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Processus de branchement avec interaction

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JURY

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RÉSUMÉ

Cette thèse se compose de quatre chapitres:

Le chapitre 1 étudie la distribution du temps de *coalescence* (plus récent ancêtre commun) de deux individus tirés au hasard (uniformément) dans la génération actuelle d'un processus de Bienaymé-Galton-Watson en temps continu.

Dans le chapitre 2, nous obtenons une représentation de la diffusion de Feller logistique en termes des temps locaux d'un mouvement brownien réfléchi H avec une dérive qui est affine en le temps local accumulé par H à son niveau actuel.

Le chapitre 3 considère la diffusion de Feller avec compétition générale. Nous donnons des conditions précises sur le terme de la concurrence, pour le but de décider si le temps d'extinction (qui est aussi la hauteur du processus) reste borné ou non lorsque la taille initiale de la population tend vers l'infini, et de même pour la masse totale du processus.

Dans le chapitre 4, nous généralisons les résultats du chapitre 3 pour le cas du processus de branchement à espace d'état continu à trajectoires discontinues avec compétition.

Mots-Clés: Processus de Bienaymé-Galton-Watson, coalescence, diffusion de Feller logistique, temps local, théorème de Ray-Knight, processus de branchement, compétition, temps d'extinction, masse totale

ABSTRACT

This thesis consists of four chapters:

Chapter 1 investigates the distribution of the coalescence time (most recent common ancestor) for two individuals picked at random (uniformly) in the current generation of a continuous time Bienaymé-Galton-Watson process.

In chapter 2 we obtain a Ray-Knight representation of Feller's branching diffusion with logistic growth in terms of the local times of a reflected Brownian motion H with a drift that is affine in the local time accumulated by H at its current level.

Chapter 3 considers the Feller's branching diffusion with general competition. We give precise conditions on the competition term, in order to decide whether the extinction time (which is also the height of the process) remains or not bounded as the initial population size tends to infinity, and similarly for the total mass of the process.

In chapter 4 we generalize the results of chapter 3 to the case of continuous state branching process with competition which has discontinuous paths.

Keywords: Bienaymé-Galton-Watson process, coalescence, Feller diffusion with logistic growth, local time, Ray-Knight theorem, branching process, competition, extinction time, total mass

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PRESENTATION

It was about one hundred forty years ago that Galton and Watson, in treating the problem of the extincsion of family names, showed how probability theory could be applied to study the effects of chance on the development of populations. Galton-Watson model and its generalizations have been treated extensively in the twentieth century. Now Galton-Watson tree (which is also called Bienaymé- Galton-Watson tree) plays a fundamental role in both the theory and applications of stochastic processes.

Our goal for the first chapter is to investigate the distributions of coalescence times for Bienaymé-Galton-Watson (BGW) tree. This goal was also that of various other works such as [1, 2, 4, 10, 11]. In Lambert [4], fine results were obtained for BGW process in the discrete setting. Here we want to extend those results to the case of continuous time BGW process. More precisely, Chapter 1 studies the distribution of the coalescence time (most recent common ancestor) for two individuals picked at random (uniformly) in the current generation of a continuous time BGW process founded t units of time ago. Here we also obtain limiting distributions as $t \to \infty$ in the subcritical case and generalize our results for two individuals to the joint distribution of coalescence times for any finite number of individuals sampled in the current generation.

We next consider a BGW tree to which we superimpose a logistic interaction between the branches (which destroys the independence, and hence also the branching property). We proceed to a renormalization of the model parameters and get in limit a diffusion which is solution of the SDE

$$dZ_t = \sigma\sqrt{Z_t}\,dW_t + (\theta Z_t - \gamma Z_t^2)\,dt, \qquad Z_0 = x.$$

This is called *Feller's branching diffusion with logistic growth* which has been studied in detail by Lambert [5]. In chapter 2 we obtain a representation of Feller's branching diffusion with logistic growth in terms of the local times of a reflected Brownian motion H with a drift that is affine in the local time accumulated by H at its current level. As in the classical Ray-Knight representation, the excursions of H are the exploration paths of the trees of descendants of the ancestors at time $t = 0$, and the local time of H at height t measures the population size at time t (see e.g. [6]). We cope with the dependence in the reproduction by introducing a pecking order of individuals: an individual explored at time s and living at time $t = H_s$ is prone to be killed by any of its contemporaneans that have been explored so far. The proof of our main result of chapter 2 relies on approximating H with a sequence of Harris paths H^N which figure in a Ray-Knight representation of the total mass of a branching particle system. We obtain a suitable joint convergence of H^N together with its local times *and* with the Girsanov densities that introduce the dependence in the reproduction.

In chapter 3 we consider a discrete model of population with interaction where the birth and death rates are non linear functions of the population size. According to [3], after proceeding to renormalization of the model parameters, we obtain in the limit of large population that the population size evolves as a *Feller's branching diffusion with general competition* which is solution of the SDE

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + 2\int_0^t \int_0^{Z_s^x} W(ds, du),$$

where $W(ds, du)$ is a time space white noise on $([0, \infty))^2$. We give precise conditions on the function f, in order to decide whether the extinction time of the process remains or not bounded as the ancestral population size tends to infinity, and similarly for the total mass of the process.

In chapter 4 we generalize the results of chapter 3 for *Feller's branching diffusion with general competition* to the case of *continuous state branching process with competition* which has discontinuous paths.

These chapters is completely written in English. In fact, Chapter 1, Chapter 2 and

Chapter 3 are three articles accepted for publication in Journal of Applied Probability, Probability Theory & Related Fields and ESAIM Probability and Statistics respectively (see [7, 8, 9]). In the first part of this thesis we present the results obtained during the thesis, which are detailed in the following four chapters. This part is completely written in French.

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INTRODUCTION

Dans l'introduction nous exposons les résultats obtenus au cours de cette thèse, lesquels sont détaillés dans les quatre chapitres suivants.

0.1 Temps de coalescence pour le processus de Bienaymé-Galton-Watson

L'objectif du premier chapitre est d'étudier la distribution du temps de coalescence pour le processus de Bienaymé-Galton-Watson (BGW). Cet objectif était également celui d'autres travaux comme [1, 2, 6, 9, 11]. Dans Lambert [6], les bons résultats ont été obtenus pour le processus de BGW en temps discret. Ici, nous voulons étendre ces résultats pour le cas du processus de BGW en temps continu.

0.1.1 La distribution du temps de coalescence

Nous considérons un processus de branchement en temps continu $Z = \{Z_t, t \geq 0\}$ à valeurs dans $\mathbb{N}$. Un tel processus est un processus de BGW dans lequel à chaque individu est attaché un vecteur aléatoire décrivant sa durée de vie et le nombre de ses descendants. Nous supposons que ces vecteurs aléatoires sont i.i.d. Le taux de reproduction est gouverné par une mesure finie μ sur $\mathbb{N}$, satisfaisant $\mu(1) = 0$. Plus précisément, chaque individu vit un temps exponentiel de paramètre $\mu(\mathbb{N})$, et est remplacé par un nombre aléatoire d'enfants selon la probabilité $\mu(\mathbb{N})^{-1}\mu$. Alors la dynamique du processus de Markov en temps continu Z est entièrement caractérisée par la mesure μ. Pour $x \in \mathbb{N}$, dénotons par $\mathbb{P}_x$ la loi de Z quand $Z_0 = x$. On a la proposition suivante, qui peut être trouvée dans [3], page 106.

