergie fixe pour transmettre le message à la destination. Ce coût est encouru, si et seulement si le relais est le premier à livrer le message à la destination, auquel cas le relais obtient la récompense. Si au contraire, le message a déjà été livré, le relais ne reoit pas de récompense mais il n'aura pas de coût de transmission à payer.

3.3 Le rôle de l'information

La récompense moyenne à payer par la source dépend de l'information qu'elle donne aux relais. Il existe plusieurs stratégies possibles pour la source. Nous distinguons les *stratégies statiques* et les *stratégies dynamiques*. Dans les stratégies statiques, l'information donnée aux relais est fixe et ne dépend pas des moments auxquels la source rencontre les relais. Nous considérons trois stratégies statiques :

- *information complète* (en anglais: full - F): chaque relais est informé par la source du nombre d'autres relais qui ont déjà reu le message, et à quels moments,
- *information partielle* (en anglais: partial - P): chaque relais est informé par la source de combien de copies du messages sont en circulation, mais la source ne révèle pas l'âge de ces copies,
- *pas d'information* (en anglais: no information - N): la source ne dit rien aux relais. Chaque relai ne connaît que le moment auquel il rencontre la source.

Dans les stratégies dynamiques, la source adapte les informations qu'elle transmet à la volée en fonction des instants auxquels elle rencontre les relais. Dans une telle stratégie, la décision de donner une information complète, une information partielle ou pas d'information du tout à un relais dépend des temps de contact avec les relais précédents. Nous adoptons le point de vue de la source et étudions la stratégie qu'elle devrait suivre afin de minimiser le prix à payer pour délivrer un message.

3.4 La probabilité de succès estimée

Soit S_i, $i = 1, \ldots, N$, l'instant aléatoire où la source rencontre le relais i. Nous désignons par $\mathbf{S}$ le vecteur $(S_1, \ldots, S_N)$. Nous désignons aussi par $\mathbf{s}$ le vecteur $(s_1, s_2, \ldots, s_N)$ des temps de contact des relais avec la source.

Définissons $p_i(\mathbf{s})$ comme la (vraie) probabilité de succès du relais i étant donné le vecteur $\mathbf{s}$ des instants de contact. C'est la probabilité pour ce relais d'être le premier à livrer le message. Soit $p_i^{(k)}(\mathbf{s})$ la probabilité de succès estimée sous le mode $k \in \{F, P, N\}$ par le relais i quand il rencontre la source.

3.5 Récompenses promises par la source aux relais individuels

Définissons $V_i^{(k)}(\mathbf{s})$ comme le coût net pour le relais i sous le mode k, et soit $R_i^{(k)}(\mathbf{s})$ la récompense demandée par ce relais à la source dans ce mode. La récompense $R_i^{(k)}(\mathbf{s})$ proposée au relais i doit compenser l'espérance du coût $\mathbb{E}[V_i^{(k)}(\mathbf{s})]$, qui est donnée par

$$\mathbb{E}[V_i^{(k)}(\mathbf{s})] = C_r + C_s\mathbb{E}[\tilde{T}_d] + [C_d - R_i^{(k)}(\mathbf{s})]p_i^{(k)}(\mathbf{s}). \tag{9}$$

Le premier terme dans ce coût net est le coût de réception, qui est toujours engagé. Le second terme représente le coût de stockage moyen. Le dernier terme est le coût de transmission du message vers la destination qui donne alors la récompense au relais. Ce terme entre en jeu seulement si le relais i est le premier à atteindre la destination, ce qui explique le facteur $p_i^{(k)}(\mathbf{s})$.

Le relais i acceptera le message à condition que la récompense proposée compense son coût moyen de livraison du message, c'est à dire si $R_i^{(k)}(\mathbf{s})$ est tel que $\mathbb{E}[V_i^{(k)}(\mathbf{s})] \leq 0$. La récompense minimale que la source doit promettre au relais i est donc

$$\begin{aligned} R_i^{(k)}(\mathbf{s}) &= C_d + \left(C_r + C_s\mathbb{E}[\tilde{T}_d]\right)\frac{1}{p_i^{(k)}(\mathbf{s})} \\ &=: C_1 + C_2\frac{1}{p_i^{(k)}(\mathbf{s})}. \end{aligned} \tag{10}$$

Notons que la récompense demandée par le relais i dépend de l'information donnée par la source uniquement au travers de la probabilité de succès estimée $p_i^{(k)}$.

Compte tenu de $S_1 = s_1, \cdots, S_N = s_N$, la récompense moyenne payée par la source dans le mode k est

$$\overline{R}^{(k)}(\mathbf{s}) = \sum_{i=1}^{N} p_i(\mathbf{s})R_i^{(k)}(\mathbf{s}) = C_1 + C_2\sum_{i=1}^{N}\frac{p_i(\mathbf{s})}{p_i^{(k)}(\mathbf{s})}. \tag{11}$$

3.6 Récompense moyenne payée par la source dans une stratégie statique

La récompense moyenne payée par la source lorsque l'espérance est prise sur tous les temps de rencontre, peut être considérée comme la récompense moyenne à long terme par message que la source devra payer si elle envoie un grand nombre de messages (et en supposant que la génération des messages se produit à une échelle de temps bien supérieure à celle du processus de contact).

La récompense attendue payée par la source dans le mode k peut être obtenue en déconditionnant (11) sur $S_1, \cdots, S_N$,

$$\overline{R}^{(k)} = \int_{\mathbf{s}} \overline{R}^{(k)}(\mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s}, \tag{12}$$

où $f_{\mathbf{S}}(\mathbf{s})$ est la distribution conjointe de $S_1, \cdots, S_N$. Nous avons prouvé le théorème suivant :

Théorème 4. *La récompense moyenne à payer par la source dans le mode $k \in \{F, P, N\}$ est*

$$\overline{R}^{(k)} = C_1 + NC_2. \tag{13}$$

Cela montre que si la source n'adapte pas l'information qu'elle donne, la récompense moyenne qu'elle devra payer reste la même indépendamment de l'information qu'elle transmet. Nous notons également que la récompense moyenne augmente linéairement avec le nombre de relais.

3.7 La stratégie adaptive

La source peut-elle faire mieux en adaptant le type d'information qu'elle donne à un relais en fonction de l'instant auquel elle le rencontre ? Nous avons montré que la source peut effectivement réduire la récompense attendue qu'elle paie si elle peut adapter le type d'information dynamiquement.

Une hypothèse clé que nous faisons dans l'analyse de la stratégie adaptative est que les relais sont naïfs: ils ne réagissent pas au fait que la source adapte sa stratégie. Un relais va calculer sa probabilité de succès uniquement en fonction de l'instant auquel il rencontre la source et de l'information supplémentaire qu'elle lui donne, s'il y en a une.

Stratégie adaptative par rapport statique

Soit $\overline{R}^{(A)}$ la récompense moyenne payée par la source quand elle utilise une stratégie adaptative. Quand elle rencontre un relais, la source peut calculer la récompense qu'elle devrait promettre à ce relais dans chaque mode, et ensuite choisir le mode minimisant la récompense à promettre à ce relais. Autrement dit,

$$\overline{R}^{(A)} = \int_{\mathbf{s}} \left(\sum_{n=1}^{N} p_n(\mathbf{s}) \min_k \left(R_n^{(k)} \right) \right) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s}. \tag{14}$$

Etant donnée la définition de la stratégie adaptative, la récompense moyenne de cette stratégie ne peut être supérieure à celle d'une stratégie statique, qui donne donc une borne supérieure. En outre, la source doit payer au minimum $C_1 + C_2$ parce que c'est le coût moyen quand il y a un seul relai, et on a ainsi une borne inférieure. Il en résulte que

Proposition 1. $C_1 + C_2 \leq \overline{R}^{(A)} \leq \overline{R}^{(k)} = C_1 + NC_2$.

Corollaire 2. $\frac{\overline{R}^{(A)}}{\overline{R}^{(k)}} \geq \frac{C_1+C_2}{C_1+NC_2} \geq \frac{1}{N}$.

Ainsi, en utilisant une stratégie adaptative la source peut réduire ses dépenses d'au plus un facteur $1/N$.

Bien que l'expression analytique exacte de la récompense moyenne de la politique adaptative soit difficile à obtenir, nous avons pu en constater numériquement tout l'intérêt de cette stratégie dans le cas de temps inter-contact exponentiellement distribués. En effet, en supposant des temps inter-contact exponentiels entre un relais et la source (destination resp.) de taux λ (μ resp.), nous avons obtenu des expressions explicites pour les probabilités de succès estimées par les relais dans chaque mode. Pour minimiser la récompense promise à un relais, la source choisit le mode d'information maximisant la probabilité de succès estimée par le relais.

Sur la figure 2, $\overline{R}^{(A)}$ est tracée en une fonction de λ pour $N = 5$, $\mu = 1$, $C_1 = 1$, et $C_2 = 5$. Nous observons que $\overline{R}^{(A)}$ croît avec λ et devient proche de $\overline{R}^{(F)}$ quand $\lambda \to \infty$. D'autre part, pour de petites valeurs de λ, $\overline{R}^{(A)}$ est proche de la récompense minimale $C_1 + C_2$. Il semble que $\overline{R}^{(A)}$ ait la forme $(C_1 + C_2) + C_2(1 - e^{-\lambda\gamma})$, pour une certaine constante γ, mais nous n'avons pas pu prouver ce résultat.

Nous avons également donné quelques propriétés structurelles de la stratégie adaptative dans le cas de $N = 2$ relais.

Deux relais, densité décroissante des temps inter-contact

Supposons que les densités de probabilités des temps inter-contacts résiduels, $\tilde{f}_s$ et $\tilde{f}_d$, sont des fonctions décroissantes.

Pour établir la structure de la stratégie adaptative, il faut déterminer quel mode d'information a la récompense la plus basse à un instant donné. La récompense d'un mode donné dépend à son tour de la probabilité du succès estimée par le relais basé sur l'information communiquée par la source (voir (10)).

Notre premier résultat montre qu'il est toujours bénéfique pour la source de donner l'information au premier relais quel que soit s_1.

Proposition 2.

$$R_1^{(F)}(\mathbf{s}) = R_1^{(P)}(\mathbf{s}) \leq R_1^{(N)}(\mathbf{s}) \tag{15}$$

Le résultat suivant concerne la récompense que la source devrait proposer au deuxième relais.

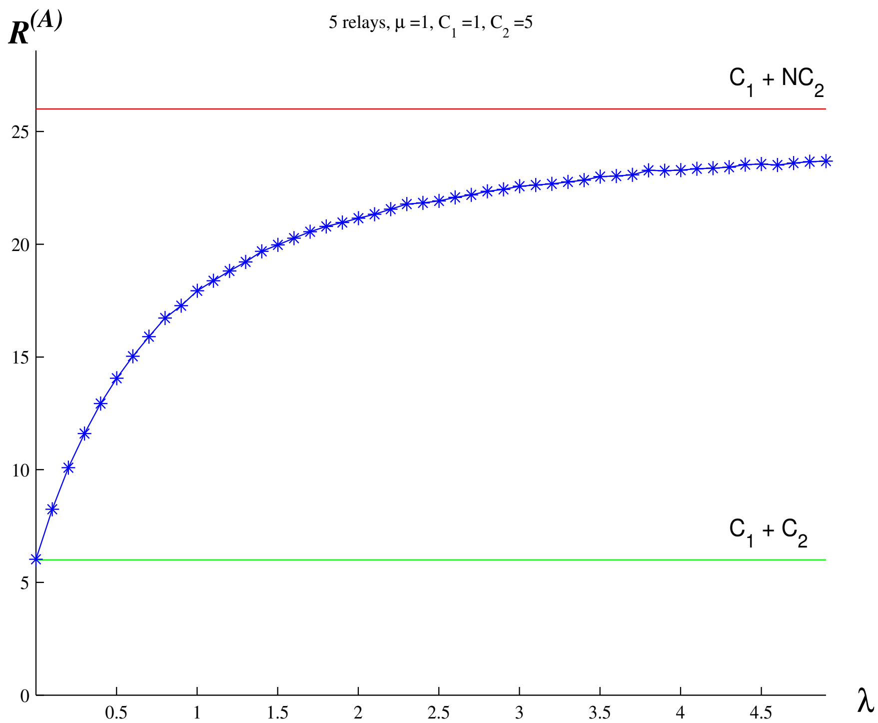

Figure 2: Récompense moyenne payée par la source dans la stratégie adaptive.

Proposition 3.

$$R_2^{(N)}(\mathbf{s}) \leq R_2^{(P)}(\mathbf{s}). \tag{16}$$

La proposition 3 dit qu'entre le choix d'informer un relais qu'il est second, et celui de ne pas donner cette information, il est préférable pour la source de ne pas donner cette information.

Le Théorème 5 compare le mode sans information avec celui correspondant à l'information complète.

Définissons la différence entre les probabilités de succès en fonction de s_1 and s_2,

$$g(s_1, s_2) = p_2^{(N)}(s_1, s_2) - p_2^{(F)}(s_1, s_2), \tag{17}$$

Pour la source, il sera préférable de donner l'information quand $g(s_1, s_2) < 0$.

Théorème 5. *Il existe* $0 \leq \theta_1 < \infty$ *tel que*

1. *si* $0 \leq s_1 < \theta_1$*, alors* $g(s_1, s_2) \geq 0$*,* $\forall s_2 \geq s_1$*;*
2. *si* $\theta_1 < s_1 < \infty$*, alors*

(a) $g(s_1, s_2) < 0$, $\forall s_2 \in [s_1, s_1 + \omega(s_1))$,

(b) $g(s_1, s_2) > 0$, $\forall s_2 \in (s_1 + \omega(s_1), \infty)$,

où θ_1 *est une solution de l'équation* $g(s_1, s_1) = 0$ *et* $\omega(s_1)$ *est une solution de* $g(s_1, s_1+v) = 0$ *en ce qui concerne* v *quand* $g(s_1, s_1) < 0$.

Donc, si la source a rencontré le premier relai à $s_1 \leq \theta_1$, alors, indépendamment de l'instant auquel elle rencontre le second relai, elle ne doit pas donner l'information à ce deuxième relai. D'autre part, si $s_1 \geq \theta_1$, alors la stratégie de la source doit être du type seuil: si elle rencontre le second relais avant $s_1 + \omega(s_1)$, alors elle devrait donner l'information complète, sinon elle ne devrait donner aucune information.

4 Récompense fixe pour inciter les noeuds mobiles à la coopération

Le mécanisme d'incitation introduit dans le modèle précédent assure une totale coopération des nœuds mobiles dans la transmission du message en promettant de couvrir le coût moyen estimé par chaque nœud relai. Cependant, un mécanisme d'incitation qui peut compenser toutes les dépenses d'un relais peut être très coûteux pour la source. Nous avons développé une stratégie adaptative pour la source qui lui permet de réduire le coût à payer pour transmettre un message. Néanmoins, la récompense qu'elle aura à payer peut être encore très élevée, à cause des paramètres de mobilité des nœuds et de la consommation d'énergie des relais. Il peut alors être intéressant de fixer une limite à la récompense que la source peut payer. Pour construire une récompense optimale, nous avons besoin de savoir comment les nœuds mobiles sont disposés à participer à la transmission du message en réponse à la récompense fixe proposée par la source, et ce qui pourrait être les stratégies de meilleure réponse des relais.

4.1 Description du problème

Nous considérons un réseau sans fil avec un nœud source fixe, un nœud destination fixe et N relais mobiles. Nous supposons un schéma de routage à deux sauts pour le DTN. Un relais qui accepte le message a un coût du réception fixe C_r, ensuite un coût de stockage C_s par unité de temps encourue pour stocker le message, et un coût fixe de livraison C_d de transmission du message à la destination.

Nous étudions un processus de décision à temps discret pour les relais. La source génère le message à l'instant 0 avec une date limite à l'instant $\tau + 1$. Les contacts des relais avec la source et la destination sont supposés intervenir à des instants i.i.d., p (resp. q) étant la probabilité qu'un relai rencontre la destination (resp. source) à l'instant suivant.

Une fois qu'un message est généré, la source le propose à chaque relais qu'elle rencontre. Lorsqu'un relais rencontre la source, il peut décider d'accepter le message ou le rejeter.

Une fois que le relais accepte le message, il peut choisir de le garder ou de le l'abandonner à chaque pas de temps ultérieur. En tant qu'incitation à la coopération, la source offre une récompense fixe, R, à chaque relai rencontré, mais la récompense est payée seulement au premier relais qui délivre le message. Un relai qui rencontre la source n'est pas informé de l'existence d'autres copies du message.

L'état de chaque relai prend l'une des cinq valeurs possibles:

Valeur	Signification	Ensemble d'actions
0	le relai n'a pas le paquet	$\emptyset$
m_s	le relai rencontre la source	$(accepter, rejeter)$
1	le relai a le paquet	$(jeter, garder)$
m_d	le relai rencontre la destination	$\emptyset$
2	le relai quitte le jeu	$\emptyset$

Chaque relai prend ses décisions afin de minimiser son coût moyen. Le coût de chaque relai dépend de ses propres actions ainsi que de celles des autres relais. Cette interaction stratégique entre les relais s'inscrit dans le cadre des jeux stochastiques introduits par Shapley, 1953. Dans notre modèle, chaque relais est conscient de son propre état, mais ne connaît pas ceux des autres. En outre, il ne sait pas si le paquet a déjà été livré à la destination ou non. Nous formulons donc ce jeu comme un jeu stochastique à information partielle (Goush et al., 2004).

4.2 Le cas d'un seul joueur

Pour le cas d'un seul joueur, nous avons d'abord obtenu la condition nécessaire suivante pour qu'un relai ayant accepté le message le conserve jusqu'à ce qu'il rencontre la destination

$$R > \frac{C_s}{\alpha p} + C_d, \tag{18}$$

où α est le facteur d'actualisation ($0 \leq \alpha < 1$). En fait, cette condition nous indique la valeur minimale que doit avoir la récompense pour garantir que le relai ne va pas jeter le message.

Ensuite, la stratégie du relai en ce qui concerne le fait d'accepter ou non le message de la source est du type seuil. A savoir, il existe un seuil t^* tel que le relais acceptera le message s'il rencontre la source avant t^*, et il le rejettera s'il la rencontre après t^*. Ce seuil est défini par :

$$t^* = \tau - 1 - \frac{\ln\left(1 + \frac{C_r(1-\overline{p}\alpha)}{C_s+\alpha p(C_d-R)}\right)}{\ln(\overline{p}\alpha)}. \tag{19}$$

4.3 Jeu avec deux joueurs

Etant donné la compléxité du problème, nous nous concentrons sur les politiques de type seuil. Dans ce type de politique, un relai accepte le message s'il rencontre la source à un instant $n \leq \theta_1$ et le rejete systématiquement après θ_1. De même, un relai ayant accepté le message le conserve au plus tard jusqu'à l'instant θ_2, et le jette passé cet instant.

En utilisant la formulation de la programmation dynamique, le coût optimal à venir à partir de l'état $x \in \{0, m_s, 1, m_d, 2\}$ à l'instant n peut être exprimé comme,

$$V_n^1(1) = \min(0, U_{n,1}, U_{n,2}, \dots, U_{n,\tau-n}), \tag{20}$$

où

$$U_{n,i} = \sum_{j=1}^{i} (\alpha\overline{p})^{j-1} \left[C_s + \alpha p V_{n+j}^1(m_d)\right], \tag{21}$$

Nous supposons qu'un des deux relais (par exemple le relais 2) suit'une politique de type seuil avec θ_1^2 et θ_2^2. La proposition suivante montre qu'une fois que le relais 1 a le message, il utilise également une stratégie de type seuil pour décider de le garder ou de le jetter.

Proposition 4. *Si $U_{\theta_2^2,1} \geq 0$ alors il existe un seuil $\theta_2^1 \leq \theta_2^2$ tel que le relai 1 garde le message jusqu'à θ_2^1 et le jette à l'instant $\theta_2^1 + 1$. Sinon, si $U_{\theta_2^2,1} < 0$, alors le relai 1 garde le message jusqu'à ce qu'il rencontre la destination ou jusqu'à l'expiration du délai.*

La Proposition 4 combinée avec la Proposition 5 montre que si un relais suit une politique de type seuil, alors l'autre utilisera également une stratégie similaire.

Proposition 5. *Il existe θ_1^1 tel que le relais 1 rejette le message s'il rencontre la source à $n > \theta_1^1$.*

Nous arrivons donc à la question suivante: existe-t-il un équilibre du jeu dans lequel chaque joueur utilise une stratégie de seuil ? Une réponse positive à cette question n'est pas évidente, mais dans l'affirmative cela donne une impulsion forte de recherche et ouvre la possibilité d'affiner notre mécanisme de récompense.

5 Conclusion et perspectives

5.1 Une approche différente pour l'étude de la convergence

Notre approche pour prouver la convergence de la dynamique de meilleure réponse est basée sur la notion de rayon spectral non-linéaire. Pour appliquer cette approche il faut montrer que l'opérateur de meilleure réponse est continu au sens de Lipschitz, et que son rayon spectral non linéaire est strictement inférieur à l'unité. Le rayon spectral non linéaire est en relation avec le rayon spectral joint d'un ensemble de matrices jacobiennes de l'opérateur. Pour notre jeu de routage, nous avons montré que la fonction de meilleure réponse est Lipschitz, et nous avons établi la structure spécifique des matrices jacobiennes. Nous avons ainsi obtenu une condition suffisante purement structurelle qui permet de réduire l'analyse de la convergence de la dynamique séquentielle de meilleure réponse à l'analyse du rayon spectral joint de certaines matrices. Nous avons montré que cette condition est respectée dans deux cas: (a) le jeu à deux joueurs pour un nombre arbitraire de liens et pour une large classe de fonctions de coût, et (b) pour un nombre arbitraire de joueurs et des liens dans le cas des fonctions de latence linéaires.

Pour les fonctions de latence satisfaisant des hypothèses de convexité raisonnables, nous conjecturons que la condition suffisante proposée est valable pour un nombre arbitraire de joueurs et des liens. Nous espérons pouvoir prouver cette conjecture. Il serait également intéressant d'envisager l'utilisation de l'approche basée sur le rayon spectral non-linéaire pour étudier la convergence des dynamiques de mailleure réponse dans des topologies de réseaux plus complexes.

5.2 Mécanisme d'incitation pour DTNs

Un problème central dans les DTN est de persuader les nœuds mobiles de participer à la transmission des messages. Dans cette thèse, nous avons proposé un mécanisme basé sur une récompense pour inciter les nœuds mobiles à sacrifier leur mémoire et leur énergie pour relayer les messages. Le mécanisme d'incitation est conu pour assurer la participation des relais dans le processus de livraison en proposant une récompense qui prend en compte les frais encourus par les relais. Cette récompense est le montant minimum qui compense le coût de livraison moyen estimé par le relais à partir des informations communiquées par la source (nombre de copies existantes du message, âge de ces copies). Nous avons d'abord montré que la récompense moyenne payée par la source reste la même indépendamment de l'information qu'elle donne, allant de l'information complète sur l'état à pas d'information du tout. Nous avons également étudié le cas dynamique dans lequel la source peut modifier les informations qu'elle transmet à la volée en fonction de quand elle rencontre le relais. Sous certaines hypothèses supplémentaires, la source peut gagner en adoptant la stratégie dynamique. Ensuite, nous avons abordé le processus de décision à temps discret pour les relais, quand le message a une durée de vie. Pour le mode "pas d'information", dans le cas d'un seul relai et en supposant une récompense fixe, nous avons étudié la politique optimale du relais. Nous avons établi jusqu'à quand il doit accepter le message de la source, et une fois le message accepté, jusqu'à quand le relais doit le garder. Nous avons ensuite considéré le cas de deux relais et démontré que si un relai suit une politique optimale du type seuil, alors l'autre se comporte de manière similaire.

Dans notre modèle, nous nous sommes limités à une paire de source-destination qui génère des paquets. Pour plusieurs paires source-destination, les buffers des nœuds peuvent déborder si aucune politique de rejet des messages n'est adoptée. Dans ce scénario, les politiques efficaces d'abandon au niveau des nœuds relais décident quels messages doivent être prioritisés sous des contraintes de capacité, indépendamment de l'algorithme de routage spécifique utilisé. Dans le futur, nous proposons de travailler sur les politiques d'Abandon/Ordonnacement intentionnelles dans les DTN en ce qui concerne notre mécanisme. Cette étude incite des sources à développer une conception du mécanisme afin de connaître les informations sur les messages qu'un relais stocke dans son buffer. Ensuite, nous proposerons un mécanisme qui peut permettre à la source de susciter des informations privées pour chaque nœud relais qu'elle rencontre.

1 INTRODUCTION

Design and management of large-scale communication networks are central problems for the research community. One of the most studied directions deals with the analysis of decentralized routing mechanisms in networks. In contrast to a centralized scheme, a decentralized routing scheme offers wide-ranging advantages including scalability, ease of deployment and robustness to failures and environmental disturbances. In the last ten years, a substantial research effort has also been devoted to Delay tolerant networking due its progressive ideas of a network architecture that can cope with intermittent connectivity and long delays in communication.

However, several challenges arise when seeking to implement decentralized routing schemes and to design mechanisms for DTNs. They are related to the selfish behaviour of participants. Decentralized routing involves autonomous agents that compete for network resources to route their own traffic through the network. In DTNs, mobile nodes that are expected to support communication between other nodes, may not be willing to do so due to their individual objectives.

Game theory provides effective tools to design and analyze such competitive environments. Game theory has already proven to be a powerful theoretical framework for understanding, controlling and designing complex dynamic networks with many agents. Game theory gives various concepts of equilibria and allows to offer mechanisms to achieve efficient and desirable global outcomes in spite of the selfish behavior of the agents. Motivated by wholesome influences of game theory, we apply its techniques to model and analyze the selfish behaviour in both decentralized routing and DTNs.

1.1 Scope of Game Theory

The birth of Game theory is usually associated with the publication of the monograph of Neumann et al., 1944, "Theory of Games and Economic Behavior". Before this, von Neumann developed an idea of a game as of a general model of abstract conflicts, and the monograph presented a mathematical approach to games as a systematic theory. In fact, in this book a complicated, important and, moreover, highly unconventional mathematical discipline was created.

Essentially, game theory attempted to mathematically describe some unsolved problems of economic behaviour. The basic premise of the theory consists in the idea that each individual seeks to maximize its gain and minimize its loss, like in chess or poker. However, game theory encompasses much more than the usual idea of maximizing, because without this new element, it would be little different from the old approaches. According to the new theory, an outcome depends not only on what one player wants to achieve, but on the intentions of other players. Thus, game theory studies the abstract model of conflict, i.e. a situation which involves at least two sides, represented by persons, groups or control systems, whose activities are purposefully directed, and interests of the parties are partially or completely opposite. Conflicting nature of such problems does not imply hostility between the parties, but attests to various interests.

In the cases where there is a clash of interests, formalization of the decision-making process and finding an optimal solution are impossible with traditional methods of optimization for decision-making. In conventional extremal problems, the matter concerns one person who makes decisions, and the result of these decisions depends on the choice that is determined by the actions of only one person. Such schemes do not apply to situations where decisions optimal for one side, are not optimal for another one and where the result of a decision depends on all the conflicting parties.

Quite significantly problems dealing with conflict situations cannot be properly formulated and fully solved without the mathematical theory of games. Similarly to how the problem of random events cannot be properly solved only by methods of classical analysis and find a solution only by a new mathematical instrument, probability theory and mathematical statistics, conflicts cannot be studied only with probability theory and require a new mathematical discipline, game theory.

The subject matter of game theory is thus interactions of individuals in a group where the actions of each individual have an effect on the outcome that is of interest to all. Game theory aims to understand and predict the behaviour of selfish individuals in a competitive environment. It thus can be applied in any field with selfish nature of interactions, where decision made by an individual influences outcomes of all participants. Such interactions are typical not only for the area of business and economics, but also arise in political and military affairs, biological systems and communication networks etc.

However, the distinctive feature of the game compared to the real conflict situation is that the first is carried out under predefined rules. This is the main limitation in the application of game theory. The main importance of game theory is that it gives the orientation when the use of another mathematical approach is not possible due to lack of information about the actions of the opponent.

Not surprisingly, game theory has been applied to networking, in most cases to solve routing and resource allocation problems in competitive environments. A subset of references is included in a survey on networking games by Altman et al., 2006. Recently, game theory was also applied to wireless communication (see book by Han et al., 2012) and mobile networks.

1.2 Steady-State in Decentralized Routing

In the management of large-scale communication networks routing traffic is a core problem. Router, or routing agent, uses routing algorithm to find a best route, or set of routes, to a destination. The best route can be defined according to some performance criterion, e.g. number of hops [1], distance, speed, time delay or communication costs of packet transmission.

A centralized approach to design a routing algorithm is based on a global network optimization. Such an approach requires full information about the traffic status of the network and implies that the routers are obedient units that follow a global optimal algorithm for traffic routing. Centralized routing thus may be represented by a scheme (Figure 1.1) where there is a single routing agent who controls allocation of all incoming traffic over the routes of the network.

However, the central control is inadequate in the conditions of scalability and growing complexity of networks. Distributed nature of a large-scale network implies a lack of coordination among its users. Instead, each user attempts to obtain maximum performance according to his own parameters and objectives. The management of such a network cannot be thus seen as a single control objective. The inability to use a central regulation raises the need for a decentralized control paradigm, where network control functions are entrusted to individual users. A user thus acts as an autonomous routing agent and independently seeks to optimize the allocation of his own traffic. Figure 1.2 depicts decentralized routing scheme with multiple agents.

An indispensable component of the design of a decentralized system is the steady state analysis. If a dynamic system achieves a steady state it will retain its properties unchanged in time. Steady state determination is important for estimating a core characteristic when shifting to a decentralized network architecture, namely the loss in the overall performance. The performance degradation is the consequence of the fact that in decentralized routing scheme, each agent performs an individual optimization without regard to the overall costs

[1] A *hop* is a trip a packet takes from one router or intermediate point to another in the network.

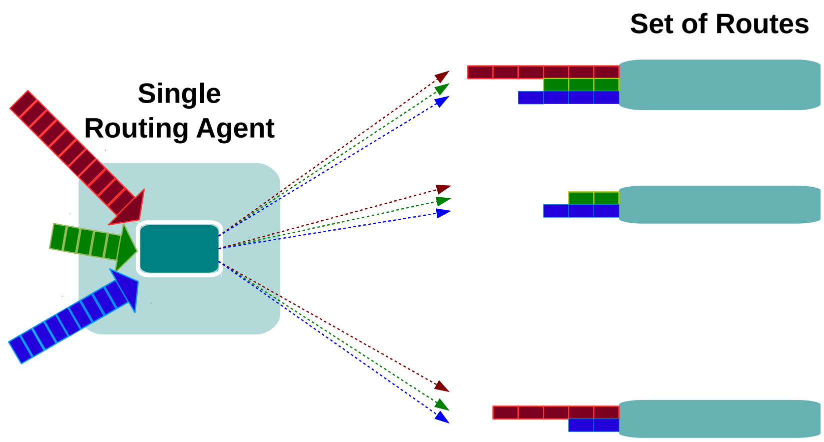

Figure 1.1: Centralized Routing Scheme: a single routing agent controls allocation of all incoming traffic over the routes of the network.

or delay of the system.

Game-theoretic modelling of a decentralized routing scenario allows to perform steady state analysis due to different equilibrium concepts and to make quantitative characteristic of the performance degradation in terms of Price of Anarchy (Koutsoupias et al., 1999). Assuming rational behavior of selfish network users that aim to optimize their own individual performance, network routing scenario can be modeled as a non-cooperative multi-player game. The resulting steady state of the traffic allocation in the selfish routing corresponds to the Nash equilibrium notion of game-theoretic scenario that is the situation when no individual deviation of an agent can improve its performance. The Price of Anarchy is a standard measure of the inefficiency of decentralized algorithms. Its small value indicates that, in the worst case, the gap between a Nash Equilibrium and the optimal solution is not significant, and thus that good performances can be achieved even without a centralized control.

A key property of the equilibrium is that once it is reached, the users will continue to use the same policy, and the system will remain in that equilibrium. Nevertheless, a main difficulty with the notion of equilibrium is that in realistic scenarios there is no justification to expect that the system is initially in equilibrium. Moreover, the users may be unable to compute the equilibrium individually, since a user is generally unaware of some parameters private to others that can influence his own benefit. A natural assumption to be made is that the users are likely to stick with greedy way in their behavior, meaning that each user would occasionally update his own decisions so as to optimize his individual performance,

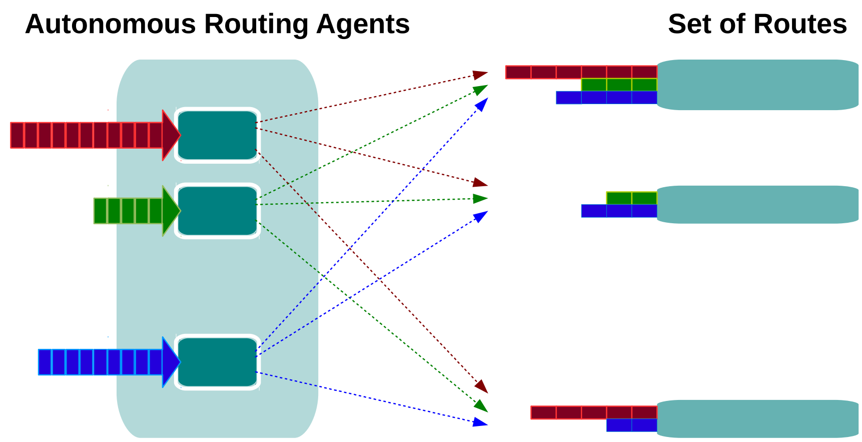

Figure 1.2: Decentralized Routing Scheme: autonomous routing agents control allocation of their own traffic over the routes of the network.

without any coordination with other users.

This thesis addresses a central question when reasoning about steady state of the decentralized routing: do uncoordinated routing agents converge to a Nash equilibrium? In our study, we shall be concerned with the convergence of autonomous routing agents to a Nash equilibrium under some "natural" dynamics. More precisely, we address this question assuming the well-known (myopic) best-response dynamics. Best-response dynamics play a central role in game theory (Berger et al., 2011) since the Nash equilibrium concept is implicitly based on the assumption that players follow best-response dynamics until they reach a state from which no player can improve his utility. In a game, the best-response of a player is defined as its optimal strategy conditioned on the strategies of the other players. It is, as the name suggests, the best response that the player can give for a given strategy of the others. Best-response dynamics then consists of players taking turns in some order to adapt their strategy based on the most recent known strategy of the others (without considering the effect on future play in the game). We shall consider the sequential (or round robin) best-response dynamics, where players play in a cyclic manner according to a pre-defined order.

The focus of our study is the convergence of sequential best-response dynamics in a network of parallel links, shared by a finite number of selfish users. Each user controls a non-negligible portion of the total traffic, and seeks to split his flow over the links of the network so as to minimize his own cost. This model was introduced in the seminal article of Orda et al., 1993, where it is shown that for the users that may have different traffic demands, there exists a unique Nash equilibrium under reasonable convexity assumptions on the edge latency functions. Since this publication, obtaining convergence results of best-response dynamics for routing games has remained a challenging problem for general cases with non-linear link latency functions. Development of a more tractable, feasible method to prove convergence of a dynamics is therefore an important goal in this area (in practical sense). In this thesis we construct such an approach for best-response dynamics in routing games and obtain a sufficient condition for convergence. We then use this approach to establish convergence results for some special cases.

1.3 Cooperation in Delay-Tolerant Networks

Delay- and Disruption-Tolerant Networking (DTN) was proposed as a communication paradigm to support connectivity in environments where end-to-end paths between sources and destinations may not be available at all time. In particular, DTN provides an architecture that can span across multiple networks coping with deficiencies of TCP/IP based Internet. It is capable of acting as an overlay on top of a heterogeneous environment consisting of different communication segments, such as wired Internet, wireless sensor/ad-hoc networks, satellite links, wireless local area networks, etc.

For reliable data transmission, the communication model based on TCP/IP and other standard Internet transport protocols assumes continuous connectivity. This requires links to be connected by end-to-end, low-delay paths between source and destination. However, in communication environments such as satellite communications, wireless networks, that are characterized by long delays, packet losses and link disruptions, TCP/IP protocol becomes ineffective. Due to intermittent connectivity it is likely that contemporaneous source-destination path may not exists from time to time, and implementation of TCP/IP protocol in this case will lead to that a packet whose destination cannot be found will be dropped (1.3).

DTN architecture bypasses the requirement for contemporaneous end-to-end connectivity (Figure 1.4). DTN offer an alternative for realizing communications by implementing a *store-carry-forward* approach, where information fragments, packets, are transiently stored in network devices to be then forwarded to the destination. In other words, DTN divides the end-to-end path into multiple DTN hops. Intermediate nodes receive packets and temporarily store them until next hop, i.e. until an opportunity to send the packet to the destination or to another intermediate node. Figure 1.5 illustrates store-carry-forward communication in DTNs: there is no direct connection between the source and the destination in a considered time period, and the packet, or message, can be forwarded from the source to the destination through intermediate mobile nodes.

The assumption that mobile nodes may serve as relays with a premise that they can store information for a long time before forwarding it reflects a main idea of DTN architecture. Due to random node mobility and uncertainty in connectivity, DTN algorithms commonly imply multi-copy routing for message delivery, when the message is delivered

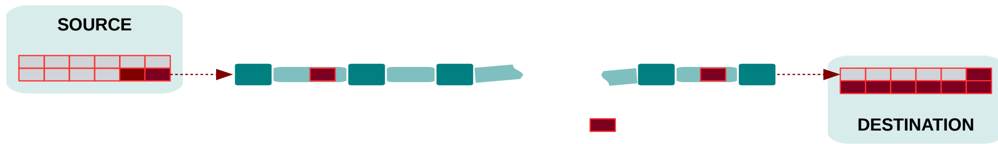

Figure 1.3: If the end-to-end connection is not perfect, a standard Internet protocol discards any packet that cannot be forwarded because a link is down.