Proposition 0.1.1. *La fonction génératrice du processus Z est donnée par*

$$\mathbb{E}_x(s^{Z_t}) = \psi_t(s)^x, \qquad s \in [0,1], x \in \mathbb{N},$$

où

$$\frac{\partial \psi_t(s)}{\partial t} = \Phi(\psi_t(s)), \qquad \psi_0(s) = s,$$

et la fonction Φ *est définie par*

$$\Phi(s) = \sum_{n=0}^{\infty} (s^n - s)\mu(n), \quad s \in [0,1].$$

Nous considérons deux individus σ_1, σ_2 à la génération actuelle $t > 0$, et demandons quand ils fusionnent, c'est-à-dire, combien de temps s'est écoulé depuis leur ancêtre commun. D'une manière plus rigoureuse, pour $0 < u \leq t$, dénotons par $\tau_u(\sigma_i)$ la mère (unique) de σ_i au temps $(t-u), i = 1, 2$. Le temps de coalescence $T(\sigma_1, \sigma_2)$ de σ_1, σ_2 est déterminé uniquement par

$$T(\sigma_1, \sigma_2) := \inf\{u : 0 < u \leq t, \tau_u(\sigma_1) = \tau_u(\sigma_2)\},$$

avec la convention $\inf \emptyset = \infty$. Nous dénotons par T le temps de coalescence de deux individus tirés au hasard (uniformément) parmi les individus présents dans la génération actuelle. Si la génération actuelle contient moins de deux individus, on pose $T = \infty$.

Si la notation $\mathbb{P}^{(t)}$ indique que t est la génération actuelle, la distribution de T est donnée dans le théorème suivant.

Théorème 0.1.1. *Pour tout* $0 < t_1 \leq t_2 \leq t, y \geq 1, y \in \mathbb{N}$,

$$\mathbb{E}^{(t)}(Z_t(Z_t - 1)s^{Z_t - 2}, T \leq t_1 \mid Z_{t-t_2} = y) = y\psi'_{t_2}(s)\psi_{t_2}(s)^{y-1}\frac{\psi''_{t_1}(s)}{\psi'_{t_1}(s)}, \quad s \in [0,1).$$

La fonction génératrice précédente peut être inversée comme suit, pour tout $p \geq 2$

$$\mathbb{P}^{(t)}(Z_t = p, T \in dt_1 \mid Z_{t-t_2} = y)/dt_1 =$$

$$y \sum_{n \geq 2} n\mu(n)\mathbb{E}\Big(\frac{Z_{t_2}^{(1)}(1)Z_{t_1}^{(2)}(n-1)}{p(p-1)}, Z_{t_2}^{(0)}(y-1) + Z_{t_2}^{(1)}(1) + Z_{t_1}^{(2)}(n-1) = p\Big),$$

où les processus $Z^{(0)}, Z^{(1)}, Z^{(2)}$ *sont des copies indépendantes de* Z, *et la notation* $Z_{t_2}^{(0)}(y-1)$ *désigne la valeur au temps* t_2 *du processus* $Z^{(0)}$ *issu de* $y - 1$.

Une conséquence du Théorème 0.1.1 est

Corollaire 0.1.2. *Pour tout* $0 < t_1 \leq t$,

$$\mathbb{P}_x^{(t)}(T \leq t_1) = x \int_0^1 ds(1-s) \frac{\psi''_{t_1}(s)}{\psi'_{t_1}(s)} \psi'_t(s) \psi_t(s)^{x-1}.$$

En particulier,
$\mathbb{P}_x^{(t)}$ *(Au moins deux individus sont en vie à l'instant* t*, une paire aléatoire n'a aucun ancêtre commun)* $=$

$$x(x-1) \int_0^1 ds(1-s) \psi'_t(s)^2 \psi_t(s)^{x-2}.$$

0.1.2 La distribution quasi-stationnaire

Dans cette sous-section, on obtient la limite de la loi conditionnelle du temps de coalescence sachant que $\{Z_t \geq 2\}$. Informellement, cette limite incarne la situation où la généalogie a été fondée il y a longtemps et n'est pas encore éteinte, avec au moins deux descendants à l'instant actuel.

Nous considérons le cas $\psi'_1(1) = \mathbb{E}_1(Z_1) < 1$ (cas sous-critique) quand $\mathbb{E}_1(Z_1 \log(Z_1)) < \infty$. À partir du Théorème 6 de [12], il existe une suite non négative $(\alpha_k, k \geq 1)$ dont la somme est égale à 1 tel que

$$\lim_{t \to \infty} \mathbb{P}_x(Z_t = j \mid Z_t > 0) = \alpha_j, \qquad \forall x \in \mathbb{N}, j \geq 1. \tag{0.1.1}$$

La suite $(\alpha_k, k \geq 1)$ s'appelle la limite de Yaglom du processus Z. Si on définit

$$g(s) = \sum_{k \geq 1} \alpha_k s^k, \qquad s \in [0,1],$$

alors (0.1.1) en déduit que

$$g(s) = \lim_{t \to \infty} \mathbb{E}_x(s^{Z_t} \mid Z_t > 0) = \lim_{t \to \infty} \frac{\psi_t(s) - \psi_t(0)}{1 - \psi_t(0)}, \qquad s \in [0,1].$$

Le résultat suivant peut être trouvé dans [3], chapitre IV (page 170).

Proposition 0.1.3. *Dans le cas sous-critique quand* $\mathbb{E}_1(Z_1 \log(Z_1)) < \infty$*, on a pour tout* $s \in [0,1]$*,*

$$\lim_{t\to\infty} \mathbb{E}_x(Z_t s^{Z_t - 1} \mid Z_t > 0) = g^{'}(s) \leq g^{'}(1) < \infty.$$

Notons par $\tilde{Z}$ la limite de Z_t conditionnée à $\{Z_t \geq 2\}$ quand $t \to \infty$. On a le

Théorème 0.1.2. *Dans le cas sous-critique quand* $\mathbb{E}_1(Z_1 \log(Z_1)) < \infty$*, la distribution quasi-stationnaire* $\mathbb{P}^{qs}$ *de* T *et* $\tilde{Z}$ *est définie par*

$$\mathbb{P}^{qs}(\tilde{Z} = p, T \in dh) = \lim_{t\to\infty} \mathbb{P}_x^{(t)}(Z_t = p, T \in dh \mid Z_t \geq 2), \qquad p \geq 2, h > 0.$$

Alors $\mathbb{P}^{qs}$ *définit une loi de probabilité qui ne dépend pas de* x *et satisfait*

$$\mathbb{E}^{qs}(\tilde{Z}(\tilde{Z}-1)s^{\tilde{Z}-2}, T \leq h) = \frac{g^{'}(s)}{1 - g^{'}(0)} \frac{\psi_h^{''}(s)}{\psi_h^{'}(s)}.$$

En particulier,

$$\mathbb{P}^{qs}(T \leq h) = \frac{1}{1 - g^{'}(0)} \int_0^1 ds(1-s) \frac{\psi_h^{''}(s)}{\psi_h^{'}(s)} g^{'}(s).$$

0.1.3 Coalescence multivariée

Supposons que la génération actuelle contient au moins $n+1$ individus, $n \geq 1$. Nous allons présenter la distribution des temps de coalescence, lorsque $n+1$ individus sont échantillonnés uniformément et indépendamment à l'instant actuel t. Pour $k = 1, 2, ..., n$, on note par T_k le temps de coalescence du premier individu avec le $(k+1)$-ième individu, et par T_k^* le k-ième temps de coalescence. On a le

Théorème 0.1.3. *Pour tout* $0 < t_1 < t_2 < ... < t_n \leq t$*, la distribution conjointe des temps de coalescence* T_k *est donnée par*

$$\mathbb{E}_x^{(t)}(Z_t(Z_t-1)...(Z_t-n)s^{Z_t-n-1}, T_1 \in dt_1, ..., T_n \in dt_n)/dt_1...dt_n =$$

$$x\psi_t^{'}(s)\psi_t(s)^{x-1} \prod_{i=1}^{n} \psi_{t_i}^{'}(s) \big[\sum_{k\geq 2} k(k-1)\mu(k)\psi_{t_i}(s)^{k-2}\big], \quad s \in [0,1).$$

Théorème 0.1.4. *Pour tout* $0 < t_1 < t_2 < ... < t_n \leq t$*, la distribution conjointe des temps de coalescence* T_k^* *est donnée par*

$$\mathbb{E}_x^{(t)}(Z_t(Z_t-1)...(Z_t-n)s^{Z_t-n-1}, T_1^* \in dt_1, ..., T_n^* \in dt_n)/dt_1...dt_n =$$
$$\frac{n!(n+1)!}{2^n} x\psi_t^{'}(s)\psi_t(s)^{x-1}\prod_{i=1}^{n}\psi_{t_i}^{'}(s)\big[\sum_{k\geq 2}k(k-1)\mu(k)\psi_{t_i}(s)^{k-2}\big], \quad s \in [0,1).$$