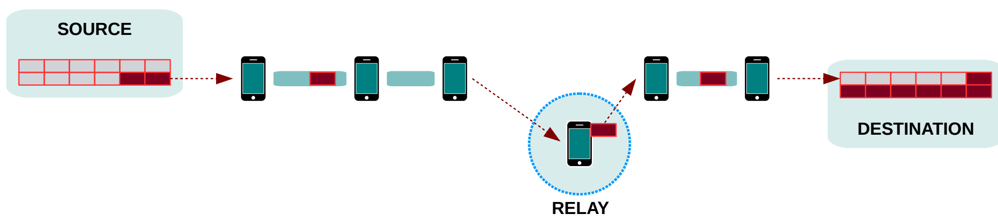

Figure 1.4: DTN nodes serve as relays supporting communication even when end-to-end paths between source and destination may not be available in a given time.

if one of the relay nodes with a copy encounters the destination. Replication of the original message by the so-called epidemic routing protocol ensures that at least some copy will reach the destination node with high probability and with a minimum delivery delay. Flooding the network with messages, Epidemic routing leads, however, significant resource consumption. To avoid the overload of the network with messages while retaining a high delivery performance, the two-hop routing scheme provides simple and more efficient variant of the epidemic-style routing. Under this scheme, forwarding of a message copy is allowed in at most two steps, when a relay received the message from the source can not transmit it to another relay node but only if it encounters the destination.

However, in DTN applications, readiness to participate in forwarding is rather uncommon. In practice, DTNs are composed of mobile devices, including smartphones, tablets or other mobile devices having multiple wireless interfaces. They constantly move and can contact with each other when they enter each others' communication range. DTN nodes are controlled by rational entities, such as people or organizations that can be expected to behave selfishly. When a mobile node needs to conserve its power or due to other individual objectives, it may not be willing to serve as a relay in data transmitting, a link may

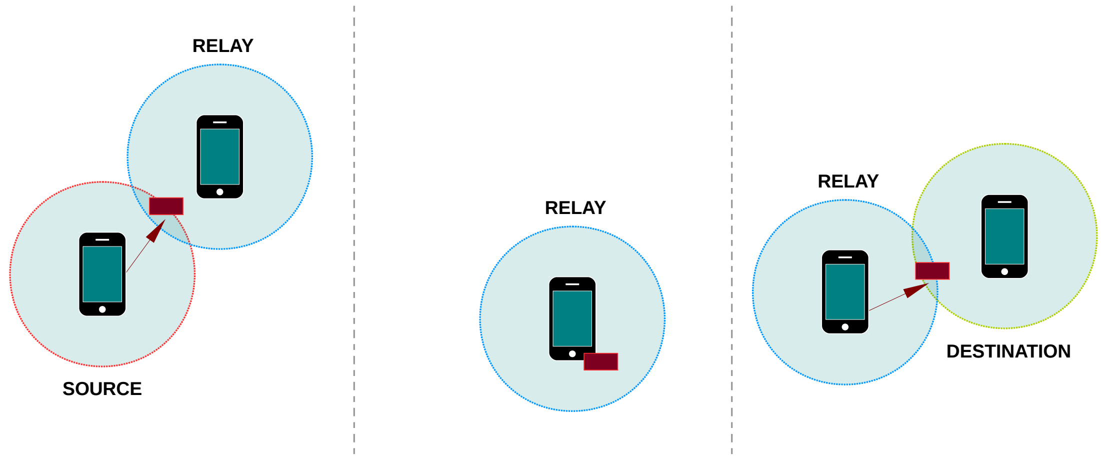

Figure 1.5: Store-Carry-Forward Approach in Delay Tolerant Networking.

then not be established and the packet will be terminated by the node. Selfish behaviour of DTN nodes and corresponding decentralized nature of their decision making requires mechanisms that should offer appropriate incentives for the nodes to behave in ways that are favourable for the network as a whole.

Game theory allows to model various interactions among selfish DTN nodes and to design equilibrium-inducing mechanisms that provide incentives for individual users to behave in socially-constructive ways. In essence, a question of interest is that of how to provide suitable incentives to discourage selfish behavior. In this thesis, we address the two-hope routing scheme in DTN and introduce a rewarding mechanism that promotes full nodal cooperation to serve as relays. This scheme mandates that a relay will receive a reward if and only if it is the first one to deliver the message to the destination. In our scheme we avoid the use of feedbacks that allow relays to know whether the message has been successfully delivered or not. This is an important technical issue in DTNs since large delays the feedback messages may incur.

The source thus has to decide the amount of reward it proposes to each potential relay that it meets, and the relays have to decide whether to accept the message or not. The success of a given relay depends on the number of relays that have already accepted the message: the bigger the number of nodes relaying the message, the higher the delivery probability for the message, but indeed the less the chance for the given relay to receive a reward from the system.

In addition to the incentive mechanism itself, a key objective of our study is to understand which information setting raises the lowest expected reward the source has to pay for message delivery and how the source would reduce this expected reward. We investigate different information settings the source may employ to inform a relay node it meets about its state: *full information* in which the source informs the meeting relay about the number of relays that have already accepted the message and at which time; *partial information* in which the source gives to each relay it meets only the information on the number of existing message copies, and *no information* where the relays do not have any information. Further, incentivising mobile nodes by a fixed reward, we aim to predict their behavior by investigating an optimal policy for a relay by which the relay has to decide whether to accept the message or not and once the message is accepted to drop or to retain it.

1.4 Thesis Organization and Overview of Results

This thesis is structured into three parts. We begin with theoretical foundations of game-theory (Chapter 2). Chapter 3 is entirely devoted to the problem we investigate in the context of decentralized routing. Chapters 4 and 5 are game-theoretic investigation for Delay tolerant networking. An overview below presents more detailed descriptions for each chapter.

Preliminaries on Game Theory.

Chapter 2 offers a game-theoretic framework necessary for applying the theory to the problems we are interested in. We concentrate on non-cooperative game theory and describe various type of games with corresponding solution concepts. We give particular attention to the basic concept of Nash equilibrium and best-response dynamics of a game. We present a definition of a stochastic game and its equilibrium concept, that provides a basis for modeling node competition in DTNs.

Convergence of the Best-Response Dynamics.

Chapter 3 opens our research part. This chapter contains convergence analysis for decentralized routing over parallel links. The chapter begins with a discussion about different models of selfish routing. Then it presents an overview of related convergence results. After an accurate description of our non-cooperative game for the model of atomic splittable routing, we introduce the sequential best-response dynamics for the game and emphasize that for this game, there exists a unique Nash equilibrium. We then present our approach to prove convergence of the best-response dynamics that is based on the concept of non-linear spectral radius of an operator. We proceed with the construction of one-round function for sequential best-response and investigate some properties of this function. We establish the specific structure of the Jacobian matrices of the best-response operator and derive a sufficient condition for the convergence. Using this structure and sufficient condition, we obtain convergence results for two-player games with general cost functions and for the game with an arbitrary number of players with linear cost functions.

Reward-Based Incentives for DTNs.

Chapter 4 directs the focus of the thesis to the study of DTN models. The beginning is a brief excursion into Delay tolerant networking. Next, an overview of research work is given with respect to DTN routing mechanisms and incentive design for DTNs. Then we give the system model of a DTN we investigate. The DTN includes a finite number of relay nodes by means of which a single source intends to send a message to its destination. We consider this model assuming the two-hop routing scheme described previously. We describe our assumptions on contact process of a relay with the source and the destination and then define different settings depending on the information the source conveys to a relay when meets, that are a *full information* setting, a *partial information* setting and a setting with *no information*. We investigate the impact of information the source shares with relays on the reward that it has to propose to them in a static scenario, i.e. following a fixed information setting. After that the extension to the dynamic scenario is provided and the analysis of an adaptive strategy is given for the network of two relays and general inter-contact time of a relay with the source (destination) and with an arbitrary number of relays assuming exponential inter-contact time distribution.

Threshold Type Policy of DTN Nodes in No Information Setting.

Chapter 5 continues studies for DTNs. We modify DTN model of the previous chapter and consider a discrete time decision process for the relays for a given lifetime of the message. A i.i.d. distribution is assumed for contact times between a relay and the source (destination).

Under the two-hop routing scheme and no information setting, we investigate the behaviour of the relays as a response to the fixed reward the source offers to a relay for successful delivery of the message. After model description we give structural elements necessary for defining a stochastic game and then formalize our DTN model in terms of stochastic game with partial information. The chapter proceeds by studying the behaviour of the relays in equilibrium, focusing on the optimal policy of a relay to accept the message from the source or not and if the message is accepted to retain or to drop it. We then establish that if one of the relays use a threshold type policy, then the other one will also use a similar policy.

Chapter 6 gives the conclusion. It contains the summary of research contributions presented in this thesis and identify interesting avenues for further research.

2

THEORETICAL PRELIMINARIES ON GAME THEORY

This chapter defines the main concepts of game theory and simultaneously introduce some key ideas from the theory related to our study. After an illustrative example of a situation to be considered as a game, theoretical foundations are presented with focus on the non-cooperative game theory. We discuss difference between games depending on a movement order of players and information available to a player, and define corresponding solution concepts such as dominant strategy solution, Nash equilibrium etc. A description of a natural best-response play for finding Nash equilibrium ends this chapter to move then to the next one with our study of the convergence of the best-response dynamics.

A Classical Example of a Game-theoretical Situation

Prisoner's Dilemma.

Game theory is best exemplified by a famous illustration of conflict situation called the Prisoner's Dilemma that was originally proposed by Merrill Flood and Melvin Dresher working at the RAND Corporation in 1950, and endowed with its name in 1992 by Albert William Tucker who has formalized this situation with prison sentence rewards. This example shows how the behaviour of rational participants in a conflict situation can affect each other's outcomes.

In the scenario of this situation, two criminals are arrested for committing a crime and imprisoned separately without means of communicating with each other. Due to lack of sufficient evidence for a conviction, the authorities offer to each suspect to make a deal.

If one of the prisoners provides convicting evidence against another one then the first one goes free, while the latter gets 3 years in prison. If both betray each other, then each of them will serve 2 years in prison. If both use the right to remain silent, then they will serve one year. The essence of the dilemma is how criminals will overcome this difficult situation.

This scenario, with choices and resulting outcomes for the two prisoners is summarized in the table below,

Prisoner's Dilemma		Prisoner B	
		remains silent	betrays
Prisoner A	remains silent	$-1,\ -1$	$-3,\ \ 0$
	betrays	$0,\ -3$	$-2,\ -2$

where the number with negative sign symbolizes the length of prison sentence, the first number in each pair is for prisoner A and the second one for the another. Based on this table, it is obviously, the higher the number the better for a prisoner.

Each prisoner will rationally attempt to minimize his jail sentence, and, thus, each prisoner being a self-interested and distrusting his partner, will not be inclined to remain silent, because, in any case, he could get more benefits from betraying his partner even hoping that the partner will keep silence. In other words, each prisoner surmising about a betrayal by his partner, would prefer to serve not three but two years, opting for the betrayal from his own side, or hoping that his partner will keep silent, the prisoner would prefer to be released, again opting for the betrayal. Therefore, the pursuit of personal benefit by each of the suspects, leads the situation to the only possible outcome forcing the prisoners to betray each other, while they would benefit more if they both cooperate.

This "Trust Game" (Kartik, 2009) analyzed in game theory shows why two purely "rational" individuals might not cooperate, even if it appears that the best solution in their interests would be to do so.

Hereinafter, we will focus on non-cooperative game theory. Further sections will give the formal description of a game, present various types of games and explain basic solution concepts for each type.

2.1 Non-cooperative Game Theory

Non-cooperative and cooperative games.

Game theory formalises an interactive situation as a *game*. According to the nature of interaction among participants, the games can be *non-cooperative*, when the participants are not allowed to enter into an agreement, to form coalitions, or *cooperative*, when the participants are allowed to and will form coalitions. In non-cooperative games, all choices are made by self-interested individuals. Cooperative games, in contrast, represent a competition between coalitions of participants, rather than between individuals. Game theory can thus be classified into two respective branches. Our study will be concerned with the individualistic approach and further consideration will be concentrated on non-cooperative game theory, also called *strategic* games.

A simple example of a non-cooperative game has just been shown by the Prisoner's Dilemma. Next, we proceed to the conceptual basis of the theory, and we will start with description of the basic game elements with Prisoner's Dilemma illustration and other examples.

2.1.1 Basic Elements and Assumptions

To be fully defined, a game model must contain the following elements.

- The *players* (also called agents) that form the *player set*, denoted $N = (1, 2, \cdots, n)$. Game-theoretic situation usually involves several players. In case, when there is only one decision-maker, the related problem reduces to an *optimization problem*.

  Players.

- The *information* and *actions* available to each player at each decision point.
  In a game model, each player has available to him two or more well-specified actions or sequences of actions.
  A player's *strategy*, s_i, is a complete plan of actions, that specifies an action the player takes at every point in the game in which the player is called on to act.
  The *strategy space*, $\mathcal{S}_i$, is the set of strategies available to a player. A *strategy combination*, or *strategy profile*, is a set of strategies, one for each player in the game, denoted by $\mathbf{s} = (s_1, s_2, \cdots, s_n)$.
  A strategy consisting of selecting and playing a single action is called a *pure strategy*, and a choice of pure strategy for each player is called *pure-strategy profile*.

  Actions and strategies.

- The *payoffs* for each outcome.
  Every possible combination of strategies available to the players leads to a well-defined outcome that terminates the game. Each possible outcome specifies an associated payoff for each player.
  The payoff can be in any quantifiable form. It can be represented as an abstract concept, *utility* (Neumann et al., 1944). Such a utility corresponds to a preference of a player and is perceived as a magnitude of subjective welfare or change in subjective welfare that a player derives from an event. The utility value shows how much the player likes the outcome. The payoff must thus reflect the motivation of the particular player.
  The player i's payoff is represented by the payoff function, or utility function, $u_i = u_i(s_1, s_2, \cdots, s_n)$.

  Payoffs.

Finite and continuous games.

A *finite game* consists of a finite number of players each with a finite strategy set (i.e., S_i is a finite set for each $i \in N$, and N is a finite set). A game is *continuous* if S_i is a continuous set.

Thus, Prisoner's Dilemma described above, is a finite two-player game. Each prisoner has choice from two actions. The prisoners make their decisions simultaneously in one step. Therefore each strategy of a player corresponds to an available action: *Betray* or *Remain silent*. The table constructed above for the Dilemma, contains players' payoffs for each possible outcome: *If one betrays, he goes free, and the other gets 3 years in jail. If both betray, both get 2 years in jail. If neither betrays. both get one year in jail.*

We call attention to the fact that the perverse outcome in the Prisoner's Dilemma is the result of the prisoners have no means of committing to cooperate. The Prisoner's Dilemma is thus the case of a non-cooperative game.

Assumptions of game theory.

Some remarks on this example related to the prisoners' behavior reveal crucial assumptions of the game theory. The prisoners act *rationally*, meaning that each prisoner strives to maximize his own benefit, i.e. his payoff according to the payoff table constructed above for the Dilemma. Players' rationality is the basic premise in game theory. It also implies that the players take into account that the other players act rationally. In other words, since the payoff of each player depends not only on his own decision but also on the other players' decisions, he must reason about how the other players would prefer to act according to their rational behaviour. The latter is related with the assumption of players' *intelligence*, that implies that each player of the game knows everything about the game that a game theorist knows and, thus, any inference that a game theorist can make about the game may be drawn by the players as well. An important implication of intelligence is the *common knowledge*. Aumann, 1976 gave a formal definition of "common knowledge" and an informal description of it. According to the latter, a fact is common knowledge among the players if every player knows it, every player knows that every player knows it, and so on... In this regard, a natural assumption for the game is that the rules of the game are common knowledge, and, summing up, the model of the game with the rules and assumption of players' rationality is the common knowledge for the players.

2.1.2 Order of Moves

Static and dynamic games.

A game is called *simultaneous*, or *static* game, if all players in the game move simultaneously, or if the later players are unaware of the earlier players' movements. If the later players have some knowledge about the earlier players' movements, the game in such case is called *sequential*, or *dynamic* game.

2.1.3 Types of Information

Common knowledge.

The information available to a player can be of different types. A piece of information is *common knowledge* in the sense just described.

In a *complete information* game, every player is aware of all other players, the timing of the game, and the set of strategies and payoffs for each player. In an *incomplete information* game, at least one player is uncertain about some relevant information about another player. In particular, a player may be uncertain about another's payoff function. The nature of the uncertainty is usually assumed to be common knowledge.

Complete and incomplete information.

Furthermore, in dynamic games, the information may be one of two particular types, perfect or imperfect. In a *perfect information* game, at each point in the game, the players who are to move know the entire history of the game to that point. In an *imperfect information* game, some player is uncertain about the history of the game when it is his turn to move.

Perfect and imperfect information.

It should be noted the importance of difference between complete and perfect information. In a game of complete information, the structure of the game and the payoff functions of the players are commonly known but players may not see all of the moves made by other players, while in games of perfect information, each player observes other players' moves, but may lack some information on others' payoffs, or on the structure of the game.

The games are analyzed regarding the movement order and the information type. Further, we shall focus on some important game types relevant to our study and reveal corresponding solution concepts. Before this, the following section will describe representation forms for static and dynamic games.

2.1.4 Game Representation

There are two distinct but related ways of describing a non-cooperative game mathematically. The *extensive form* is the most detailed game representation, it is used to formalize games with some important order. The extensive form is often applied for dynamic games. It describes play by means of a *game tree* that explicitly indicates when players move, which moves are available, and what they know about the moves of other players and nature when they move. Most importantly it specifies the payoffs that players receive at the end of the game.

Extensive form of a game.

An alternative to the extensive form is the *normal* form, that is also known as the *strategic form*, the most fundamental in game theory. This is less detailed than the extensive one, and it specifies only the list of strategies available to each player. It is presumed that the players act simultaneously or, at least, that an acting player is unaware of the movements of the others. Since the players' strategies determine how each player is to play in each circumstance, a strategy profile can be associated with payoffs received by each player under their strategies of this profile. This map from strategy profiles to payoffs is called the *normal* or strategic form.

Normal form of a game.

More formally, the *normal*, or *strategic form* of a finite n-player game is the set of the players' strategy spaces, $(\mathcal{S}_1, \cdots, \mathcal{S}_n)$ and their payoff functions $(u_1, \cdots, u_n)$. Such a game is denoted as a tuple, $G = (N, \mathcal{S}, \mathbf{u})$, with N being a finite set of n players, $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ and $\mathbf{u} = (u_1, \cdots, u_n)$, where $u_i : \mathcal{S} \mapsto \mathbb{R}$ is a real-valued utility function for player $i \in N$. *Notation*: For any $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, $\mathbf{x}_{-i}$ denotes the vector obtained from x by excluding x_i, i.e. $\mathbf{x}_{-i} = \mathbf{x} \setminus (x_i) = (x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$.

The strategic form of a game is frequently represented by a game matrix. An example of a strategic form game was already met before, the Prisoner's Dilemma described at the beginning of this chapter.

2.2 Static Games of Complete Information. Basic Solution Concepts

This section presents basic solution concepts for study of strategic form games with complete information, introducing the notions of dominant strategies, pure strategy Nash equilibrium and Nash equilibrium in mixed strategies. The narration of this section follows the book by Nisan et al., 2007.

2.2.1 Dominant Strategy Solution

The Prisoner's Dilemma has a very special property: it is obvious how each prisoner should play, since each of them has a unique best strategy, independent of the strategy played by the other player. This game has a so called *dominant strategy solution*.

Dominant strategy solution.

More formally, a strategy vector $\mathbf{s} \in \mathcal{S}$ is a *dominant strategy solution* , if for each player i, and alternate strategy vector $\mathbf{s}' \in \mathcal{S}$,

$$u_i(s_i, \mathbf{s}'_{-i}) \geq u_i(s'_i, \mathbf{s}'_{-i}).$$

Furthermore, a strategy available to a player is strictly dominated if there is another available strategy that is better for every combination of the other players' strategies.

Formally, a strategy $s''_i \in \mathcal{S}_i$ is *strictly dominated* by another strategy $s'_i \in \mathcal{S}_i$ if

$$u_i(s'_i, \mathbf{s}_{-i}) > u_i(s''_i, \mathbf{s}_{-i}), \ \forall \mathbf{s}_{-i} \in \mathcal{S}_{-i}.$$

A rational player would not play a strictly dominated strategy, since he can obtain a higher payoff by switching to a strategy that dominates it. The other players know that he is rational and will not play his dominated strategy. In the smaller game without this strategy there might be a player who also has a strictly dominated strategy and thus will not play it. This process of elimination is called *iterated elimination of strictly dominated strategies* and it provides a method to find a game solution best for each player, a dominant strategy solution.

However, existence of a dominant strategy solution is valid only for very few games. Most often, there are no strictly dominated strategies in a game, and in such cases the elimination method described above becomes inappropriate for finding an outcome that satisfy all players, this process will not identify such outcome.

2.2.2 Pure Strategy Nash Equilibria

A less stringent than a dominant strategy solution and more widely applicable solution concept provided by game theory is a Nash equilibrium, the central solution concept in game theory. For individual players that act maximizing their own payoffs, the Nash equilibrium reflects a steady state from which no single player can individually improve his benefit by deviating.

Formally, a strategy vector $\mathbf{s} \in \mathcal{S}$ is said to be a *Nash equilibrium* if for all players i and each alternate strategy $s_i' \in \mathcal{S}_i$,

Pure strategy Nash equilibrium.

$$u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i}).$$

In other words, strategies in $\mathbf{s}$ represent choices of all players such that no player i can improve his payoff by changing his strategy from s_i to s_i' assuming that all other players adhere to their strategies in $\mathbf{s}$. Such equilibrium is called pure strategy Nash equilibrium, since each player deterministically plays his chosen strategy. Nash equilibrium is self-enforcing in the sense that once the players are playing such a solution, it is in every player's best interest to stick to his strategy (Nisan et al., 2007). However, Nash equilibrium may not be unique.

Consider the game with the following payoff matrix,

<table>
<tr><td colspan="2" rowspan="2">Battle of the Sexes</td><td colspan="2">Boy</td></tr>
<tr><td>Baseball</td><td>Softball</td></tr>
<tr><td rowspan="2">Girl</td><td>Baseball</td><td>5, 6</td><td>1, 1</td></tr>
<tr><td>Softball</td><td>2, 2</td><td>6, 5</td></tr>
</table>

that corresponds to the game "Battle of the Sexes", an example of a so-called "coordination game", where two players ought to choose the same option between two. The matrix expresses the player's preferences via payoffs. The solutions where the players choose different events are not stable since in each case, either of the two players can improve his payoff by switching his action. The two remaining options, where both players choose the same event, are stable solutions; the girl prefers the first and the boy prefers the second. Thus, coordination game have multiple equilibria.

2.2.3 Mixed Strategy Nash Equilibria

A game, however, need not possess any pure strategy Nash equilibrium. Consider the game "Matching Pennies" with the following payoff matrix,

Matching Pennies		Payer 2	
		Head	Tail
Payer 1	Head	1, −1	−1, 1
	Tail	−1, 1	1, −1

Here, two payers, each having a penny, are asked to choose from among two strategies heads (H) and tails (T). The row payer wins if the two pennies match, while the column payer wins if they do not match, and the number −1 indicates win and −1 indicates loss. It is easy to see that this game has no stable solution.

A three-strategy generalization of the "Matching Pennies" game is the popular children's game of Rock, Paper, Scissors, also known as "Rochambeau". The payoff matrix of the game is shown below.

Rock-Paper-Scissors		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0, 0	−1, 1	1, −1
	Paper	1, −1	0, 0	−1, 1
	Scissors	−1, 1	1, −1	0, 0

In this game, each of the two players can play by one of the tree strategies. If both players choose the same action, there is no winner. Otherwise, each of the actions wins over one of the other actions and loses to the remaining action, i.e. Rock wins against Scissors, Scissors wins against Paper, Paper wins over Rock. The strategies cyclically dominate each other, and anyone who has played Rock-Paper-Scissors knows that the best way is to use a random choice. Indeed, if the players are allowed to randomize and each player picks each of his actions with probability 1/3 (1/2 for the Matching Pennies game), then the game will obtain a stable solution, since the expected payoff of each player will be equal 0 and neither player can improve on this by choosing a different randomization. The expected payoff, or *expected utility*, of a player that he maximizes by randomizing his choice, is the basic notion of decision theory.

Mixed strategies.

A choice of a player by randomizing over the set of available actions according to some probability distribution is called *mixed strategy*. In other words, "A *mixed strategy* of player i will be a collection of non-negative numbers which have unit sum and are in one to one correspondence with his pure strategies" (Nash, 1951). It is assumed that players independently choose strategies using the probability distribution. The independent random choices of players leads to a probability distribution of strategy vectors s.

Nash, 1951 proved that under this extension, *every game with a finite number of players, each having a finite set of strategies, has a Nash equilibrium of mixed strategies.*

Mixed strategy Nash equilibrium.

2.3 Dynamic Games of Complete Information

Many interactive situations involve agents choosing actions over time. The natural way to translate a (finite) dynamic interactive decision situation (of complete information) in a game is the extensive-form representation, using a game tree. In addition to the players, actions, outcomes, and payoffs, the game tree will provide a history of play or a path of play. In a dynamic game, a strategy of a player is a complete plan of actions that specifies a feasible action for the player in every contingency in which the player might be called on to act.

A game tree consists of an *initial node* (the starting point of the game), from which there are *branches* (the actions that the first mover can take) and at the end of each branch is a node. The end node of a branch is a *terminal node* if no more actions can be taken, otherwise it is a *decision node*. The game tree extends until all the nodes are terminal nodes, and at the terminal nodes, the payoffs to the players are listed. An important aspect of the game tree is the *information set*. It is a set that for a particular player, establishes all the possible moves that could have made in the game so far, given what that player has observed. It is possible that a game is being played and a player is uncertain as to which of a few decision nodes the player is at. In this case, the collection of decision nodes is that player's information set.

One way to make a prediction on what path in the extensive-form representation of a dynamic game will be played is first to translate the extensive-form in the associated normal-form and then to apply the concept of Nash equilibrium.

The set of Nash equilibria in a dynamic game of complete information is the set of Nash equilibria of its normal-form. Thus, the finding of the Nash equilibria in a dynamic game of complete information consists in constructing the normal-form of the dynamic game and calculating the Nash equilibria of the normal-form game.

However, two difficulties arise with this approach. First, dynamic games of complete information typically have many Nash equilibria. Secondly, many Nash equilibria in dynamic games involve players choosing non-credible strategies, i.e. when a player adopts his choice of strategy at his stage in order to manipulate the behaviour of other player in the next move in a dynamic game and to generate a different Nash equilibrium.

A stronger solution concept, named Subgame Perfect Nash Equilibrium, allows to eliminate non-credible strategies. The central idea underlying the concept of subgame perfect Nash equilibrium is the principle of sequential rationality: equilibrium strategies should specify optimal behaviour from any (reached or not reached) point in the game onward, not only along the equilibrium path.

Subgame.

To formally define this solution concept, first we define a notion of subgame. A subgame of an extensive-form game is a subset of the game with the following properties: it begins with an information set containing a single decision node; it contains all decision nodes and terminal nodes that are successors (both immediate and later) of this node, and contains only these nodes; and it does not cut any information sets (Nisan et al., 2007).

A subgame considered in isolation, is a game in its own right, and the concept of a Nash equilibrium can therefore be applied to subgames.

Subgame Perfect Nash Equilibrium.

A Nash equilibrium of a dynamic game is *subgame perfect* if the strategies of the Nash equilibrium constitute or induce a Nash equilibrium in every subgame of the game. Subgame perfect Nash equilibrium is a Nash equilibrium since the game as a whole is a subgame of itself, but not every Nash equilibrium is subgame perfect.

Every finite dynamic game of complete information (i.e., any dynamic game in which each of a finite number of players has a finite set of feasible strategies) has a subgame perfect Nash equilibrium, possibly involving mixed strategies.

A special class of dynamic games of complete information is that of *perfect information*. Thus, an extensive-form game is a game of perfect information if each information set contains a single decision node. Otherwise, it is a game of *imperfect information*.

2.3.1 Dynamic Games of Complete and Perfect Information

In a (finite) game of perfect information, when it is a player's turn to move, he observes previous moves of all players. A player is aware about previous moves of all other players.

In a (finite) game of perfect information, every decision node initiates a subgame, and the smallest subgames are always single-player decision problems.

Every finite dynamic game of complete and perfect information has a pure strategy subgame-perfect Nash equilibrium that can be derived through Backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium. The proof of these two statements can be found, for example, in (Mas-Colell et al., 1995, p.272).

2.3.2 Dynamic Games of Complete and Imperfect Information

In a game with imperfect information, some player does not know the action taken by at least one of the other players. To identify the set of subgame perfect Nash equilibria in more general (finite) dynamic games with incomplete information, the generalisation of the backward induction procedure is used.

While the original backward induction procedure (when applied to finite games of perfect information) always yields at least one pure strategy subgame perfect Nash equilibrium this is no longer true for the generalised backward induction procedure (when applied to games of imperfect information).

However, as was already mentioned before, every finite game of complete information has a (sub)game perfect Nash equilibrium, possibly involving mixed strategies. Moreover, if no (sub)game encountered in any step of the Generalised Backward Induction Procedure has multiple Nash equilibria, then there is a unique subgame perfect Nash equilibrium. The proof of these statements can be also found in (Mas-Colell et al., 1995).

2.4 Stochastic Games

A special class of extensive form games is repeated games. When players interact by playing a similar stage game numerous times, the game is called a *repeated game*. A stochastic game is a repeated game with probabilistic transitions. Stochastic games were introduced by Shapley, 1953: "In a stochastic game the play proceeds by steps from position to position, according to transition probabilities controlled jointly by the two players". In other words, a stochastic game is a collection of normal-form games that the players play repeatedly, and a particular game played at any given iteration depends probabilistically on the previous game played and on the actions taken by all players in that game.

Repeated game.

The game is played in a sequence of stages. At the beginning of each stage the game is in some state. A payoff of a player depends on the current state and the actions chosen by the players. In the next stage, the game moves to a new random state, whose distribution depends on the previous state and the actions chosen by the players. The procedure may be continued for a finite or infinite number of stages. The total payoff of a player is often represented by the discounted sum of the stage payoffs or the limit inferior of the averages of the stage payoffs.

Stochastic games generalize both Markov decision processes (MDPs) and repeated games. An MDP can be considered as a stochastic game with only one player, while a repeated game can be seen as a stochastic game with only one state.

Formally, a *stochastic game*, also known as a *Markov game*, is defined by the following elements:

Stochastic, or Markov, game.

- A finite set of players, N.
- A state space, Q.
- For each player $i \in N$, an action set A_i. An action profile, $\mathbf{a}$, of the game is the element of the action space $A = \times_{i \in N} A_i$.
- A *transition probability* $P : Q \times A \times Q \to [0, 1]$, where $P(q, \mathbf{a}, q')$ is a probability of transitioning to the state q' if the action profile $\mathbf{a}$ is used in state q. Transition probabilities thus depend upon current state of the game and actions of players.
- For each player $i \in N$, a payoff $r_i(q, \mathbf{a})$ is a real-valued function of the state q and the action profile $\mathbf{a} \in A$. The vector $\mathbf{r}(q, \mathbf{a}) = (r_i(q, \mathbf{a}))_i$ is composed of payoffs to

each player.

The game starts at some initial state q_1. At stage t, players observe state q_t and then choose simultaneously their actions, $a_{i,t} \in A_i$ forming the action profile $\mathbf{a}_t = (a_{i,t})_i$. A next state q_{t+1} occurs according to the probabilities $P(q_t, \mathbf{a}_t, \cdot\,)$. The play sequence of the stochastic game, $q_1, \mathbf{a}_1, \cdots, q_t, \mathbf{a}_t, \cdots$, defines a sequence of payoffs, $\mathbf{r}_1, \cdots, \mathbf{r}_t, \cdots$, where $\mathbf{r}_t = \mathbf{r}(q_t, \mathbf{a}_t)$. The sequence $h_t = (q_0, \mathbf{a}_0, q_1, \mathbf{a}_1, \cdots, \mathbf{a}_{t-1}, q_t)$ forms a history up to the stage t, and H_t is the set of all possible histories of this length.

In a stochastic game, for a player i, a *pure strategy* specifies a choice of action for i at every stage of every possible history, and a *mixed strategy* of i is a probability distribution over his pure strategies. There are several restricted classes of strategies. A *behavioral strategy* is a mixed strategy in which the mixing takes place at each history independently. A *Markov strategy* is a behavioral strategy such that for each time t, the distribution over actions depends only on the current state, but the distribution may be different at time t than at time $t' \neq t$. Next, a *stationary strategy* is a Markov strategy in which the distribution over actions depends only on the current state and not on the time t.

A player chooses his strategy to maximize his overall payoff. The most common method to aggregate payoffs into an overall payoff are *average reward* and *future discounted reward*. The case of average rewards is more complicated since the limit average may not exist, however under some conditions on strategy profile, average reward stochastic game has a Nash equilibrium for the two-player case (see Shoham et al., 2008, Th.6.2.6). For the discounted-reward case, a strategy profile is a *Markov-perfect equilibrium* (MPE) if it consists of only Markov strategies and it is a Nash equilibrium regardless of the starting state. Every n-player, general-sum, discounted-reward stochastic game has a MPE (Shoham et al., 2008, Th.6.2.5). Markov perfect equilibria can be obtained using backward induction, whose advantage is that instead of searching for equilibrium in the (large) space of strategies, one only need to find Nash equilibrium in a succession of static games of complete information.

2.5 Games of Incomplete Information

The game types considered so far, the games of complete information, have an important assumption that the game played is common knowledge. Particularly, the players in a game of complete information are aware about order of playing the game, possible actions of each other and how outcomes of the game translate into payoffs. This knowledge of a game has been assumed to be itself common knowledge that allowed to develop such solution method and concepts as iterated elimination of dominated strategies, Nash equilibrium and Subgame Perfect Nash equilibrium.

In contrast, in a game of incomplete information, or *Bayesian game*, not all players possess full information about their opponents. Namely, in a game of incomplete information, initially at least one player does not know the payoff function of another one, meaning that for the player, some information that influences an opponent's payoff is unavailable. For example, a firm may not know the cost of production of its competitor, insurance companies may not be sure how careful a driver is etc. Games of incomplete information are called Bayesian games (see Zamir, 2012 and references therein).

A player who does not know the private information of an opponent, may, however, have some beliefs about this information. These beliefs are assumed to be common knowledge.

There are several ways to define a Bayesian game. Harsanyi, 1967; Harsanyi, 1968a; Harsanyi, 1968b proposed to transform a game of incomplete information by introducing Nature as a new player of the game. Nature does not have a payoff function, or its payoff function can be viewed as a constant, and Nature has the unique strategy of randomizing in a commonly known way. The Harsanyi transformation involves introducing a prior move by Nature that determines players' types. Namely, Nature randomly sets the state of the world and then reveals some information regarding the state of the world to each player, but not the same information. Some or all players can have private information. Harsanyi suggested to characterize the different states of the world and a players private information by defining player types. Nature thus in its move determines a player's type and reveals this type to the player, but not to his opponents. It is assumed that the probability according to which Nature moves is common knowledge. A player's payoff function depends on his type.

A static game of incomplete information is then described by the following elements.

- Player set, $N = \{1, 2, \cdots, n\}$.

- For player $i \in N$, T_i is the set of all his possible types. The state of the world is defined as a vector of types, $\mathbf{t} = (t_1, t_2, \cdots, t_n)$, $t_i \in T_i$. The set of all possible states of the world is $T = \times_{i \in N} T_i$. Excluding the player i's type, the state of the world is defined as $\mathbf{t}_{-i} = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n)$, and all possible states of the world as $T_{-i} = \times_{i \neq j \in N} T_j$.

- The probability of any given state of the world, $p = p(\mathbf{t})$, $p(\mathbf{t}) > 0$, $\sum_{t \in T} p(\mathbf{t}) = 1$. The probability that i's type is t_i is defined as $p_i(t_i) = \sum_{\mathbf{t}_{-i} \in T_{-i}} p(t_i, \mathbf{t}_{-i})$, and the conditional probability of the state of the world given i's type is t_i is $p(\mathbf{t}|t_i) = p(\mathbf{t})/p_i(t_i)$. $P = \{p(\mathbf{t})|\mathbf{t} \in T\}$ represent the probability distribution over all states of the world.

- For player $i \in N$, $s_i(t_i)$ is a pure strategy given t_i, and S_i is the set of possible pure strategies given type t_i. The strategy profile conditional on the state of the world is $\mathbf{s}(\mathbf{t}) = (s_1(t_1), s_2(t_2), \cdots, s_n(t_n))$. A pure strategy for player i is a collection of strategies, one for each type, $s_i = \{\mathbf{s}(t_i)|t_i \in T_i\}$.

- For player $i \in N$, the utility $u_i(\mathbf{s}) = \sum_{\mathbf{t} \in T} u_i(\mathbf{s}(\mathbf{t}), t_i) p(\mathbf{t})$ is the expected utility, or expected payoff conditional on the state of the world and player i's type. $U = \{u_1(s), u_1(s), \cdots, u_n(s)\}$ represents the payoff space.

A static game of incomplete information is thus defined as $G = \{N, S, U, T, P\}$. Such a description transforms a game of incomplete information into a game of imperfect information since the move of Nature is not observed perfectly by each player and hence the players are not aware about whole history of the game when they have to move.