0.2 "Les arbres sous attaque": une représentation de Ray-Knight de la diffusion de Feller logistique

La diffusion de Feller logistique est gouvernée par l'EDS

$$dZ_t = \sigma\sqrt{Z_t}\,dW_t + (\theta Z_t - \gamma Z_t^2)\,dt, \qquad Z_0 = x > 0, \tag{0.2.1}$$

avec σ, θ et γ des constantes positives. Elle a été étudié en détail par Lambert [7], et modélise l'évolution de la taille d'une population avec compétition. Dans le chapitre 2, nous obtenons une représentation de la diffusion de Feller logistique en termes des temps locaux d'un mouvement brownien réfléchi H avec une dérive qui est affine en le temps local accumulé par H à son niveau actuel. Plus précisément, on considère l'EDS

$$H_s = \frac{2}{\sigma}B_s + \frac{1}{2}L_s(0,H) + \frac{2\theta}{\sigma^2}s - \gamma\int_0^s L_r(H_r, H)dr, \qquad s \geq 0, \tag{0.2.2}$$

où B est un mouvement brownien standard, et pour $s, t \geq 0$, $L_s(t,H)$ désigne le (semi–martingale) temps local accumulé par H au niveau t à l'instant s. D'après [10], l'EDS (0.2.2) admet une unique solution en loi. On définit

$$S_x := \inf\{s > 0 : (\sigma^2/4)L_s(0,H) > x\}. \tag{0.2.3}$$

Le résultat principal du chapitre 2 est

Théorème 0.2.1. *Supposons que* H *est solution de l'équation* (0.2.2)*, et soit, pour* $x > 0$*,* S_x *défini par* (0.2.3)*. Alors* $(\sigma^2/4)L_{S_x}(t,H)$*,* $t \geq 0$*, est solution de l'équation* (0.2.1).

0.2.1 Une approximation discrète

Le but de cette sous-section est de donner des approximations discrètes de (0.2.1) et (0.2.2).

Pour $x > 0$ et $N \in \mathbb{N}$ l'approximation de (0.2.1) sera donnée par la masse totale $Z^{N,x}$ d'une population dont chaque individu a une masse de $1/N$. La masse initiale est $Z_0^{N,x} = \lfloor Nx \rfloor /N$, et $Z^{N,x}$ est un processus de Markov: si à un instant t, $Z_t^{N,x} = k/N$,

$$Z^{N,x} \text{ saute de } k/N \text{ à } \begin{cases} (k+1)/N \text{ au taux } kN\sigma^2/2 + k\theta \\ (k-1)/N \text{ au taux } kN\sigma^2/2 + k(k-1)\gamma/N. \end{cases} \tag{0.2.4}$$

Pour $\gamma = 0$, c'est (à une constante multiplicative près) comme un processus de Galton-Watson en temps continu: chaque individu, indépendamment des autres donne naissance au taux $N\sigma^2/2 + \theta$, et meurt au taux $N\sigma^2/2$. Pour $\gamma \neq 0$, le taux de mort quadratique détruit l'indépendance, et par conséquent détruit aussi la propriété de branchement. Cependant, en regardant les individus vivants à l'instant t comme étant arrangés "de gauche à droite", et en décrétant que chacun des combats deux à deux (qui se passe au taux 2γ) est gagné par l'individu à la gauche, on obtient le taux de mort supplémentaire $2\gamma\mathcal{L}_i(t)/N$ pour l'individu i, où $\mathcal{L}_i(t)$ désigne le nombre d'individus situés à gauche de l'individu i à l'instant t.

La dynamique de reproduction qui vient d'être décrite donne lieu à une forêt $F^{N,x}$ d'arbres planaires (voir la Figure 1). À tout point de branchement, on imagine la "nouvelle branche" étant placée à la droite de la branche mère. En raison du massacre asymétrique, les arbres plus loins à droite ont une tendance à rester plus petits: ils sont "sous attaque" par les arbres à leur gauche. On note que, avec la construction décrite ci-dessus, les $F^{N,x}$, $x > 0$, sont couplées: quand x est augmenté de $1/N$, un nouvel arbre est ajouté à la droite. On note l'union des $F^{N,x}$, $x > 0$, par F^N.

On peut associer à la forêt F^N un processus $H^N = (H_s^N)$ continu et linéaire par morceaux à valeurs dans $\mathbb{R}_+$ (qui s'appelle la trajectoire d'exploration de F^N) de la manière suivante:

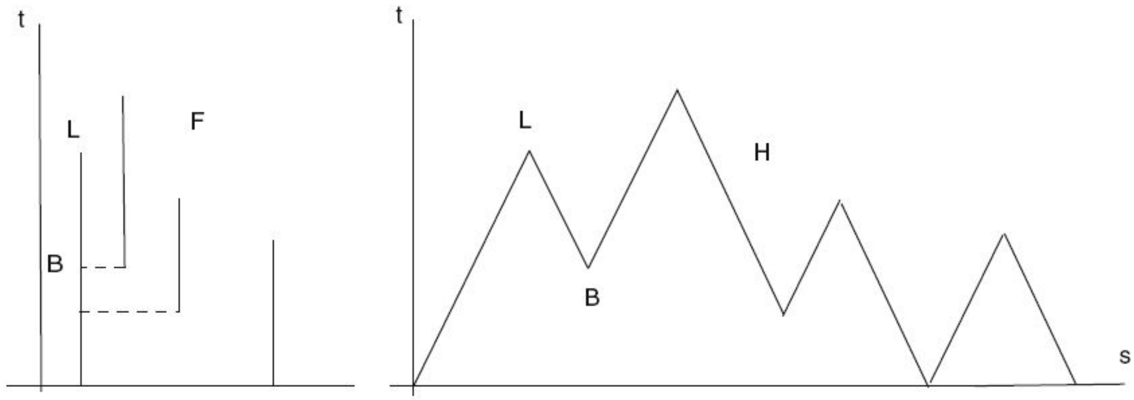

Figure 1: Une réalisation de (les deux premiers arbres de) F^N et (les deux premières excursions de) son exploration H^N.

À partir de la racine de l'arbre le plus à gauche, on va vers le haut à vitesse de $2N$ jusqu'à ce qu'on touche le haut de la première branche mère (c'est la feuille marquée par **L** dans Figure 1). Ensuite on tourne et va vers le bas, à même vitesse de $2N$, jusqu'à ce qu'on arrive au prochain point de branchement (qui est marqué par **B** dans Figure 1). De là, on va vers le haut à la branche suivante, et procède d'une manière similaire jusqu'à ce qu'on revient à la hauteur 0, ce qui signifie que l'exploration de l'arbre le plus à gauche est terminée. Ensuite on explore l'arbre suivant, et ainsi de suite.

Pour $x > 0$ on note par S_x^N l'instant auquel l'exploration de la forêt $F^{N,x}$ est terminée. Évidemment, pour chaque $t \geq 0$, le nombre de branches dans $F^{N,x}$ qui sont vivantes à l'instant t est égal à la moitié du nombre d'intersections de la trajectoire d'exploration de F^N arrêtée à S_x^N avec la droite horizontale à la hauteur t. Rappelant que la pente de H^N est $\pm 2N$, on définit

$$\Lambda_s^N(t) := \frac{1}{2N} \# \text{ de } t\text{-intersections de } H^N \text{ entre les temps d'exploration } 0 \text{ et } s, \tag{0.2.5}$$

où nous comptons un minimum local de H^N à t comme deux t-intersections, et un maximum local comme zéro. Notons que, par notre convention, tous les deux $s \mapsto \Lambda_s^N(t)$ et $t \mapsto \Lambda_s^N(t)$ sont continues à droite, et en particulier $\Lambda_0^N(0) = 0$. On appelle $\Lambda_s^N(t)$ le temps local non normalisé accumulé par H^N au niveau t à l'instant s. Le temps d'exploration S_x^N peut être exprimé comme

$$S_x^N = \inf\{s : \Lambda_s^N(0) \geq \lfloor Nx \rfloor / N\}. \tag{0.2.6}$$

On a

Proposition 0.2.1. *La trajectoire d'exploration $s \mapsto H_s^N$ obéit à la dynamique stochastique suivante:*

- *À l'instant $s = 0$, H^N commence à zéro et avec une pente $2N$.*
- *Alors que H^N va vers le haut, sa pente saute de $2N$ à $-2N$ au taux $N^2\sigma^2 + 4\gamma N\ell$, où $\ell = \Lambda_s^N(H_s^N)$ est le temps local accumulé par H^N au niveau actuel H_s^N à l'instant actuel s.*
- *Alors que H^N va vers le bas, sa pente saute de $-2N$ à $2N$ au taux $N^2\sigma^2 + 2N\theta$.*
- *Chaque fois que H^N revient à zéro, il est réfléchi au-dessus de zéro.*

Le corollaire suivant est une version discrète du Théorème 0.2.1, et sera utilisé pour la démonstration du Théorème 0.2.1 en prenant $N \to \infty$.

Corollaire 0.2.2. *Soit H^N le processus stochastique suivant la dynamique spécifiée dans la Proposition 0.2.1, et Λ^N son temps local tel que défini par* (0.2.5)*. Pour $x > 0$, soit S_x^N le temps d'arrêt défini par* (0.2.6)*. Alors $t \mapsto \Lambda_{S_x^N}^N(t)$ suit la dynamique* (0.2.4)*.*

0.2.2 La convergence des processus $Z^{N,x}$ quand $N \to \infty$

Dans cette sous-section, nous montrons que

Proposition 0.2.3. *Quand $N \to \infty$, $Z^{N,x} \Rightarrow Z^x$, où Z^x est l'unique solution de l'EDS* (0.2.1) *et est donc une diffusion de Feller logistique.*

0.2.3 La convergence de la trajectoire d'exploration dans le cas $\theta = \gamma = 0$.

Soit H^N le processus stochastique comme dans la Proposition 0.2.1 avec $\theta = \gamma = 0$. On définit le temps local normalisé accumulé par H^N au niveau t à l'instant s comme

$$L_s^N(t) := \frac{4}{\sigma^2} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t < H_u^N < t+\varepsilon\}} du$$

Notons que ce processus n'est pas une fonction continue ni à droite ni à gauche de s. Toutefois, parce que les sauts sont de taille $O(1/N)$, la limite de L^N quand $N \to \infty$ sera continue. Il est facile de vérifier que $L_{0+}^N(0) = \frac{2}{N\sigma^2}$, et

$$L_s^N(t) = \frac{4}{\sigma^2} \Lambda_s^N(t), \quad \forall s \geq 0, t \geq 0,$$

où Λ^N a été défini par (0.2.5). Alors avec S_x^N défini par (0.2.6), nous pouvons récrire

$$S_x^N = \inf\{s > 0 : L_s^N(0) \geq \frac{4}{\sigma^2} \lfloor Nx \rfloor / N\}. \tag{0.2.7}$$

Soit $\{V_s^N, s \geq 0\}$ le processus càdlàg à valeurs dans $\{-1, 1\}$ qui satisfait pour presque tout $s > 0$,

$$\frac{dH_s^N}{ds} = 2NV_s^N. \tag{0.2.8}$$

À partir de la Proposition 0.2.1 on a

$$V_s^N = 1 + 2\int_0^s \mathbf{1}_{\{V_{r-}^N = -1\}} dP_r^N - 2\int_0^s \mathbf{1}_{\{V_{r-}^N = 1\}} dP_r^N + \frac{N\sigma^2}{2}(L_s^N(0) - L_{0+}^N(0)), \tag{0.2.9}$$

où $\{P_s^N, s \geq 0\}$ est un processus de Poisson d'intensité $N^2\sigma^2$. On écrit V_r^N dans (0.2.8) comme

$$\mathbf{1}_{\{V_{r-}^N = +1\}} - \mathbf{1}_{\{V_{r-}^N = -1\}}$$

et désigne par M_s^N la martingale $P_s^N - N^2\sigma^2 s$, on déduit de (0.2.9) que

$$H_s^N + \frac{V_s^N}{N\sigma^2} = M_s^{1,N} - M_s^{2,N} + \frac{1}{2} L_s^N(0) - \frac{1}{2} L_{0+}^N(0),$$

où

$$M_s^{1,N} = \frac{2}{N\sigma^2}\int_0^s \mathbf{1}_{\{V_{r-}^N=-1\}} dM_r^N \quad \text{et} \quad M_s^{2,N} = \frac{2}{N\sigma^2}\int_0^s \mathbf{1}_{\{V_{r-}^N=1\}} dM_r^N \qquad (0.2.10)$$

sont deux martingales orthogonales. Le résultat principal de cette sous-sectione est le théorème suivant.

Théorème 0.2.2. *Pour tout $x > 0$, quand $N \to \infty$,*

$$\begin{aligned}&(\{H_s^N, M_s^{1,N}, M_s^{2,N},\ s \geq 0\}, \{L_s^N(t), s, t \geq 0\}, S_x^N)\\ &\quad \Rightarrow \left(\left\{H_s, \frac{\sqrt{2}}{\sigma}B_s^1, \frac{\sqrt{2}}{\sigma}B_s^2,\ s \geq 0\right\}, \{L_s(t), s, t \geq 0\}, S_x\right)\end{aligned}$$

pour la topologie de la convergence localement uniforme. B^1 et B^2 sont deux mouvements browniens standards indépendants, H est la solution de l'EDS

$$H_s = \frac{2}{\sigma}B_s + \frac{1}{2}L_s(0),$$

dont le mouvement brownien B est donné par

$$B_s = \frac{1}{\sqrt{2}}(B_s^1 - B_s^2),$$

L est le temps local de H, et S_x a été défini par (0.2.3).

Une conséquence immédiate de ce résultat est

Corollaire 0.2.4. *Pour tout $x > 0$, quand $N \to \infty$,*

$$(\{H_s^N, M_s^{1,N}, M_s^{2,N},\ s \geq 0\}, \{L_{S_x^N}^N(t), t \geq 0\}) \Rightarrow (\{H_s, \frac{\sqrt{2}}{\sigma}B_s^1, \frac{\sqrt{2}}{\sigma}B_s^2,\ s \geq 0\}, \{L_{S_x}(t), t \geq 0\})$$

dans $C([0,\infty)) \times (D([0,\infty)))^3$.

0.2.4 Changement de mesure et la preuve du Théorème 0.2.1

Comme dans la section précédente, pour $N \in \mathbb{N}$ fixe, soit H^N le processus suivant la dynamique spécifiée dans la Proposition 0.2.1 avec $\theta = \gamma = 0$. On note la mesure de probabilité par $\mathbb{P}$, et la filtration par $\mathcal{F} = (\mathcal{F}_s)$. Notre premier objectif est de construire une mesure $\tilde{\mathbb{P}}^N$ sous laquelle H^N suit la dynamique spécifiée dans la Proposition 0.2.1 pour $\theta, \gamma > 0$.

Ici, un rôle crucial est joué par le processus ponctuel P^N des maxima et minima locaux successifs de H^N, excluant les minima à hauteur 0. Sous $\mathbb{P}$, il est un processus de Poisson d'intensité $\sigma^2 N^2$. Plus précisément, le processus $Q^{1,N}$ qui compte les minima locaux successifs de H^N (hormis ceux à la hauteur 0) est un processus ponctuel avec une intensité prévisible $\lambda_s^{1,N} := N^2\sigma^2\mathbf{1}_{\{V_{s-}^N=-1\}}$, et le processus $Q^{2,N}$ qui compte les maxima locaux successifs de H^N est un processus ponctuel avec une intensité prévisible $\lambda_s^{2,N} := N^2\sigma^2\mathbf{1}_{\{V_{s-}^N=+1\}}$.