In a Bayesian game $G = \{N, S, U, T, P\}$, a strategy profile $s^* = (s_1^*, \cdots, s_n^*)$ is a pure-strategy Bayesian Nash equilibrium if for each player $i \in N$ and for each i's types $t_i \in T_i$,

$$s_i^*(t_i) = \arg\max_{s_i \in S_i} \sum_{t_{-i}} p(t_{-i}|t_i) u_i(s_1^*(t_1), \cdots, s_i, \cdots, s_n^*(t_n); t_i).$$

In other words, the Bayesian Nash equilibrium requires that no type of an individual player can do better by unilaterally changing its strategy. Harsanyi, 1967; Harsanyi, 1968a; Harsanyi, 1968b proved that *in a game with incomplete information in which the number of types of each player is finite, each Bayesian equilibrium is a Nash equilibrium, and conversely every Nash equilibrium is a Bayesian equilibrium.*

The concept of a Bayesian Nash equilibrium can be applied to analyze dynamic games of incomplete information, where players take turns sequentially. However, Bayesian Nash equilibria in dynamic games of incomplete information suffer from the same flaw as Nash equilibria in dynamic games of complete information, such as incredible strategies. In dynamic games of complete and perfect information, such implausible equilibria might be eliminated by applying subgame perfect Nash equilibrium, where subgame contains complete information set. However, dynamic games of incomplete information contain non-singleton information sets, and usually do not have any subgames other than the game as a whole, and subgame perfection becomes unfeasible.

To refine the equilibria generated by the Bayesian Nash solution concept or subgame perfection, one can extend the concept of subgame perfection and apply the Perfect Bayesian equilibrium solution concept. Specifically, the idea of a subgame is replaced by the more general idea of a *continuation game*, a game that can begin at any information set rather than only at a singleton information set. The players strategies are then required not only to constitute a Bayesian Nash equilibrium for the entire game, but also a Bayesian Nash equilibrium in every continuation game. In order to behave rationally at an information set that contains more than one node, the player who moves at that set has to form a belief on the relative likelihoods of being at each of various decision nodes in the the information set he is at, conditional upon play having reached that information set. The belief is a probability, and the probabilities (beliefs) for all decision nodes within an information set sum to 1. A Perfect Bayesian equilibrium consists therefore of a strategy profile and also of a belief profile. It requires, that strategies are optimal given beliefs and that beliefs are consistent with the strategies being played. A detailed description of dynamic Bayesian games and corresponding solution concepts may be found in the book by Gibbons, 1992 and references therein.

2.6 Finding Equilibria via Natural Game Play: Best Response Dynamics

Feasibility of Nash equilibrium concept to an interaction circumstance is based on the conjecture that players will learn to play an equilibrium if they interact repeatedly. A Nash equilibrium is thus expected as a result of rational adaptation of players in the game. The most natural strategy for playing a game is the "best response". In general, the best response dynamics proceeds as follows. At each stage, every player chooses the best-response to the actions of all the other players in the *previous round*, ignoring all history before this round.

More formally, the procedure is performed as described below. While the current strategy profile s is not a pure Nash equilibrium, consider an arbitrary player i . Its utility under the strategy profile s is $u_i(s)$. Assuming that all other players adhere to their strategies in s_{-i}, player i can beneficially change his utility by unilateral deviation from his strategy s_i to some other strategy $s'_i \in S_i$. A deviation from the strategy s_i to s'_i is said to be an *improving response for player* i if $u_i(s'_i, s_{-i}) > u_i(s)$ and it is said to be a *best response* if s'_i maximizes the player i's utility, $\max_{s'_i \in S_i} u_i(s'_i, s_{-i})$.

Proving that natural dynamics converge quickly to an equilibrium lends plausibility to the predictive power of an equilibrium concept. Best-response dynamics provides a straightforward procedure by which players search for a pure Nash equilibrium (PNE) of a game. In some games, such as the Prisoners Dilemma or the Coordination Game, best-response dynamics leads the players to a Nash equilibrium in a few steps. There are some games, where the players will not reach the equilibrium in a finite number of steps, but the strategy vector will converge to the equilibrium. If best-response dynamics reaches a steady state, it is clearly a PNE. It cycles in any game without one. It can also cycle and not converge in games that have a PNE. A simple example when best-response dynamics does not converge, is matching pennies, where the players will cycle through the 4 possible strategy vectors if they alternate in making best responses.

3

CONVERGENCE OF THE BEST-RESPONSE DYNAMICS IN ROUTING GAMES OVER PARALLEL LINKS

This chapter focuses on the convergence of sequential best-response dynamics in a network of parallel links, shared by a finite number of selfish users, where each user controls a non-negligible portion of the total traffic, and seeks to split his flow over the links of the network so as to minimize his own cost.

The chapter begins by considering the notion of selfish routing and describing different types of routing games with corresponding equilibrium concepts. Then, in Section 3.2, focusing on the routing problems that interest us and on the convergence issue to an equilibrium in selfish routing, we trace the earlier convergence results for some special cases. Section 3.3 gives the statement of our routing problem. Formalizing it as a non-cooperative game, we define equilibrium concept for the game and the best response dynamic for it. Section 3.4 explains the Non-Linear Spectral Radius Approach for our convergence problem. Sections 3.5 shows our convergence results.

3.1 Routing Games

Routing problems arising in a transportation or communication network, where self- interested users share network resources to send their flows and each user aims to get his own flow to the destination in minimal delay or with minimal cost, can be modeled as non-cooperative games. The mathematical model of selfish routing has been extensively studied in Transportation Science literature (Pigou, 1920; Wardrop, 1952; Beckmann et al., 1956; Yang et al., 2004; Boyce et al., 2005; Yang et al., 2008) and widely explored in Computer Sciences (Cantor et al., 1974; Gallager, 1977; Orda et al., 1993; Bertsekas et al., 1997; Qiu et al., 2003; Friedman, 2004), with a large number of works on routing games (e.g. Fleischer et al., 2004; Roughgarden et al., 2002; Roughgarden, 2005a; Nisan et al., 2007).

Selfish routing models assume usually the flow to be time-invariant, or static (Roughgarden, 2005a). A dynamic flow implies that the state of a network is conditioned by the time when the users employ the network, and a user choosing his best route does not consider the congestion on any link based on the total number of users that traverse it, but the user considers the congestion on the link that will be experienced on it when the user reaches this link. Dynamic selfish routing has been investigated by Anshelevich et al., 2009. In our study, we are concerned with a static flow in selfish routing.

A routing game is generally described via a directed graph representing the underlying network. Each player, or network user, has a volume of traffic to be routed from its source node to its destination node in the graph through available paths consisting of network links, the edges of the graph. The flow on a link of the network faces a delay, and the delay is characterized by a latency function, or cost function. The link latency function, in the static flow context, depends on the total amount of flow this link contains, it is usually non-decreasing and convex. Each user is able to choose how to route his traffic flow over the network so as to minimize his own incurred cost. User's cost function corresponds to the latencies of the links the user employs for his flow and is specified for a particular game model.

There are different models of routing games depending on the amount of flow the players control. In a network routing game with nonatomic players, there is a continuum of players each controlling a negligible amount of the overall traffic flow. This type of games were first studied by Wardrop, 1952 in the road traffic context. Exemplified by transportation network, such a game involves a large number of players representing drivers. Each player is insignificant in that it cannot individually influence the congestion level of any road in the network. The Wardrop equilibrium notion is concerned with this type of games.

Another type of routing games is that with atomic players. In contrast to non-atomic case, atomic games describe situations in which players have significant influence since each player controls a non-infinitesimal amount of flow. Moreover, in the context of atomic routing, players may or may not be able to split their flow along several paths. There are *non-atomic* (Wardrop, 1952; Aumann et al., 1969; Roughgarden et al., 2002; Roughgarden, 2005a; Awerbuch et al., 2009), *atomic unsplittable* (Fotakis et al., 2004; Awerbuch et al., 2004) and *atomic splittable* (Orda et al., 1993; Roughgarden, 2005b; Cominetti et al., 2009) routing games, respectively. Our investigations is devoted to atomic splittable games.

In an atomic splittable routing game, each player is given a non-negligible amount of flow that the player can route fragmenting over available paths. Each player routes his flow to minimize his own average delay, or own total costs, that is the sum over links of the product of his flow on the link and the delay on the link. The challenge with atomic splittable routing model is that each player has an infinite strategy space consisting of all possible ways of routing his flow. Another challenge is that the players, unlike in non-atomic routing games, are commonly asymmetric since each of them is given a different flow value.

In a routing game, an equilibrium flow is characterized by a traffic allocation at steady state, wherein no user may change his flow assignment to reduce his total incurred cost of routing his flow. In atomic splittable routing games, equilibria exist under some moderate assumptions on the delay functions (Rosen, 1965). Equilibria in atomic splittable games will be refered as Nash equilibria.

Our study will focus on the convergence issue of uncoordinated users in selfish routing to a Nash equilibrium, when the users implement an asynchronous best-response dynamics. The next section gives a survey of the convergence results relevant to our work.

3.2 Review on Related Convergence Results

The model we shall study was introduced in the seminal article of Orda et al., 1993 in the communication network context. They have addressed the routing problem in networks from a game theoretical viewpoint. Namely, the article starts with consideration of the network of parallel links interconnecting a common source to a common destination. The set of links shared by the finite number of selfish users each of which has a nonnegligible portions of flow to ship by splitting through the links and seeks to minimize own incurred cost. The cost of each user is described by cost function that is the sum of the costs the user incurred on each link. The authors formalized the problem as a non-cooperative game, and under the special assumption on user's link cost function, such as continuity, convexity and continuous differentiability whenever the function is finite, the considered routing game is shown to be a convex game (Rosen, 1965) and to have thus a Nash equilibrium point according to the theorem in (Rosen, 1965, Th.1, p.522). Further, user's link cost function is endowed with additional assumptions. Particularly, user's link cost is taken to be dependent on user's flow on the link as also on the total flow on that link and increasing in these two arguments. A link's marginal cost for a user is thus a function of two arguments and it is assumed to be strictly increasing in each of its two arguments. With these additional assumptions, the user's link cost functions were referred to as *type-A* functions, and the uniqueness of the Nash equilibrium point has been established for *type-A* functions using Kuhn-Tucker conditions for cost minimization and based on the monotonicity and increasing properties of the user's marginal link cost functions.

The special case of two users in a network of two parallel links was investigated for the stability of the Nash equilibrium point, i.e. for the convergence of adjustments dynamics to this point. Assuming the players make best response in turns, Orda et al., 1993 referred this dynamics to the *Elementary Stepwise System*. First, for the flow configuration, that is the flow of each user over each link, they proved that each component of the flow configuration increases or decreases monotonically with each step of the dynamics. With this and due to that the flows are bounded, convergence of the dynamics has been established. The authors point out however that this convergence result is not readily extendible to more general cases. Indeed, in (Altman et al., 2001), for the two-link case and more than two users, some asynchronous as well as synchronous best response schemes have been shown not to converge to the equilibrium.

However, Altman et al., 2001, studying the two-link case and assuming linear latency functions for the links, prove that a round robin adjustment scheme, or sequential best-response dynamics, converges for any number of players. To do so, they began by showing that for the case of several users and a network of several parallel links with linear costs, there exists a unique Nash equilibrium. Moreover, an explicit expression for the Nash equilibrium was obtained. Then, for the two-link case, they focused on the best response of one user. They take the equilibrium solution for the user and introduce a deviation of the optimal flow over each link in the best-response of this user from that in the equilibrium solution. Expressing updates of the user in round-robin best-response through these deviations, the authors obtained a recursive updating formula for the deviation of the best-response policies from their equilibrium values. In this formula, analyzing the matrix coefficient of transition to the subsequent round, they have shown that all its eigenvalues are in the interior of the unit disk, i.e. spectral radius of the matrix is lower than unity, which implies convergence of the Round Robin update scheme to the unique Nash equilibrium.

Following a similar analysis, Altman et al., 2001 also showed convergence result for the *Round Robin in blocks of two* that is a Round Robin scheme in which the users (of even number) update their policies in pairs, i.e., first players 1 and 2 update, then players 3 and 4, and so on.

The type of formulation of a routing game used by Orda et al., 1993 and Altman et al., 2001, assumed that the users may have different traffic demands. For routing games with several players when the players have the same amount of traffic to route through the network from the same source to the same destination and when the players use the same *type-A cost functions*, Orda et al., 1993 showed that there exists a unique Nash equilibrium. In this case, the convergence of the best-response dynamics to the Nash equilibrium follows from the fact that the symmetric game is a potential game (Monderer et al., 1996). A fascinating property of a potential game is that *the incentive of all players to change their strategies can be expressed using a single global function called the potential function* (Han, 2007, p.235). First, this concept was proposed by Rosenthal, 1973, and then Monderer et al., 1996 made a characterization of games that have a potential function. In a potential game, the Nash equilibrium corresponds to the minimum of a convex optimization problem. Specifically, if one consider the first order optimality conditions (Karush-Kuhn-Tucker (KKT) conditions, Kuhn et al., 1951; Karush, 1939) of each of the player problems under an equilibrium and sum them up, one gets the KKT conditions of the problem expressed by a potential function of the game. Reasoning so, Cominetti et al., 2009 have provided a potential function for symmetric game with atomic players and shown that the game is a potential one (Monderer et al., 1996).

Meanwhile, Orda et al., 1993 provide counterexamples for non- uniqueness of the equilibrium. The routing problems formulated by Orda et al., 1993 are therefore not pliable in general, and they may not always enjoy the structure of a potential games. The powerful properties of a potential game motivates to define conditions on the structure of a player's cost function that allow one to construct a potential function of the game. Altman et al., 2007 have shown that such conditions are provided by the case of linear link costs. By identifying a potential structure for the game they obtained convergence of the *Asynchronous Best-Response Update* to the unique Nash equilibrium. Under this update rule, at each time one player updates its strategy to be the best response against the current strategy of the other players, and the set of times at which a player updates its strategy is infinite.

More recently, Mertzios has proven that, for the large class of edge latency functions introduced in (Orda et al., 1993), the two-player splittable routing game converges to the unique Nash equilibrium in a logarithmic number of steps (Mertzios, 2009). His proof of convergence also relies on a potential-based argument. Namely, he shows that the amount of flow that is reallocated in the network at each step is strictly decreasing. Unfortunately, this argument does not seem to readily extend to more than two players. We also refer to Goemans et al., 2005; Fabrikant et al., 2004; Even-Dar et al., 2003 for convergence results on related, but different, problems.

We propose a different approach to study the convergence of best-response dynamics. The oncoming sections describe the model and explain the Non-Linear Spectral Radius Approach for the convergence problem. Then the convergence results are presented with use of this approach.

3.3 Problem Statement

3.3.1 Notations

In the following, $\mathbb{R}_+$ denotes the set of non-negative real numbers. Recall that the 1-norm of a vector $\mathbf{x} \in \mathbb{R}^S$ is $\|\mathbf{x}\|_1 = \sum_{i=1}^{S} |x_i|$. For $\mathbf{x} \in \mathcal{X}$, $\mathcal{B}_o(\mathbf{x}, r)$ will denote the open ball of radius r centered at point $\mathbf{x}$, i.e., $\mathcal{B}_o(\mathbf{x}, r) = \{\mathbf{z} \in \mathcal{X} : \|\mathbf{x} - \mathbf{z}\|_1 < r\}$. Let $\mathbf{1}$ denote the column vector $(1, 1, \ldots, 1)^T$.

We let I and 0 denote the identity and the zero matrices, respectively (their sizes will be clear from the context). A matrix A is positive, and we write $A \geq 0$, if and only if $a_{i,j} \geq 0$, $\forall i, j$, and that it is negative if $-A$ is positive. We recall that the 1-norm of a matrix A is $\|A\|_1 = \max_j \sum_i |a_{ij}|$. Denote by $\sigma(A)$ the spectrum of the matrix A, i.e., $\sigma(A) = \{\lambda \in \mathbb{R} : \exists \mathbf{x} \neq 0, A\mathbf{x} = \lambda\mathbf{x}\}$, by $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$ its spectral radius, and we recall that $\rho(A) \leq \|A\|_1$. If $A_1, \dots, A_n$ is a collection of matrices, we denote by $\prod_{i=1}^n A_i$ the product $A_n A_{n-1} \dots A_1$.

For any function f that is differentiable at point $\mathbf{x}$, we denote by $\mathrm{D}f(\mathbf{x})$ its Jacobian matrix at $\mathbf{x}$.

3.3.2 Non-cooperative routing game

We investigate a non-cooperative routing game with K routing agents and S links in which each routing agent can control how its own traffic is routed over the parallel links. This routing game is depicted on Figure 3.1.

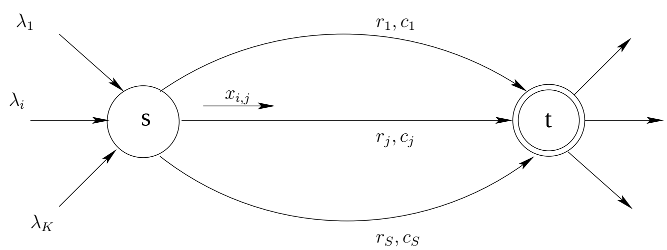

Figure 3.1: Traffic classes route their packets over parallel links.

Denote by $\mathcal{S} = \{1, \dots, S\}$ the set of links. Link $j \in \mathcal{S}$ has capacity r_j and a holding cost c_j per unit time is incurred for each packet sent on this link. We let $\pi_j = c_j / r_j$ denote the cost per unit capacity for link j.

We let $\mathcal{C} = \{1, \dots, K\}$ be the set of routing agents and λ_i be the traffic intensity of routing agent i. We shall also refer to routing agent i as traffic class i, or user i. Each class can control how its own traffic is splitted over the parallel links and seeks to minimize its own cost. Let $\mathbf{x}_i = (x_{i,j})_{j \in \mathcal{S}}$ denote the routing strategy of class i, with $x_{i,j}$ being the amount of traffic it sends over link j. We let $\mathcal{X}_i$ denote the set of routing strategies for class i, i.e., the set of vectors $\mathbf{x}_i \in \mathbb{R}^S$ such that $0 \leq x_{i,j} < r_j$ for all $j \in \mathcal{S}$, and $\sum_{j \in \mathcal{S}} x_{i,j} = \lambda_i$.

A strategy profile is a choice of a routing strategy for each user such that the stability condition $\sum_{i \in \mathcal{C}} x_{i,j} < r_j$ is satisfied for all links $j \in \mathcal{S}$. It is thus a vector $\mathbf{x} = (\mathbf{x}_i)_{i \in \mathcal{C}}$ belonging to the product strategy space $\mathcal{X} = \bigotimes_{i \in \mathcal{C}} \mathcal{X}_i$ such that $\sum_{i \in \mathcal{C}} x_{i,j} < r_j$, for all $j \in \mathcal{S}$. It will be assumed that $\sum_{i \in \mathcal{C}} \lambda_i < \sum_{j \in \mathcal{S}} r_j$, so that $\mathcal{X} \neq \emptyset$.

Finally, let $\mathbf{x}_{-i}$ denote the vector $(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_K)$. This vector gives the strategies of all players other than player i, and belongs to the set $\mathcal{X}_{-i}$ of vectors $\mathbf{x}$ such that $\mathbf{x} \in \bigotimes_{k \neq i} \mathcal{X}_k$ and $\sum_{k \neq i} x_{k,j} < r_j$ for all $j \in \mathcal{S}$.

The optimization problem solved by class i, which depends on the routing decisions of the other classes, can be formulated as follows:

$$\text{minimize } T_i(\mathbf{x}, \mathbf{x}_{-i}) = \sum_{j \in \mathcal{S}} \pi_j \, x_{i,j} \, \phi(\rho_j) \tag{BR-i}$$

subject to

$$\mathbf{x} \in \mathcal{X}_i, \tag{3.1}$$

$$y_j = x_{i,j} + \sum_{k \neq i} x_{k,j}, \quad \forall j \in \mathcal{S}, \tag{3.2}$$

$$\rho_j = y_j / r_j, \quad \forall j \in \mathcal{S}, \tag{3.3}$$

$$\rho_j < 1, \quad \forall j \in \mathcal{S}, \tag{3.4}$$

$$\tag{3.5}$$

In the above formulation, y_j represents the total traffic offered to link j, ρ_j is the utilization rate of this link, and ϕ is the cost associated to the link when there is a traffic of y_j flowing through it. In transportation or communication networks, ϕ models the delay on the road or the link. The total cost incurred by user i is then the sum of the cost of individual links weighted by the amount of traffic the user sends on each of the links. Thus, given the strategies of the others, user i seeks to minimize its total cost subject to flow conservation and stability constraints.

Assumption 1. *We shall make the following assumptions on the cost function ϕ:*

(A_1) $\phi : [0, 1) \to [0, \infty)$,

(A_2) $\lim_{\rho \to 1^-} \phi(\rho) = +\infty$,

(A_3) *continuous, strictly increasing, convex function, and is twice continuously differentiable.*

Remark 1. *At first glance, it appears that the assumptions are not loose enough to include polynomial cost functions, which are widely used in transportation networks. However, it will be shown in that any function satisfying*

(B_1) $\phi : [0, \infty) \to [0, \infty)$,

(B_2) $\lim_{\rho \to \infty} \phi(\rho) = +\infty$, *and*

(B_3) (A_3),

has an equivalent function which satisfies assumptions (A_1)–(A_3)*. Two functions are said to be equivalent if the solution of* (BR-i) *with one function is also the solution of* (BR-i) *with the other. Thus, results obtained for functions satisfying* (A_1)–(A_3) *will be applicable to functions that satisfy* (B_1)–(B_3)*.*

We note that $\forall \mathbf{x}_{-i} \in \mathcal{X}_{-i}$, there exists a non-empty subset of $\mathcal{X}_i$ on which Problem (BR-i) is well-defined. It follows from the assumption that $\sum_{i \in \mathcal{C}} \lambda_i < \sum_{j \in \mathcal{S}} r_j$.

3.3.3 Nash equilibrium

A Nash equilibrium of the routing game is a strategy profile from which no class finds it beneficial to deviate unilaterally. Hence, $\mathbf{x}^* \in \mathcal{X}$ is a Nash Equilibrium Point (NEP) if $\mathbf{x}_i^*$ is an optimal solution of problem (BR-i) for all classes $i \in \mathcal{C}$, that is, if

$$\mathbf{x}_i^* = \arg\min_{\mathbf{z} \in \mathcal{X}_i} T_i(\mathbf{z}, \mathbf{x}_{-i}^*), \quad \forall i \in \mathcal{C},$$

where $\mathbf{x}_{-i}^*$ is the vector of strategies of all players other than player i at the NEP.

It follows from our assumptions on the function ϕ, that the link cost functions are a special case of type-B functions, as defined in (Orda et al., 1993). As proved in Theorem 2.1 of this reference, this implies the existence of a unique NEP for our routing game. In the following, we shall denote by $\mathbf{x}^*$ this Nash equilibrium point.

3.3.4 Best response dynamics

The best-response of player is defined as its optimal strategy conditioned on the strategies of the other players. It is, as the name suggests, the best response that the player can give for a given strategy of the others. Let $x^{(u)} : \mathcal{X} \to \mathcal{X}$, defined as

$$x^{(u)}(\mathbf{x}) = \left(\arg\min_{\mathbf{z} \in \mathcal{X}_u} T_u(\mathbf{z}, \mathbf{x}_{-u}), \mathbf{x}_{-u} \right), \tag{3.6}$$

be the best-response of user u to the strategy $\mathbf{x}_{-u}$ of the other players. From the definition of T_u, it can be shown that for each $\mathbf{x} \in \mathcal{X}$, there is a unique $x^{(u)}(\mathbf{x})$. Given a point $\mathbf{x} \in \mathcal{X}$, the strategy profile $x^{(u)}(\mathbf{x})$ describes the strategies of all the players after the best response of user u.

Best-response dynamics then consists of players taking turns in some order to adapt their strategy based on the most recent known strategy of the others (without considering the effect on future play in the game).

Define a *round* to be a sequence of best-responses in which each player plays exactly once. Once an order is fixed in the first round, it is assumed to be the same in each subsequent round. The order in which the players best-respond in the first-round can be arbitrary. Let us fix this order to be 1, 2, …, K.

Define $\hat{x}^{(1)} : \mathcal{X} \to \mathcal{X}$ as

$$\hat{x}^{(1)}(\mathbf{x}) = x^{(K)} \circ x^{(K-1)} \circ \ldots \circ x^{(1)}(\mathbf{x}), \tag{3.7}$$

be the point reached from $\mathbf{x}$ after one round of play. One can recursively define

$$\hat{x}^{(n)}(\mathbf{x}) = \hat{x}^{(1)} \circ \hat{x}^{(n-1)}(\mathbf{x}), \tag{3.8}$$

which is the point reached after n rounds.

The best-response dynamics can then be defined as the sequence $\{\hat{x}^{(n)}(\mathbf{x}_0)\}_{n\geq 1}$ corresponding to the strategy of players after each round of best-response when $\mathbf{x}_0$ is the initial strategy. A NEP has the property that each player's strategy is a best-response to strategies of the other players. Therefore if $\mathbf{x}_0$ is a NEP then sequence will remain at $\mathbf{x}_0$.

The main question we seek to answer is: do the best-response dynamics for the routing game converge from any starting point? If it converges, then it converges to the Nash equilibrium point.

3.4 The Non-linear Spectral Radius Approach

A usual method to prove the convergence of iterates of an operator $\hat{x}^{(1)} : \mathcal{X} \to \mathcal{X}$ is to show that this operator is a contraction. For this, one needs to find a suitable norm, say $\|\cdot\|$, for which there exists a constant $c \in [0, 1)$ such that

$$\|\hat{x}^{(1)}(\mathbf{x}) - \hat{x}^{(1)}(\mathbf{y})\| \leq c\|\mathbf{x} - \mathbf{y}\|,$$

for every pair of points $\mathbf{x}$ and $\mathbf{y}$ in the set $\mathcal{X}$. The contraction condition says that the distance between iterates of the function starting from two different points decreases with each iteration. The constant c depends on the norm, and for a continuously differentiable operator, it can be computed as $\sup_{\mathbf{x}}\|D\hat{x}^{(1)}(\mathbf{x})\|$, which is the supremum of the Jacobian over all points in the domain of the operator. It is then sufficient to find a norm in which the above condition is satisfied.

For the best-response function, it turns out that it is non-trivial to find such a norm, independently of the starting point, in which the distance decreases with every iteration. Instead, as will be seen later it will be sufficient to find a norm in which the distance decreases asymptotically and not with every iteration. This weaker condition can be formalized using the notion of the *non-linear spectral radius* described below.

For a function $f : \mathcal{X} \to \mathcal{X}$, define the set

$$\mathcal{J}(f) = \{Df(x) : f \text{ is differentiable at } x\}. \tag{3.9}$$

which is the set of Jacobian matrices of the function f evaluated at all points at which f

is differentiable.

Definition 1. *The non-linear spectral radius of a function $f : \mathcal{X} \to \mathcal{X}$ is defined as (Mak et al., 2007):*

$$\bar{\rho}(f) = \lim_{n\to\infty} \sup \sup_{A_i \in \mathcal{J}(f)} \left\| \prod_{i=1}^{n} A_i \right\|^{1/n}.$$

The non-linear spectral radius of f is related to the notion of *joint spectral radius* of a set $\mathcal{M}$ of matrices which is defined as:

$$\hat{\rho}(\mathcal{M}) = \lim_{n\to\infty} \sup \sup_{M_i \in \mathcal{M}} \left\| \prod_{i=1}^{n} M_i \right\|^{1/n}, \tag{3.10}$$

and is independent of the induced matrix norm. It measures the worst case growth rate of a sequence of linear transformations that are taken from the set $\mathcal{M}$. It can been seen that the non-linear spectral radius of f is in fact the joint spectral radius of the set of Jacobian matrices of f, $\mathcal{J}(f)$.

When there is only one matrix in $\mathcal{M}$, from Gelfand's formula it follows that the joint spectral radius is equal to the spectral radius of that matrix. For a set with several matrices, there is an equivalent result in terms of the *generalized spectral radius* of $\mathcal{M}$ which is defined as:

$$\rho(\mathcal{M}) = \limsup_{n\to\infty} \sup_{M_i \in \mathcal{M}} \rho \left(\prod_{i=1}^{n} M_i \right)^{\frac{1}{n}}, \tag{3.11}$$

where $\rho(A)$ is the spectral radius of the matrix A. If $\mathcal{M}$ is bounded then the generalized spectral radius and the joint spectral radius of $\mathcal{M}$ are equal (Berger et al., 1992).

Consider a linear dynamical system of the form

$$x_{n+1} = A_{i(n)} x_n,$$

where the matrices $A_i \in \mathcal{M}$ can be chosen differently in each step. Such a system is called a switched linear system. When all the matrices are the same, one can determine the stability of such a system by checking whether the spectral radius of this matrix is less than 1 or not. In case of switched linear systems, the same condition with the joint spectral radius in place of the spectral radius can be used to ascertain the stability of the system, see for example (Theys, 2005).

For non-linear operators, the following convergence criterion was stated in (Mak et al., 2007).

Theorem 1 (Mak et al., 2007, Theorem 1)**.** *If $f : \mathcal{X} \to \mathcal{X}$ is Lipschitz-continuous and has a non-linear spectral radius smaller than* 1*, then the iterates of f are globally asymptotically stable. Moreover, the rate of exponential decay, r, satisfies $0 < r \leq -\log(\bar{\rho}(f))$.*

Thus, instead of requiring the best-response to be a contraction, one can show the convergence of the best-response dynamics by showing that:

1. $\hat{x}^{(1)}$ is Lipschitz-continuous; and

2. $\bar{\rho}(\hat{x}^{(1)}) < 1$.

In the rest of this section, first we shall show a few properties of the best-response function, and then compute the structure of its Jacobian matrices, before arriving at our main result.

3.4.1 Properties of the best-response function

The purpose of this section is to establish various properties of best-response function, mainly related to its continuity and differentiability. Let us define

$$\mathcal{S}_u(\mathbf{x}) = \{j \in \mathcal{S} : x_{u,j}^{(u)}(\mathbf{x}) > 0\} \tag{3.12}$$

as the set of links used by player u in its best-response to the strategies $\mathbf{x}_{-u}$ of other players. We have the following result.

Theorem 2. *The best-response function $x^{(u)}$ of player u is Lipschitz-continuous on $\mathcal{X}$ with*

$$\|x^{(u)}(\mathbf{z}) - x^{(u)}(\mathbf{w})\|_1 < 2\,\|\mathbf{z} - \mathbf{w}\|_1, \quad \forall \mathbf{z}, \mathbf{w} \in \mathcal{X}. \tag{3.13}$$

Proof. Consider two points $\mathbf{z}$ and $\mathbf{w}$ in $\mathcal{X}$. Let the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}_+^S$ be such that $a_j = \sum_{i \neq u} z_{i,j}$ and $b_j = \sum_{i \neq u} w_{i,j}$ for all $j \in \mathcal{S}$. In other words, a_j and b_j are the total traffic sent on link j by users other than u in configurations $\mathbf{z}$ and $\mathbf{w}$, respectively. To simplify notations, we denote by $x_{u,j}^z$ and $x_{u,j}^w$ the traffic sent on link j by player u after his best-response at points $\mathbf{z}$ and $\mathbf{w}$, respectively, that is $x_{u,j}^z = x_{u,j}^{(u)}(\mathbf{z})$ and $x_{u,j}^w = x_{u,j}^{(u)}(\mathbf{w})$. For the purpose of the proof, we also define

$$f_j(x, y) = \pi_j \left(\phi\left(\frac{x+y}{r_j}\right) + \frac{x}{r_j} \phi'\left(\frac{x+y}{r_j}\right) \right),$$

for all links $j \in \mathcal{S}$. Then the marginal costs of player u on link j after the best-response of that player at points $\mathbf{z}$ and $\mathbf{w}$ can be written as $g_{u,j}(x^{(u)}(\mathbf{z})) = f_j(x_{u,j}^z, a_j)$ and $g_{u,j}(x^{(u)}(\mathbf{w})) = f_j(x_{u,j}^w, b_j)$. From the KKT conditions, there exist μ_z and μ_w such that $f_j(x_{u,j}^z, a_j) \geq \mu_z$, with equality if $j \in \mathcal{S}_u(\mathbf{z})$, and $f_j(x_{u,j}^w, b_j) \geq \mu_w$, with equality if $j \in \mathcal{S}_u(\mathbf{w})$. Without loss of generality, we assume that $\mu_z \geq \mu_w$. As a consequence, we have

$$f_j(x_{u,j}^z, a_j) \geq f_j(x_{u,j}^w, b_j), \quad \forall j \in \mathcal{S}_u(\mathbf{w}). \tag{3.14}$$

Consider now the sets

$$\mathcal{S}^- = \left\{ j \in \mathcal{S} \ : x_{u,j}^z < x_{u,j}^w \ \right\}, \tag{3.15}$$

and

$$\mathcal{S}^+ = \left\{ j \in \mathcal{S} \ : x_{u,j}^z \geq x_{u,j}^w \ \right\}. \tag{3.16}$$

Assume first that $\mathcal{S}^- = \emptyset$. Then $x_{u,j}^z \geq x_{u,j}^w$ for all $j \in \mathcal{S}$. However, since

$$\sum_{j \in \mathcal{S}} x_{u,j}^z = \sum_{j \in \mathcal{S}} x_{u,j}^w = \lambda_u, \tag{3.17}$$

this implies that $x_{u,j}^z = x_{u,j}^w$ for all $j \in \mathcal{S}$. It yields

$$\sum_{j \in \mathcal{S}} \left| x_{u,j}^z - x_{u,j}^w \right| = 0 \tag{3.18}$$

Assume now that $\mathcal{S}^- \neq \emptyset$. Since $\mathcal{S} = \mathcal{S}^- \bigcup \mathcal{S}^+$, we obtain from (3.17) that

$$\sum_{j \in \mathcal{S}^+} \left(x_{u,j}^z - x_{u,j}^w \right) = - \sum_{j \in \mathcal{S}^-} \left(x_{u,j}^z - x_{u,j}^w \right), \tag{3.19}$$

which leads to

$$\sum_{j \in \mathcal{S}} \left| x_{u,j}^z - x_{u,j}^w \right| = 2 \sum_{j \in \mathcal{S}^-} \left| x_{u,j}^z - x_{u,j}^w \right|. \tag{3.20}$$

For $j \in \mathcal{S}^-$, we have by definiton $0 \leq x_{u,j}^z < x_{u,j}^w$, and hence $j \in \mathcal{S}_u(\mathbf{w})$. Thus, $\mathcal{S}^- \subset \mathcal{S}_u(\mathbf{w})$. With (3.14), it yields $f_j(x_{u,j}^z, a_j) \geq f_j(x_{u,j}^w, b_j)$, and thus

$$\phi\left(\frac{x_{u,j}^z + a_j}{r_j}\right) + \frac{x_{u,j}^z}{r_j}\phi'\left(\frac{x_{u,j}^z + a_j}{r_j}\right) \geq \phi\left(\frac{x_{u,j}^w + b_j}{r_j}\right) + \frac{x_{u,j}^w}{r_j}\phi'\left(\frac{x_{u,j}^w + b_j}{r_j}\right),$$

for all $j \in \mathcal{S}^-$. However, since for $j \in \mathcal{S}^-$ we have $x_{u,j}^z < x_{u,j}^w$ and since ϕ and ϕ' are strictly increasing, this implies that $x_{u,j}^z + a_j > x_{u,j}^w + b_j$, from which we deduce that

$$0 < x_{u,j}^w - x_{u,j}^z < a_j - b_j \quad \forall j \in \mathcal{S}^-. \tag{3.21}$$

It yields

$$\sum_{j \in \mathcal{S}^-} \left| x_{u,j}^z - x_{u,j}^w \right| < \sum_{j \in \mathcal{S}^-} |a_j - b_j| \tag{3.22}$$

With (3.20), we thus obtain

$$\sum_{j \in \mathcal{S}} \left| x_{u,j}^z - x_{u,j}^w \right| < 2 \sum_{j \in \mathcal{S}^-} |a_j - b_j| \leq 2 \sum_{j \in \mathcal{S}} |a_j - b_j| \tag{3.23}$$

From (3.18) and (3.23), we obtain that, whether $\mathcal{S}^-$ be empty or not, we have

$$\begin{aligned}\sum_{j\in\mathcal{S}} \left|x^z_{u,j} - x^w_{u,j}\right| &< 2\sum_{j\in\mathcal{S}}\left|\sum_{i\neq u} z_{i,j} - \sum_{i\neq u} w_{i,j}\right| \\ &< 2\sum_{j\in\mathcal{S}}\sum_{i\neq u}|z_{i,j} - w_{i,j}| \\ &< 2\sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{S}}|z_{i,j} - w_{i,j}| \end{aligned} \tag{3.24}$$

Since $x^{(u)}_{i,j}(\mathbf{z}) = x^{(u)}_{i,j}(\mathbf{w})$ for all $j \in \mathcal{S}$ and all $i \neq u$, we also have

$$\sum_{i\neq u}\sum_{j\in\mathcal{S}}\left|x^{(u)}_{i,j}(\mathbf{z}) - x^{(u)}_{i,j}(\mathbf{w})\right| = 0 \tag{3.25}$$

Finally, from (3.24) and (3.25), we conclude that

$$\sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{S}}\left|x^{(u)}_{i,j}(\mathbf{z}) - x^{(u)}_{i,j}(\mathbf{w})\right| < 2\sum_{i\in\mathcal{C}}\sum_{j\in\mathcal{S}}|z_{i,j} - w_{i,j}|, \tag{3.26}$$

that is,

$$\|x^{(u)}(\mathbf{z}) - x^{(u)}(\mathbf{w})\|_1 < 2\,\|\mathbf{z} - \mathbf{w}\|_1, \tag{3.27}$$

as claimed. □

Corollary 1. *Since the best-response over one round, $\hat{x}^{(1)}$, is a composition of best-responses of each of the players (cf. (3.6)), it then follows that $\hat{x}^{(1)}$ is Lipschitz continuous.*

Remark 2. *The continuity of the best-response functions is a direct consequence of Berge's Theorem on the continuity of correspondences (Berge, 1959, see also page 64 of Border, 1985). However, Lipschitz continuity requires some more work than that.*

Once the Lipschitz continuity of $\hat{x}^{(1)}$ has been established, it remains to be shown that its non-linear spectral radius is smaller than 1. For this, we shall investigate the points at which the $\hat{x}^{(1)}$ is differentiable and compute the structure of its Jacobian.

We note that, according to Rademacher's theorem (Evans et al., 1992), a consequence of Theorem 2 is that the best-response function $x^{(u)}$ is Fréchet-differentiable almost everywhere in $\mathcal{X}$; that is, the points in $\mathcal{X}$ at which $x^{(u)}$ is not differentiable form a set of Lebesgue measure zero. To compute the points at which the derivative is defined, we shall need the following definitions:

- Let
$$g_{i,j}(\mathbf{x}) = \frac{\partial T_i}{\partial x_{i,j}}(\mathbf{x}) = \pi_j\left(\phi\left(\frac{y_j}{r_j}\right) + \frac{x_{i,j}}{r_j}\phi'\left(\frac{y_j}{r_j}\right)\right), \tag{3.28}$$

where $y_j = \sum_k x_{k,j}$, be the marginal cost of player i on link j under strategy profile $\mathbf{x}$.

We say that link j is *marginally used* by user u at point $\mathbf{x}$ whenever the flow of user u on that link is 0 although the marginal cost of that player on that link is minimum, that is

$$x_{u,j} = 0 \text{ and } g_{u,j}(\mathbf{x}) = \min_{k \in \mathcal{S}} g_{u,k}(\mathbf{x}). \tag{3.29}$$

- we say that the set $\mathcal{S}_u(\mathbf{x})$ is *locally stable* at point $\mathbf{x}$ if it does not change for an infinitesimal variation on the strategies of the other players, that is

$$\exists \epsilon > 0, \forall \mathbf{z} \in \mathcal{B}_o(\mathbf{x}, \epsilon), \mathcal{S}_u(\mathbf{x}) = \mathcal{S}_u(\mathbf{z}). \tag{3.30}$$

From our assumptions on the function ϕ, the continuity of the best-response functions imply that of the marginal costs $g_{i,j}$ defined in (3.28) under the best-response dynamics. In the following, we say that no link is marginally used by user u in its best-response at point $\mathbf{x}$ if there is no link that is marginally used by user u at point $x^{(u)}(\mathbf{x})$. The two notions introduced above are related through the following result.

Lemma 1. *if there is no link that is marginally used by player u in its best-response at point $\mathbf{x}$, then the set of links $\mathcal{S}_u(\mathbf{x})$ is locally stable at point $\mathbf{x}$.*

Proof. Let Ω_u be the set of points $\mathbf{x} \in \mathcal{X}$ where $\mathcal{S}_u(\mathbf{x})$ is locally stable. Let us define

$$f_j(x, y) = \pi_j \left(\phi(\frac{x+y}{r_j}) + \frac{x}{r_j} \phi'(\frac{x+y}{r_j}) \right),$$

for all links $j \in \mathcal{S}$. Note that $f_j(x, y)$ is continuous and strictly increasing in both x and y. Then the marginal cost of player u on link j after the best-response of that player can be written as $g_{u,j}(x^{(u)}(\mathbf{x})) = f_j(x^{(u)}_{u,j}(\mathbf{x}), \sum_{k \neq u} x_{k,j})$. From the KKT conditions, the function $\mu : \mathcal{X}_{-u} \to \mathbb{R}$ defined by

$$\mu(\mathbf{x}_{-u}) = \min_{j \in \mathcal{S}} g_{u,j}(x^{(u)}(\mathbf{x}))$$

is such that

$$j \in \mathcal{S}_u(\mathbf{x}) \iff f_j(0, \sum_{k \neq u} x_{k,j}) < \mu(\mathbf{x}_{-u}). \tag{3.31}$$

Note that the continuity of the best-response function $x^{(u)}$ on $\mathcal{X}$ (cf. Theorem 2) implies that of the marginal costs, and therefore the continuity of μ on $\mathcal{X}_{-u}$.

Let $\mathbf{x}$ be a point such that no link is marginally used by player u in its best-response at point $\mathbf{x}$. Let us first consider $j \in \mathcal{S}_u(\mathbf{x})$. From (3.31), there exists $\delta > 0$ such that

$f_j(0, \sum_{k \neq u} x_{k,j}) \leq \mu(\mathbf{x}_{-u}) - \delta$. Since $f_j(x, y)$ is continuous in y and $\mu(\mathbf{x}_{-u})$ is continuous on $\mathcal{X}_{-u}$, there exists $\epsilon_1 > 0$ such that, for all $\mathbf{z} \in \mathcal{B}_o(\mathbf{x}, \epsilon_1)$,

$$f_j(0, \sum_{k \neq u} z_{k,j}) < f_j(0, \sum_{k \neq u} x_{k,j}) + \frac{\delta}{2} < \mu(\mathbf{x}_{-u}) - \frac{\delta}{2},$$

and $\mu(\mathbf{z}_{-u}) > \mu(\mathbf{x}_{-u}) - \frac{\delta}{2}$. It yields

$$f_j(0, \sum_{k \neq u} z_{k,j}) < \mu(\mathbf{x}_{-u}) - \frac{\delta}{2} < \mu(\mathbf{z}_{-u}), \quad \forall \mathbf{z} \in \mathcal{B}_o(\mathbf{x}, \epsilon_1),$$

and thus, according to (3.31), we have $j \in \mathcal{S}_u(\mathbf{z})$ for all $\mathbf{z} \in \mathcal{B}_o(\mathbf{x}, \epsilon_1)$ if $j \in \mathcal{S}_u(\mathbf{x})$. As a consequence, if $\mathcal{S}_u(\mathbf{x}) = \mathcal{S}$, then $\mathcal{S}_u(\mathbf{z}) = \mathcal{S}$ for all $\mathbf{z}$ sufficiently close to $\mathbf{x}$, and thus $\mathbf{x} \in \Omega_u$.

Otherwise we can find $j \in \mathcal{S} \setminus \mathcal{S}_u(\mathbf{x})$. Since no link is marginally used by player u in its best-response at point $\mathbf{x}$, there exists $\beta > 0$ such that

$$f_j(0, \sum_{k \neq u} x_{k,j}) \geq \mu(\mathbf{x}_{-u}) + \beta, \quad \forall j \in \mathcal{S} \setminus \mathcal{S}_u(\mathbf{x}). \tag{3.32}$$

Proceeding as above, we can show that there exists $\epsilon_2 > 0$ such that, for all $\mathbf{z} \in \mathcal{B}_o(\mathbf{x}, \epsilon_2)$, $\mu(\mathbf{z}_{-u}) < \mu(\mathbf{x}_{-u}) + \frac{\beta}{2}$ and

$$f_j(0, \sum_{k \neq u} z_{k,j}) > f_j(0, \sum_{k \neq u} x_{k,j}) - \frac{\beta}{2} > \mu(\mathbf{x}_{-u}) + \frac{\beta}{2},$$

from which we conclude that $f_j(0, \sum_{k \neq u} z_{k,j}) > \mu(\mathbf{z}_{-u})$, for all $\mathbf{z} \in \mathcal{B}_o(\mathbf{x}, \epsilon_2)$. This implies that if $j \notin \mathcal{S}_u(\mathbf{x})$, then $j \notin \mathcal{S}_u(\mathbf{z})$ for all $\mathbf{z} \in \mathcal{B}_o(\mathbf{x}, \epsilon_2)$.

Choosing $\epsilon = \min(\epsilon_1, \epsilon_2)$, we thus conclude that for all $\mathbf{z} \in \mathcal{B}_o(\mathbf{x}, \epsilon)$, $\mathcal{S}_u(\mathbf{x}) \subset \mathcal{S}_u(\mathbf{z})$ and $\mathcal{S} \setminus \mathcal{S}_u(\mathbf{x}) \subset \mathcal{S} \setminus \mathcal{S}_u(\mathbf{z})$, which is equivalent to $\mathcal{S}_u(\mathbf{x}) = \mathcal{S}_u(\mathbf{z})$. This shows that if no link is marginally used by player u in its best-response at point $\mathbf{x}$, then $\mathcal{S}_u(\mathbf{x})$ is locally stable. □

Our first result regarding the differentiability of best-response functions is the following.

Proposition 1. *The best-response function $x^{(u)}$ is differentiable at every point $\mathbf{x} \in \mathcal{X}$ such that no link is marginally used by player u in its best-response at point $\mathbf{x}$.*

Proof. From Lemma 1, we know that if $\mathbf{x}$ is such that no link is marginally used by user u in its best-response at point $\mathbf{x}$, then the set of links $\mathcal{S}_u(\mathbf{x})$ is locally stable at $\mathbf{x}$. As shown in Theorem 3, this condition is sufficient to compute the partial derivatives of $x^{(u)}$ at point $\mathbf{x}$. It can be seen from (3.60) and (3.61) that the partial derivatives $\frac{\partial x_i^{(u)}}{\partial \mathbf{x}_v}(\mathbf{x})$, $i \neq u, v \in \mathcal{C}$, and $\frac{\partial x_u^{(u)}}{\partial \mathbf{x}_u}(\mathbf{x})$ are contiuous at $\mathbf{x}$. According to Lemma 2, the continuity of the partial derivatives $\frac{\partial x_{u,i}^{(u)}}{\partial \mathbf{x}_v}(\mathbf{x})$ at $\mathbf{x}$ for $i \notin \mathcal{S}_u(\mathbf{x})$ follows from the local stability of $\mathcal{S}_u(\mathbf{x})$ at $\mathbf{x}$. Finally, a closed-form formula is given in (3.35) for the partial derivatives $\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}}$ for $v \neq u$ and for $i \in \mathcal{S}_u(\mathbf{x}), k \in \mathcal{S}$. In view of equations (3.43)-(3.47), the continuity of these partial derivatives follows from our assumptions on ϕ and from the continuity of $x^{(u)}$ at $\mathbf{x}$. Thus, all partial derivatives of $x^{(u)}$ exist and are continuous at $\mathbf{x}$, and therefore $x^{(u)}$ is continuously differentiable at $\mathbf{x}$.

□

3.4.2 Structure of the Jacobian matrices

The Jacobian matrix of $\hat{x}^{(1)}$ is the product of Jacobian matrices of best-responses of individual players. So, we shall start by computing the Jacobian of the best-response functions of individual players.

Consider a player u and a point $\mathbf{x} \in \mathcal{X}$ at which $x^{(u)}$ is differentiable. The Jacobian matrix of this function is then the block matrix

$$\mathrm{D}x^{(u)}(\mathbf{x}) = \begin{pmatrix} \frac{\partial x_1^{(u)}}{\partial \mathbf{x}_1}(\mathbf{x}) & \dots & \frac{\partial x_1^{(u)}}{\partial \mathbf{x}_K}(\mathbf{x}) \\ \vdots & & \vdots \\ \frac{\partial x_K^{(u)}}{\partial \mathbf{x}_1}(\mathbf{x}) & \dots & \frac{\partial x_K^{(u)}}{\partial \mathbf{x}_K}(\mathbf{x}) \end{pmatrix},$$

where the (i, j)-block $\frac{\partial x_i^{(u)}}{\partial \mathbf{x}_j}(\mathbf{x})$ measures the sensitivity of the strategy of player i obtained after the best response of player u with respect to a change in the strategy of player j.

The best-response of a player u is sensitive only to the strategies of the other players $v \neq u$, and these sensitivities are reflected by the block matrices $\frac{\partial x_u^{(u)}}{\partial \mathbf{x}_v}$ which appear in the uth row of the Jacobian matrix. Recalling that

$$\frac{\partial x_u^{(u)}}{\partial \mathbf{x}_v}(\mathbf{x}) = \left(\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,j}}(\mathbf{x})\right)_{i\in\mathcal{S}, j\in\mathcal{S}}, \tag{3.33}$$

we shall distinguish between links $i \notin \mathcal{S}_u(\mathbf{x})$ and links $i' \in \mathcal{S}_u(\mathbf{x})$. We assume in the following that the set $\mathcal{S}_u(\mathbf{x})$ is locally stable (cf. Section 3.4.1), and thus that it does not change for an infinitesimal variation on the strategy $\mathbf{x}_v$ of player $v \in \mathcal{C}$.

Lemma 2. *For all links $i \notin \mathcal{S}_u(\mathbf{x})$,*

$$\frac{\partial x_{u,i}^{(u)}}{\partial \mathbf{x}_v}(\mathbf{x}) = 0, \quad \forall v \in \mathcal{C}, \tag{3.34}$$

Proof. The proof follows from the assumption that $\mathcal{S}_u(\mathbf{x})$ is locally stable at $\mathbf{x}$. We have $x_{u,i}^{(u)}(\mathbf{x} + h\mathbf{y}) = x_{u,i}^{(u)}(\mathbf{x}) = 0$ for any vector $\mathbf{y}$ and $h > 0$ sufficiently small. This implies that the directional derivatives of $x_{u,i}^{(u)}$, and thus its partial derivatives, are 0. $\square$

For links $i \in \mathcal{S}_u(\mathbf{x})$, we have:

Lemma 3. *There exist a vector $\boldsymbol{\theta} \in \mathbb{R}_+^S$ and a vector $\boldsymbol{\gamma} \in \mathbb{R}_+^S$ satisfying $\gamma_i = 0$ for all $i \notin \mathcal{S}_u(\mathbf{x})$ and $\sum_{i \in \mathcal{S}} \gamma_i = 1$ such that*

$$\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}} = \begin{cases} \theta_i \left(\gamma_i - 1\right) & \text{if } k = i, \\ \theta_k \gamma_i & \text{otherwise}, \end{cases} \tag{3.35}$$

for all players $v \neq u$ and all links $i \in \mathcal{S}_u(\mathbf{x})$ and $k \in \mathcal{S}$.

Proof. The proof is based on two observations: *(i)* at a best-response strategy, the change in marginal cost of player u due to a change in the strategy of player v is the same in every link that is used at the best-response strategy; and *(ii)* the total flow is conserved for player u irrespective of the change in the strategy of player 1.

Recall that

$$g_{u,i}(x^{(u)}(\mathbf{x})) := \frac{\partial T_u}{\partial x_{u,i}}(x^{(u)}(\mathbf{x})).$$

is the marginal cost of player u at link i under strategy profile $x^{(u)}(\mathbf{x})$, i.e., after the best-response of player u.

For $i \in \mathcal{S}_u(\mathbf{x})$, from the KKT conditions, the best-response strategy of player u, $\mathbf{x}_u^{(u)}$, is such that the marginal cost is the same in all the links that receive a positive traffic at this strategy. That is,

$$g_{u,i}(x^{(u)}(\mathbf{x})) = \mu(\mathbf{x}_{-u}) \quad \forall i \in \mathcal{S}_u(\mathbf{x}), \tag{3.36}$$

where the constant μ depends upon the strategies of the players but not on the index of the link. The local stability of $\mathcal{S}_u(\mathbf{x})$ implies that the set of links used by user u does not change for an infinitesimal variation on the strategies of the other players. This leads to our first observation which is that the change in the marginal cost of player u at its best-response strategy due to the change in the strategy of player $v \neq u$ at link k is the same at all links that receive a positive flow of player u. Thus,

$$\frac{\partial g_{u,i}}{\partial x_{v,k}}(x^{(u)}(\mathbf{x})) = \mu_2, \quad \forall i \in \mathcal{S}_u(\mathbf{x}), \tag{3.37}$$

where μ_2 depends upon the strategies of the players. We have not made this dependence explicit in order to simplify the notation.

For a function of the form $h(f(x), x)$, its derivative with respect to x is given by

$$\frac{dh(f(x), x)}{dx} = \frac{dh(f, x)}{df}\frac{df}{dx} + \frac{dh(f, x)}{dx},$$

where in the first term on the RHS, h is treated to as a function of f only, whereas in the second term it is treated as a function of x only.

Since $x_{u,i}^{(u)}$ is a function of $x_{v,k}$, we can use the above formula to rewrite (3.37) as

$$\frac{dg_{u,i}}{dx_{u,i}}\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}}+\frac{dg_{u,i}}{dx_{v,k}}=\mu_2, \quad \forall i\in\mathcal{S}_u(\mathbf{x}), \tag{3.38}$$

where the partial derivates are replaced by full derivates in order to indicate that the function is differentied in one variable while treating the other as constant.

The particular form of the cost function given in problem ((BR-i)) permits a simplification of the LHS of the above equation by noting that the marginal cost in a link depends only on the traffic that is routed to that link. Thus,

$$\frac{dg_{u,i}}{dx_{u,i}}\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}}+\delta_k(i)\frac{dg_{u,i}}{dx_{v,k}}=\mu_2, \quad \forall i\in\mathcal{S}_u(\mathbf{x}), \tag{3.39}$$

where $\delta_k(i)$ is unity if $i=k$, and is zero otherwise.

The value of μ_2 can be computed using the second observation that the total flow of player u is conserved irrespective of the strategy of player v. That is,

$$\sum_{i\in\mathcal{S}_u(\mathbf{x})}\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}}=0 \tag{3.40}$$

We thus obtain

$$\begin{aligned}\mu_2 &= \left(\sum_{l\in\mathcal{S}_u(\mathbf{x})}\delta_k(l)\frac{dg_{u,l}}{dx_{v,k}}\left(\frac{dg_{u,l}}{dx_{u,l}}\right)^{-1}\right)\left(\sum_{l\in\mathcal{S}_u(\mathbf{x})}\left(\frac{dg_{u,l}}{dx_{u,l}}\right)^{-1}\right)^{-1}\\ &= \left(\frac{dg_{u,k}}{dx_{v,k}}\left(\frac{dg_{u,k}}{dx_{u,k}}\right)^{-1}\right)\left(\sum_{l\in\mathcal{S}_u(\mathbf{x})}\left(\frac{dg_{u,l}}{dx_{u,l}}\right)^{-1}\right)^{-1},\end{aligned} \tag{3.41}$$

and

$$\frac{\partial x_{u,i}^{(u)}}{\partial x_{v,k}}=\theta_k\left(\gamma_i-\delta_k(i)\right), \quad \forall i\in\mathcal{S}_u(\mathbf{x}), \tag{3.42}$$

where

$$\theta_k=\frac{dg_{u,k}}{dx_{v,k}}\left(\frac{dg_{u,k}}{dx_{u,k}}\right)^{-1}, \tag{3.43}$$

and

$$\gamma_i = \left(\sum_{l \in \mathcal{S}_u(\mathbf{x})} \left(\frac{dg_{u,l}}{dx_{u,l}} \right)^{-1} \right)^{-1} \left(\frac{dg_{u,i}}{dx_{u,i}} \right)^{-1}. \tag{3.44}$$

We will now show that $0 < \theta_k < 1$ and $0 < \gamma_i < 1$. We have

$$g_{u,k} = \pi_k \left(\phi(\rho_k) + \frac{x_{u,k}}{r_k} \phi'(\rho_k) \right). \tag{3.45}$$

Thus, since ϕ is an increasing and convex function,

$$\frac{dg_{u,k}}{dx_{v,k}} = \frac{\pi_k}{r_k} \left(\phi'(\rho_k) + \frac{x_{u,k}}{r_k} \phi''(\rho_k) \right) > 0, \tag{3.46}$$

independently of the player $v \neq u$, and

$$\frac{dg_{u,k}}{dx_{u,k}} = \frac{\pi_k}{r_k} \left(2\phi'(\rho_k) + \frac{x_{u,k}}{r_k} \phi''(\rho_k) \right) > 0. \tag{3.47}$$

Thus, from (3.43), $\theta_k > 0$ and

$$\theta_k = \frac{\phi'(\rho_k) + \frac{x_{u,k}}{r_k} \phi''(\rho_k)}{2\phi'(\rho_k) + \frac{x_{u,k}}{r_k} \phi''(\rho_k)} < 1. \tag{3.48}$$

We thus obtain that θ_k is independent of v and that $0 < \theta_k < 1$. Similarly, we note that γ_i is positive and smaller than unity due to the fact that $\frac{dg_{0,l}}{dx^{(1)}_{0,k}}$ is positive for all l. To conclude the proof, we note that $\sum_{i \in \mathcal{S}_u(\mathbf{x})} \gamma_i = 1$ from the definition of the vector $\boldsymbol{\gamma}$ in (3.44). Thus, letting $\gamma_i = 0$ for $i \notin \mathcal{S}_u(\mathbf{x})$, we obtain $\sum_{i \in \mathcal{S}} \gamma_i = 1$. □

Remark 3. *The vectors $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$ depend upon the strategy profile $\mathbf{x}$ and upon the player u that updates its strategy. We have not made this dependence explicit in order to simplify the notation.*

Further, the vector $\boldsymbol{\theta}$ has the following important property which will be helpful in establishing the desired inequality on the non-linear spectral radius of $\hat{x}^{(1)}$.

Lemma 4. *There exists a constant $q < 1$ such that*

$$\frac{1}{2} \leq \theta_k \leq q, \quad \forall k \in \mathcal{S}, \ \forall \mathbf{x} \in \mathcal{X}, \ \forall u \in \mathcal{C}. \tag{3.49}$$

In order to prove Lemma 4, we need the following result.

Lemma 5. *There exists a strictly positive constant $\rho_{\max} < 1$, independant of u and $\mathbf{x}$, such that the utilization rate of each and every link $j \in \mathcal{S}_u(\mathbf{x})$ is upper bounded by $\rho_{\max}$*

after the best-response of user u at point $\mathbf{x}$, that is

$$\rho_j^{(u)}(\mathbf{x}) \leq \rho_{\max}, \quad \forall j \in \mathcal{S}_u(\mathbf{x}),\ \forall \mathbf{x} \in \mathcal{X},\ \forall u \in \mathcal{C}, \tag{3.50}$$

where $\rho_j^{(u)}(\mathbf{x}) = \frac{1}{r_j}\sum_{i\in\mathcal{C}} x_{i,j}^{(u)}(\mathbf{x})$.

Proof of Lemma 5. Observe that $\mathbf{x} \in \mathcal{X}$ implies that $\sum_{k\neq u} x_{k,j} < r_j$ for all links j, and thus that the optimization problem for player u is well-defined. Let $\mathbf{z} = x^{(u)}(\mathbf{x})$ be the point reached after the best response of player u. To simplify notations, we let $\rho_j = \rho_j^{(u)}(\mathbf{x})$. From the KKT conditions, there exists $\mu_u(\mathbf{x}_{-u})$ such that

$$\pi_j\left[\phi(\rho_j) + \frac{z_{u,j}}{r_j}\phi'(\rho_j)\right] = \mu_u(\mathbf{x}_{-u}),\ \forall j \in \mathcal{S}_u(\mathbf{x}) \tag{3.51}$$

$$\pi_j\phi(\rho_j) \geq \mu_u(\mathbf{x}_{-u}),\ \forall j \notin \mathcal{S}_u(\mathbf{x}) \tag{3.52}$$

Since $0 \leq z_{u,j}/r_j \leq \rho_j$, $\forall j \in \mathcal{S}_u(\mathbf{x})$, (3.51) leads to the inequalities

$$\pi_j\phi(\rho_j) \leq \mu_u(\mathbf{x}_{-u}), \tag{3.53}$$

$$\mu_u(\mathbf{x}_{-u}) \leq \pi_j(\rho_j\phi'(\rho_j) + \phi(\rho_j)). \tag{3.54}$$

Moreover, ρ_j and $\phi'(\rho_j)$ are non-negative. Thus, (3.52) leads to

$$\mu_u(\mathbf{x}_{-u}) \leq \pi_j(\rho_j\phi'(\rho_j) + \phi(\rho_j)),\ \forall j \notin \mathcal{S}_u(\mathbf{x}),$$

which combined with (3.54) gives the inequality

$$\mu_u(\mathbf{x}_{-u}) \leq \pi_j(\rho_j\phi'(\rho_j) + \phi(\rho_j)),\ \forall j \in \mathcal{S}. \tag{3.55}$$

Let $f_j : [0,1) \to [c_j, \infty)$ be defined by $f_j(x) := \pi_j(x\phi'(x) + \phi(x))$. Note that f is increasing and non-negative, and hence invertible. The inverse on f_j is defined on $[c_j, \infty)$. Let us define $h_j : [0,\infty) \to [0,1)$ in the following way :

$$h_j(x) = \begin{cases} f_J^{-1}(x) & \text{if} \quad x \in [c_j, \infty); \\ 0 & \text{if} \quad x \in [0, c_j). \end{cases}$$

The function h_j is continuous and non-decreasing. Further, from (3.55),

$$h_j(\mu_u(\mathbf{x}_{-u})) \leq \rho_j = \frac{\sum_k z_{k,j}}{r_j}.$$

Summing over all the links, we obtain the following functional inequality on $\mu_u(\mathbf{x}_{-u})$:

$$\bar{h}(\mu_u(\mathbf{x}_{-u})) := \sum_j r_j h_j(\mu_u(\mathbf{x}_{-u})) \leq \sum_k \sum_j z_{k,j} = \sum_i \lambda_i, \tag{3.56}$$

that is $\mu_u(\mathbf{x}_{-u})$ is such that the above inequality is satisfied. A bound on $\mu_u(\mathbf{x}_{-u})$ itself can now be obtained by making use of the following observations. Since h_j is continuous and non-decreasing for all $j \in \mathcal{S}$, $\bar{h}$ is continuous and non-decreasing. It has $[0,\infty)$ as its domain and $[0, \sum_j r_j)$ as its image. Further, $\lim_{x\to\infty} \bar{h}(x) = \sum_j r_j$. From the stability condition, $\sum_i \lambda_i < \sum_j r_j$. Using these properties and (3.56), we can conclude that $\mu_u(\mathbf{x}_{-u}) \leq \mu_{max} < \infty$.

It then follows from (3.53) that

$$\rho_j \leq \beta_j = \phi^{-1}\left(\frac{\mu_{max}}{\pi_j}\right), \quad \forall j \in \mathcal{S}_u(\mathbf{x}),$$

and the upper bound β_j depends neither upon u nor upon $\mathbf{x}$. Moreover, ϕ is such that $x < \infty \Leftrightarrow \phi^{-1}(x) < 1$, and hence $\beta_j < 1$. By definition of $\mathcal{S}_u(\mathbf{x})$, we also have $\rho_j > 0$ and thus $\beta_j > 0$. Taking $\rho_{\max} = \max_{j\in\mathcal{S}_u(\mathbf{x})} \beta_j$ yields the proof. □

Proof of Lemma 4. We note from (3.48) that, since $\frac{x_{u,k}}{r_k}\phi''(\rho_k) \geq 0$, we have $\theta_k \geq (\phi'(\rho_k) + \frac{x_{u,k}}{r_k}\phi''(\rho_k))/(2\phi'(\rho_k) + 2\frac{x_{u,k}}{r_k}\phi''(\rho_k))$, implying that

$$\theta_k \geq \frac{1}{2}. \tag{3.57}$$

Since ϕ is increasing and convex , θ_k is an increasing function of $x_{u,k}$ (considering $\rho_k = \rho_k^{(u)}(\mathbf{x})$ as fixed), and since $x_{u,k}/r_k \leq \rho_k$, we also have the following inequality:

$$\begin{aligned} \theta_k &\leq \frac{\phi'(\rho_k) + \rho_k\phi''(\rho_k)}{2\phi'(\rho_k) + \rho_k\phi''(\rho_k)} \\ &\leq 1 - \frac{\phi'(\rho_k)}{2\phi'(\rho_k) + \rho_k\phi''(\rho_k)}. \end{aligned} \tag{3.58}$$

Since the numerator and the denominator of the fraction appearing on the right-hand side of (3.58) are strictly increasing in ρ_k, Lemma 5 implies that

$$\theta_k \leq q = 1 - \frac{\phi'(0)}{2\phi'(\rho_{\max}) + \rho_{\max}\phi''(\rho_{\max})} < 1. \tag{3.59}$$

□

The structure of the Jacobian matrices of the best-response functions is summarized in the following result.

Theorem 3. *The Jacobian matrix of the best response function $x^{(u)}$ of player $u \in \mathcal{C}$ has the following form*

$$\mathrm{D}x^{(u)}(\mathbf{x}) = \begin{pmatrix} I & \dots & 0 & \dots & 0 \\ \vdots & \ddots & & & \vdots \\ M_u(\mathbf{x}) & \dots & 0 & \dots & M_u(\mathbf{x}) \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & 0 & \dots & I \end{pmatrix},$$

and $M_u(\mathbf{x}) = \Psi(\Gamma B - I)\Theta$, where

- *B is the $S \times S$ matrix with 1 in every entry, i.e., $B = \mathbf{1}^T\mathbf{1}$,*
- *$\Gamma = diag(\boldsymbol{\gamma})$ and $\Theta = diag(\boldsymbol{\theta})$, the vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\theta}$ being those defined in Lemma 3,*
- *Ψ a positive diagonal matrix such that $\Psi_{i,i} = 1$ if $i \in \mathcal{S}_u(\mathbf{x})$, and $\Psi_{i,i} = 0$ otherwise.*

Proof. The proof is broken down in three steps. Firstly, the uth row follows directly from Lemma 3. Secondly, the strategies of all players except player u do not change following the best response of player i. Therefore, for all $i \neq u$ and all $v \in \mathcal{C}$, we have

$$\frac{\partial x_i^{(u)}}{\partial \mathbf{x}_v}(\mathbf{x}) = \begin{cases} I & \text{if } v = i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.60}$$

This explains the appearance of the identity matrix on the diagonal and the 0 matrix in other columns of each row except the row corresponding to the player doing the best-response (that is, row u).

Finally, since the best response of player u at point $\mathbf{x}$ is insensitive to her strategy at that point and depends only on the strategies of the other player, we can conclude that, for all $u \in \mathcal{C}$,

$$\frac{\partial x_u^{(u)}}{\partial \mathbf{x}_u}(\mathbf{x}) = 0. \tag{3.61}$$

This explains why the diagonal block in the uth row is 0. $\square$

A consequence of Theorem 3 and Lemma 4.

Proposition 2. *The set $\mathcal{J}_i$ of Jacobian matrices is bounded.*

Proof of Proposition 2. Consider a player $u \in \mathcal{C}$ and a point $\mathbf{x}$ where the best-response function $x^{(u)}$ is differentiable. Theorem 3 implies that

$$\begin{aligned} \left\| \mathrm{D}x^{(u)}(\mathbf{x}) \right\|_1 &\leq \|I\|_1 + \|\mathrm{M}_u(\mathbf{x})\|_1 \\ &\leq 1 + \max_{m \in \mathcal{S}} \sum_{n \in \mathcal{S}_u(\mathbf{x})} |\theta_m(\gamma_n - \delta_m(n))| \end{aligned} \tag{3.62}$$

For $m \notin \mathcal{S}_u(\mathbf{x})$, we have

$$\sum_{n \in \mathcal{S}_u(\mathbf{x})} |\theta_m(\gamma_n - \delta_m(n))| = \theta_m < 1, \tag{3.63}$$

while for $m \in \mathcal{S}_u(\mathbf{x})$, we have

$$\begin{aligned} \sum_{n \in \mathcal{S}_u(\mathbf{x})} |\theta_m(\gamma_n - \delta_m(n))| &= \theta_m \left(\sum_{n \in \mathcal{S}_u(\mathbf{x}), n \neq m} \gamma_n + |\gamma_m - 1| \right) \\ &= 2\theta_m(1 - \gamma_m) \\ &< 2. \end{aligned} \tag{3.64}$$

With (3.62), (3.63) and (3.64), we obtain that $\left\| \mathrm{D}x^{(u)}(\mathbf{x}) \right\|_1 < 3$ for all players $u \in \mathcal{C}$ and all points $\mathbf{x}$ where $x^{(u)}$ is differentiable. From its definition in (3.9), we thus conclude that the set $\mathcal{J}$ is bounded. $\square$

From the submultiplicativity of norms and relation (3.7), it follows that

Corollary 2. *The set $\mathcal{J}$ is bounded.*

Corollary 3. *The Jacobian matrix of $\hat{x}^{(1)}$ has the form*

$$\mathrm{D}\hat{x}^{(1)}(\mathbf{x}) = \prod_{u=K}^{1} \mathrm{D}x^{(u)}(\mathbf{x}),$$

where the index u goes down from K to 1.

3.5 Convergence of best-response dynamics

In this section, we shall first formulate a conjecture on the non-linear spectral radius of $\hat{x}^{(1)}$ on which the main result of this study hinges. Then, this conjecture will be shown to be true for two particular cases : *(a)* two-player routing games; *(b)* K player routing games with linear link cost function, ϕ.

Conjecture 1. *For a fixed K and S, let $\hat{\mathcal{J}}$ be the set of matrices of the form given in Corollary 3. Then, the joint spectral radius of $\hat{\mathcal{J}}$ is strictly less than 1.*

On the extensive numerical experiments that we conducted, the above conjecture was indeed true.

The main result of this study is then:

Theorem 4. *If Conjecture 1 is true, then the best-response dynamics (3.8) for the routing game (BR-i) converges to the unique Nash equilibrium of the game.*

While we were unable to prove the conjecture, and hence the convergence of best-response dynamics, in its generality, we can show its validity for two non-trivial cases – the two player game, and the K player game with linear link cost function, which we show below.

3.5.1 Two-player routing game

First, we shall prove a general result related to the Joint spectral radius of a certain class of matrices. The claimed result on the convergence of the best-response for the two-player game will then follow directly from that result.

Let $\mathcal{D}^+$ be the set of positive diagonal matrices, and $\mathcal{G}$ be the set of diagonal matrices $\Gamma \in \mathcal{D}^+$ whose diagonal entries satisfy in addition

$$\sum_{i=1}^{S} \gamma_i = 1. \tag{3.65}$$

For any natural number $k \geq 0$, the above two types of diagonal matrices are used to define the set $\mathcal{M}$ of $S \times S$ matrices as follows. $\mathcal{M}$ is the set of matrices M that can be written as $M = (\Gamma B - I)\,\Theta$ for some matrices $\Gamma \in \mathcal{G}$ and $\Theta \in \mathcal{D}^+$. We also define $\mathcal{M}^{(k)}$ for $k \geq 0$ as the set of matrices that can be written as the product of k matrices belonging to $\mathcal{M}$, where by convention $\mathcal{M}^{(0)}$ contains only the identity matrix.

For $q \in (0,1)$, we say that a matrix M is in the set $\mathcal{M}_q$ if $M = (\Gamma B - I)\,\Theta \in \mathcal{M}$ and in addition $\|\Theta\|_1 \leq q$. We similarly define $\mathcal{M}_q^{(k)}$ as the set of matrices that can be written as the product of k matrices belonging to $\mathcal{M}_q$. We note that the set $\mathcal{M}_q$ is obviously bounded.

According to Theorem 3 and Lemma 4, the Jacobian matrices of the best-response functions of players 1 and 2 have the following simple form:

$$\mathrm{D}x^{(1)}(\mathbf{x}) = \begin{pmatrix} 0 & \Psi_1 M_1 \\ 0 & I \end{pmatrix}, \quad \text{and} \quad \mathrm{D}x^{(2)}(\mathbf{x}) = \begin{pmatrix} I & 0 \\ \Psi_2 M_2 & 0 \end{pmatrix}, \tag{3.66}$$

where $M_1, M_2 \in \mathcal{M}_q$ for some $q < 1$ and where Ψ_1, Ψ_2 are diagonal matrices with 0-1 entries on the diagonal. Using Corollary 3, the Jacobian of the best-response function over one round has the form

$$\mathrm{D}\hat{\mathbf{x}}^{(1)} = \begin{pmatrix} 0 & \Psi_1 M_1 \\ 0 & \Psi_2 M_2 M_1 \end{pmatrix},$$

where $M_1, M_2 \in \mathcal{M}_q$. It then follows that the structure of the product of n Jacobian matrices has the following form.

Lemma 6. *If $J_1, J_2, \ldots, J_n \in \mathcal{J}$, then*

$$\prod_{i=1}^{n} J_i = \begin{pmatrix} 0 & \Psi_1 X_1^{(2n-1)} \\ 0 & \Psi_2 X_2^{(2n)} \end{pmatrix}, \tag{3.67}$$

where Ψ_1, Ψ_2 are positive diagonal matrices with 0-1 entries on the diagonal, $X_1^{(2n-1)} \in \mathcal{M}_q^{(2n-1)}$, and $X_2^{(2n)} \in \mathcal{M}_q^{(2n)}$.

Proof. The proof is by induction. The claim is true for $n = 1$. Given that the form is true for some n, it will be shown that the form holds for $n + 1$. By definition,

$$\begin{aligned} \prod_{i=1}^{n+1} J_i &= J_{n+1} \prod_{i=1}^{n} J_i \\ &= \begin{pmatrix} 0 & \Psi_1 M_1 \\ 0 & \Psi_2 M_2 M_1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \Psi_3 X_1^{(2n-1)} \\ 0 & \Psi_4 X_2^{(2n)} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \Psi_1 M_1 \Psi_3 X_2^{(2n)} \\ 0 & \Psi_2 M_2 M_1 \Psi_4 X_2^{(2n)} \end{pmatrix} \end{aligned}$$

Since $M_1 \in \mathcal{M}_q$ and Ψ_4 is a 0-1 diagonal matrix, it follows that $M_1\Psi_4 \in \mathcal{M}_q$. Using the previous fact and the definition $\mathcal{M}_q^{(2n)}$ and the fact that $X_2^{(2n)} \in \mathcal{M}_q^{(2n)}$, one can deduce that $(M_1\Psi_3)X_2^{(2n)} \in \mathcal{M}_q^{(2(n+1)-1)}$, and $M_2 M_1 \Psi_4 X_2^{(2n)} \in \mathcal{M}_q^{(2(n+1))}$. □

Lemma 6 shows that the behaviour of a large product of Jacobian matrices is governed by the asymptotic behaviour of the matrices $X_1^{(n)}, X_2^{(n)}$. These matrices are themselves the product of matrices that belong to $\mathcal{M}_q$. This suggests to first characterize the asymptotic growth rate of products of matrices in $\mathcal{M}_q$. Our key result regarding this characterization is stated in theorem 5.

Theorem 5. *For any $k \geq 1$ and any matrix $M = \prod_{i=1}^{k} (\Gamma^{(i)} B - I)\, \Theta^{(i)}$ in $\mathcal{M}^{(k)}$, it holds that*

$$\rho(M) \leq \prod_{i=1}^{k} \theta_{max}^{i}, \tag{3.68}$$

where $\theta_{max}^{i} = \max\limits_{1 \leq j \leq S} \theta_j^{(i)}$ for all $i = 1, \ldots, k$.