À partir de la Proposition 0.2.1 nous voulons changer le taux $\lambda_s^{1,N}$ en $\lambda_s^{1,N}(1+\frac{2\theta}{N\sigma^2})$ et le taux $\lambda_s^{2,N}$ en $\lambda_s^{2,N}(1+\frac{4\gamma\Lambda_s^N(H_s)}{N\sigma^2})$. On aura besoin des deux versions du théorème de Girsanov et quelques résultats concernant l'exponentielle de Doléans et "*goodness*" qui peuvent être trouvés dans [8]. On considère les martingales locales

$$X_s^{N,1} := \int_0^s \frac{2\theta}{N\sigma^2}\mathbf{1}_{\{V_{r-}^N=-1\}}dM_r^N, \quad X_s^{N,2} := \int_0^s \frac{\gamma L_r^N(H_r^N)}{N}\mathbf{1}_{\{V_{r-}^N=1\}}dM_r^N,$$
$$X^N := X^{N,1} + X^{N,2}.$$

Soit $Y^N := \mathcal{E}(X^N)$ l'exponentielle de Doléans de X^N. On a

Proposition 0.2.5. *Y^N est une $(\mathcal{F}, \mathbb{P})$-martingale. Soit $\tilde{\mathbb{P}}^N$ la mesure de probabilité sur $\mathcal{F}$ dont la restriction à $\mathcal{F}_s$, $s > 0$, a la densité Y_s^N par rapport à $\mathbb{P}|_{\mathcal{F}_s}$. Alors sous $\tilde{\mathbb{P}}^N$ le processus H^N suit la dynamique spécifiée dans la Proposition 0.2.1 pour $\theta, \gamma > 0$.*

Ensuite, nous allons analyser le comportement des densités de Girsanov quand $N \to \infty$. Pour cela on utilise les martingales $M^{1,N}$ et $M^{2,N}$ définis dans (0.2.10), et note que

$$X_s^N = \int_0^s \left\{\theta dM_r^{1,N} + \frac{\gamma\sigma^2 L_r^N(H_r^N)}{2}dM_r^{2,N}\right\}.$$

Il est facile de vérifier que deux suites $\{M^{1,N}\}_{N\geq 1}$ et $\{M^{2,N}\}_{N\geq 1}$ sont "*good*". À partir du Théorème 0.2.2 on obtient

$$X^N_\cdot \Rightarrow \int_0^\cdot \{\frac{\sqrt{2}\theta}{\sigma} dB^1_r + \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2} dB^2_r\} := X_\cdot.$$

De plus, $\{X^N_s\}_{N\geq 1}$ est "*good*" aussi, alors

$$Y^N = \mathcal{E}(X^N) \Rightarrow \mathcal{E}(X) =: Y.$$

En combinant ces faits avec le Corollaire 0.2.4, on déduit que

$$(H^N, L^N_{S^N_x}, Y^N) \Rightarrow (H, L_{S_x}, Y).$$

Parce que B^1 et B^2 sont orthogonales,

$$\begin{aligned} Y_s &= \mathcal{E}\Big(\int_0^\cdot \{\frac{\sqrt{2}\theta}{\sigma} dB^1_r + \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2} dB^2_r\}\Big)_s \\ &= \mathcal{E}\Big(\frac{\sqrt{2}\theta}{\sigma} B^1\Big)_s \, \mathcal{E}\Big(\int_0^\cdot \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2} dB^2_r\Big)_s \\ &= \exp\Big\{\frac{\sqrt{2}\theta}{\sigma} B^1_s + \int_0^s \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2} dB^2_r - \int_0^s \big[\frac{\theta^2}{\sigma^2} + \frac{\gamma^2\sigma^2}{4} L_r(H_r)^2\big] dr\Big\} \end{aligned}$$

On a les résultats suivants

Proposition 0.2.6. *Y est une $(\mathcal{F}, \mathbb{P})$-martingale. Soit $\tilde{\mathbb{P}}$ la mesure de probabilité sur $\mathcal{F}$ dont la restriction à $\mathcal{F}_s$, $s > 0$, a la densité Y_s par rapport à $\mathbb{P}|_{\mathcal{F}_s}$. Alors sous $\tilde{\mathbb{P}}$ le processus H est la solution de l'EDS* (0.2.2) *avec B_s est remplacé par*

$$\tilde{B}_s := \frac{1}{\sqrt{2}}(B^1_s - B^2_s) - \frac{\theta}{\sigma} s + \frac{\gamma\sigma}{2} \int_0^s L_r(H_r) dr,$$

qui est un mouvement brownien standard sous $\tilde{\mathbb{P}}$.

Théorème 0.2.3. *Soit H^N le processus suivant la dynamique spécifiée dans la Proposition 0.2.1, et soit H l'unique solution faible de l'EDS* (0.2.2)*. On a*

$$(H^N, L^N_{S^N_x}) \Rightarrow (H, L_{S_x}) \quad in \quad C([0,\infty]) \times D([0,\infty]), \tag{0.2.11}$$

où S^N_x et S_x sont définis dans (0.2.7) *et* (0.2.3).

On peut maintenant démontrer le Théorème 0.2.1.

Preuve du Théorème 0.2.1 : On définit $Z_t^{N,x} := \frac{\sigma^2}{4} L_{S_x^N}^N(t)$. D'après le Corollaire 0.2.2, $Z^{N,x}$ suit la dynamique (0.2.4). À partir de (0.2.11), $\frac{\sigma^2}{4} L_{S_x}$ est la limite en loi de $Z^{N,x}$ quand $N \to \infty$. D'après la Proposition 0.2.3, $t \mapsto \frac{\sigma^2}{4} L_{S_x}(t)$ est une solution faible de l'EDS (0.2.1), qui complète la preuve du Théorème 0.2.1.

□

0.3 La hauteur et la masse totale de la forêt d'arbres généalogiques d'une grande population avec compétition

Considérons un processus de branchement en temps continu, qui est à valeurs soit dans $\mathbb{Z}_+$ soit dans $\mathbb{R}_+$ (dans ce dernier cas on parle de processus de branchement à espace d'état continu, et dans ce cas nous nous limitons aux diffusions de Feller). De tels processus peuvent être utilisés comme des modèles d'évolution de population. Ici, nous voulons modéliser les interactions entre les individus (par exemple compétition pour des ressources limitées) qui détruit la propriété de branchement. Cettes interactions peuvent augmenter le nombre de naissances, ou augmenter le nombre de décès. Dans le chapitre 3, on définit cette interaction par une fonction f satisfaisant l'hypothèse suivante.

Hypothèse (H1): $f \in C(\mathbb{R}_+, \mathbb{R})$, $f(0) = 0$, et il existe $\theta \geq 0$ tel que

$$f(x+y) - f(y) \leq \theta x \qquad \forall x, y \geq 0.$$

Nous donnons des conditions explicites sur f qui entraînent que le temps d'extinction (qui est aussi la hauteur de la forêt d'arbres généalogiques) reste ou non borné quand la taille initiale de la population tend vers l'infini, et nous discutons la même question en ce qui concerne la masse totale de la forêt d'arbres généalogiques.

0.3.1 Le modèle de population discret avec interaction

On considère un processus de population en temps continu $\{X_t^m, t \geq 0\}$ à valeurs dans $\mathbb{Z}_+$ partant à l'instant zéro de $X_0^m = m \geq 1$, c'est-à-dire, m est le nombre d'ancêtres de la population. $\{X_t^m, t \geq 0\}$ est un processus de Markov en temps continu à valeurs dans $\mathbb{Z}_+$, qui évolue de la manière suivante. Si $X_t^m = 0$, alors $X_s^m = 0$ pour tout $s \geq t$. Si à un instant t, $X_t^m = k \geq 1$

$$X_t^m \quad \text{saute de } k \text{ à} \begin{cases} k+1, & \text{au taux} \quad \lambda k + F^+(k) \\ k-1, & \text{au taux} \quad \mu k + F^-(k), \end{cases}$$

où f est une fonction satisfaisant (H1), λ, μ sont des constantes positives, et

$$F^+(k) := \sum_{\ell=1}^{k} (f(\ell) - f(\ell-1))^+, \qquad F^-(k) := \sum_{\ell=1}^{k} (f(\ell) - f(\ell-1))^-.$$

On décrit maintenant une évolution conjointe de tous $\{X_t^m, t \geq 0\}_{m \geq 1}$, ou en d'autres termes du processus à deux paramètres $\{X_t^m, t \geq 0, m \geq 1\}$. Supposons que les m ancêtres sont arrangés de gauche à droite. Cet ordre est transmis à leurs descendants: Les descendants d'un individu sont placés à sa droite et si à un instant t donné l'individu i est placé à droite de l'individu j, alors toute la descendance de l'individu i à partir de cet instant t sera placée à droite de toute la descendance de l'individu j. Cela veut dire que la forêt d'arbres généalogiques de cette population est une forêt d'arbres planaires où l'arbre issu du premier ancêtre (dans l'ordre gauche-droite) est placé tout à fait à gauche, l'arbre du deuxième ancêtre est placé à droite du premier et ainsi de suite.