The main difficulty in proving Theorem 5 is that the matrices M of $\mathcal{M}^{(k)}$ are neither positive nor negative. To circumvent this difficulty, we shall construct a positive or negative matrix A such that $\rho(M) \leq \rho(A)$ and $\|A\|_1 \leq \prod_{i=1}^{k} \theta_{max}^{i}$. Before showing how to construct such a matrix, we state two basic properties of the matrices in $\mathcal{M}^{(k)}$ in the following lemma.

Lemma 7. *For any matrix $M \in \mathcal{M}^{(k)}$, the following two assertions hold:*

(a) for each and every column j, $\sum_{i=1}^{S} m_{i,j} = 0$,

(b) if $\lambda \neq 0$ is an eigenvalue of M and if $\mathbf{x}$ is the associated eigenvector, then $\sum_{i=1}^{S} x_i = 0$.

Proof. Let us first prove assertion (a). Consider $M \in \mathcal{M}^{(k)}$ and write M as $M = (\Gamma B - I)\,\Theta\, Y$ with $Y \in \mathcal{M}^{(k-1)}$. Then,

$$\begin{aligned}\left(\sum_i m_{i,1}, \ldots, \sum_i m_{i,S}\right) = \mathbf{1}^T M &= \mathbf{1}^T (\Gamma B - I)\,\Theta\, Y \\ &= \left(\sum_i \gamma_i - 1, \ldots, \sum_i \gamma_i - 1\right) Y \\ &= \mathbf{0}^T,\end{aligned}$$

which proves the result. Let us now prove assertion (b). Let $M \in \mathcal{M}^{(k)}$ be written in the form $M = (\Gamma B - I)\Theta Y$ and consider $\lambda \in \sigma(M)$, $\lambda \neq 0$ and $\mathbf{x} \neq \mathbf{0}$ such that $\lambda\mathbf{x} = M\mathbf{x}$. Multiplying on both sides by $\mathbf{1}^T$, we obtain

$$\lambda \sum_{i=1}^{S} x_i = 0 = \mathbf{1}^T\,\mathbf{x} = \mathbf{1}^T\, M\,\mathbf{x} = 0,$$

where the last equality follows from assertion (a). Since $\lambda \neq 0$, this implies that $\sum_{i=1}^{S} x_i = 0$. □

We will now use property (b) of Lemma 7 to show that, for any matrix $M \in \mathcal{M}^{(k)}$, if we choose the matrix A to be of the form $A = DB + M$, where D is any diagonal matrix, then $\rho(M) \leq \rho(A)$.

Lemma 8. *For any matrix $M \in \mathcal{M}^{(k)}$ and for any diagonal matrix D, $\rho(M) \leq \rho(DB + M)$.*

Proof. Let $\lambda \neq 0$ be an eigenvalue of M and $\mathbf{x}$ be the associated eigenvector. We have

$$(DB + M)\mathbf{x} = DB\mathbf{x} + \lambda\mathbf{x} = \left(\sum_i x_i\right) D\mathbf{1} + \lambda\mathbf{x} = \lambda\mathbf{x}, \tag{3.69}$$

where the last equality is obtained using property (b) of Lemma 7. Since this can be done for all non-zero eigenvalues of M, we conclude that $\sigma(M) - \{0\} \subset \sigma(DB + M)$. This clearly implies that

$$\max_{\lambda \in \sigma(M)} |\lambda| \leq \max_{\lambda \in \sigma(DB+M)} |\lambda|\,,$$

i.e., $\rho(M) \le \rho(DB + M)$. □

Given a matrix $M \in \mathcal{M}^{(k)}$, we shall now consider two specific choices of the diagonal matrix D : the first choice allows to obtain a matrix $A \ge 0$ such that $\rho(M) \le \rho(A)$, while the second one produces a matrix $A \le 0$ with the same property. Since the two choices lead to a positive or negative matrix A, the evaluation of $\|A\|_1$ is greatly simplified, allowing to obtain useful upper bounds on $\rho(M)$. These bounds are proven in the following proposition.

Proposition 3. *For any matrix $M \in \mathcal{M}^{(k)}$, the two following inequalities on $\rho(M)$ are valid:*

$$\rho(M) \le -\sum_{i=1}^{S} \min_{1\le k\le S} (m_{i,k}), \tag{3.70}$$

$$\rho(M) \le \sum_{i=1}^{S} \max_{1\le k\le S} (m_{i,k}), \tag{3.71}$$

Proof. Let us first consider the diagonal matrix D defined as

$$D = -diag\left(\min_k (m_{1,k}), \min_k (m_{2,k}), \dots, \min_k (m_{S,k})\right),$$

and consider the matrix $A = DB + M$. Since $a_{i,j} = m_{i,j} - \min_k (m_{i,k})$, $\forall i, j$, we have $A \ge 0$. We know from Lemma 8 that $\rho(M) \le \rho(A) \le \|A\|_1$. Hence

$$\begin{aligned}\rho(M) &\le \max_{1\le j\le S}\left(\sum_i a_{i,j}\right), \\ &\le \max_{1\le j\le S}\left(\sum_i m_{i,j} - \sum_i \min_k (m_{i,k})\right), \\ &\le -\sum_i \min_k (m_{i,k}),\end{aligned}$$

where the last inequality is obtained using property (a) of Lemma 7.

To prove the second inequality, we now define the matrix D as follows

$$D = -diag\left(\max_k (m_{1,k}), \max_k (m_{2,k}), \dots, \max_k (m_{S,k})\right),$$

and obtain a matrix $A = DB + M \le 0$ since $a_{i,j} = m_{i,j} - \max_k (m_{i,k})$, $\forall i, j$. Again, using

$\rho(M) \leq \rho(A) \leq \|A\|_1$, we obtain

$$\begin{aligned}
\rho(M) &\leq \max_{1\leq j\leq S}\left(-\sum_i a_{i,j}\right), \\
&\leq \max_{1\leq j\leq S}\left(\sum_i \max_k (m_{i,k}) - \sum_i m_{i,j}\right), \\
&\leq \sum_i \max_k (m_{i,k}),
\end{aligned}$$

and both inequalities on $\rho(M)$ are proved. □

We will now prove that we can recursively obtain upper bounds on the terms appearing on the right-hand sides of (3.70) and (3.71).

Lemma 9. *Let the matrix M be in $\mathcal{M}^{(k)}$ and let $X \in \mathcal{M}^{(k-1)}$, $\Theta \in \mathcal{D}^+$ and $\Gamma \in \mathcal{G}$ be such that $M = X(\Gamma B - I)\,\Theta$. Then*

$$-\sum_{i=1}^{S} \min_{1\leq j\leq S}(m_{i,j}) \leq \theta_{max} \sum_{i=1}^{S} \max_{1\leq j\leq S}(x_{i,j}), \tag{3.72}$$

$$\sum_{i=1}^{S} \max_{1\leq j\leq S}(m_{i,j}) \leq -\theta_{max} \sum_{i=1}^{S} \min_{1\leq j\leq S}(x_{i,j}), \tag{3.73}$$

where $\theta_{max} = \max_i \theta_i$.

Proof. We have $m_{i,j} = \theta_j\left(\sum_k x_{i,k}\gamma_k - x_{i,j}\right)$, $\forall i,j$. Since $\max_j(x_{i,j}) \geq \sum_k x_{i,k}\gamma_k$ for all i, we have

$$m_{i,j} \geq \theta_j\left(\sum_k x_{i,k}\gamma_k - \max_j(x_{i,j})\right) \geq \theta_{max}\left(\sum_k x_{i,k}\gamma_k - \max_j(x_{i,j})\right), \tag{3.74}$$

for all $i,j = 1,\ldots,S$, and thus $\min_j(m_{i,j}) \geq \theta_{max}\left(\sum_k x_{i,k}\gamma_k - \max_j(x_{i,j})\right)$. As a consequence

$$\begin{aligned}
\sum_i \min_j(m_{i,j}) &\geq \theta_{max}\left(\sum_i\sum_k x_{i,k}\gamma_k - \sum_i \max_j(x_{i,j})\right), \\
&\geq \theta_{max}\left(\sum_k\left(\sum_i x_{i,k}\right)\gamma_k - \sum_i \max_j(x_{i,j})\right),
\end{aligned}$$

and since $\sum_i x_{i,k} = 0$ for all k according to property (a) of Lemma 7, it yields

$$\sum_i \min_j(m_{i,j}) \geq -\theta_{max} \sum_i \max_j(x_{i,j}) \tag{3.75}$$

which proves that $-\sum_i \min_j(m_{i,j}) \leq \theta_{max} \sum_i \max_j(x_{i,j})$, as claimed.

The proof of the second inequality is similar. We observe that

$$m_{i,j} \leq \theta_j \left(\sum_k x_{i,k}\gamma_k - \min_j(x_{i,j}) \right) \leq \theta_{max} \left(\sum_k x_{i,k}\gamma_k - \min_j(x_{i,j}) \right), \tag{3.76}$$

for all $i, j = 1, \ldots, S$, and thus $\max_j(m_{i,j}) \leq \theta_{max} \left(\sum_k x_{i,k}\gamma_k - \min_j(x_{i,j}) \right)$. It yields

$$\begin{aligned}
\sum_i \max_j(m_{i,j}) &\leq \theta_{max} \left(\sum_i \sum_k x_{i,k}\gamma_k - \sum_i \min_j(x_{i,j}) \right), \\
&\leq \theta_{max} \left(\sum_k \left(\sum_i x_{i,k} \right) \gamma_k - \sum_i \min_j(x_{i,j}) \right), \\
&\leq -\theta_{max} \sum_i \min_j(x_{i,j}),
\end{aligned}$$

as claimed. □

We are now in position to prove Theorem 5.

Proof of Theorem 5. Consider a matrix $M = \prod_{i=1}^{k} (\Gamma^{(i)}B - I)\,\Theta^{(i)}$ in $\mathcal{M}^{(k)}$. Define the matrices $X^{(n)} = \prod_{i=1}^{n} (\Gamma^{(i)}B - I)\,\Theta^{(i)}$ for $n = 1, 2, \ldots, k$. Note that $X^{(n)} \in \mathcal{M}^{(n)}$, that $M = X^{(k)}$ and that $X^{(n)} = X^{(n-1)}(\Gamma^{(n)}B - I)\Theta^{(n)}$ for $1 < n \leq k$.

We have $X^{(1)} = (\Gamma^{(1)}B - I)\,\Theta^{(1)}$. With (3.71) we have $\rho(X^{(1)}) \leq \sum_{i=1}^{S} \max_{1\leq k\leq S} \left(x_{i,k}^{(1)} \right)$. However

$$\begin{aligned}
\sum_{i=1}^{S} \max_{1\leq k\leq S} \left(x_{i,k}^{(1)} \right) &\leq \sum_{i=1}^{S} \max \left((\gamma_i^{(1)} - 1)\theta_i^{(1)}, \max_{k\neq i} \left(\gamma_i^{(1)}\theta_k^{(1)} \right) \right), \\
&\leq \theta_{max}^{(1)} \sum_{i=1}^{S} \gamma_i^{(1)} = \theta_{max}^{(1)},
\end{aligned} \tag{3.77}$$

from wich we conclude that $\rho(X^{(1)}) \leq \theta_{max}^{(1)}$. If $k = 1$, we have $M = X^{(1)}$ and thus $\rho(M) \leq \theta_{max}^{(1)}$. For $k > 1$, we consider separately the case when it is even and the case when it is odd. If k is even, Proposition 3 states that

$$\rho(M) \leq -\sum_{i=1}^{S} \min_{1\leq k\leq S} (m_{i,k}) \,, \tag{3.78}$$

and the repeated application of Lemma 9 yields

$$\begin{aligned}
\rho(M) &\leq \theta_{max}^{(k)} \sum_{i=1}^{S} \max_{1\leq j\leq S} \left(x_{i,j}^{(k-1)}\right) \\
&\leq -\theta_{max}^{(k)}\, \theta_{max}^{(k-1)} \sum_{i=1}^{S} \min_{1\leq j\leq S} \left(x_{i,j}^{(k-2)}\right) \\
&\vdots \\
&\leq \prod_{i=2}^{k} \theta_{max}^{(i)} \sum_{i=1}^{S} \max_{1\leq k\leq S} \left(x_{i,k}^{(1)}\right),
\end{aligned}$$

and we conclude with (3.77) that $\rho(M) \leq \prod_{i=1}^{k} \theta_{max}^{(i)}$. If on the contrary k is odd, we use the second inequality in Proposition 3 to obtain

$$\rho(M) \leq \sum_{i=1}^{S} \max_{1\leq k\leq S} (m_{i,k}) \,. \tag{3.79}$$

Applying again repeatedly Lemma 9 yields

$$\begin{aligned}
\rho(M) &\leq -\theta_{max}^{(k)} \sum_{i=1}^{S} \min_{1\leq j\leq S} \left(x_{i,j}^{(k-1)}\right) \\
&\leq \theta_{max}^{(k)}\, \theta_{max}^{(k-1)} \sum_{i=1}^{S} \max_{1\leq j\leq S} \left(x_{i,j}^{(k-2)}\right) \\
&\vdots \\
&\leq \prod_{i=2}^{k} \theta_{max}^{(i)} \sum_{i=1}^{S} \max_{1\leq k\leq S} \left(x_{i,k}^{(1)}\right),
\end{aligned}$$

and with (3.77) it proves that $\rho(M) \leq \prod_{i=1}^{k} \theta_{max}^{(i)}$. We therefore conclude that $\rho(M) \leq \prod_{i=1}^{k} \theta_{max}^{(i)}$ for all matrices $\mathcal{M}^{(k)}$, and for all $k \geq 1$. □

We prove below that there exist some matrices in $\mathcal{M}^{(k)}$ for which the upper bound on the spectral radius of Theorem 5 is tight.

Lemma 10. *For any $k \geq 1$, there exists $M \in \mathcal{M}^{(k)}$ such that $\rho(M) = \prod_{i=1}^{k} \theta_{max}^{(i)}$.*

Proof. Consider a matrix $M = \prod_{i=1}^{k} (\Gamma^{(i)}B - I)\,\Theta^{(i)} \in \mathcal{M}^{(k)}$ such that $\Theta^{(i)} = \theta^{(i)}_{max} I$ for all $1 \leq i \leq k$. Obviously, $M = \left(\prod_{i=1}^{k} \theta^{(i)}_{max}\right) \prod_{i=1}^{k} (\Gamma^{(i)}B - I)$. Observe now that for all m, n

$$\Gamma^{(m)}B\Gamma^{(n)}B = \left(\sum_i \gamma_i^{(n)}\right) \Gamma^{(m)}B = \Gamma^{(m)}B,$$

which implies that

$$\begin{aligned}
\left(\Gamma^{(m)}B - I\right)\left(\Gamma^{(n)}B - I\right) &= \Gamma^{(m)}B\Gamma^{(n)}B - \Gamma^{(m)}B - \Gamma^{(n)}B + I \\
&= \Gamma^{(m)}B - \Gamma^{(m)}B - \Gamma^{(n)}B + I \\
&= -\left(\Gamma^{(n)}B - I\right). \qquad (3.80)
\end{aligned}$$

Hence

$$\prod_{i=1}^{k} (\Gamma^{(i)}B - I) = (-1)^k \left(\Gamma^{(k)}B - I\right), \qquad (3.81)$$

and thus we obtain $M = (-1)^k \left(\prod_{i=1}^{k} \theta^{(i)}_{max}\right) \left(\Gamma^{(k)}B - I\right)$, which implies that $\rho(M) = \left(\prod_{i=1}^{k} \theta^{(i)}_{max}\right) \rho\left(\Gamma^{(k)}B - I\right)$. We note that $\Gamma^{(k)}B$ is a matrix of rank 1, since all its columns are the same. Moreover, the sum of each column is 1. Thus, the spectrum of $\Gamma^{(k)}B$ is $\{1, 0, 0 \ldots, 0\}$, which implies that the spectrum of $\Gamma^{(k)}B - I$ is $\sigma\left(\Gamma^{(k)}B - I\right) = \{0, -1, -1 \ldots -1\}$. We conclude that $\rho\left(\Gamma^{(k)}B - I\right) = 1$, which implies that $\rho(M) = \left(\prod_{i=1}^{k} \theta^{(i)}_{max}\right)$. □

The above theorem holds for any product of matrices belonging to $\mathcal{M}$. If we now restrict our attention to matrices belonging to $\mathcal{M}_q$, we obtain the following immediate corollary.

Corollary 4. *For any product $M_n M_{n-1} \ldots M_1$ of matrices belonging to $\mathcal{M}_q$, we have $\rho(M_n M_{n-1} \ldots M_1) \leq q^n$, implying that $\rho(\mathcal{M}_q) \leq q$.*

Proof. Consider $M_1, M_2 \ldots M_n \in \mathcal{M}_q$. Each matrix M_i can be written as $M_i = (\Gamma^{(i)}B - I)\,\Theta^{(i)}$, where $\theta^i_{max} = \|\Theta^{(i)}\|_1 \leq q$. From theorem 5, we thus obtain $\rho(M_n M_{n-1} \ldots M_1)^{\frac{1}{n}} \leq q$. As a consequence,

$$\sup_{M_1, \ldots, M_n \in \mathcal{M}_q} \rho\left(\prod_{i=1}^{n} M_i\right)^{\frac{1}{n}} \leq q$$

Since $\mathcal{M}_q$ is bounded, its joint spectral radius and its generalized spectral radius coincide. From the definition in (3.11), we immediately obtain that $\rho(\mathcal{M}_q) \leq q$. □

We are now in position to prove that sequential best-response dynamics converges to the unique Nash equilibrium $\mathbf{x}^*$.

Theorem 6. *For the two player routing game over parallel links, the sequential best-response dynamics converges to the unique Nash equilibrium for any initial point* $\mathbf{x}_0 \in \mathcal{X}$.

Proof. Since $\mathcal{J}$ is bounded (see Corollary 2), and the Generalized spectral radius is equal to the Joint spectral radius of a bounded set of matrices, it suffices to prove that $\rho(\mathcal{J}) < 1$. From Lemma 6, we have

$$det\left(\prod_{i=1}^{n} J_i - \lambda I\right) = det\left(-\lambda I\right)\, det\left(\Psi_2 X_2^{(2n)} - \lambda I\right) = (-\lambda)^S\, det\left(\Psi_2 X_2^{(2n)} - \lambda I\right),$$

implying that $\lambda \neq 0$ is an eigenvalue of $\prod_{i=1}^{n} J_i$ if and only if it is an eigenvalue of $\Psi_2 X_2^{(2n)}$. Thus, $\rho\left(\prod_{i=1}^{n} J_i\right) = \rho\left(\Psi_2 X_2^{(2n)}\right)$. Further,

$$\rho\left(\Psi_2 X_2^{(2n)}\right) \leq \left\|\Psi_2 X_2^{(2n)}\right\|_1 \leq \|\Psi_2\|_1 \left\|X_2^{(2n)}\right\|_1 = \left\|X_2^{(2n)}\right\|_1,$$

and thus, since $X_2^{(2n)} \in \mathcal{M}_q^{(2n)}$,

$$\rho\left(\prod_{i=1}^{n} J_i\right) \leq \sup_{M \in \mathcal{M}_q^{(2n)}} \|M\|_1 = \rho_{2n}\left(\mathcal{M}_q\right),$$

where the last equality is obtained using the definition of the Joint spectral radius (3.10). Let $\epsilon = \frac{1-q}{2} > 0$. Since $\rho_n\left(\mathcal{M}_q\right)^{\frac{1}{n}} \to \rho\left(\mathcal{M}_q\right)$ as $n \to \infty$, there exists N such that for all $n \geq N$,

$$\rho\left(\prod_{i=1}^{n} J_i\right)^{\frac{1}{n}} \leq \rho\left(\mathcal{M}_q\right) + \epsilon \leq q + \frac{1-q}{2} = \frac{1+q}{2},$$

where the last inequality follows from Corollary 4. Since the right hand-side is independant of $J_1, \ldots, J_n$, we deduce that

$$\sup_{J_1,\ldots,J_n \in \mathcal{J}} \rho\left(\prod_{i=1}^{n} J_i\right)^{\frac{1}{n}} \leq \frac{1+q}{2}, \quad \forall n \geq N,$$

and, according to 3.11, it yields $\rho\left(\mathcal{J}\right) \leq \frac{1+q}{2} < 1$. □

3.5.2 K player games with linear link cost functions

Consider $\phi(x) = x$, a delay function which is often used in congestion games to model delays in road networks. From (3.48), it follows that $\theta_k = 1/2$. Thus, the matrix M_u in Theorem 3 is of the form $\frac{1}{2}(\Gamma B - I)$ for some $\Gamma \in \mathcal{G}$.

Theorem 7. *For the K player routing game over parallel links and linear delay function, the sequential best-response dynamics converges to the unique Nash equilibrium for any initial point $\mathbf{x}_0 \in \mathcal{X}$.*

Proof. We shall show that the product of Jacobian matrices over n rounds goes to 0 as $n \to \infty$. This shows that their JSR is less than 1, and hence best-response converges.

First we shall show this for the three player game as the proof follows the same steps for any number of players. Omitting the multiplier Ψ, the one-round Jacobian matrix for three players has the form :

$$J^{(1)} = J_3 J_2 J_1 = \begin{pmatrix} 0 & M_1 & M_1 \\ 0 & M_2 M_1 & M_2 + M_2 M_1 \\ 0 & M_3(M_2 M_1 + M_1) & M_3(M_2 + M_2 M_1 + M_1) \end{pmatrix}.$$

Note that $M_v M_u = (\Gamma_v B - I)\Theta_v(\Gamma_u B - I)\Theta_u = \theta^2(\Gamma_v B - I)(\Gamma_u B - I)$ where $\theta = 1/2$.

Denote $\Gamma_u B - I = H_u$, then $M_v M_u = -\theta^2 H_u$, and the Jacobian matrix for one round, is as follows

$$\begin{aligned} J^{(1)} &= \begin{pmatrix} 0 & H_1^{(1)}\theta & H_1^{(1)}\theta \\ 0 & -H_1^{(1)}\theta^2 & H_2^{(1)}\theta - H_1^{(1)}\theta^2 \\ 0 & H_1^{(1)}\theta^3 - H_1^{(1)}\theta^2 & -H_2^{(1)}\theta^2 + H_1^{(1)}\theta^3 - H_1^{(1)}\theta^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & H_1^{(1)}\theta & H_1^{(1)}\theta \\ 0 & -H_1^{(1)}\theta^2 & -H_1^{(1)}\theta^2 + H_2^{(1)}\theta \\ 0 & -H_1^{(1)}\theta^2 + H_1^{(1)}\theta^3 & H_1^{(1)}\theta^3 - H_1^{(1)}\theta^2 - H_2^{(1)}\theta^2 \end{pmatrix}. \end{aligned}$$

Note here that for any round n, $\left(H_u^{(n)}\right)^2 = -H_u^{(n)}$ and for different rounds n, m, $H_v^{(m)} H_u^{(n)} = -H_u^{(n)}$. With notation $p_{i,j}^{(n)}(\theta)$ and $q_{i,j}^{(n)}(\theta)$, or more simply $p_{i,j}^{(n)}$ and $q_{i,j}^{(n)}$, for polynomial coefficients of $H_1^{(n)}$ and $H_2^{(n)}$, respectively, the Jacobian matrix $J^{(n)}$ after n rounds, it will take the following form

$$J^{(n)} = \begin{pmatrix} 0 & H_1^{(1)} p_{1,2}^{(n)} + H_2^{(1)} q_{1,2}^{(n)} & H_1^{(1)} p_{1,3}^{(n)} + H_2^{(1)} q_{1,3}^{(n)} \\ 0 & H_1^{(1)} p_{2,2}^{(n)} + H_2^{(1)} q_{2,2}^{(n)} & H_1^{(1)} p_{2,3}^{(n)} + H_2^{(1)} q_{2,3}^{(n)} \\ 0 & H_1^{(1)} p_{3,2}^{(n)} + H_2^{(1)} q_{3,2}^{(n)} & H_1^{(1)} p_{3,3}^{(n)} + H_2^{(1)} q_{3,3}^{(n)} \end{pmatrix}.$$

To find recurrence relation between the polynomial coefficients in successive rounds, write

$$
\begin{aligned}
J^{(n+1)} &= \begin{pmatrix} 0 & H_1^{(n+1)}\theta & H_1^{(n+1)}\theta \\ 0 & -H_1^{(n+1)}\theta^2 & -H_1^{(n+1)}\theta^2 + H_2^{(n+1)}\theta \\ 0 & H_1^{(n+1)}\theta^3 - H_1^{(n+1)}\theta^2 & H_1^{(n+1)}\theta^3 - H_1^{(n+1)}\theta^2 - H_2^{(n+1)}\theta^2 \end{pmatrix} J^{(n)} \\
&= \begin{pmatrix} 0 & H_1^{(n+1)}\theta & H_1^{(n+1)}\theta \\ 0 & -H_1^{(n+1)}\theta^2 & -H_1^{(n+1)}\theta^2 + H_2^{(n+1)}\theta \\ 0 & H_1^{(n+1)}\theta^3 - H_1^{(n+1)}\theta^2 & H_1^{(n+1)}\theta^3 - H_1^{(n+1)}\theta^2 - H_2^{(n+1)}\theta^2 \end{pmatrix} \times \\
&\quad \times \begin{pmatrix} 0 & H_1^{(1)}p_{1,2}^{(n)} + H_2^{(1)}q_{1,2}^{(n)} & H_1^{(1)}p_{1,3}^{(n)} + H_2^{(1)}q_{1,3}^{(n)} \\ 0 & H_1^{(1)}p_{2,2}^{(n)} + H_2^{(1)}q_{2,2}^{(n)} & H_1^{(1)}p_{2,3}^{(n)} + H_2^{(1)}q_{2,3}^{(n)} \\ 0 & H_1^{(1)}p_{3,2}^{(n)} + H_2^{(1)}q_{3,2}^{(n)} & H_1^{(1)}p_{3,3}^{(n)} + H_2^{(1)}q_{3,3}^{(n)} \end{pmatrix}.
\end{aligned}
$$

One can then deduce the following recursive expressions for the vectors of polynomial coefficients in the second column.

$$
\begin{pmatrix} p_{1,j}^{(n+1)} \\ p_{2,j}^{(n+1)} \\ p_{3,j}^{(n+1)} \end{pmatrix} = \begin{pmatrix} 0 & -\theta & -\theta \\ 0 & \theta^2 & \theta^2 - \theta \\ 0 & -\theta^3 + \theta^2 & -\theta^3 + 2\theta^2 \end{pmatrix} \begin{pmatrix} p_{1,j}^{(n)} \\ p_{2,j}^{(n)} \\ p_{3,j}^{(n)} \end{pmatrix},
$$

and

$$
\begin{pmatrix} q_{1,j}^{(n+1)} \\ q_{2,j}^{(n+1)} \\ q_{3,j}^{(n+1)} \end{pmatrix} = \begin{pmatrix} 0 & -\theta & -\theta \\ 0 & \theta^2 & \theta^2 - \theta \\ 0 & -\theta^3 + \theta^2 & -\theta^3 + 2\theta^2 \end{pmatrix} \begin{pmatrix} q_{1,j}^{(n)} \\ q_{2,j}^{(n)} \\ q_{3,j}^{(n)} \end{pmatrix}.
$$

A similar relation can be deduced for the vector of polynomials in the third column.

If it can be shown that the spectral radius of the matrix

$$
A_3 = \begin{pmatrix} 0 & -\theta & -\theta \\ 0 & \theta^2 & \theta^2 - \theta \\ 0 & -\theta^3 + \theta^2 & -\theta^3 + 2\theta^2 \end{pmatrix}
$$

is less than 1, then we can conclude that the any product of Jacobian matrices will go to 0 in any norm as $n \to \infty$, and thus conclude that the JSR of $\mathcal{J}$ is smaller than 1.

For a K player game, it turns out that the matrix A_K has the form

$$
[A_K]_{i,j} = \begin{cases} (1-\theta)^{i-1} & \text{for } j > i, \\ (1-\theta)^{i-1} - (1-\theta)^{i-j} & \text{for } j \leq i. \end{cases} \tag{3.82}
$$

which is expanded form is

$$A_K = \begin{pmatrix} 0 & -\theta & -\theta & \cdots & -\theta \\ 0 & -\theta(1-\theta)+\theta & -\theta(1-\theta) & \cdots & -\theta(1-\theta) \\ 0 & -\theta(1-\theta)^2+\theta(1-\theta) & -\theta(1-\theta)^2+\theta & \cdots & -\theta(1-\theta)^2 \\ 0 & -\theta(1-\theta)^3+\theta(1-\theta)^2 & -\theta(1-\theta)^3+\theta(1-\theta) & \cdots & -\theta(1-\theta)^3 \\ 0 & -\theta(1-\theta)^4+\theta(1-\theta)^3 & -\theta(1-\theta)^4+\theta(1-\theta)^2 & \cdots & -\theta(1-\theta)^4 \\ \vdots & & & & \\ 0 & -\theta(1-\theta)^{k-1}+\theta(1-\theta)^{k-2} & -\theta(1-\theta)^{k-1}+\theta(1-\theta)^{k-3} & \cdots & -\theta(1-\theta)^{k-1}+\theta \end{pmatrix}.$$

In Proposition 4 stated just after this proof, it is shown that the spectral radius of A_K is less than θ which is less than 1. We can thus conclude that the product of Jacobians will tend to 0 as $n \to \infty$, and hence the best-response will converge. $\square$

Proposition 4. *The spectral radius of the matrix A_K defined in (3.82) is less than θ.*

Proof. We shall show that the zeros of $\det(A_K - \lambda I)$ are in the unit circle. The expanded form of $A_K - \lambda I$ is as follows,

$$A_K - \lambda I = \begin{pmatrix} 0-\lambda & -\theta & -\theta & \cdots & -\theta \\ 0 & -\theta(1-\theta)+\theta-\lambda & -\theta(1-\theta) & \cdots & -\theta(1-\theta) \\ 0 & -\theta(1-\theta)^2+\theta(1-\theta) & -\theta(1-\theta)^2+\theta-\lambda & \cdots & -\theta(1-\theta)^2 \\ 0 & -\theta(1-\theta)^3+\theta(1-\theta)^2 & -\theta(1-\theta)^3+\theta(1-\theta) & \cdots & -\theta(1-\theta)^3 \\ 0 & -\theta(1-\theta)^4+\theta(1-\theta)^3 & -\theta(1-\theta)^4+\theta(1-\theta)^2 & \cdots & -\theta(1-\theta)^4 \\ \vdots & & & & \\ 0 & -\theta(1-\theta)^{k-1}+\theta(1-\theta)^{k-2} & -\theta(1-\theta)^{k-1}+\theta(1-\theta)^{k-3} & \cdots & -\theta(1-\theta)^{k-1}+\theta-\lambda \end{pmatrix}.$$

Transform the $A_K - \lambda I$ by multiplying each row i by $-(1-\theta)$ and adding it to row $i+1$, for $i = K-1, K-2, \ldots, 1$, to get

$$\det(A_K - \lambda I) = \det \begin{pmatrix} -\lambda & -\theta & -\theta & -\theta & \cdots & -\theta & -\theta \\ \lambda(1-\theta) & \theta-\lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda(1-\theta) & \theta-\lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda(1-\theta) & \theta-\lambda & \cdots & 0 & 0 \\ 0 & 0 & 0 & \lambda(1-\theta) & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & \lambda(1-\theta) & \theta-\lambda \end{pmatrix}.$$

Computing the determinant along the last column, one obtains the polynomial

$$\det(A_K - \lambda I) = (-1)^k \left[(\lambda-\theta)^k + \theta \sum_{i=0}^{k-1} (\lambda-\theta)^{k-1-i} \lambda^i (1-\theta)^i \right].$$

Denote the expression in the square brackets by $\mathcal{P}(\lambda)$, i.e.

$$\mathcal{P}(\lambda) = \frac{(1-\theta)^k}{1-\lambda}\left[(\lambda-\theta)^k + \theta\sum_{i=0}^{k-1}(\lambda-\theta)^{k-1-i}\lambda^i(1-\theta)^i\right].$$

For $\lambda \neq 1$, after some algebra, we obtain

$$\mathcal{P}(\lambda) = \frac{(1-\theta)^k}{1-\lambda}\left[\lambda^k - \frac{\lambda(\lambda-\theta)^k}{(1-\theta)^k}\right].$$

Since $0 < \theta < 1$, for $\lambda < 0$, $\mathcal{P}(\lambda)$ is positive for even k and negative for odd k.

For $\lambda > 1$, the denominator $(1-\lambda)$ is negative. Note that for $\lambda > 1$, $\theta < \theta\lambda < \lambda$ and then $0 < \lambda - \theta\lambda < \lambda - \theta$. Then for the expression in the numerator the following holds, $\lambda^k(1-\theta)^k - \lambda(\lambda-\theta)^k = (\lambda-\theta\lambda)^k - \lambda(\lambda-\theta)^k < (\lambda-\theta\lambda)^k - (\lambda-\theta)^k < 0$. Hence $\mathcal{P}(\lambda) > 0$.

For $\theta < \lambda < 1$, the denominator $(1-\lambda)$ is positive. Note that for $\theta < \lambda < 1$, $0 < \theta\lambda < \theta < \lambda < 1$ and then $0 < \lambda - \theta < \lambda - \theta\lambda$. Then for the expression in the numerator,
$\lambda^k(1-\theta)^k - \lambda(\lambda-\theta)^k = (\lambda-\theta\lambda)^k - \lambda(\lambda-\theta)^k > (\lambda-\theta\lambda)^k - (\lambda-\theta)^k > 0$. Hence for $\theta < \lambda < 1, \quad \mathcal{P}(\lambda) > 0$.

Moreover, $\mathcal{P}(1) > 0$.

Thus, the zeros of the function $\mathcal{P}(\lambda)$ are in $[0, \theta]$. □

3.6 Conclusion

We have proposed a different approach to study the convergence of the best-response dynamics based on the notion of non-linear spectral radius. By following this approach we established a sufficient condition for the convergence.

The key idea to prove the convergence is to study the Jacobian matrices of best-response functions, and to analyze how long products of such matrices grow as a function of the number of best-response updates. The growth rate of matrix products is characterized by the so-called joint or generalized spectral radius.

For sequential best-response dynamics in a routing game over parallel links, we have constructed a best-response operator as a function for one-round of play and shown that it is Lipschitz-continuous. We have established the specific structure of Jacobian matrix of the best-response function. Then we have obtained a sufficient condition for the convergence of the best-response dynamics as that the joint spectral radius of matrices of this form be strictly less than unity. We thus obtain a purely structural sufficient condition that allows to reduce the analysis of the convergence of the sequential best-response dynamics to the analysis of the joint spectral radius of certain matrices.

Based on the specific structure of the Jacobian matrices in our game and the sufficient condition we prove the convergence for two non-trivial cases: the two-player game for an arbitrary number of links, and with arbitrary number of players and links in the case of linear latency functions. Furthermore, although we were not able to prove it, we conjecture that the proposed sufficient condition is valid for any numbers of players and links with general latency functions.

The results of these studies are published in the proceedings of the 7^{th} International Conference on Performance Evaluation Methodologies and Tools, ValueTools-2013 (Brun et al., 2013).

4

REWARD-BASED INCENTIVES FOR NODAL COLLABORATION IN DELAY TOLERANT NETWORKS

This chapter is concerned with another important area of competitive interaction crucial in nowadays communications, Delay/Disruption Tolerant Networking. We focus on mechanism design principles arising from the need to provide communications within environments where continuous end-to-end connectivity cannot be assumed and connection between a source and destination is performed through mobile nodes serving as relays.

The beginning of the chapter is devoted to the description of the Delay-Tolerant communication paradigm and routing in DTNs. Section 4.3 discusses related work on incentive mechanisms for DTNs. Section 4.4 introduces the system model and the assumptions used in our study. In section 4.5 we investigate the impact of information that the source share with relays on the reward that it has to propose to them as composition in static scenario. The extension to the dynamic scenario is provided in section 4.6.

4.1 Delay Tolerant Networking

Delay-tolerant networking (DTN) (Fall, 2003; Farrell et al., 2005; Cerf et al., 2007; MacMahon et al., 2009) is an approach to network architecture that aims to support connection in environments characterised by very long latencies due to extreme distances or frequent interference in connectivity. Examples of such environments are sparse mobile networks, extreme terrestrial areas, or space communications.

Prior to the emergence of the delay-tolerant communication paradigm most studies were concerned with developing technology of wireless connection between non-fixed locations of computers. In 1990s studies in the field of wireless communications proposed the area of mobile ad hoc networking (MANET) and vehicular ad hoc networking. Investigations for the realization of an InterPlaNetary (IPN) Internet (Travis, 2001; Akyildiz et al., 2003) related to the necessity of networking technologies that can cope with the significant delays and packet corruption of deep-space communications, gave ideas for design of terrestrial networks. In 2002, Kevin Fall generalized challenged interplanetary and terrestrial networks by introducing the notion of Delay-Tolerant Networks (DTNs) (see publication, Fall, 2003).

4.2 DTN Routing Schemes

Communication support and message delivery in DTN is performed through routing protocols that are based on store, carry, and forward paradigm (Cerf et al., 2007), where a mobile node carries the message until it encounters the destination node or any other node that has high probability of meeting the destination node. Based on this paradigm various DTN routing schemes have been proposed. Some of them seeking to minimize the message delivery time by replicating many copies of the message (Vahdat et al., 2000), whereas for other ones the emphasis is more on resource and energy consumption. Based on the number of created copies of the same message, DTN routing protocols can be of the following types: forwarding, quota-replication, and flooding (Lo et al., 2011). In a forwarding scheme, such as MEED (Jones et al., 2007), a single-copy message is forwarded through successive path of intermediate nodes to the destination. A quota-replication scheme, such as Spray&Wait (Spyropoulos et al., 2005), involves creation of a specific number of message copies called message quota. Under a flooding scheme, such as Epidemic routing (Vahdat et al., 2000), a network is flooded with an extremely large number of message copies.

In our work, we focus on the so-called two-hop routing scheme, which is known to provide a good tradeoff between message delivery time and resource consumption (Al-Hanbali et al., 2008). With two-hop routing, the communication is basically in 3 phases:

- First, the source gives the message to each and every mobile nodes it meets. These nodes act as relays for delivering the message to its destination.
- A relay cannot forward the message to another relay, so it will store and carry the message until it is in radio range of the destination.
- Once this happens, the relay delivers the message to the destination.

4.3 Review on Related Work on Incentive Mechanisms for DTNs

The core objective for the design of DTNs is to support communications even when end-to-end connectivity fails. In most studies for DTNs, it is assumed that relays are willing to cooperate with the source node. In practice, DTNs are composed of mobile devices, including smartphones, tablets or other mobile devices having multiple wireless interfaces. The delivery of a message thus can incur a certain number of costs for a relay. First, there are energy costs for receiving the message from the source and transmitting it to the destination. It is also natural to assume that there is some cost per unit time for storing the message in the buffer of the relay. A central question is whether owners of relay devices are willing to have battery depleted to sustain DTNs communications. The selfish behavior of relays becomes a core threat which hinders any possible attempt to optimize network performance. In different contexts, user participation in network operations is assured by means of appropriate incentive mechanisms.

In the literature on DTNs (El-Azouzi et al., 2012; Zhang et al., 2007), several incentive schemes have been recently proposed. For example, Shevade et al., 2008 uses Tit-for-Tat (TFT) to design an incentive-aware routing protocol that allows selfish DTN nodes to maximize their individual utilities while conforming to TFT constraints. Mobicent (Chen et al., 2010) is a credit-based incentive system which integrates credit and cryptographic technique to solve the edge insertion and edge hiding attacks among nodes. PI (Lu et al., 2010) attaches an incentive on the sending bundle to stimulate the selfish nodes to cooperate in message delivery. SMART (Zhu et al., 2009) is a secure multilayer credit-based incentive scheme for DTNs. In SMART, layered coins are used to provide incentives to selfish DTN nodes for bundle forwarding. MobiGame (Wei et al., 2011) is a user-centric and social-aware reputation based incentive scheme for DTNs. In addition, Li et al., 2010 proposes socially selfish routing in DTNs, where a node exploits social willingness to determine whether or not to relay packets for others. Authors in (Ning et al., 2011) formulate nodal communication as a two-person cooperative game for a credit-based incentive scheme to promote nodal collaboration. RELICS (Uddin et al., 2010) is another cooperative based energy-aware incentive mechanism for selfish DTNs, in which a rank metric was defined to measure the transit behavior of a node. In (Wang et al., 2012), authors proposed an incentive driven dissemination scheme that encourages nodes to cooperate and chooses delivery paths that can reach as many nodes as possible with fewest transmissions. A fundamental aspect that is usually ignored in DTN literature is the feedback message, which may incur into a large delay. In fact, the exchange of rewards between relays should not require feedback messages. In order to overcome lack of feedback, our mechanism assumes that a relay receives a positive reward if and only if it is the first one to deliver the message to the corresponding destination. (Chahin et al., 2013) is a credit-based incentive system using the theory of Minority Games (Moro, 2004) in order to attain coordination in distributed fashion. This mechanism considers the realistic case when the cost for taking part in the forwarding process varies with the devices technology or the users habits.

The mechanism we shall propose relates to the field of mechanism design that concerns itself with how to develop incentive mechanism that will lead to a desirable solution from a system-wide point of view. In recent years mechanism design has found many important applications in the computer sciences; e.g., in security design problems (Feigenbaum et al., 2002), in distributed scheduling resource allocation (Johari, 2004) and cooperation routing in ad-hoc networks (P. Michiardi, 2002).

4.4 System Model and Objectives

We consider a wireless network with one source node, one destination node and N relays. We shall assume that the source and the destination nodes are fixed and not in radio range of each other, whereas other nodes are moving according to a given mobility model.

At time 0, the source generates a message for the destination. The source wants this message to be delivered to the destination as fast as possible. However, it cannot transmit it directly to the destination since both nodes are not in radio range of each other. Instead, the source proposes to each relay it meets a reward for delivering the message[1]. It is assumed that the network is two-hop, that is a relay has to deliver the message by itself to the destination (it cannot forward the message to another relay). An important assumption we shall make is that relays are not seeking to make profit: a relay accepts the message provided the reward promised to it by the source offsets its expected cost for delivering the message to the destination, *as estimated by the relay* when it meets the source.

This expected cost has several components. A relay that accepts the message from the source always incurs a *reception cost* C_r. This is a fixed energy cost for receiving the message from the source. The relay will then store the message into its buffer and carry it until it is in radio range of the destination. We assume here that there is an incurred *storage cost* C_s per unit time the message is stored in the buffer of the relay. Hence, the expected storage cost depends on the expected time it takes to reach the destination. Once the relay meets the destination, it can deliver the message. This incurs an additionnal *transmission cost* C_d which is a fixed energy cost for transmitting the message to the destination. This cost is incured if and only if the relay is the first one to deliver the message to the destination, in which case the relay gets the reward. If on the contrary, the message has already been delivered, the relay gets nothing but save the transmission cost.

[1]Note that since the source is not informed when the message reaches the destination, it can still propose the message to a relay even if the message has already been delivered by another relay.

4.4.1 The Role of Information

As should be apparent from the above discussion, the reward asked by a relay to the source depends both on the expected time it will take for the relay to reach the destination and on the *probability of success* it estimates at the time it meets the source. The latter represents the probability of this relay to be the first one to deliver the message. The crucial observation here is that this probability notably depends on the information given by the source to the relay. Intuitively, if a relay is told by the source that there are already many message copies in circulation, it will correctly infer that it has a higher risk of failure than if it was the first one to meet the source, and it will naturally ask for a higher reward. The source can of course choose not to disclose the information on the number of existing message copies, in which case relays estimate their success probabilities based solely on the time at which they meet the source and on the number of competitors. In that case, the first relay to meet the source will certainly underestimate its success probability, and again ask for a higher reward than if it was told it was the first one.

It is thus clear that the expected reward to be paid by the source depends on the information it gives to the relays. There are several feasible strategies for the source. We shall distinguish between *static strategies* and *dynamic strategies*. In static strategies, the information given to the relays is fixed in that it does not depend on the times at which the source meets the relays. We shall consider three static strategies:

- *full information*: each relay is told by the source how many other relays have already received the message, and at what times,
- *partial information*: each relay is told by the source how many message copies there are in circulation, but the source does not reveal the age of these copies,
- *no information*: each relay is told nothing by the source; it only knows at what time it meets the source.

In dynamic strategies, the source adapts the information it conveys on the fly as and when it meets the relays. In such a strategy, the decision to give full information, only partial information or no information at all to a relay depends on the contact times with previous relays.

4.4.2 Assumptions on Contact Processes

As mentionned before, the N relays are moving according to a given mobility model. This model represents the movement of relays, and how their location, velocity and acceleration change over time. However, rather than assuming a specific mobility model, we instead characterize the movements of relays solely through their contact processes with the source and the destination. Our main assumption here is that inter-contact times between a relay and the source (resp. destination) are independent and identically distributed (*i.i.d.*) random variables with finite first and second moments. In the following, we let T_s (resp. T_d) be the random time between any two consecutive contacts between a relay and the source (resp. destination). We shall moreover assume that the random variables T_s and T_d are independent. In addition, we shall assume that contacts between relays and any of the fixed nodes are instantaneous, i.e., that the duration of these contacts can be neglected.

At this point, we make two important observations:

- For a given relay, the time instant at which the message is generated by the source can be seen as a random point in time with respect to the contact process of this relay with the source. Hence, the random time between the instant at which the message is generated and the instant at which the relay will meet the source corresponds to what is called the *residual life* of the inter-contact times distribution with the source in the language of renewal theory. In the sequel, we shall refer to this time as the residual inter-contact time with the source.

- Similarly, the time instant at which a given relay receives the message from the source can be considered as a random point in time with respect to the contact process of this relay with the destination. Hence, residual inter-contact time with the destination is given by the residual life of the inter-contact times distribution with the destination.

Let $F_s(x) = \mathbb{P}(T_s > x)$ (resp. $F_d(x) = \mathbb{P}(T_d > x)$) be the complementary cumulative distribution function of T_s (resp. T_d). As a consequence of the above, the density functions of the residual inter-contact times with the source and the destination are given by

$$\tilde{f}_s(x) = \frac{F_s(x)}{\mathbb{E}[T_s]} \quad \text{and} \quad \tilde{f}_d(x) = \frac{F_d(x)}{\mathbb{E}[T_d]}, \tag{4.1}$$

respectively. We also note that the mean residual inter-contact times with the source and the destination are given by $\mathbb{E}[\tilde{T}_s] = \mathbb{E}[T_s^2]/(2\,\mathbb{E}[T_d])$ and $\mathbb{E}[\tilde{T}_d] = \mathbb{E}[T_d^2]/(2\,\mathbb{E}[T_d])$, respectively.

4.4.3 Objectives

In the following, we adopt the point of view of the source and investigate the strategy it should follows in order to minimize the price to be paid for delivering a message. We first analyze the case of static strategies in Section 4.5, and then consider dynamic strategies in Section 4.6.

4.5 Expected Reward Under a Static Strategy

In this section, we assume that the source follows a static strategy, i.e., it does not adapt the information it conveys to as and when it meets the relays. More precisely, we consider the three following settings: (a) the source always gives full information to the relays, (b) it always gives only partial information to the relays or (c) it always gives no information at all to the relays. In the sequel, the superscript F (resp. P, N) will be used to denote quantities related to the *full information* (resp. *partial information*, *no information*) setting. Also, we shall use relay i and the i^{th} relay interchangeably to refer to the relay that is the i^{th} one to meet the source in chronological order.

4.5.1 Estimated Probability of Success

Let S_i, $i = 1, \ldots, N$, be the random time at which the source meets the i^{th} relay. We denote by $\mathbf{S}$ the vector $(S_1, \ldots, S_N)$. In order to simplify notations, we shall write $\mathbf{S}_{-n}$ to denote the vector $(S_1, \ldots, S_{n-1}, S_{n+1}, \ldots, S_N)$ and $\mathbf{S}_{\overline{m:n}}$ to denote the vector $(S_m, \ldots, S_n)$. Similarly, for fixed $s_1, s_2, \ldots, s_N$, we denote by $\mathbf{s}$ the vector $(s_1, s_2, \ldots, s_N)$. We shall also use the notations $\mathbf{s}_{-n}$ and $\mathbf{s}_{\overline{m:n}}$ with the same interpretation as for vectors of random variables.

Define $p_i(\mathbf{s})$ as the (real) probability of success of the i^{th} relay for the given vector $\mathbf{s}$ of contact times, that is the probability of this relay to be the first one to deliver the message. Let also $p_i^{(k)}(\mathbf{s})$ be the probability of success estimated under setting k by relay i when it meets the source[2]. Note that in general $p_i^{(k)}(\mathbf{s})$ and $p_i(\mathbf{s})$ are different. Indeed, the probability of success $p_i(\mathbf{s})$ depends on all contact times. On the contrary, it is obvious that for $i < N$, $p_i^{(k)}(\mathbf{s})$ does not depend on $s_{i+1}, \cdots, s_N$, since, when it meets the source, relay i does not know at what time the source will meet relays $i+1, \ldots, N$. Similarly, for $i > 1$, $p_i^{(k)}(\mathbf{s})$ depends on $s_1, \ldots, s_{i-1}$ only in the *full information* setting. Besides, we also note that

$$p_1^{(P)}(\mathbf{s}) = p_1^{(F)}(\mathbf{s}), \tag{4.2}$$

since the first relay obtains exactly the same information from the source in the partial information and in the full information settings. Finally, we note that

$$p_N^{(F)}(\mathbf{s}) = p_N(\mathbf{s}), \tag{4.3}$$

since in the full information setting, the last relay knows the contact times of all relays with the source.

[2] We remind the reader that relay i refers to the ith relay in chronological order of meeting times with the source.

4.5.2 Expected Cost for a Relay

Define $V_i^{(k)}(\mathbf{s})$ as the net cost for relay i under setting k, and let $R_i^{(k)}(\mathbf{s})$ be the reward asked by this relay to the source under this setting. The reward $R_i^{(k)}(\mathbf{s})$ proposed to relay i has to offset its expected cost $\mathbb{E}[V_i^{(k)}(\mathbf{s})]$, which is given by

$$\mathbb{E}[V_i^{(k)}(\mathbf{s})] = C_r + C_s\mathbb{E}[\tilde{T}_d] + [C_d - R_i^{(k)}(\mathbf{s})]p_i^{(k)}(\mathbf{s}). \tag{4.4}$$

The first term in the net expected cost is the reception cost, which is always incurred. The second term represents the expected storage cost. It is directly proportional to the mean of the residual inter-contact time with the destination. The last term is the cost of transmitting the message to the destination which then gives the reward to the relay. This term enters into play only if relay i is the first one to reach the destination, which explains the factor $p_i^{(k)}(\mathbf{s})$.

4.5.3 Rewards Promised by the Source to Individual Relays: General Inter-Contact Times

Relay i will accept the message provided the proposed reward offsets its expected cost, that is, if $R_i^{(k)}(\mathbf{s})$ is such that $\mathbb{E}[V_i^{(k)}(\mathbf{s})] \leq 0$. Thus, the minimum reward that the source has to promise relay i is

$$\begin{aligned} R_i^{(k)}(\mathbf{s}) &= C_d + \Big(C_r + C_s\mathbb{E}[\tilde{T}_d]\Big)\frac{1}{p_i^{(k)}(\mathbf{s})} \\ &=: C_1 + C_2\frac{1}{p_i^{(k)}(\mathbf{s})}. \end{aligned} \tag{4.5}$$

Note that the reward asked by relay i depends on the information given by the source only through the estimated probability of success $p_i^{(k)}$.

Given $S_1 = s_1, \cdots, S_N = s_N$, the expected reward paid by the source under setting k is

$$\overline{R}^{(k)}(\mathbf{s}) = \sum_{i=1}^{N} p_i(\mathbf{s})R_i^{(k)}(\mathbf{s}). \tag{4.6}$$

With (4.5), it yields

$$\overline{R}^{(k)}(\mathbf{s}) = C_1 + C_2\sum_{i=1}^{N}\frac{p_i(\mathbf{s})}{p_i^{(k)}(\mathbf{s})}. \tag{4.7}$$

While the reward promised to the relays in different information settings can be computed using the above equations, we now give explicit expressions for these rewards for exponential inter-contact times which are observed in certain mobility models.

4.5.4 Rewards Promised by the Source to Individual Relays: Exponential Inter-Contact Times

Let us assume that the inter-contact times between a relay and the source (resp. destination) follows an exponential distribution with rate λ (resp. μ)

We shall first compute the probability of success of each of relays given all the contact times, and then use this expression to compute the probability of success of each of relays in the three information settings. The rewards to be promised to relays can then be computed using (4.5).

Proposition 5. *For a given vector* $\mathbf{s} = (s_1, \cdots, s_N)$*, the success probability of* n^{th} *relay is,*

$$p_n(\mathbf{s}) = \sum_{i=n}^{N} \frac{1 - \left(e^{-\mu(s_{i+1}-s_i)}\right)^i}{i} \prod_{j=1}^{i} e^{-\mu(s_i - s_j)}. \tag{4.8}$$

Proof. Consider relay n that met the source at time s_n and first compute its probability to deliver the message to the destination for each time interval $(s_i, s_{i+1}]$, $n \leq i < N$. The probability that a relay does not meet the destination in $(s_i, s_{i+1}]$ is $e^{-\mu(s_{i+1}-s_i)}$, and the probability that the n^{th} relay will be the first one to meet the destination in $(s_i, s_{i+1}]$ among i relays that have the message at time s_i, is $\frac{1-\left(e^{-\mu(s_{i+1}-s_i)}\right)^i}{i}$.

Next, take into account the probability that none of the relays that received the message before time s_i have not yet meet the destination, which is $\prod_{j=1}^{i} e^{-\mu(s_i-s_j)}$.

The probability of success of the n^{th} relay is then the sum of success probabilities in each interval $(s_i, s_{i+1}]$, $i \geq n$,

$$p_n(\mathbf{s}) = \sum_{i=n}^{N} \frac{1 - \left(e^{-\mu(s_{i+1}-s_i)}\right)^i}{i} \prod_{j=1}^{i} e^{-\mu(s_i - s_j)}. \tag{4.9}$$

□

Next, for each setting $k \in \{F, P, N\}$, write the success probability, $p_i^{(k)}$, estimated by relay i when it receives the message from the source.

Full Information Case

Proposition 6. *For given times* $\mathbf{s} = (s_1, \cdots, s_n)$*,* n^{th} *relay computes its probability of success as*

$$p_n^{(F)}(\mathbf{s}) = \mu \prod_{k=1}^{n-1} e^{-\mu(s_n - s_k)} \sum_{i=n}^{N} \tfrac{(N-n)!}{(N-i)!} \lambda^{i-n} \prod_{j=n}^{i} \tfrac{1}{(N-j)\lambda + j\mu}. \tag{4.10}$$

Proof. In order to derive the formula for success probability, $p_n^{(F)}$, estimated by a relay in the full information setting, we shall use the expression of its real success probability given all the contact times with the source, which is given in Proposition 5, and uncondition future meeting-times of the relays with the source. That is,

$$p_n^{(F)}(\mathbf{s}) = \int p_n(\mathbf{s}) f_{\mathbf{S}_{\overline{n+1:N}}|\mathbf{S}_{\overline{1:n}}}(\mathbf{s}_{\overline{n+1:N}}) ds_{\overline{n+1:N}}, \; n = 1, 2, \ldots, N-1, \tag{4.11}$$

and $p_N^{(F)}(\mathbf{s}) = p_N(\mathbf{s})$.

From (4.8), one can infer that $p_n(\mathbf{s})$ satisfies the following recursion on n:

$$p_n(\mathbf{s}) = p_{n+1}(\mathbf{s}) + \frac{1 - e^{-\mu(s_{n+1}-s_n)n}}{n} \prod_{j=1}^{n} e^{-\mu(s_n - s_j)}. \tag{4.12}$$

Also, since the inter-contact times with the source are i.i.d., the order statistics of the future meeting-times with the source has the product form

$$f_{\mathbf{S}_{\overline{n+1:N}}|\mathbf{S}_{\overline{1:n}}}(\mathbf{s}_{\overline{n+1:N}}) = (N-n)! \prod_{j=n+1}^{N} \frac{\tilde{f}_s(s_j)}{\tilde{F}_s(s_n)}, \tag{4.13}$$

where $\tilde{f}_s$ in the residual inter-contact time density function and $\tilde{F}$ is the corresponding complementary cumulative distribution function. For exponentially distributed random variables with parameter λ, the order statistics takes the form

$$f_{\mathbf{S}_{\overline{n+1:N}}|\mathbf{S}_{\overline{1:n}}}(\mathbf{s}_{\overline{n+1:N}}) = (N-n)! \prod_{j=n}^{N-1} \lambda e^{-(N-j)\lambda(s_{j+1}-s_j)}, \tag{4.14}$$

from which it follows that

$$f_{\mathbf{S}_{\overline{n+1:N}}|\mathbf{S}_{\overline{1:n}}}(\mathbf{s}_{\overline{n+1:N}}) = (N-n)\lambda e^{-(N-n)\lambda(s_{n+1}-s_n)} f_{\mathbf{S}_{\overline{n+2:N}}|\mathbf{S}_{\overline{1:n+1}}}(\mathbf{s}_{\overline{n+2:N}}) \tag{4.15}$$

Substituting (4.15) and (4.12) in (4.11), we

$$\begin{aligned} p_n^{(F)}(\mathbf{s}) = \int_{\mathbf{s}_{\overline{n+1:N}}} & \left(p_{n+1}(\mathbf{s}) + \frac{1 - e^{-\mu(s_{n+1}-s_n)n}}{n} \prod_{j=1}^{n} e^{-\mu(s_n - s_j)} \right) \\ & (N-n)\lambda e^{-(N-n)\lambda(s_{n+1}-s_n)} f_{\mathbf{S}_{\overline{n+2:N}}|\mathbf{S}_{\overline{1:n+1}}}(\mathbf{s}_{\overline{n+2:N}}) \, d\mathbf{s}_{\overline{n+1:N}} \end{aligned} \tag{4.16}$$

Note that the second term in the above sum does not depend upon $s_{n+2}, s_{n+3}, \ldots, s_N$, and the first term can be rewritten in terms of $p_{n+1}^{(F)}(\mathbf{s})$ using (4.11), which gives

$$\begin{aligned} p_n^{(F)}(\mathbf{s}) = &\int_{s_{n+1}} p_{n+1}^{(F)}(\mathbf{s})(N-n)\lambda e^{-(N-n)\lambda(s_{n+1}-s_n)} ds_{n+1} \\ &+ \int_{s_{n+1}} \frac{1-e^{-\mu(s_{n+1}-s_n)n}}{n} \prod_{j=1}^{n} e^{-\mu(s_n-s_j)}(N-n)\lambda e^{-(N-n)\lambda(s_{n+1}-s_n)} ds_{n+1} \end{aligned} \tag{4.17}$$

Equation (4.17) gives a recursion for $p_n^{(F)}$ in terms of $p_{n+1}^{(F)}$. The proof of the claimed result will follow if we show that (4.10) satisfies this recursion. The base case is $n = N$, for which we note that $p_N^{(F)}(\mathbf{s})$ given in (4.10) is equal to $p_N(\mathbf{s})$ given in 4.8. Now, assume that for all $j = n+1, \ldots, N$, $p_j^{(F)}$ is given by (4.10).

Consider the first term in the RHS of (4.17). From (4.10),

$$\begin{aligned} p_{n+1}^{(F)}(\mathbf{s}) &= \mu\theta_{n+1} \prod_{k=1}^{n} e^{-\mu(s_{n+1}-s_k)} \\ &= \mu\theta_{n+1} \left(\prod_{k=1}^{n} e^{-\mu(s_n-s_k)}\right) e^{-n\mu(s_{n+1}-s_n)}, \end{aligned}$$

where

$$\theta_{n+1} = \sum_{i=n+1}^{N} \frac{(N-(n+1))!}{(N-i)!} \lambda^{i-(n+1)} \prod_{j=n+1}^{i} \frac{1}{(N-j)\lambda + j\mu}. \tag{4.18}$$

Therefore,

$$\begin{aligned} (N-n)\int_{s_{n+1}=s_n}^{\infty} p_{n+1}^{(F)}(\mathbf{s})\lambda e^{-\lambda(N-n)(s_{n+1}-s_n)} ds_{n+1} = \\ \mu\theta_{n+1}\left(\prod_{k=1}^{n} e^{-\mu(s_n-s_k)}\right) \frac{\lambda(N-n)}{\lambda(N-n)+\mu n}. \end{aligned}$$

Similarly, the second term becomes

$$\mu \left(\prod_{k=1}^{n} e^{-\mu(s_n-s_k)}\right) \frac{1}{\lambda(N-n)+\mu n}.$$

Thus we can rewrite (4.17) as

$$p_n^{(F)}(\mathbf{s}) = \mu \left(\prod_{k=1}^{n} e^{-\mu(s_n-s_k)}\right) \left(\theta_{n+1}\frac{\lambda(N-n)}{\lambda(N-n)+\mu n} + \frac{1}{\lambda(N-n)+\mu n}\right). \tag{4.19}$$

We can verify from (4.18) that θ_n follows the recursion

$$\theta_n = \theta_{n+1}\frac{\lambda(N-n)}{\lambda(N-n)+\mu n} + \frac{1}{\lambda(N-n)+\mu n},$$

which allows to conclude that, as claimed,

$$p_n^{(F)}(\mathbf{s}) = \mu\theta_n \prod_{k=1}^{n-1} e^{-\mu(s_n - s_k)},$$

where the term corresponding to $k = n$ in the product in (4.19) is just 1 and can be omitted. □

Partial Information Case

Proposition 7. *Given the time s_n with the number, n, of already existing copies, the n^{th} relay computes its success probability as*

$$\begin{aligned} p_n^{(P)}(\mathbf{s}) = & \left(\tfrac{\lambda}{\lambda-\mu}\tfrac{e^{-\mu s_n}-e^{-\lambda s_n}}{1-e^{-\lambda s_n}}\right)^{n-1} \\ & \times\mu \sum_{i=n}^{N} \tfrac{(N-n)!}{(N-i)!}\lambda^{i-n} \prod_{j=n}^{i} \tfrac{1}{(N-j)\lambda+j\mu}, \quad \text{if } \lambda \neq \mu, \end{aligned} \tag{4.20}$$

and

$$p_n^{(P)}(\mathbf{s}) = \left(\lambda s_n \tfrac{e^{-\lambda s_n}}{1-e^{-\lambda s_n}}\right)^{n-1} \sum_{i=n}^{N} \tfrac{(N-n)!}{(N-i)!N^{i-n+1}}, \quad \text{if } \lambda = \mu. \tag{4.21}$$

Proof. The probability that after time s_n, the n^{th} relay is the first one to deliver the message to the destination is given by

$$\sum_{i=n}^{N} \frac{(N-n)!}{(N-i)!}\frac{\mu}{\lambda}\prod_{j=n}^{i}\frac{\lambda}{(N-j)\lambda+j\mu}. \tag{4.22}$$

Consider a relay that received the copy of the message before time s_n. For $\lambda \neq \mu$, the probability that the relay does not meet the destination before s_n is

$$\int_0^{s_n} \frac{\lambda e^{-\lambda s}e^{-\mu(s_n-s)}}{1-e^{-\lambda s_n}}ds = \frac{\lambda}{\lambda-\mu}\frac{e^{-\mu s_n}-e^{-\lambda s_n}}{1-e^{-\lambda s_n}}. \tag{4.23}$$

Then the probability that none of the $n-1$ relays that received the message before time s_n did not deliver it to the destination before s_n is

$$\left(\frac{\lambda}{\lambda-\mu}\frac{e^{-\mu s_n}-e^{-\lambda s_n}}{1-e^{-\lambda s_n}}\right)^{n-1}, \quad \text{for } \lambda \neq \mu. \tag{4.24}$$

The product of this probability with the probability (4.22) that after time s_n, n^{th} relay is the first one to deliver the message to the destination, gives the claimed result.

Similarly reasoning, the claimed result for $\lambda = \mu$ is obtained after substituting λ instead of μ in (4.22) and with that the integral in (4.23) gives $\lambda s_n \frac{e^{-\lambda s_n}}{1-e^{-\lambda s_n}}$. □

Corollary 5. *For the given times* $\mathbf{s} = (s_1, \cdots, s_n)$*, the success probability of the* n^{th} *relay in the full information setting,* $p_n^{(F)}$*, can be represented through* $p_n^{(P)}$ *as follows,*

$$p_n^{(F)}(s_1, \cdots, s_n) = \frac{\prod_{k=1}^{n-1} e^{-\mu(s_n - s_k)}}{\left(\frac{\lambda}{\lambda-\mu} \frac{e^{-\mu s_n} - e^{-\lambda s_n}}{1-e^{-\lambda s_n}}\right)^{n-1}} p_n^{(P)}(\mathbf{s}), \;\; \text{if } \lambda \neq \mu. \tag{4.25}$$

No Information Case

Proposition 8. *Given only the time* s_n*, the* n^{th} *relay computes its success probability as*

$$\begin{aligned} & p_n^{(N)}(\mathbf{s}) = \\ & = \sum_{m=1}^{N} \frac{(N-1)!}{(N-m)!(m-1)!} \left(1 - e^{-\lambda s_n}\right)^{m-1} \left(e^{-\lambda s_n}\right)^{N-m} p_m^{(P)}. \end{aligned} \tag{4.26}$$

Proof. Consider the relay n that meets the source at time s_n and informed only this meeting time and not the number of already existing copies of the message. The probability that any relay does not meet the source before time s_n is $e^{-\lambda s_n}$ and that it meets the source is $1 - e^{-\lambda s_n}$. Then the n^{th} relay can compute its probability of success as

$$\begin{aligned} & p_n^{(N)}(\mathbf{s}) = \\ & = \sum_{m=1}^{N} C_{N-1}^{m-1} \left(1 - e^{-\lambda s_n}\right)^{m-1} \left(e^{-\lambda s_n}\right)^{N-m} p_m^{(P)}(\mathbf{s}) \\ & = \sum_{m=1}^{N} \frac{(N-1)!}{(N-m)!(m-1)!} \left(1 - e^{-\lambda s_n}\right)^{m-1} \left(e^{-\lambda s_n}\right)^{N-m} p_m^{(P)}(\mathbf{s}). \end{aligned} \tag{4.27}$$

□

Thus, the source when it meets a relay can compute the reward it should promise to this relay within each setting based on the corresponding success probability estimated by the relay.

4.5.5 Expected Reward Paid by the Source

Until now, we have computed the reward the source should offer to each of the relays as a function of the time it meets them and the information offered to them. We now turn our attention to the expected reward paid by the source when the expectation is taken over all possible meeting times. This quantity can be thought of as the long-run average reward per message the source will have to pay if it sends a large number of messages (and assuming that message generation occurs at a much slower time scale than that of the contact process).

The expected reward paid by the source under setting k can be obtained by unconditioning (4.6) on $S_1, \cdots, S_N$,

$$\begin{aligned}\overline{R}^{(k)} &= \int_{\mathbf{s}} \overline{R}^{(k)}(\mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s} \\ &= \int_{s_1=0}^{\infty} \int_{s_2=s_1}^{\infty} \cdots \int_{s_N=s_{N-1}}^{\infty} \overline{R}^{(k)}(\mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) ds_N \cdots ds_2 ds_1,\end{aligned} \tag{4.28}$$

where $f_{\mathbf{S}}(\mathbf{s})$ is the joint distribution of $S_1, \cdots, S_N$. Since the residual inter-contact times between the relays and the source are i.i.d. random variables, $f_{\mathbf{S}}(\mathbf{s})$ is the joint distribution of the order statistics of the N random variables $S_1, \cdots, S_N$. That is,

$$f_{\mathbf{S}}(\mathbf{s}) = N!\, \tilde{f}_s(s_1) \ldots \tilde{f}_s(s_N). \tag{4.29}$$

With (4.7), (4.28) and (4.29), we obtain the expected reward paid by the source in terms of the probabilities of success estimated by the relays,

$$\overline{R}^{(k)} = C_1 + C_2\, N! \sum_{n=1}^{N} \int_{\mathbf{s}} \frac{p_n(\mathbf{s})}{p_n^{(k)}(\mathbf{s})} \tilde{f}_s(s_1) \ldots \tilde{f}_s(s_N) d\mathbf{s}. \tag{4.30}$$

From the probability of success estimated by the relays in the three settings, we can prove that the expected reward to be paid by the source for delivering its message is the same in all three settings, as stated in Theorem 8.

Theorem 8. *The expected reward to be paid by the source under setting $k \in \{F, P, N\}$ is*

$$\overline{R}^{(k)} = C_1 + NC_2. \tag{4.31}$$

Proof. Since $p_n^{(k)}$ does not depend on $s_{n+1}, \cdots, s_N$, we can rewrite (4.30) as follows

$$\begin{aligned}&\overline{R}^{(k)} = C_1 + C_2 \times \\ &\times \sum_{n=1}^{N} \int_{s_{1:n}} \frac{f_{\mathbf{S}_{1:n}}(\mathbf{s}_{1:n})}{p_n^{(k)}(\mathbf{s}_{1:n})} \\ &\qquad \int_{\mathbf{s}_{n+1:N}} p_n(\mathbf{s}) f_{\mathbf{S}_{n+1:N}|\mathbf{S}_{1:n}}(\mathbf{s}_{n+1:N}|\mathbf{s}_{1:n}) d\mathbf{s}_{N:n+1} \\ &\quad d\mathbf{s}_{n:1},\end{aligned} \tag{4.32}$$

where $d\mathbf{s}_{N:n+1}$ is to be read as $ds_N ds_{N-1} \cdots ds_{n+1}$, and

$$f_{\mathbf{S}_{n+1:N}|\mathbf{S}_{1:n}}(\mathbf{s}_{n+1:N}|\mathbf{s}_{1:n}) = \frac{f_{\mathbf{S}_{1:N}}(\mathbf{s}_{1:N})}{f_{\mathbf{S}_{1:n}}(\mathbf{s}_{1:n})}. \tag{4.33}$$

We now proceed to the analysis of the success probabilities estimated by the relays in each of the three settings.

Full Information Setting The success probability of the n^{th} relay in the full information setting can be expressed as

$$p_n^{(F)}(\mathbf{s}_{1:n}) = \int\limits_{\mathbf{s}_{n+1:N}} p_n(\mathbf{s}) f_{\mathbf{S}_{n+1:N}|\mathbf{S}_{1:n}}(\mathbf{s}_{n+1:N}|\mathbf{s}_{1:n}) d\mathbf{s}_{N:n+1}. \tag{4.34}$$

With (4.32), it yields

$$\begin{aligned} \overline{R}^{(k)} &= C_1 + C_2 \sum_{n=1}^{N} \int\limits_{\mathbf{s}_{1:n}} \frac{f_{\mathbf{S}_{1:n}}(\mathbf{s}_{1:n})}{p^{(F)}(\mathbf{s}_{1:n})} p^{(F)}(\mathbf{s}_{1:n}) d\mathbf{s}_{n:1} \\ &= C_1 + C_2 \sum_{n=1}^{N} 1 = C_1 + NC_2. \end{aligned}$$

Partial Information Setting With (4.32) and (4.34), we can write the expected reward under the partial information setting as follows

$$\overline{R}^{(P)} = C_1 + C_2 \sum_{n=1}^{N} \int\limits_{\mathbf{s}_{1:n}} f_{\mathbf{S}_{1:n}}(\mathbf{s}_{1:n}) \frac{p_n^{(F)}(\mathbf{s})}{p_n^{(P)}(\mathbf{s})} d\mathbf{s}_{n:1}. \tag{4.35}$$

Since $p_n^{(P)}$ depends only on s_n, we can change the integration order in (4.35) to obtain

$$\begin{aligned} \overline{R}^{(P)} &= C_1 + C_2 \times \\ &\times \sum_{n=1}^{N} \int\limits_{s_n=0}^{\infty} \frac{f_{S_n|\mathbf{S}_{1:n-1}}(s_n|\mathbf{s}_{1:n-1})}{p_n^{(P)}(\mathbf{s})} \\ &\qquad \int\limits_{\mathbf{s}_{n-1:1}} p_n^{(F)} f_{\mathbf{S}_{1:n-1}}(\mathbf{s}_{1:n-1}) d\mathbf{s}_{1:n-1} ds_n. \end{aligned} \tag{4.36}$$

Now, observe that the success probability of the n^{th} relay can be expressed as

$$p_n^{(P)}(\mathbf{s}) = \frac{\int\limits_{\mathbf{s}_{n-1:1}} p_n^{(F)}(\mathbf{s}) f_{\mathbf{S}_{1:n-1}}(\mathbf{s}_{1:n-1}) d\mathbf{s}_{1:n-1}}{\int\limits_{\mathbf{s}_{n-1:1}} f_{\mathbf{S}_{1:n-1}}(\mathbf{s}_{1:n-1}) ds_1 \cdots ds_{n-1}}, \tag{4.37}$$

where the integral $\int\limits_{\mathbf{s}_{n-1:1}}$ is to be read $\int\limits_{s_{n-1}=0}^{s_n} \cdots \int\limits_{s_1=0}^{s_2}$.

With (4.36), it yields

$$
\begin{aligned}
\overline{R}^{(P)} = &\ C_1 + C_2 \times \\
&\times \sum_{n=1}^{N} \int_{s_n=0}^{\infty} f_{S_n|\mathbf{S}_{1:n-1}}(s_n|\mathbf{s}_{1:n-1}) \\
&\qquad\qquad \int_{\mathbf{s}_{n-1:1}} f_{\mathbf{S}_{1:n-1}}(\mathbf{s}_{1:n-1}) d\mathbf{s}_{1:n-1} ds_n \\
&= C_1 + C_2 \sum_{n=1}^{N} \int_{\mathbf{s}_{n:1}} f_{\mathbf{S}_{1:n}}(\mathbf{s}_{1:n}) d\mathbf{s}_{1:n} \\
&= C_1 + C_2 \sum_{n=1}^{N} 1 = C_1 + NC_2. \qquad (4.38)
\end{aligned}
$$

No Information Case Since the success probability of the n^{th} relay in the no information setting depends only on s_n, we can rewrite the expression for the expected reward paid by the source as

$$
\begin{aligned}
&\overline{R}^{(N)} = C_1 + C_2 \times \qquad (4.39)\\
&\times \sum_{n=1}^{N} \int_{s_n=0}^{\infty} \frac{1}{p_n^{(N)}(s_n)} \int_{\substack{\mathbf{s}_{1:n-1} \le s_n \\ \mathbf{s}_{n+1:N}}} p_n(\mathbf{s}) f_{\mathbf{S}_{1:N}}(\mathbf{s}_{1:N}) d\mathbf{s}_{-n} ds_n
\end{aligned}
$$

where the integral $\int_{\substack{\mathbf{s}_{1:n-1} \le s_n \\ \mathbf{s}_{n+1:N}}}$ is to be read as

$$
\int_{s_1=0}^{s_n} \cdots \int_{s_{n-1}=s_{n-2}}^{s_n} \int_{s_{n+1}=s_n}^{\infty} \cdots \int_{s_N=s_{N-1}}^{\infty} .
$$

Observe that the joint distribution $f_{\mathbf{S}_{1:N}}(\mathbf{s}_{1:N})$ can be eqivalently written as follows

$$
\begin{aligned}
&f_{\mathbf{S}_{1:N}}(\mathbf{s}_{1:N}) \qquad (4.40)\\
&= (N-1)! \tilde{f}_s(s_1) \cdots \tilde{f}_s(s_{n-1}) \tilde{f}_s(s_{n+1}) \cdots \tilde{f}_s(s_N) N \tilde{f}_s(s_n) \\
&= f_{\mathbf{S}_{-n}}(\mathbf{s}_{-n}) N \tilde{f}_s(s_n).
\end{aligned}
$$

Note that the outer summation in (4.39) specifies only the ordinal position of the time s_n for each member of summation, and thus can be put under the integral by removing the ordinal dependence as follows,

$$
\begin{aligned}
&\overline{R}^{(N)} = C_1 + C_2 \times \qquad (4.41)\\
&\times\ N \int_{s_n=0}^{\infty} \frac{\tilde{f}_s(s_n)}{p_n^{(N)}(s_n)} \sum_{m=1}^{N} \int_{\substack{\mathbf{s}_{1:m-1} \le s_n \\ \mathbf{s}_{m+1:N}}} p_m(\mathbf{s}) f_{\mathbf{S}_{-m}}(\mathbf{s}_{-m}) d\mathbf{s}_{-m} ds_n.
\end{aligned}
$$

Now the sum represents the success probability of the n^{th} relay in the no information setting, namely,

$$p_n^{(N)}(s_n) = \sum_{m=1}^{N} \int\limits_{\substack{\mathbf{s}_{1:m-1} \le s_n \\ \mathbf{s}_{m+1:N}}} p_m(\mathbf{s}) f_{\mathbf{S}_{-m}}(\mathbf{s}_{-m}) d\mathbf{s}_{-m}. \tag{4.42}$$

Thus,

$$\begin{aligned} \overline{R}^{(N)} &= C_1 + NC_2 \int\limits_{s_n=0}^{\infty} \frac{p_n^{(N)}(s_n)}{p_n^{(N)}(s_n)} \tilde{f}_s(s_n) ds_n \\ &= C_1 + NC_2. \end{aligned} \tag{4.43}$$

□

Theorem 8 shows that if the source does not adapt the information it gives, the expected reward it will have to pay remains the same irrespective of the information it conveys. We also note that the expected reward grows linearly with the number of relays.