On décrète que à tout instant t, l'individu i dans la population donne naissance par interaction au taux $(f(\mathcal{L}_i(t)+1) - f(\mathcal{L}_i(t)))^+$ et meurt à cause de la compétition au taux $(f(\mathcal{L}_i(t)+1) - f(\mathcal{L}_i(t)))^-$, où $\mathcal{L}_i(t)$ désigne le nombre d'individus situés à gauche de l'individu i à l'instant t. Cela signifie que l'individu i est sous attaque par les autres situés à sa gauche si $f(\mathcal{L}_i(t)+1) - f(\mathcal{L}_i(t)) < 0$ tandis que l'interaction améliore sa fertilité si $f(\mathcal{L}_i(t)+1) - f(\mathcal{L}_i(t)) > 0$.

On définit la hauteur et la longueur de la forêt des arbres généalogiques par

$$H^m = \inf\{t > 0, X_t^m = 0\}, \qquad L^m = \int_0^{H^m} X_t^m dt, \qquad \text{pour} \quad m \geq 1.$$

Notons que notre couplage des $\{X^m, m \geq 1\}$ entraîne que H^m et L^m sont p.s. croissantes comme fonctions de m. On veut étudier les limites de H^m et L^m quand $m \to \infty$.

0.3.2 Modèle avec interaction dans le cas continu

On fait une renormalisation adéquate du modèle discret ci-dessus. Si on choisit $m = [Nx]$, remplace λ par $\lambda_N = 2N$, μ par $\mu_N = 2N$, f par $f_N(x) = Nf(x/N)$, et définit le processus $Z_t^N = N^{-1}X_t^N$, il est montré dans [4] que Z^N converge en loi vers l'unique solution de l'EDS (voir Dawson, Li [5])

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + 2\int_0^t \int_0^{Z_s^x} W(ds, du),$$

où W est un bruit blanc sur $\mathbb{R}_+ \times \mathbb{R}_+$. Cette EDS couple l'évolution de tous $\{Z_t^x,\ t \geq 0\}$ conjointement pour toutes les valeurs de $x > 0$.

À partir de [4], $\{Z_\cdot^x, x \geq 0\}$ est un processus de Markov à valeurs dans $C(\mathbb{R}_+, \mathbb{R}_+)$ (l'espace des fonctions continues de $\mathbb{R}_+$ à valeurs dans $\mathbb{R}_+$) partant de 0 à $x = 0$. De plus, on a que pour $0 < x \leq y, Z_t^y \geq Z_t^x$ pour tout $t \geq 0$ p.s. Pour $x > 0$, on définit T^x le temps d'extinction du processus Z^x (il est aussi appelé la hauteur du processus Z^x) par

$$T^x = \inf\{t > 0, Z_t^x = 0\}.$$

Et définit S^x la masse totale de Z^x par

$$S^x = \int_0^{T^x} Z_t^x dt.$$

Notons que notre couplage des $\{Z^x, x > 0\}$ entraîne que T^x et S^x sont p.s. croissants. On veut étudier les limites de T^x et S^x quand $x \to \infty$.

0.3.3 Les résultats principaux

Les résultats principaux du chapitre 3 sont suivants.

Théorème 0.3.1. *Supposons que f est une fonction satisfaisant (H1) et il existe $a_0 > 0$ tel que $f(x) \neq 0$ pour tout $x \geq a_0$. On a*

1) *Si $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx = \infty$, alors*

$$\sup_{m>0} H^m = \infty \quad p.s., \qquad \sup_{x>0} T^x = \infty \quad p.s.$$

2) *Si $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx < \infty$, alors*

$$\sup_{m>0} H^m < \infty \quad p.s., \qquad \sup_{x>0} T^x < \infty \quad p.s.$$

De plus, il existe une constante positive c tel que

$$\sup_{m>0} \mathbb{E}\big(e^{cH^m}\big) < \infty, \qquad \sup_{x>0} \mathbb{E}\big(e^{cT^x}\big) < \infty.$$

Théorème 0.3.2. *Supposons que la fonction $\frac{f(x)}{x}$ satisfait (H1) et il existe $a_0 > 0$ tel que $f(x) \neq 0$ pour tout $x \geq a_0$. On a*

1) *Si $\int_{a_0}^{\infty} \frac{x}{|f(x)|} dx = \infty$, alors*

$$\sup_{m>0} L^m = \infty \quad p.s., \qquad \sup_{x>0} S^x = \infty \quad p.s.$$

2) *Si $\int_{a_0}^{\infty} \frac{x}{|f(x)|} dx < \infty$, alors*

$$\sup_{m>0} L^m < \infty \quad p.s., \qquad \sup_{x>0} S^x < \infty \quad p.s.$$

De plus, il existe une constante positive c tel que

$$\sup_{m>0} \mathbb{E}\big(e^{cL^m}\big) < \infty, \qquad \sup_{x>0} \mathbb{E}\big(e^{cS^x}\big) < \infty.$$

0.4 *Temps d'extinction et la masse totale du processus de branchement à espace d'état continu avec compétition*

Dans le chapitre 4, nous généralisons les résultats du chapitre 3 obtenus dans le cas de la diffusion de Feller avec compétition pour le cas du processus de branchement à espace d'état continu (CSBP) avec compétition (dont les trajectoires sont discontinues). Plus précisément, supposons que $\sigma \geq 0$ est une constante, et $(r \wedge r^2)m(dr)$ est une mesure finie sur $(0, \infty)$. Soit ψ la fonction donnée par

$$\psi(\lambda) = \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r)m(dr), \qquad \lambda \geq 0.$$

Soit $W(ds, du)$ le bruit blanc sur $(0, \infty)^2$ muni de la mesure de Lebesgue $dsdu$, et soit $N(ds, dz, du)$ la mesure aléatoire de Poisson sur $(0, \infty)^3$ d'intensité $dsm(dz)du$. Soit $\tilde{N}(ds, dz, du)$ la mesure compensée de $N(ds, dz, du)$. Nous allons considérer le CSBP avec compétition caractérisé par le mécanisme de branchement ψ, qui est gouverné par l'EDS

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + \sigma \int_0^t \int_0^{Z_s^x} W(ds, du) + \int_0^t \int_0^\infty \int_0^{Z_{s-}^x} z\tilde{N}(ds, dz, du), \tag{0.4.1}$$

où la fonction f satisfait (H1). D'après Dawson, Li [5], l'équation (0.4.1) a une unique solution forte. Cette EDS couple l'évolution des $\{Z_t^x,\ t \geq 0\}$ conjointement pour toutes les valeurs de $x > 0$.

Pour $x > 0$, on définit T^x le temps d'extinction du processus Z^x par

$$T^x = \inf\{t > 0, Z_t^x = 0\}.$$

Et définit S^x la masse totale de Z^x par

$$S^x = \int_0^{T^x} Z_t^x dt.$$

Notons que notre couplage des $\{Z^x, x > 0\}$ entraîne que T^x et S^x sont p.s. croissants. On étudie ensuite les limites de T^x et S^x quand $x \to \infty$.