The result in Theorem 8 has the following intuitive explanation. It says that the expected reward paid by the source is equal to expected total cost incurred by all the relays in the process of delivering the message. Each relay accepts and stores the message until it meets the destination, and a cost of $C_2 = C_r + C_s \mathbb{E}[\tilde{T}_d]$ in the process. Since there are N relays which carry the message, the expected total cost for carrying the message is NC_2. Of these N, one relay will be successful in delivering the message and will incur an additional delivery cost of $C_1 = C_d$. Thus, the expected total cost incurred by the relays is $C_1 + NC_2$. Since on the long run the relays make neither a profit nor a loss, the expected total costs incurred by the relays should be offset by the reward paid by the source, which explains the result in Theorem 8. What is less intuitive though is that the expected reward paid does not depend on the type of information given to the relays.

4.6 Adaptive Strategy

The analysis in the previous section shows that as long as the information given to all the relays is of the same type, the source has to pay the same reward. Could the source do better by changing the type of information it gives to relays based on and when it meets them? We show in this section that the source can indeed reduce the expected reward it pays if it can adapt the type of information dynamically. Consider the following situation in which the source encounters the second relay a long time after it encountered the first one. If the source discloses the time when it met the first relay to the second one, then the second relay will correctly compute its probability of success to be small and will ask for a high reward. If instead the source were not to disclose this information, then the probability of success computed by the relay would be higher and the source could propose a lower reward. Thus, source stands to gain by changing the type of information based on the time instants it encounters the relays.

In this we shall investigate the benefits that an adaptive strategy can procure for the source, and bring to light certain structural properties concerning of the optimal adaptive strategy for some particular cases of the model.

A key assumption we shall make in the analysis of the adaptive strategy is that the relays do not react to the fact that the source is adapting its strategy. A relay will compute its success probability based only on its contact time with the source and additional information, if any, received from the source. In practice, if the relay knows that the source will adapt its strategy as a function of time, then the relay will also react accordingly, to which the source will react, and so on *ad infinitum*. As a first approximation, we shall restrict the analysis of the adaptive strategy assuming that the relays are naive.

4.6.1 Adaptive Versus Static Strategies

We shall first give bounds on the expected reward paid by the source when it uses the adaptive strategy.

Let $\overline{R}^{(A)}$ denote the expected reward paid by the source when it uses the adaptive strategy. The decision of the source to either give or not information to a relay it meets will depend upon the reward it has to propose in each of the three settings. Thus, the source when it meets a relay can compute the reward it should promise to this relay within each setting based on the corresponding success probability estimated by the relay and then to choose the setting of least reward to be paid to this relay. That is,

$$\overline{R}^{(A)} = \int\limits_{\mathbf{s}} \left(\sum_{n=1}^{N} p_n(\mathbf{s}) \min_k \left(R_n^{(k)} \right) \right) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s}. \tag{4.44}$$

From the definition of the adaptive strategy, it can do no worse than any static strategy which gives an upper bound. Also, the source has to pay at least $C_1 + C_2$ because this is the average cost when there is only one relay, which gives a lower bound. It follows that

Proposition 9. $C_1 + C_2 \leq \overline{R}^{(A)} \leq \overline{R}^{(k)} = C_1 + NC_2$.

Corollary 6. $\frac{\overline{R}^{(A)}}{\overline{R}^{(k)}} \geq \frac{C_1+C_2}{C_1+NC_2} \geq \frac{1}{N}$.

By using an adaptive strategy the source can reduce its expenses at most by a factor of $1/N$.

Although the exact analytical expressions for an adaptive policy is difficult to compute, an advantage of the adaptive strategy can be seen from the numerical results. In Figures 4.1 and 4.2, $\overline{R}^{(A)}$ is plotted as a function of λ for $N = 5$, $\mu = 1$, $C_1 = 1$, and $C_2 = 5$ ($C_2 = 0.5$ in Figure 4.2). It is observed that $\overline{R}^{(A)}$ increases with λ and is gets close to $\overline{R}^{(F)}$ when $\lambda \to \infty$. On the other hand, for small values of λ, $\overline{R}^{(A)}$ is close to the minimal

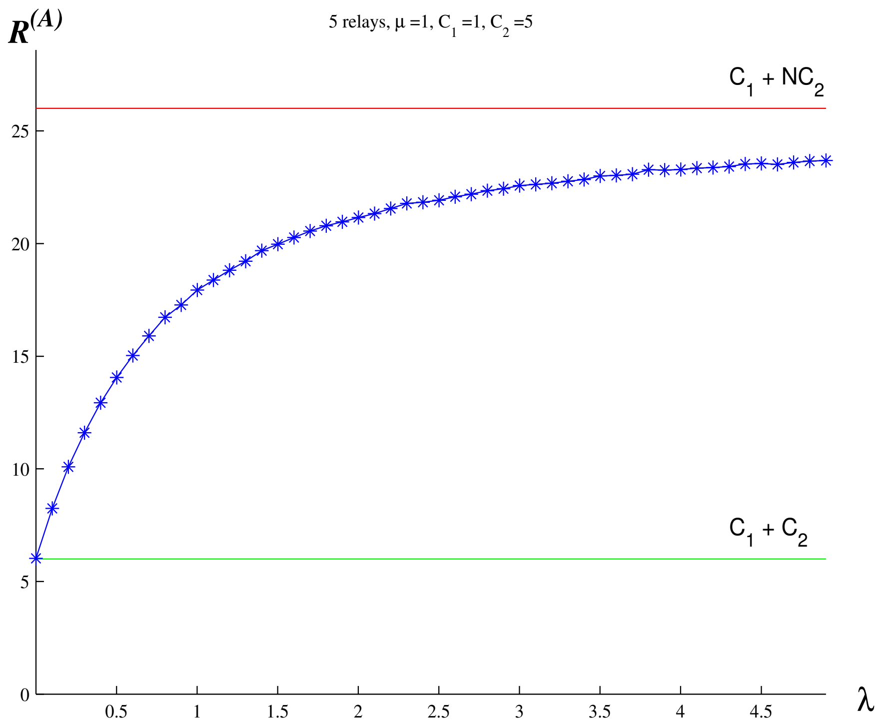

Figure 4.1: Expected reward paid by the source for the adaptive strategy. $N = 5$, $\mu = 1$, $C_1 = 1$, $C_2 = 5$.

reward $C_1 + C_2$. It appears that $\overline{R}^{(A)}$ has the form $(C_1 + C_2) + C_2(1 - e^{-\lambda\gamma})$, for some constant γ, but we are unable to prove this result.

The exact analytical expression of $\overline{R}^{(A)}$ is difficult to compute unlike the expression for $\overline{R}^{(k)}$. Nonetheless, we shall give some structural properties of the adaptive strategy. In particular, for $N = 2$, it will be shown that the adaptive strategy is of threshold type in which the second relay is given either full information or no information depending on how late it meets the source after the first one.

4.6.2 Two Relays, Decreasing Density Function of Inter-Contact Times

Let us consider a network of a fixed single source, a fixed single destination, and two relays with an underlying mobility model described in the Section 4.4.2. Futher assume that densities of residual inter-contact times, $\tilde{f}_s$ and $\tilde{f}_d$, are decreasing functions.

In order to establish the structure of the adaptive strategy, one needs to determine which information setting has the lowest reward at any given instant. The reward of a given setting depends in turn on the probability of success estimated by the relay based

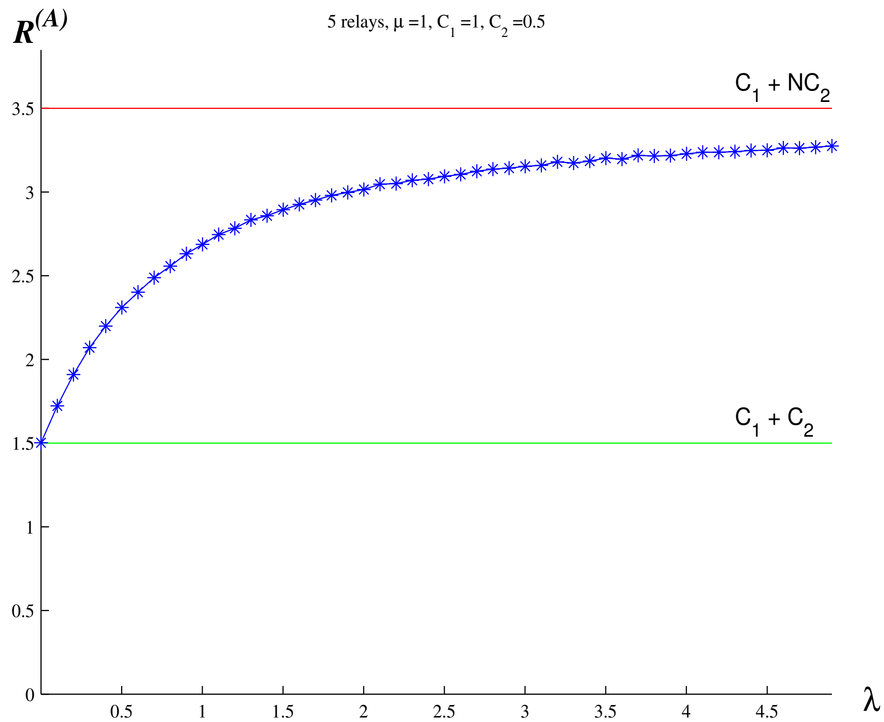

Figure 4.2: Expected reward paid by the source for the adaptive strategy. $N = 5$, $\mu = 1$, $C_1 = 1$, $C_2 = 0.5$.

on the information given by source (see (4.5)). For the comparison of the rewards, we shall need a few results on the probabilities of success, which we give now.

Lemma 11. *1.*

$$p_2(\mathbf{s}) \leq \frac{1}{2} \leq p_1(\mathbf{s}), \tag{4.45}$$

2. for fixed s_2, $p_1(s_1, s_2)$ decreases ($p_2(s_1, s_2)$ increases) with s_1.

Proof. Prove the first inequality in the first part of the lemma. Then the second inequality will follow from the fact that $p_1(\mathbf{s}) + p_2(\mathbf{s}) = 1$.

The probability of success of the second relay given vector of meeting times with the source, $\mathbf{s}$,

$$p_2(\mathbf{s}) = \int\limits_{y_2=s_2}^{\infty} \tilde{f}_d(y_2 - s_2) \int\limits_{y_1=y_2}^{\infty} \tilde{f}_d(y_1 - s_1) dy_1 dy_2. \tag{4.46}$$

Change the variables and using the properties of the integration of non-negative functions obtain,

$$p_2(\mathbf{s}) = \int\limits_{y_2=0}^{\infty} \tilde{f}_d(y_2) \int\limits_{y_1=y_2+s_2-s_1}^{\infty} \tilde{f}_d(y_1)dy_1dy_2 \qquad (4.47)$$
$$\leq \int\limits_{y_2=0}^{\infty} \tilde{f}_d(y_2) \int\limits_{y_1=y_2}^{\infty} \tilde{f}_d(y_1)dy_1dy_2.$$

The last expression gives 1/2. Show it thoroughly.

Consider probability density function $f(\cdot)$. Thus, for the function f, by the changing of integration order obtain,

$$\int\limits_{u=0}^{\infty} f(u) \int\limits_{v=u}^{\infty} f(v)dvdu = \int\limits_{v=0}^{\infty} f(v) \int\limits_{u=0}^{v} f(u)dudv. \qquad (4.48)$$

Note also, that the integration in the left hand side does not depend of the choice of the integration variables and thus can be rewritten as

$$\int\limits_{u=0}^{\infty} f(u) \int\limits_{v=u}^{\infty} f(v)dvdu = \int\limits_{v=0}^{\infty} f(v) \int\limits_{u=v}^{\infty} f(u)dudv. \qquad (4.49)$$

Summation of this two equalities gives one in the right hand side due to the properties of the probability density function and thus,

$$\int\limits_{u=0}^{\infty} f(u) \int\limits_{v=u}^{\infty} f(v)dvdu = \tfrac{1}{2}. \qquad (4.50)$$

Since $p_1(\mathbf{s}) + p_2(\mathbf{s}) = 1$, then for the second statement of the lemma to hold, show only that for fixed s_2, the probability $p_2(s_1, s_2)$ is increasing function of s_1. This directly follows from (4.46) due to the decreasing property of the function $\tilde{f}_d$. □

The above result states that the real probability of success of the first relay decreases when its meeting time with the source gets closer to that of the second relay. It gives a similar monotonicity result for the probability of success of the second relay. The assumption of decreasing density function comes into play in the proof of these results.

The next lemmas shows the similar inequalities for the success probabilities in the full information setting and the partial information setting.

Lemma 12.

$$p_2^{(F)}(\mathbf{s}) \leq \frac{1}{2} \leq p_1^{(F)}(\mathbf{s}). \qquad (4.51)$$

Proof. The first inequality follows from Lemma 11 and equality (4.3).

For the second inequality, note that the probability of success of the first relay in the full information setting can be represented as follows,

$$p_1^{(F)}(\mathbf{s}) = \int\limits_{s_2=s_1}^{\infty} p_1(\mathbf{s})\tilde{f}_s(s_2 - s_1)ds_2. \qquad (4.52)$$

Using Lemma 11 for $p_1(\mathbf{s})$, we obtain

$$p_1^{(F)}(\mathbf{s}) \geq \tfrac{1}{2} \int\limits_{s_2=s_1}^{\infty} \tilde{f}_s(s_2 - s_1) ds_2 = \tfrac{1}{2}, \tag{4.53}$$

since $\int_{s_2=s_1}^{\infty} \tilde{f}_s(s_2 - s_1) ds_2 = 1$ due to the property of probability density function. □

Lemma 13.

$$p_2^{(P)}(\mathbf{s}) \leq \frac{1}{2} \leq p_1^{(P)}(\mathbf{s}). \tag{4.54}$$

Proof. From Lemma 12 for $p_1^{(F)}$, along with equation (4.2), it follows that $p_1^{(P)}(\mathbf{s}) \geq 1/2$. It is now sufficient to show that $p_2^{(P)}(\mathbf{s}) \leq 1/2$.

The success probability of the second relay in the partial information setting satisfies

$$p_2^{(P)}(\mathbf{s}) = \frac{\int\limits_{s_1=0}^{s_2} p_2^{(F)}(\mathbf{s}) \tilde{f}_s(s_1) ds_1}{\int\limits_{s_1=0}^{s_2} \tilde{f}_s(s_1) ds_1} \tag{4.55}$$

$$\leq \frac{\tfrac{1}{2} \int\limits_{s_1=0}^{s_2} \tilde{f}_s(s_1) ds_1}{\int\limits_{s_1=0}^{s_2} \tilde{f}_s(s_1) ds_1} = \frac{1}{2}. \tag{4.56}$$

where the inequality follows from Lemma 12 according to which $p_2^{(F)} \leq 1/2$. □

We now proceed to the main results on the comparison of the rewards in various information settings. The first results shows that it is always beneficial for the source to give information to the first relay independently of s_1.

Proposition 10.

$$R_1^{(F)}(\mathbf{s}) = R_1^{(P)}(\mathbf{s}) \leq R_1^{(N)}(\mathbf{s}) \tag{4.57}$$

Proof. The equality $R_1^{(F)} = R_1^{(P)}$ follows from (4.5) and (4.2). For the inequality, from (4.5), it is sufficient to establish that

$$p_1^{(N)}(\mathbf{s}) \leq p_1^{(P)}(\mathbf{s}), \ \forall s_1 \geq 0.$$

The probability,

$$\begin{aligned} p_1^{(N)}(\mathbf{s}) &= p_2^{(P)}(s_1)\mathbb{P}(S_2 < s_1) + p_1^{(P)}(s_1)(1 - \mathbb{P}(S_2 < s_1)) \\ &= \mathbb{P}(S_2 < s_1)\left[p_2^{(P)}(s_1) - p_1^{(P)}(s_1)\right] + p_1^{(P)}(s_1) \\ &\leq p_1^{(P)}(s_1), \end{aligned} \tag{4.58}$$

where the last inequality follows from (4.54). □

The next result in favour of an adaptive strategy pertains to the reward the source should propose to the second relay.

Proposition 11.

$$R_2^{(N)}(\mathbf{s}) \leq R_2^{(P)}(\mathbf{s}). \tag{4.59}$$

Proof. The success probability of the second relay in the no information setting, $p_2^{(N)}(\mathbf{s})$, can be expressed as

$$p_2^{(N)}(\mathbf{s}) = p_2^{(P)}(\mathbf{s})\mathbb{P}(S_1 < s_2) + p_1^{(P)}(\mathbf{s})(1 - \mathbb{P}(S_1 < s_2)), \tag{4.60}$$

with S_1 being the random time when the source gives the copy of the message to the first relay it meets.

With (4.54), the following inequality holds,

$$\begin{aligned} p_2^{(N)}(\mathbf{s}) &\geq p_2^{(P)}(\mathbf{s})\mathbb{P}(S_1 < s_2) + p_2^{(P)}(\mathbf{s})(1 - \mathbb{P}(S_1 < s_2)) \\ &= p_2^{(P)}(\mathbf{s}), \end{aligned} \tag{4.61}$$

and the statement of the proposition follows. □

Proposition 11 says that between the choice of informing a relay that it is the second one and not giving this information, it is better for the source not to give this information.

Before proceeding to the next result, we prove another lemma.

Lemma 14. *$p^{(N)}(s)$ decreases with s.*

Proof. The probability,

$$p^{(N)}(s) = p_2^{(P)}(s)\mathbb{P}(\hat{S} < s) + p_1^{(P)}(s)(1 - \mathbb{P}(\hat{S} < s)).$$

Find its derivative on s,

$$\begin{aligned} \frac{dp^{(N)}(s)}{ds} &= p_2^{(P)}(s)\tilde{f}_s(s) + \tfrac{dp_2^{(P)}(s)}{ds}\mathbb{P}(\hat{S} < s) \\ &\quad -p_1^{(P)}(s)\tilde{f}_s(s) + \tfrac{dp_1^{(P)}(s)}{ds}(1 - \mathbb{P}(\hat{S} < s)). \\ &= \left[p_2^{(P)}(s) - p_1^{(P)}(s)\right]\tilde{f}_s(s) + \tfrac{dp_2^{(P)}(s)}{ds}\mathbb{P}(\hat{S} < s) \\ &\quad +\tfrac{dp_1^{(P)}(s)}{ds}(1 - \mathbb{P}(\hat{S} < s)). \end{aligned}$$

The first term of the last sum is negative due to (4.54). To complete the proof, we show the negativity of two last terms of this sum.

From (4.55), find the derivative,

$$\begin{aligned}\frac{dp_2^{(P)}(s)}{ds} &= \frac{p_2^{(F)}(s,s)\tilde{f}_s(s)\int\limits_{\hat{s}=0}^{s}\tilde{f}_s(\hat{s})d\hat{s} - \int\limits_{\hat{s}=0}^{s}p_2^{(F)}(\hat{s},s)\tilde{f}_s(\hat{s})d\hat{s}\tilde{f}_s(s)}{\left(\int\limits_{\hat{s}=0}^{s}\tilde{f}_s(\hat{s})d\hat{s}\right)^2}\\ &= \frac{\tilde{f}_s(s)\int\limits_{\hat{s}=0}^{s}\left[p_2^{(F)}(s,s)-p_2^{(F)}(\hat{s},s)\right]\tilde{f}_s(\hat{s})d\hat{s}}{\left(\int\limits_{\hat{s}=0}^{s}\tilde{f}_s(\hat{s})d\hat{s}\right)^2} \leq 0,\end{aligned}$$

since $p_2^{(F)}(s,s) - p_2^{(F)}(\hat{s},s) \leq 0$ due to the second statement of the Lemma 11 and the equation (4.3).

With (4.2) and from (4.52), the derivative,

$$\frac{dp_1^{(P)}(s)}{ds} = -p_1(s,s)\tilde{f}_s(s) < 0.$$

Thus, the derivative $\frac{dp^{(N)}(s)}{ds}$ is negative and the claimed result follows. □

Until now we have shown that it is optimal to give the full information to the first relay, and for the second relay it is giving no information is always better that giving partial information. We now compare the settings of no information with that of full information.

Our main result for this section, stated in Theorem 9 shows that there is a threshold, which depends on the meeting time with the first relay, before which it is optimal to give full information to the second relay and beyond which it is optimal to give no information. Once, the source meets the first relay, it can compute this threshold, and based on when it meets the second relay decide to give or not the information.

Define the difference of the success probabilities as a function of s_1 and s_2,

$$g(s_1,s_2) = p_2^{(N)}(s_1,s_2) - p_2^{(F)}(s_1,s_2), \tag{4.62}$$

then for the source, it will be better to give information when $g(s_1,s_2) < 0$.

Theorem 9. *There exists $0 \leq \theta_1 < \infty$ such that*

1. *if $0 \leq s_1 < \theta_1$, then $g(s_1,s_2) \geq 0$, $\forall s_2 \geq s_1$;*
2. *if $\theta_1 < s_1 < \infty$, then*
   (a) $g(s_1,s_2) < 0$, $\forall s_2 \in [s_1, s_1 + \omega(s_1))$,
   (b) $g(s_1,s_2) > 0$, $\forall s_2 \in (s_1 + \omega(s_1), \infty)$,

where θ_1 is a solution of the equation $g(s_1,s_1) = 0$ and $\omega(s_1)$ is a solution of $g(s_1,s_1+v) = 0$ with respect to v when $g(s_1,s_1) < 0$.

Before going to the proof of the above result, we give some consequences. If the source met the first relay at $s_1 \leq \theta_1$, then irrespective of the time instant at which it meets the second relay, it should not give any information to the second relay. On the other hand, if $s_1 \geq \theta_1$, then the strategy of the source should be of threshold type: if it meets the second relay before $s_1 + \omega(s_1)$, then it should give full information, otherwise it should not give any information.

Proof of Theorem 9. First, note that for fixed s_2, $g(s_1, s_2)$ decreases with s_1, since in this case $p_2^{(F)}(s_1, s_2)$ increases with s_1 (Lemma 11 with equality 4.3), whereas $p_2^{(N)}(\mathbf{s})$ does not depend on s_1.

Thus, the closer s_1 is to s_2 the smaller $g(s_1, s_2)$ is. This also implies that for fixed s_1, $g(s_1, s_1 + v)$ will increase with v, for $v \geq 0$.

Let us show that $g(0, s_2) = p_2^{(N)}(0, s_2) - p_2^{(F)}(0, s_2)$ is non-negative. Using the expression

$$p_2^{(N)}(s_1, s_2) = p_2^{(P)}(s_1, s_2)\mathbb{P}(S_1 < s_2) + p_1^{(P)}(s_2, s_2)(1 - \mathbb{P}(S_1 < s_2)), \quad (4.63)$$

we obtain,

$$\begin{aligned} g(0, s_2) = \ & [p_1^{(P)}(s_2, s_2) - p_2^{(F)}(0, s_2)] \\ & -[p_1^{(P)}(s_2, s_2) - p_2^{(P)}(0, s_2)]\mathbb{P}(S_1 < s_2). \end{aligned} \quad (4.64)$$

With (4.2), and that $p_2^{(F)}(s_1, s_2)$ increases with s_1 (Lemma 11 with equality 4.3), the difference,

$$p_1^{(P)}(s_2, s_2) - p_2^{(F)}(0, s_2) \geq p_1^{(F)}(s_2, s_2) - p_2^{(F)}(s_2, s_2) \geq 0, \quad (4.65)$$

where the last inequality follows from the Lemma 12.

Now due to the non-negativity of the first difference in (4.64) the following inequality can be obtained,

$$\begin{aligned} g(0, s_2) \ \geq \ & [p_1^{(P)}(s_2, s_2) - p_2^{(F)}(0, s_2)]\mathbb{P}(S_1 < s_2) \\ & -[p_1^{(P)}(s_2, s_2) - p_2^{(P)}(0, s_2)]\mathbb{P}(S_1 < s_2) \\ = \ & \mathbb{P}(S_1 < s_2)[p_2^{(P)}(0, s_2) - p_2^{(F)}(0, s_2)]. \end{aligned} \quad (4.66)$$

The success probability, $p_2^{(P)}(s_1, s_2)$, can be represented as

$$p_2^{(P)}(s_1, s_2) = \frac{\int_{\hat{s}_1=0}^{s_2} p_2^{(F)}(\hat{s}_1, s_2)\tilde{f}_s(\hat{s}_1)d\hat{s}_1}{\int_{\hat{s}_1=0}^{s_2} \tilde{f}_s(\hat{s}_1)d\hat{s}_1}. \quad (4.67)$$

Again, due to the increasing property of $p_2^{(F)}(s_1, s_2)$ on s_1, $p_2^{(F)}(\hat{s}_1, s_2) \geq p_2^{(F)}(0, s_2)$. Then, since $p_2^{(F)}(0, s_2)$ does not depend on s_1, we obtain,

$$p_2^{(P)}(0, s_2) \geq \frac{p_2^{(F)}(0, s_2) \int_{\hat{s}_1=0}^{s_2} \tilde{f}_s(\hat{s}_1) d\hat{s}_1}{\int_{s_1=0}^{s_2} \tilde{f}_s(s_1) ds_1} = p_2^{(F)}(0, s_2), \tag{4.68}$$

and hence, $g(0, s_2) \geq 0$. Since, for fixed s_2, the function $g(s_1, s_2)$ is non-negative at $s_1 = 0$ and decreases in s_1, we can conclude that the equation $g(s_1, s_2) = 0$ has at most one real solution with respect to s_1.

Thus, if for s_1 and s_2 close to each other, $g(s_1, s_2) < 0$, i.e. if $g(s_1, s_1) < 0$ then there exists $\omega(s_1)$ such that $g(s_1, s_2) < 0$ if $s_2 \in [s_1, s_1 + \omega(s_1))$ and $g(s_1, s_2) > 0$ for $s_2 \in (s_1 + \omega(s_1), \infty)$ since $g(s_1, s_1 + v)$ increases with v as was seen before. Meanwhile, in case $g(s_1, s_1) \geq 0$, the difference $g(s_1, s_1 + v)$ will be positive $\forall v \geq 0$.

Now let us find out when the condition $g(s_1, s_1) < 0$ holds. As was shown before, for fixed s_2, $g(0, s_2) \geq 0$, and hence, $g(0, 0) \geq 0$. Consider the behaviour of $g(s_1, s_1)$ with increasing of s_1.

Note that $p_2^{(F)}(s_1, s_1) = 1/2$, since,

$$p_2^{(F)}(s_1, s_1) = \int\limits_{y_2=0}^{\infty} \tilde{f}_d(y_2) \int\limits_{y_1=y_2}^{\infty} \tilde{f}_d(y_1) dy_1 dy_2 = 1/2, \tag{4.69}$$

proof of which can be found in the proof of Lemma 11. Thus,

$$g(s_1, s_1) = p_2^{(N)}(s_1, s_1) - \tfrac{1}{2}, \tag{4.70}$$

and it decreases with s_1 since $p_2^{(N)}$ decreases with time (Lemma 14).

Thus, the equation $g(s_1, s_1) = 0$ has at most one real solution θ with respect to s_1, such that if $0 \leq s_1 \leq \theta$ then $g(s_1, s_1) > 0$. If $s_1 > \theta$ then $g(s_1, s_1) < 0$ and the threshold $\omega(s_1)$ for the meeting time s_2 holds. □

4.6.3 Two relays, exponentially distributed inter-contact times

Let us illustrate the result in Theorem 9 for exponentially distributed inter-contact times.

The difference in (4.62)) can be written as

$$g(s_1, s_1 + v) = a(s_1) e^{-\mu v} - b(s_1) e^{-\lambda v},$$

where

$$a(s_1) = \frac{1}{2} \left(\frac{\lambda}{\lambda - \mu} e^{-\mu s_1} - 1 \right), \text{ and}$$
$$b(s_1) = \frac{\mu^2}{\lambda^2 - \mu^2} e^{-\lambda s_1}.$$

First, consider the case $\lambda > \mu$.

Proposition 12. *For $\lambda > \mu$, there exist $0 \leq \theta_1 \leq \theta_2 < \infty$ such that*

1. *if* $0 \leq s_1 \leq \theta_1$*, then* $g(s_1, s_1 + v) \geq 0$, $\forall v \geq 0$;
2. *if* $s_1 \geq \theta_2$*, then* $g(s_1, s_1 + v) < 0$, $\forall v \geq 0$;
3. *if* $\theta_1 < s_1 < \theta_2$*, then*
   (a) $g(s_1, s_2) < 0$, $\forall s_2 \in [s_1, s_1 + \omega(s_1))$;
   (b) $g(s_1, s_2) > 0$, $\forall s_2 \in (s_1 + \omega(s_1), \infty)$;

where

$$\theta_2 = -\frac{1}{\mu} \log\left(1 - \frac{\mu}{\lambda}\right),$$
$$\omega(s_1) = \frac{1}{\lambda - \mu} \log\left(\frac{b(s_1)}{a(s_1)}\right),$$

and θ_1 is the solution of $a(\theta_1) = b(\theta_1)$. Moreover, ω is an increasing and convex function.

For this case, the threshold $\omega(s_1)$ becomes infinity for $s_1 \geq \theta_2$. So, the adaptive strategy is of following form: if $s_1 < \theta_1$, then give no information to the second relay irrespective of when it meets the source. On the other hand, if $s_1 > \theta_2$, then give full information to the second relay irrespective of s_2. For $\theta_1 < s_1 < \theta_2$, give full information if $s_2 < s_1 + \omega(s_1)$, otherwise do not give any information. The adaptive strategy in Proposition 12 is illustrated in Figure 4.3.

The other case $\lambda \leq \mu$ is similar with the difference that $\theta_2 = \infty$. For any s_1 there will always be some values of s_2 when the source will not give information to the second relay. The formal result is as follows.

Proposition 13. *For $\lambda \leq \mu$, there exist $0 \leq \theta_1 < \infty$ such that*

1. *if* $0 \leq s_1 \leq \theta_1$*, then* $g(s_1, s_1 + v) \geq 0$, $\forall v \geq 0$;
2. *if* $\theta_1 < s_1 < \infty$*, then*
   (a) $g(s_1, s_1 + v) < 0$, $\forall s_2 \in [s_1, s_1 + \omega(s_1))$;
   (b) $g(s_1, s_1 + v) > 0$, $\forall s_2 \in (s_1 + \omega(s_1), \infty)$;

where θ_1 and $\omega(s_1)$ are as defined in Proposition 12.

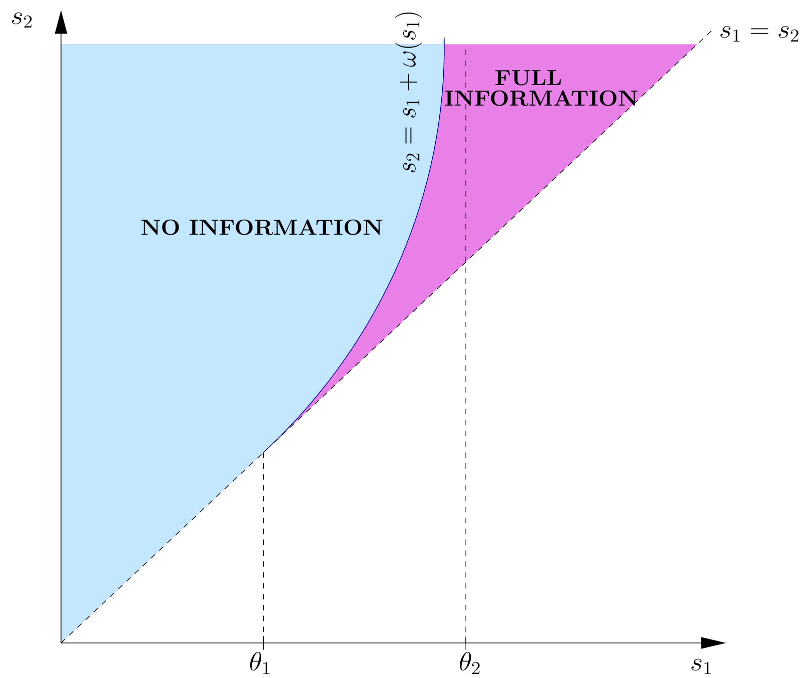

Figure 4.3: Optimal strategy for the source for $\lambda > \mu$.

The adaptive strategy for $\lambda < \mu$ for the source is illustrated in Figure 4.4. As a special case, for $\lambda = \mu$,

$$\theta_1 = \frac{-LW(-e^{-1.5}) - 1.5}{\lambda},$$
$$\omega(s_1) = \frac{2e^{\lambda s_1} - (3 + 2\lambda s_1)}{2\lambda},$$

where LW is the LambertW function.

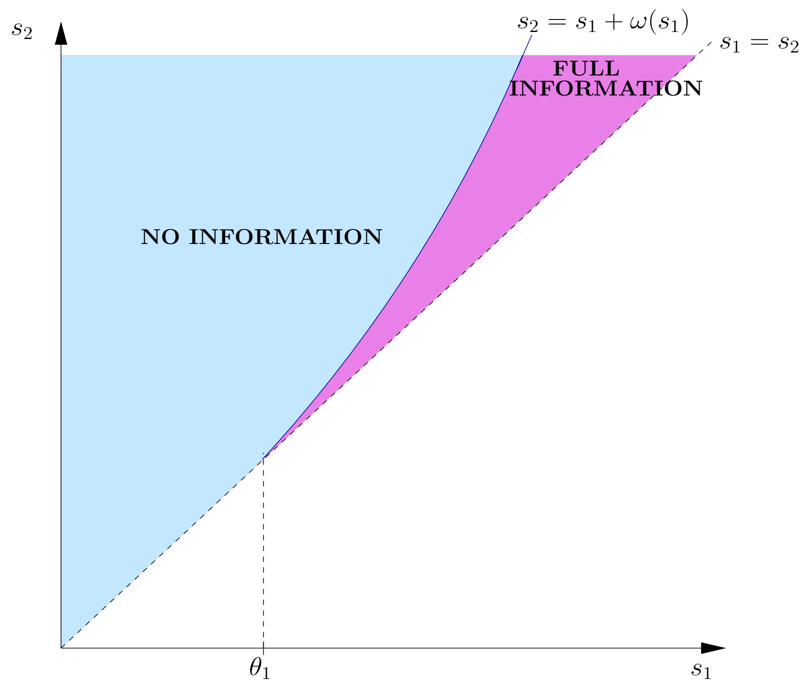

Figure 4.4: Optimal strategy for the source for $\lambda < \mu$.

4.7 Conclusion

We proposed a reward mechanism to incentive relays in message forwarding in DTNs. Furthermore we argue that such a coordination scheme should not rely on end to end control message exchange. To this respect, we have provided a novel key contribution: the reward mechanism in fact is designed to secure the participation of relays in the delivery process by proposing a reward that takes into account the costs incurred by the relays and the risk they are exposed to during the delivery process. This reward is the minimum amount that offsets the expected delivery cost, as estimated by the relay from the information given by the source (number of existing copies of the message, age of these copies). We first showed that the expected reward paid by the source remains the same irrespective of the information it conveys, provided that the type of information does not vary dynamically over time. On the other hand, the source can gain by adapting the information it conveys to a meeting relay, and we gave the structural results of the optimal adaptive policy for the source in cases of two relays or exponentially distributed inter-contact times.

Some results of this study have been published in the proceedings of the IEEE International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks, WiOpt-2014 (Brun et al., 2014).

5

A THRESHOLD TYPE POLICY OF A DTN NODE UNDER REWARD INCENTIVE MECHANISM

This chapter is also devoted to studying of DTN model with the same structure of the costs for the relay nodes and the reward mechanism introduced in the previous chapter. We focus on the optimal policies of the relays as best-response on the fixed reward promoted by the source that is to participate in message forwarding or not and if so then to drop the message or to retain it. First, we give a description of the new model and highlight differences with the previously considered one. Next, we describe the structure of stochastic games to be used later in formalization of our DTN model. Then the case with only one relay is considered in order to understand the form of relay's optimal policy, after which we examine the network assuming two relays.