0.4.1 Temps d'extinction du CSBP avec compétition

On montre que

Théorème 0.4.1. *Supposons que f est une fonction vérifiant (H1) tel que $\lim_{y\to 0^+} \frac{f(y)}{y} > -\infty$ et $\int^{\infty} d\lambda/\psi(\lambda) = \infty$. On a pour tout $x > 0$, $T^x = \infty$ p.s.*

Théorème 0.4.2. *Supposons que f est une fonction vérifiant (H1) et il existe $a_0 > 0$ tel que $f(y) \neq 0$ pour tout $y \geq a_0$. Si*

$$\int^{\infty} d\lambda/\psi(\lambda) < \infty, \qquad \int^{\infty} \frac{1}{| f(y) |} dy < \infty,$$

alors on a

$$\sup_{x>0} \mathbb{E}(T^x) < \infty.$$

0.4.2 La masse totale du CSBP avec compétition

Dans cette sous-section, nous supposerons que

Hypothèse (H2): f est une fonction satisfaisant (H1) tel que

$$\lim_{u\to 0^+} \frac{f(u)}{u} = \alpha,$$

pour un certain $-\infty < \alpha \leq \theta$, et la fonction $f_1(u) := \frac{f(u)}{u} - \alpha$ satisfait (H1).

On a le

Théorème 0.4.3. *Supposons que f est une fonction vérifiant $(H2)$ et il existe $a_0 > 0$ tel que $f(u) \neq 0$ pour tout $u \geq a_0$. Si $\int_{a_0}^{\infty} \frac{u}{|f(u)|} du = \infty$, alors*

$$S^x \to \infty \quad p.s. \quad quand \quad x \to \infty.$$

Nous considérons maintenant le cas $\int_{a_0}^{\infty} \frac{u}{|f(u)|} du < \infty$. Nous allons voir que dans ce cas $\sup_{x>0} S^x < \infty$ p.s. En fait, nous pouvons montrer qu'il a des moments finis.

Soit γ la constante tel que $f_2(u) := \gamma u - f_1(u)$ est une fonction positive et croissante (on peut choisir par exemple $\gamma > \theta$, par Hypothèse (H2)). Il est facile de voir que

$$\int_{a_0}^{\infty} \frac{1}{f_2(u)} du < \infty.$$

Soit $g(y) := \int_y^{\infty} \frac{1}{f_2(u)} du$ pour $y \geq a_0$. Alors g est décroissante et $g(y) \to 0$ quand $y \to \infty$. Dans la suite, nous ferons l'hypothèse suivante:

Hypothèse (H3): La fonction f_2 est C^1 sur (a_0, ∞) et il existe des constantes $d > 0, c > a_0$ tel que

$$g(y) f_2^{'}(y) \geq 1 + d \quad \text{pour tout} \quad y \geq c.$$

Définit la fonction $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ comme suit.

$$h(y) \quad = \quad \begin{cases} \frac{1}{g(c)^d}, & 0 \leq y \leq c \\ \frac{1}{g(y)^d}, & y > c. \end{cases}$$

Alors h est croissante et est C^2 sur (c, ∞), $h(y) \to \infty$ quand $y \to \infty$, et

$$h^{''}(y) = \frac{-d[g(y) f_2^{'}(y) - d - 1]}{f_2(y)^2 g(y)^{d+2}} \leq 0 \quad \text{pour tout} \quad y > c.$$

On a

Théorème 0.4.4. *Supposons qu'il existe $a_0 > 0$ tel que $f(u) \neq 0$ pour tout $u \geq a_0$ et que $(H2), (H3)$ sont valides. Si*

$$\int_{a_0}^{\infty} \frac{u}{\mid f(u) \mid} du < \infty \quad \textit{et} \quad \lim_{\lambda \to \infty} \frac{\psi(\lambda)}{\lambda} = \infty,$$

alors

$$\mathbb{E}\big(h(\sup_{x>0} S^x)\big) < \infty.$$

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Chapter 1

COALESCENCE TIMES FOR THE BIENAYMÉ-GALTON-WATSON PROCESS

1.1 Introduction

Random trees are mathematical objects that play an important role in many areas of mathematics and other sciences. One of the most celebrated random trees is the Bienaymé- Galton-Watson (BGW) tree, where the offspring of each vertex of the tree are independent and indentically distributed (i.i.d) random integers. BGW tree plays a fundamental role in both the theory and applications of stochastic processes. For more details, see e.g. [1, 12].

One interesting and important approach to random trees is coalescence. In [7], Lambert has investigated the distribution of coalescence time for two individuals picked at random (uniformly) in the current generation of a BGW process in the discrete setting. The purpose of this note is to extend these results of Lambert to the case of continuous time BGW process. The basic idea is the same as used in Lambert [7], but we need some other techniques. We start a continuous time BGW process from a number x of individuals at time 0. Its law is denoted by $\mathbb{P}_x$ and $\mathbb{P}_x^{(t)}$ indicates that the current time is time t. If the current time contains at least two individuals, we pick uniformly within it two individuals, without replacement. We then compute the distribution of their coalescence time T (if the current time contains less than two individuals, T is set to ∞). In the subcritical case, the law P^{qs} denoting the limit of the distributions $\mathbb{P}_x^{(t)}(\cdot \mid T < \infty)$ as $t \to \infty$ does not depend on x and is called the quasi-stationary distribution. In section 1.3, we specify the law of T under P^{qs}. In section 1.4, we extend our results to multivariate coalescence when n individuals are sampled at the current time.

In this chapter, the Lambert's results are not recalled. The reader should read again [7] to compare the results in the discrete and continuous time cases. We also refer the reader to several interesting closely related results [5, 9, 10, 13, 14].

1.2 Distribution of the coalescence time

Let $\mathbb{N}$ be the set of all natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$. We consider a continuous time $\mathbb{N}$-valued branching process $Z = \{Z_t, t \geq 0\}$, where t denotes time. Such a process is a Bienaymé-Galton-Watson process in which to each individual is attached a random vector describing its lifetime and its numbers of offspring. We assume that those random vectors are i.i.d.. The rate of reproduction is governed by a finite measure μ on $\mathbb{N}$, satisfying $\mu(1) = 0$. More precisely, each individual lives for an exponential time with parameter $\mu(\mathbb{N})$, and is replaced by a random number of children according to the probability $\mu(\mathbb{N})^{-1}\mu$. Hence the dynamics of the continuous time Markov process Z is entirely characterized by the measure μ. For $x \in \mathbb{N}$, denote by $\mathbb{P}_x$ the law of Z when $Z_0 = x$. We have the following proposition, which can be seen in [1], chapter III (page 106).

Proposition 1.2.1. *The generating function of the process Z is given by*

$$\mathbb{E}_x(s^{Z_t}) = \psi_t(s)^x, \qquad s \in [0, 1], x \in \mathbb{N},$$

where

$$\frac{\partial \psi_t(s)}{\partial t} = \Phi(\psi_t(s)), \qquad \psi_0(s) = s,$$

and the function Φ is defined by

$$\Phi(s) = \sum_{n=0}^{\infty} (s^n - s)\mu(n), \quad s \in [0, 1].$$

The continuous time BGW process Z is called immortal if $\mu(0) = 0$. In this chapter, we always assume that $\mu(0) > 0$. Let $\eta := \inf\{u > 0 : \Phi(u) = 0\}$. Since

$\Phi(0) = \mu(0) > 0$, then we have $\eta > 0$. Put

$$F(t) := \int_0^t \frac{du}{\Phi(u)}, \qquad t < \eta.$$

Then the mapping $F : (0, \eta) \to (0, \infty)$ is bijective. We call φ to be its inverse mapping. Moreover, $t \mapsto \psi_t(s)$ is the unique nonnegative solution of the integral equation

$$v(t) - \int_0^t \Phi(v(u))du = s, \qquad s \in [0,1], t \geq 0,$$

so that

$$\int_s^{\psi_t(s)} \frac{dv}{\Phi(v)} = t, \qquad s \in [0,1], s < \eta, t \geq 0.$$

Hence

$$\psi_t(s) = \varphi(t + F(s)), \qquad s \in [0,1], s < \eta, t \geq 0.$$

Note that the branching property implies that $\psi_{t_1+t_2} = \psi_{t_1} \circ \psi_{t_2}$.
Now, assume that the current generation is generation $t, t > 0$. We consider two individuals σ_1, σ_2 at the present time, and ask when they coalesce, that is, how much time has elapsed since their common ancestor. In a more rigorous way, for $0 < u \leq t$, denote by $\tau_u(\sigma_i)$ the (unique) parent of σ_i at time $(t-u), i = 1, 2$. The coalescence time $T(\sigma_1, \sigma_2)$ of σ_1, σ_2 is uniquely determined by

$$T(\sigma_1, \sigma_2) := \inf\{u : 0 < u \leq t, \tau_u(\sigma_1) = \tau_u(\sigma_2)\},$$

with the convention $\inf \emptyset = \infty$. We denote by T the coalescence time of two individuals picked at random (uniformly) among the individuals which present in the current generation. If the current generation contains less than two individuals, T is set to ∞.