5.1 On the Reward Configuration

The store-carry-forward approach by which DTN maintains connectivity, is based on the assumption that a transient node can store a message for relatively long periods of time before forwarding it to the destination or other transient node. In addition to this approach, the probability of message delivery is increased due to the implementation of multi-copy routing. However, plenty of message copies may cause a large resource consumption for DTN nodes even for a single message to be delivered. The situation is aggravated due to a technical problem with feedbacks so that the source may continue to generate copies within a message lifetime and the DTN nodes to accept them, even if the message has already been delivered. A reward mechanism that can compensate all expenses of a relay may however be very costly for the source. Indeed, the one we introduced in the previous chapter ensures full cooperation of the mobile nodes in message forwarding by promising to cover the expected cost estimated by a relay node. We have developed an adaptive strategy for the source that allows it to reduce the expected cost. Nevertheless, the expected reward may be still quite costly for the source depending on the parameters of the node mobility and energy consumption of relays' batteries. To construct an optimal reward, first we need to know how willing the mobile nodes are to participate in message forwarding in response to the fixed reward, and what could be the best-response strategies of the relays. We address these questions in our study assuming DTN model with two-hop routing scheme.

In the next section, we give a full description of the model and proceed then to analysis.

5.2 Problem description

Consider a set of nodes in which there is one source, one destination, and N relays. The relays are mobile and meet the source or the destination every once in a while. It shall be assumed that the inter-meeting time between a relay and the source (resp., destination) is a sequence of i.i.d. random variables with distribution function F_s (resp., F_d). Two nodes can exchange data only when they meet. It is assumed that the source and the destination are fixed, and thus cannot communicate directly.

After a message is generated, the source proposes it to every relay that it meets. A relay can choose to either accept the message or reject it. As an incentive, the source offers a fixed reward, say R, to be claimed by the first relay that delivers the message to the destination. We emphasize that only the first relay to deliver the message gets the reward, R. The other relays are not entitled to any share of the proposed reward. A relay that accepts the message incurs certain costs:

1. cost related to the energy spent in receiving the message from the source. This is fixed cost and will be denoted by C_r;

2. energy cost of transmitting the message to the destination in case this relay is the first one to do so. This cost is also fixed, and will be denoted by C_d.

3. and the cost of storing the message while the relay is searching for the destination. We denote by C_s the cost per unit time incurred for storing the message.

Associated with each message is a deadline before which the message remains useful to the destination. Once the deadline has passed, the destination will no longer accept the message from the relays.

The decision problem for a relay, when it meets the source, is whether to accept the message or not. In case it accepts the message, the relay can drop the message at any time if it has not yet delivered it to the destination, and if it is no longer profitable to keep the message. The precise optimization problem for the relays is described next.

5.3 Stochastic game with partial information

We shall study a discrete-time model of this game. The source generates the message at time instant 0 with a deadline at instant $\tau + 1$. It is assumed that the reception of the message from the source and its transmission to the destination each takes one time slot, so that a relay has to meet the destination before time τ in order to get the reward. When a relay meets the source it can decide whether to accept or reject the message (assuming it does not already have it). Once the relay accepts the message it can choose to retain or to drop it in each subsequent time slot until it meets the destination or the deadline of the message expires. Thus, the potential decision epochs for every relay are in the set $\{0, 1, \ldots, \tau - 1\}$. Each relay has to make decision over multiple stages and its cost depends upon its own actions as well as those of the other relays. The objective of each relay is to minimize expected cost it incurs for participating in the game. This strategic interaction between the relays falls within the framework of stochastic games introduced by Shapley, 1953. In our model, each relay is aware of its own state but does not know that of the others. Furthermore, it does not know whether the packet has already been delivered to the destination or not. Our game is thus a stochastic game with partial information (Goush et al., 2004). We now give some background on this type of games. These games are defined by:

- τ : time horizon (message deadline, in our case)
- $\mathcal{R} = \{1, 2, \ldots, N\}$ set of players (relays)
- $\mathcal{E}_j$, $j \in \mathcal{R}$: state space of relay j. We denote by X_n^j the state of player j at time n.
- $\mathcal{A}_j$, $j \in \mathcal{R}$: action space of relay j. We denote by A_n^j the action taken by player j at time n.
- $\mathcal{E} := \bigotimes_{j \in \mathcal{R}} \mathcal{E}_j$.
- $\mathcal{A} := \bigotimes_{j \in \mathcal{R}} \mathcal{A}_j$.
- $\mathcal{B}_j : \mathcal{E}_j \times \{0, 1, \ldots, \tau - 1\} \to \mathcal{D}(\mathcal{A}_j)$, where $\mathcal{D}(\mathcal{A})$ is the set of probability measures on $\mathcal{A}$. The set $\mathcal{B}_j(t)$ is the set of mixed strategies available to relay j at every time instants. In other words, an element $\sigma_n^j(x)$ is the probability distribution over the

set of actions $\mathcal{A}_j$ used by player j to choose its action when it is in state x at time n.

- $P_j,\ j \in \mathcal{R}$: transition probability matrix of relay j on the space of its state-action pairs.

- $\mathcal{E}_0$: state space of the packet. This can be 0 or 1 which indicates whether the packet has been delivered or not. We denote by X_n^0 the state of the packet at time n.

- $g_j : \mathcal{E}_j \times \mathcal{A}_j \times \mathcal{E}_0 \to \mathbb{R},\ j \in \mathcal{R}$: cost function for relay j.

Fix $\sigma := \left(\sigma^j\right)_{j\in\mathcal{R}} \in \bigotimes_{j\in\mathcal{R}} \mathcal{B}_j$. Let $\left\{\mathbf{Z}_n^\sigma := (X_n^j, A_n^j)_{j\in\mathcal{R}}^{(\sigma)}\right\}_{n=0,\dots,\tau-1}$ be the stochastic process of state-action pairs generated by σ. And, assume that the process $X_n^0,\ n \geq 1$ is adapted to the natural filtration of $\mathbf{Z}_n^\sigma$. By this we mean that, at every time instant, X_n^0 is measurable with respect to the history of the state-action pairs.

Let $\mathbf{b}_{-j} \in \mathcal{D}(\mathcal{E}_{-j})$ be the distribution of the initial state of the relays other than j. The expected cost of relay j for σ can then be defined as:

$$V_j(\sigma^j, \sigma^{-j}; x_0^0, x_0^j, \mathbf{b}_{-j}) = \mathbb{E}_{\mathbf{x}_0,\mathbf{b}_{-j}} \sum_{n=0}^{\tau-1} \alpha^n g_j(X_n^j, A_n^j, X_n^0), \tag{5.1}$$

where α is the discount factor. The terminal cost is assumed to be 0 in every state.

The objective of relay j is to minimize its cost given the strategy of the others. That is,

$$W_j(\sigma^{-j}; x_0^0, x_0^j, \mathbf{b}_{-j}) = \min_{s\in\mathcal{B}_j} V_j(s, \sigma^{-j}; x_0^0, x_0^j, \mathbf{b}_{-j}), \tag{5.2}$$

and compute

$$\beta_j(\sigma^{-j}; x_0^0, \mathbf{b}_{-j}) = \arg\min_{s\in\mathcal{B}_j} V_j(s, \sigma^{-j}; x_0^0, x_0^j, \mathbf{b}_{-j}), \tag{5.3}$$

which is the best-response of relay to σ_{-j} given the intial conditions.

This is a partially observable stochastic game (see for example, Goush et al., 2004) since each relay knows only its state but not that of the others. A consequence of the lack of information is that the concept of Markov strategies and Markov equilibrium is not applicable to this setting. The optimal action of a relay in a given state depends on the state of the other relays which is not known to this relay. The probability distribution over the states of the other relays will depend upon the actions they have been taking in the past. This means that a relay will have to keep track of the past actions of the others in order to compute its own action in a given state. The probability of arriving in a given state depends on the actions taken in the past because the action in the current state will depend upon the state of the other relays which is not known.

A policy σ is said to be an equilibrium if

$$\beta_j(\sigma^{-j}; x_0^0, \mathbf{b}_{-j}) = \sigma^j, \ \forall j. \tag{5.4}$$

The values of different parameters for our model are as follows.

State and action spaces

The state of each relay takes one of the five possible values:

Value	Significance	Action set
0	relay does not have the packet	$\emptyset$
m_s	relay meets the source	$(accept, reject)$
1	relay has the packet	$(drop, keep)$
m_d	relay meets the destination	$\emptyset$
2	relay quits the game	$\emptyset$

In states 0 and 2 the relay does not have a non-trivial action. In state 0 it is waiting to meet the source, while in state 2 it has already quit the game.

Transition matrix

Regarding the contact process that keeps track of the contacts of the relay with the source and the destination, we shall assume i.i.d. contact times. As a consequence, a relay needs to know only the current state of the contact process, and not its entire history to take its decision. In the following, we let p be the probability that a relay meets the destination at the next time step, and q be the probability that it meets the source. The state of each relay evolves according to a time-homogeneous Markov chain whose transition probabilities depend on the action chosen in each state, and is given by:

$$P_j = \begin{array}{c} \\ 0 \\ m_s \\ 1 \\ m_d \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & m_s & 1 & m_d & 2 \end{array} \\ \begin{bmatrix} 1-q & q & 0 & 0 & 0 \\ \mathbb{1}_{reject} & 0 & \mathbb{1}_{accept} & 0 & 0 \\ \mathbb{1}_{drop} & 0 & (1-p)\mathbb{1}_{keep} & p\mathbb{1}_{keep} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The transition diagram of the Markov chain is shown in Figure 5.1.

State of the packet

The state of the packet can take two values: 0 (it has not been delivered) or 1 (it has been delivered). The transition probabilities between these two states depends upon the state

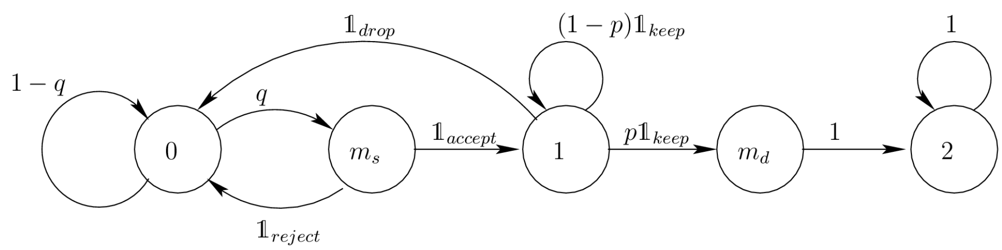

Figure 5.1: Transition diagram for the Markov chain governing the state of each relay.

of the relays.

$$
\begin{aligned}
&P(X^0_{n+1} = 1 | X^0_n = 0, \mathbf{X}_n) = P((\cup_{j\in\mathcal{R}}\{X^j_n = 2\} = \emptyset) \cap (\cup_{j\in\mathcal{R}}\{X^j_n = m_d\} \neq \emptyset)),\\
&P(X^0_{n+1} = 1 | X^0_n = 1, \mathbf{X}_n) = 1.
\end{aligned}
$$

Cost function

The one-step cost incurred by the relay depends on its current state and the action it takes (whether it accepts the packet or not, whether it drops the packet or not). Further, when it meets the destination (that is, in state m_d) the cost incurred depends upon whether any other relay has already delivered the packet or not. Hence

$$
\begin{aligned}
g(m_s, accept, \cdot) &= C_r,\\
g(1, keep, \cdot) &= C_s,\\
g(m_d, \cdot, 0) &= R - C_d,
\end{aligned}
$$

and is 0 for all other arguments.

5.4 The Single Player Case

In order to get some insights into the structure of the best-response policy of a relay, we shall first consider the case of a single player. In order to simplify notations, we drop the index j of the relay. Since no other relay can deliver the message, the state of the packet is $X^0_n = 0$ until the relay meets the destination, and thus we can further simplify notations by writing $g(x, a)$ instead of $g(x, a, 0)$.

5.4.1 Dynamic Programming Formulation

Assume that the relay meets the source at instant $t \in [0, \tau]$. For epochs $0, \ldots, t-1$, thus, there are no decisions to be made. For the remaining epochs, the optimal policy can be computed using Dynamic Programming.

Let $V_n(x)$ be the optimal cost-to-go starting in state $x \in \{0, m_s, 1, m_d, 2\}$ at instant n. From the dynamic programming equation,

$$V_n(X_n) = \min_{a \in \mathcal{A}(X_n)} g(X_n, a) + \alpha \mathbb{E} V_{n+1}(X_{n+1}), \tag{5.5}$$

where α is the discount factor ($0 \leq \alpha < 1$).

At time n, if the relay is in contact with the destination, its terminal cost is

$$V_n(m_d) = C_d - R, \quad n = 1, 2, \ldots, \tau. \tag{5.6}$$

In particular, we have $V_\tau(m_d) = C_d - R$ at time τ. If at that time the relay has the message and is not in contact with the destination, then it is optimal to drop the message since it is no longer useful, so that $V_\tau(1) = 0$. On the other hand, if the relay does not have the message at instant τ, then it incurs no costs, so that $V_\tau(0) = V_\tau(m_s) = 0$. To summarize, the terminal costs at the instant $n = \tau$ are:

$$V_\tau(m_d) = C_d - R. \tag{5.7}$$

$$V_\tau(x) = 0, \quad \forall x \neq m_d. \tag{5.8}$$

The optimal policy at different decision epochs and states can be computed recursively by rolling back (5.5). If the contact process is history dependent, then the optimal policy is usually computed numerically. However, as we shall see below, the assumption of an i.i.d. contact process enables the derivation of structural properties of the optimal policy.

5.4.2 To Drop or to Retain

Assume that the relay is in state 1 at instant $\tau - 1$, that is it has the message and it is not in contact with the destination. The relay has to decide whether to drop the message or not. Taking $n = \tau - 1$ in (5.5), we obtain

$$\begin{aligned} V_{\tau-1}(1) &= \min_{a \in \{keep, drop\}} [g(1, a) + \alpha \mathbb{E} V_\tau(X_\tau)] \\ &= \min\left(0, C_s + \alpha(p V_\tau(m_d) + \overline{p} V_\tau(1))\right), \\ &= \min\left(0, C_s + p\alpha(C_d - R)\right), \end{aligned} \tag{5.9}$$

where we have used the short-hand notation $\overline{p} = 1 - p$, and the last equality follows from (5.7)–(5.8). Thus, if the first term is the minimum, then it is optimal to drop the message at $\tau - 1$, otherwise it is optimal to keep it.

One can recursively develop (5.9) to compute the optimal policy at step n given that the relay has the message and has not yet encountered the destination. For $n = \tau - 2$, we obtain

$$\begin{aligned} V_{\tau-2}(1) &= \min\left\{0, C_s + \alpha(pV_{\tau-1}(m_d) + \overline{p}V_{\tau-1}(1))\right\}, \\ &= \min\{\ 0,\ C_s + p\alpha(C_d - R), \\ &\qquad C_s(1 + \overline{p}\alpha) + (C_d - R)\alpha(p + \overline{p}p\alpha)\}. \end{aligned} \tag{5.10}$$

Here, the second and the third terms in the minimum correspond to the cost of retaining the message at instant $\tau - 2$. Thus, if either term is negative, then it is optimal to retain the message. Otherwise, it is optimal to drop the message at instant $\tau - 2$.

More generally, the i^{th} component in the min corresponds to the cost obtained if the action *keep* is played i consecutive times starting from the current decision epoch n, until the relay meets the destination or decides to drop the message. This i^{th} component can be represented as follows,

$$U_{n,i} = \sum_{j=1}^{i} (\alpha\overline{p})^{j-1} \left(C_s + (C_d - R)\alpha p\right), \tag{5.11}$$

$$= (C_s + \alpha p(C_d - R))\frac{1 - (\overline{p}\alpha)^i}{1 - \overline{p}\alpha}. \tag{5.12}$$

The recursion (5.9) can be developed in terms of $U_{n,i}$ as:

$$V_n(1) = \min(0, U_{n,1}, U_{n,2}, \ldots, U_{n,\tau-n}). \tag{5.13}$$

The optimal policy at instant n is to retain the message if either of $U_{n,i}$ is negative. Otherwise it is optimal to drop the message at time n. Note from (5.12) that if $C_s + \alpha p(C_d - R) < 0$, then $U_{n,i} < 0$, $\forall n$ and $\forall i$, and the sequence decreases with i. From (5.13), one can conlude that if $C_s + \alpha p(C_d - R) < 0$, then the relay will retain the message until it is delivered to the destination or the deadline expires. Otherwise, the relay will drop the message immediately. Thus,

$$R > \frac{C_s}{\alpha p} + C_d, \tag{5.14}$$

is a necessary condition for the relay to attempt the delivery of the message.

5.4.3 To Accept or to Reject

Assume that the relay is in state m_s at instant t, that is it in contact with the source and has not the message. The relay has to decide whether to accept the message or not. The optimal cost at t is:

$$\begin{aligned} V_t(m_s) &= \min(0, g(m_s, accept) + \alpha V_{t+1}(1)) \\ &= \min(0, C_r + \alpha V_{t+1}(1)), \end{aligned} \tag{5.15}$$

where $V_{t+1}(1)$ can be computed from (5.13). Thus, if at time t the second term is negative, then it is optimal to accept the message from the source. Otherwise, it is optimal to reject it. In particular, if condition (5.14) is satisfied, $U_{n,i}$ is a decreasing function of i and equation (5.13) yields

$$V_n(1) = U_{n,\tau-n} = (C_s + \alpha p(C_d - R))\frac{1-(\overline{p}\alpha)^{\tau-n}}{1-\overline{p}\alpha}. \tag{5.16}$$

We thus obtain that the expected cost for the relay if it accepts the message is

$$g(m_s, accept) + \alpha V_{t+1}(1) = C_r + U_{t+1,\tau-t-1}. \tag{5.17}$$

We conclude from (5.15) and (5.17) that if the relay meets the source at time t, it will accept the message provided that

$$C_r + U_{t+1,\tau-t-1} < 0. \tag{5.18}$$

Note that (5.16) implies that $U_{t+1,\tau-t-1}$ increases with t. Since $U_{t+1,\tau-t-1}$ is negative and increases with t, there exists a threshold t^* such that for $t \leq t^*$, the relay will accept the message, and it will reject the message after t^*. The threshold can be easily computed using the above inequality,

$$t^* = \tau - 1 - \frac{\ln\left(1 + \frac{C_r(1-\overline{p}\alpha)}{C_s+\alpha p(C_d-R)}\right)}{\ln(\overline{p}\alpha)}. \tag{5.19}$$

5.5 Game with two relays

We now consider the newtork with two relays. We shall restrict our attention to threshold type policies, that is policies such $\sigma_n^j(m_s) = accept$ if $n \leq \theta_1$ and *reject* otherwise, and $\sigma_n^j(1) = drop$ if $n \geq \theta_2$, and *keep* otherwise. The threshold θ_2 could depend on the meeting time with the source. We shall show that if one relay follows a threshold type policy then the best-response of the other relay is also a policy of threshold type.

We shall thus assume that one of the two relays – say relay 2, follows a threshold type policy. That is, there exist θ_1^2 and $\theta_2^2 > \theta_1^2$ such that

$$\sigma_n^2(m_s) = \begin{cases} accept & \text{if } n \leq \theta_1^2, \\ reject & \text{if } n > \theta_1^2, \end{cases} \tag{5.20}$$

and

$$\sigma_n^2(1) = \begin{cases} keep & \text{if } n \leq \theta_2^2, \\ drop & \text{if } n > \theta_2^2. \end{cases} \tag{5.21}$$

As in Section 5.4, we shall use dynamic programming to derive the best-response policy of the first player to the above policy of relay 2. We let $V_n^1(x)$ be the optimal cost-to-go starting in state $x \in \{0, m_s, 1, m_d, 2\}$ at instant n. As we shall see below, the optimal cost-to-go starting in states m_s and 1 can be expressed in terms of the expected costs when the destination is reached.

5.5.1 Expected costs when the destination is reached

If at time n relay 1 has the message and is in contact with the destination, then its expected cost is

$$V_n^1(m_d) = \frac{1}{2}(C_d - R)\,\mathbb{P}\big(X_n^2 = m_d\big) + (C_d - R)\,\mathbb{P}\big(X_n^0 = 0, X_n^2 \neq m_d\big)\,, \tag{5.22}$$

for all $n \in \{1, 2, \ldots, \tau\}$, where it is assumed that if both relays meet the destination at the same time, then each one wins the reward with probability $\frac{1}{2}$.

Define $1 - \delta_n$ as the probability that relay 2 delivers the message at a time $t \leq n$, as estimated by relay 1 . Note that $\delta_{n-1} - \delta_n$ is the probability that the second relay meets the destination with the message precisely at time n. The expected cost $V_n^1(m_d)$ can be written as follows

$$\begin{aligned} V_n^1(m_d) &= \frac{1}{2}(C_d - R)\,(\delta_{n-1} - \delta_n) + (C_d - R)\,\delta_n, \\ &= \frac{\delta_{n-1} + \delta_n}{2}\,(C_d - R). \end{aligned} \tag{5.23}$$

Lemma 15 proves two fundamental properties of the sequence $V_1^1(m_d), V_2^1(m_d), \ldots$ that will be required to establish the structure of the optimal policy of relay 1.

Lemma 15. *The sequence $V_1^1(m_d), V_2^1(m_d), \ldots$ is such that*

(a) it is non-decreasing with n, and

(b) it is constant for all $n \geq \theta_2^2 + 1$.

Proof. To prove assertion (a), observe that since $\delta_{n-1} - \delta_n$ is the probability was defined before, we have $\delta_{n-1} \geq \delta_n$. Hence the sequence $\delta_1, \delta_2, \ldots$ is non-increasing. With (5.23), it yields $V^1_{n+1}(m_d) - V^1_n(m_d) = \frac{1}{2}(C_d - R)(\delta_{n+1} - \delta_{n-1}) \geq 0$, which concludes the proof.

Let us now prove assertion (b). Since at time $\theta_2^2 + 1$ the second relay drops the message if it has it, the probability that it delivers the message after that time is 0, implying that $\delta_n = \delta_{\theta_2^2+1}$ for all $n > \theta_2^2$. For $k > \theta_2^2 + 1$, it yields

$$V^1_k(m_d) = \frac{\delta_{k-1} + \delta_k}{2}\,(C_d - R) = \delta_{\theta_2^2+1}\,(C_d - R) = V^1_{\theta_2^2+1}(m_d), \tag{5.24}$$

which concludes the proof. □

5.5.2 To drop or to retain

Let us assume that relay 1 is in state 1, that is it has the message but it is not in contact with the destination. It has to decide whether to retain it or to drop it. Proceeding backward in time, we have

$$\begin{aligned} V^1_{\tau-1}(1) &= \min_{a \in \{keep, drop\}} [g(1,a) + \alpha \mathbb{E} V^1_\tau (X^1_\tau)], \\ &= \min\left(0, C_s + \alpha p V^1_\tau(m_d) + \alpha \overline{p} V^1_\tau(1)\right), \\ &= \min\left(0, C_s + \alpha p V^1_\tau(m_d)\right), \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} V^1_{\tau-2}(1) &= \min_{a \in \{keep, drop\}} [g(1,a) + \alpha \mathbb{E} V^1_{\tau-1} (X^1_{\tau-1})], \\ &= \min\left(0, C_s + \alpha \left[p V^1_{\tau-1}(m_d) + \overline{p} V^1_{\tau-1}(1)\right]\right), \\ &= \min\Big(0, C_s + \alpha p V^1_{\tau-1}(m_d)\,, \\ &\qquad C_s + \alpha p V^1_{\tau-1}(m_d) + \alpha \overline{p}\left[C_s + \alpha p V^1_\tau(m_d)\right]\Big). \end{aligned} \tag{5.26}$$

More generally, we have

$$V^1_n(1) = \min(0, U_{n,1}, U_{n,2}, \ldots, U_{n,\tau-n}), \tag{5.27}$$

where

$$U_{n,i} = \sum_{j=1}^{i} (\alpha \overline{p})^{j-1} \left[C_s + \alpha p V^1_{n+j}(m_d)\right]. \tag{5.28}$$

The optimal policy at instant n is to retain the message if $\min\limits_{i=1,\ldots,\tau-n} U_{n,i} < 0$. Otherwise it is optimal to drop the message at n.

We establish below two properties of the $U_{n,i}$.

Lemma 16. *The sequence $\{U_{n,1}\}_{n=1,2,\cdots}$ is a non-decreasing sequence, which is constant starting from $n = \theta_2^2$.*

Proof. We first show that the sequence is non-decreasing. With (5.28) we have

$$\begin{aligned} U_{n+1,1} - U_{n,1} &= C_s + \alpha p V_{n+2}^1(m_d) - C_s - \alpha p V_{n+1}^1(m_d), \\ &= \alpha p \left(V_{n+2}^1(m_d) - V_{n+1}^1(m_d)\right), \end{aligned}$$

and with Lemma 15 we can conclude that $U_{n+1,1} \geq U_{n,1}$ that corresponds to the first assertion of the lemma. In order to show that $U_{n,1} = U_{\theta_2^2,1}$ for all $n \geq \theta_2^2$, we use Lemma 15.(b) to obtain

$$\begin{aligned} U_{n,1} &= C_s + \alpha p V_{n+1}^1(m_d) \\ &= C_s + \alpha p V_{\theta_2^2+1}^1(m_d) \\ &= U_{\theta_2^2,1} \end{aligned}$$

□

Lemma 17. *For all $n \in \{1, 2, \ldots, \tau\}$, if $U_{n,1} \geq 0$, then $\min\limits_{i=1,\ldots,\tau-n} U_{n,i} = U_{n,1}$.*

Proof. Fix $n \in \{1, 2, \ldots, \tau\}$ and assume $U_{n,1} \geq 0$. It is enough to prove that the sequence $U_{n,1}, U_{n,2}, \ldots$ is a non-decreasing sequence. Observing from (5.28) that $U_{n,i}$ can also be written as follows

$$U_{n,i} = \sum_{j=0}^{i-1} (\alpha \overline{p})^j U_{n+j,1}, \tag{5.29}$$

we obtain with Lemma 16 that $U_{n,i+1} - U_{n,i} = (\alpha \overline{p})^i U_{n+i,1} \geq (\alpha \overline{p})^i U_{n,1}$. We thus conclude that $U_{n,1} \geq 0$ implies that $U_{n,1}, U_{n,2}, \ldots$ is a non-decreasing sequence, which yields the proof. □

We now show the following result.

Proposition 14. *At time n, $V_n^1(1) < 0$ if and only if $U_{n,1} < 0$.*

Proof. From (5.27), it is obvious that $U_{n,1} < 0$ implies that $V_n^1(1) < 0$. By contraposition, in order to show that the converse is true, it is enough to show that $U_{n,1} \geq 0$ implies that $V_n^1(1) \geq 0$, which is a direct consequence of Lemma 17. □

According to Lemma 16, the $U_{n,1}$ are non-decreasing with n. Thus, Proposition 14 implies that relay 1 will retain the message as long as $U_{n,1} < 0$, and will drop it once $U_{n,1}$ becomes positive. We are now in position to show that once relay 1 has the message, it uses a threshold type strategy to decide whether to retain it or to drop it.

Proposition 15. *If $U_{\theta_2^2,1} \geq 0$ then there exists threshold $\theta_2^1 \leq \theta_2^2$ such that relay 1 retains the message until θ_2^1 and drops it at time $\theta_2^1 + 1$. Otherwise, if $U_{\theta_2^2,1} < 0$, relay 1 retains the message until it meets the destination or the deadline expires.*

Proof. Let us first consider the case $U_{\theta_2^2,1} \geq 0$. Let t be the time at which relay 1 accepts the message from the source. Since

$$V_t^1(m_s) = min\left(0, C_r + V_{t+1}^1(1)\right),$$

has to be negative for relay 1 to accept the message, this implies that $V_{t+1}^1(1) < -C_r$. According to Proposition 14, $V_{t+1}^1(1) < 0$ in turn implies that $U_{t+1,1} < 0$. Since from Lemma 16 the sequence $U_{1,1}, U_{2,1}, \ldots$ is non-decreasing, $U_{t+1,1} < 0$ and $U_{\theta_2^2,1} \geq 0$ imply that there exists $\theta_2^1 \in [t+1, \theta_2^2]$ such that $U_{n,1} < 0$ for all $n \leq \theta_2^1$ and $U_{\theta_2^1+1,1} \geq 0$. We thus conclude that $V_n^1(1) < 0$ for all $n \leq \theta_2^1$ and $V_{\theta_2^1+1}^1(1) \geq 0$. In other words, relay 1 retains the message until time θ_2^1, and drops it at time $\theta_2^1 + 1$.

Let us now consider the case $U_{\theta_2^2,1} < 0$. According to Lemma 16, the sequence $U_{1,1}, U_{2,1}, \ldots$ is non-decreasing and constant starting from $n = \theta_2^2$. We thus conclude that $U_{n,1} < 0$ for all $n \in \{1, 2, \ldots, \tau\}$. With Proposition 14, it yields $V_n^1(1) < 0$ for all $n \in \{1, 2, \ldots, \tau\}$, implying that the optimal strategy for relay 1 is to retain the message until it meet the destination or the deadline expires. □

According to Proposition 15, the best-response policy of player 1 to the strategy of player 2 is therefore as follows:

$$\sigma_n^1(1) = \begin{cases} keep & \text{if } n \leq \theta_2^1, \\ drop & \text{if } n > \theta_2^1, \end{cases} \tag{5.30}$$

where the threshold θ_2^1 can be greater than τ.

5.5.3 To Accept or to Reject

Let t be the time at which relay 1 meets the source. The optimal expected cost at t is:

$$\begin{aligned} V_t^1(m_s) &= \min(0, g(m_s, accept) + \alpha V_{t+1}^1(1)), \\ &= \min(0, C_r + \alpha V_{t+1}^1(1)), \end{aligned} \tag{5.31}$$

where $V_{t+1}^1(1)$ can be computed from (5.27). Thus, if at time t the second term is negative, then it is optimal to accept the message from the source. Otherwise, it is optimal to reject it.

Proposition 16. *There exists θ_1^1 such that relay 1 rejects the message if it meets the source at a time $n > \theta_1^1$.*

Proof. Observe that (5.31) can be written as follows

$$V_t^1(m_s) = \min(0, C_r + \min_{i=1,\dots,\tau-t-1} U_{t+1,i}).$$

Since Lemma 16 implies that $\min_{i=1,\dots,\tau-t-1} U_{t+1,i}$ increases with t, we can assert that if at time θ_1^1 the relay rejects the message, i.e., if $\min_{i=1,\dots,\tau-\theta_1^1} U_{\theta_1^1,i} \geq 0$, then it will also reject it at all subsequent contact times $k > \theta_1^1$ with the source. □

We note that the threshold θ_1^1 can be larger than τ, in which case relay 1 always accepts the message when it meets the source. Similarly, the threshold θ_1^1 can be smaller than 1, in which case relay 1 never accepts the message when it meets the source.

5.6 Conclusion

We studied the selfish behaviour of DTN nodes incentivised by a reward for participating in message forwarding. The reward is proposed by the source to every relay it meets, but is paid only to the first one that delivers the message. A relay meeting the source is not informed of the existence of other message copies. Assuming a given lifetime for the message, we considered the (discrete-time) decision problem faced by a relay. When it meets the source, a relay has to decide whether to accept the message or not, and once the relay has the message it has to choose to retain or to drop it at subsequent decision epochs. Each relay makes its decisions in order to minimize the expected cost it incurs for participating. We modelled the interaction between mobile nodes as a stochastic game with partial information. For the single player case, we first obtained a necessary condition for the relay to attempt the delivery of the message that reflects a minimal value of the reward. In fact it implies the minimal reward sufficient to ensure that the player will not drop the message. We then saw that the relay's strategy to accept the message from the source is of a threshold type. Extending the model to the case of two players, we established that if one of the players follows a threshold type policy then the other one will also use a similar strategy. We thereby have come to the question whether such threshold strategies are an equilibrium of the game. A positive answer to these question is not obvious, however if so it gives strong research impetus and opens up a possibility to fine-tune our reward mechanism.

6 SUMMARY AND DISCUSSIONS

Communication networks is an actively researched area, and a number of studies is related to network environments where multiple self-interested parties interact. For analytical investigations of such competitive interactions, a conceptual framework is provided by game theory. Game-theoretic approach is widely used for analysis of decentralized network settings and has found applications in as diverse areas as load-balancing in server farms, power control and spectrum allocation in wireless networks, or congestion control in the Internet. In our study, we have focused on two leading research directions in communication networking that are decentralized routing and Delay tolerant networking, and investigated game scenarios therein. Primarily, we have addressed to the problem of uncoordinated routing and proposed a different approach for establishing its convergence property. For DTNs, we have modelled selfish behaviour of DTN nodes and developed a mechanism for nodal cooperation.

6.1 A Different Approach to Study of the Convergence

Convergence to invariant traffic allocation is an important property for uncoordinated routing in multi-agent networks that reflects stability of steady state. In a non- cooperative game model of competitive routing, best-response dynamics provides a natural play to reach an equilibrium distribution of traffic. A commonly used potential-based method is powerful to prove the convergence of uncoordinated dynamics in non-cooperative games. However, it faces significant technical difficulties in the construction of a suitable potential function. We aimed to develop a universal approach to establish the convergence of best-response dynamics in routing games. Our focus was on a network of parallel links shared by a finite number of selfish users, where each user controls a non-negligible portion of the total traffic, and seeks to split his flow over the links of the network so as to minimize his own cost. We have investigated convergence assuming the well-known (myopic) best-response dynamics. We have analysed the sequential (or round robin) variant of it, where players play in a cyclic manner according to a pre-defined order.

Our approach to prove the convergence of the best-response dynamics is based on the notion of non-linear spectral radius. To apply this approach one has to construct an operator for the dynamics, then show Lipschitz continuity of the operator and that its non-linear spectral radius is lower than unity. The non-linear spectral radius is related to the joint spectral radius of a set of Jacobian matrices of the operator. For our routing game, we have shown that the best-response function is Lipschitz, and established the specific structure of their Jacobian matrices. We have thus obtained a purely structural sufficient condition that allows to reduce the analysis of the convergence of the sequential best-response dynamics to the analysis of the joint spectral radius of certain matrices. We have shown that this condition is met in two cases: two-player game for an arbitrary number of links and for a wide class of cost functions; and for arbitrary numbers of players and links in the case of linear latency functions. For latency functions satisfying reasonable convexity assumptions, we conjecture that the proposed sufficient condition is valid for arbitrary numbers of players and links.

Proposed approach sheds light on convergence issues of best-response dynamics in other settings where Potential-based reasoning is not effective. We expect successful use of non-linear spectral-radius approach along with matrix analysis in studying stability of the equilibrium in the settings more complex than parallel link network topologies, starting from the cases when existence of equilibria is known.

6.2 Reward-Based Intensive Mechanism for DTNs

A central problem in Delay Tolerant Networks (DTNs) is to persuade mobile nodes to participate in relaying messages. In this thesis we have proposed a reward mechanism to incentive relays to sacrifice their memory and battery on DTNs relaying operation. The reward mechanism in fact is designed to secure the participation of relays in the delivery process by proposing a reward that takes into account the costs incurred by the relays and the risk they are exposed to during the delivery process. This reward is the minimum amount that offsets the expected delivery cost, as estimated by the relay from the information given by the source (number of existing copies of the message, age of these copies). We first showed that the expected reward paid by the source remains the same irrespective of the information it conveys, ranging from full state information to no information. We also studied the dynamic case in which the source can change the information that it conveys on the fly as and when meets the relay. Under some additional assumptions, the source can gain by adopting the dynamic strategy. Next, we have addressed to the discrete time decision process for the relays, when the message is endowed with a lifetime. For the no information setting in case of two relays and assuming a fixed incentive reward, we have studied an optimal policy for a relay that is when it should accept the message from the source and if it has accepted it when the relay should drop it. We then have shown that if a relay follows a threshold type optimal policy then another relay will behave in similar way.

A key challenge in developing our results has been to make general assumptions about the mobility of DTN nodes. In particular, the properties derived for our incentive mechanism hold under any homogeneous mobility pattern. Indeed, the large majority of analytical studies are typically assumed that the cumulative distribution function of inter contact time decays exponentially over time such as in random waypoint models. But many extensive empirical mobility traces have been showed that cumulative distribution function of inter contact time follows approximately a power law over large time range with exponent less than unit (Chaintreau et al., 2007). By investigating a general assumption about the mobility, in future works, we plan to evaluate our scheme on realistic traces (*RAWDAD: A Community Resource for Archiving Wireless Data At Dartmouth*) in order to evaluate the robustness of our proposed mechanism. Another aspect that we want to take into account is the heterogeneous models. Existing analytical studies in the literature strongly rely on the assumption that nodes identical and uniformly visit the entire network space. Experimental data, however, have shown that mobility patterns of individuals are typically restricted to a given area, and the overall node density is often largely inhomogeneous. Such models allow studying how DTN routing mechanisms are affected by highly inhomogeneous node density and differences in mobility patterns and transmission technologies.

In our model we have restricted consideration only for one source-destination pair that generates packet into DTN. For several source-destination pairs, node buffers may well overflow if no message discarding policy is adopted. In this scenario, efficient drop policies at relay nodes decide which messages should prioritised under capacity constraints regardless of the specific routing algorithm used. In the future, we propose to work on intentional DTN Drop/Scheduling policies with respect to our mechanism. Such study engenders sources to develop a mechanism design in order to know the information about the messages that relay stores in his buffer. Then we will propose a mechanism that can allow the source to truthfully elicit private information from each and every relay nodes it meet. However, information elicitation is most challenging when it is most useful: when there is no ground truth available to evaluate answers.

LIST OF PUBLICATIONS

On the convergence of the best-response algorithm in routing games.

O. BRUN, B. J. PRABHU, T. SEREGINA

In the proceedings of the 7^{th} International Conference on Performance Evaluation Methodologies and Tools, ValueTools-2013, December 2013, Turin, Italy, ICST, pp. 136–144, DOI: 10.4108/icst.valuetools.2013.254405

Modeling Rewards and Incentive Mechanisms for Delay Tolerant Networks.

O. BRUN, R. EL-AZOUZI, B. J. PRABHU, T. SEREGINA

In the proceedings IEEE 12^{th} International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks, WiOpt-2014, May 2014, Hammamet, Tunisia, pp. 233–240, DOI: 10.1109/WIOPT.2014.6850304

Reward-based Incentive Mechanisms for Delay Tolerant Networks.

T. SEREGINA

In the proceedings of the 10^{th} Workshop on Performance Evaluation, AEP-10, Juin 2014, Sophia Antipolis, France, pp. 25–26,
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