With the notation $\mathbb{P}^{(t)}$ indicates that t is the current time, the distribution of T is given in the following statement.

Theorem 1.2.1. *For any* $0 < t_1 \leq t_2 \leq t, y \geq 1, y \in \mathbb{N}$,

$$\mathbb{E}^{(t)}(Z_t(Z_t - 1)s^{Z_t - 2}, T \leq t_1 \mid Z_{t-t_2} = y) = y\psi'_{t_2}(s)\psi_{t_2}(s)^{y-1}\frac{\psi''_{t_1}(s)}{\psi'_{t_1}(s)}, \qquad s \in [0,1).$$

The previous p.g.f can be inverted as follows, for any $p \geq 2$

$$\mathbb{P}^{(t)}(Z_t = p, T \in dt_1 \mid Z_{t-t_2} = y)/dt_1 =$$
$$y \sum_{n \geq 2} n\mu(n) \mathbb{E}\Big(\frac{Z_{t_2}^{(1)}(1) Z_{t_1}^{(2)}(n-1)}{p(p-1)}, Z_{t_2}^{(0)}(y-1) + Z_{t_2}^{(1)}(1) + Z_{t_1}^{(2)}(n-1) = p \Big),$$

where $Z^{(0)}, Z^{(1)}, Z^{(2)}$ are i.i.d branching processes distributed as Z, and the notation $Z_{t_2}^{(0)}(y-1)$ denotes the value taken by $Z^{(0)}$ at time t_2 when started at $y-1$.

Remark 1.2.2. When $t_2 = t_1$, the above equation can be interpreted as follows. The amount p of population at time t is divided in three parts. An individual is marked at generation $t - t_1$ (y possible choices), which is the candidate for the common ancestor of two random individuals of generation t on $\{T \in dt_1\}$. The first part is the descendance at the current time of the $y-1$ remaining individuals. On $\{T \in dt_1\}$ the marked individual must be replaced immediately by n offspring, $n \geq 2$. Then an individual is marked among the n possible offspring of the previously marked ancestor. The descendance of this individual is the second part, and the descendance of the $n-1$ remaining others is the third part. On $\{T \in dt_1\}$, one of the two individuals sampled must be in the second part, and the other in the third part.

Proof. To get the first equation, we use the same argument used in the proof of Theorem 1 in [7]. The second equation of the theorem is equivalent to

$$\mathbb{E}^{(t)}(Z_t(Z_t-1)s^{Z_t-2}, T \in dt_1 \mid Z_{t-t_2} = y)/dt_1 =$$

$$y \sum_{n \geq 2} n\mu(n) \mathbb{E}\Big(Z_{t_2}^{(1)}(1) Z_{t_1}^{(2)}(n-1) s^{Z_{t_2}^{(0)}(y-1) + Z_{t_2}^{(1)}(1) + Z_{t_1}^{(2)}(n-1) - 2} \Big) \quad \forall s \in (0,1). \quad (1.2.1)$$

Using the first result of the theorem, the left-hand side of (1.2.1) equals

$$\mathbb{E}^{(t)}(Z_t(Z_t-1)s^{Z_t-2}, T \in dt_1 \mid Z_{t-t_2} = y)/dt_1 = y\psi_{t_2}'(s)\psi_{t_2}(s)^{y-1} \frac{\partial}{\partial t_1}\left(\frac{\psi_{t_1}''(s)}{\psi_{t_1}'(s)} \right).$$

From the Proposition 1.2.1 we have

$$\frac{\partial \psi_{t_1}(s)}{\partial t_1} = \Phi(\psi_{t_1}(s))$$
$$\frac{\partial \psi'_{t_1}(s)}{\partial t_1} = \Phi'(\psi_{t_1}(s))\psi'_{t_1}(s)$$
$$\frac{\partial \psi''_{t_1}(s)}{\partial t_1} = \Phi''(\psi_{t_1}(s))\psi'_{t_1}(s)^2 + \Phi'(\psi_{t_1}(s))\psi''_{t_1}(s),$$

so that

$$\frac{\partial}{\partial t_1}\left(\frac{\psi''_{t_1}(s)}{\psi'_{t_1}(s)}\right) = \frac{\psi'_{t_1}(s)\frac{\partial \psi''_{t_1}(s)}{\partial t_1} - \psi''_{t_1}(s)\frac{\partial \psi'_{t_1}(s)}{\partial t_1}}{\psi'_{t_1}(s)^2} = \Phi''(\psi_{t_1}(s))\psi'_{t_1}(s).$$

Then

$$\begin{aligned}\mathbb{E}^{(t)}(Z_t(Z_t-1)s^{Z_t-2}, T \in dt_1 \mid Z_{t-t_2} = y)/dt_1 \\ &= y\psi'_{t_2}(s)\psi_{t_2}(s)^{y-1}\Phi''(\psi_{t_1}(s))\psi'_{t_1}(s) \\ &= y\psi'_{t_2}(s)\psi_{t_2}(s)^{y-1}\psi'_{t_1}(s)\sum_{n\geq 2} n(n-1)\mu(n)\psi_{t_1}(s)^{n-2}.\end{aligned}$$

Finally, the right-hand side of (1.2.1) equals

$$\begin{aligned}&y\sum_{n\geq 2} n\mu(n)\mathbb{E}(s^{Z^{(0)}_{t_2}(y-1)})\mathbb{E}(Z^{(1)}_{t_2}(1)s^{Z^{(1)}_{t_2}(1)-1})\mathbb{E}(Z^{(2)}_{t_1}(n-1)s^{Z^{(2)}_{t_1}(n-1)-1}) \\ &= y\sum_{n\geq 2} n\mu(n)\mathbb{E}_{y-1}(s^{Z_{t_2}})\mathbb{E}_1(Z_{t_2}s^{Z_{t_2}-1})\mathbb{E}_{n-1}(Z_{t_1}s^{Z_{t_1}-1}) \\ &= y\sum_{n\geq 2} n\mu(n)\psi_{t_2}(s)^{y-1}\psi'_{t_2}(s)(n-1)\psi_{t_1}(s)^{n-2}\psi'_{t_1}(s),\end{aligned}$$

which ends the proof. ∎

Corollary 1.2.3. *For any* $0 < t_1 \leq t$*,*

$$\mathbb{P}^{(t)}_x(T \leq t_1) = x\int_0^1 ds(1-s)\frac{\psi''_{t_1}(s)}{\psi'_{t_1}(s)}\psi'_t(s)\psi_t(s)^{x-1}.$$

In particular,

$\mathbb{P}^{(t)}_x$ *(At least two extant individuals, a random pair has no common ancestor)* =

$$x(x-1)\int_0^1 ds(1-s)\psi'_t(s)^2\psi_t(s)^{x-2}.$$

Proof. See the proof of the corollary 1 in [7]. ∎

1.3 Quasi-stationary distribution

In this section, we consider the limiting distribution of the coalescence time when the process is conditioned on $\{Z_t \geq 2\}$ and $t \to \infty$. Informally, this limit embodies the situation where the genealogy was founded a long time ago and is still not extinct, with at least two descendants at the present time. We will need some results on quasi-stationary distributions for the continuous time BGW process, which can be found in [1, 4, 16]. The reader may see more general results on quasi-stationary distributions, which have been obtained for continuous time Markov chains by [15] and for semi-Markov processes by [3]. We also refer the reader to [2, 8, 11] for the results on quasi-stationary distributions for population processes.

We consider the case $\psi_1'(1) = \mathbb{E}_1(Z_1) < 1$ (subcritical case) when $\mathbb{E}_1(Z_1 \log(Z_1)) < \infty$. According to Theorem 6 in [16], there is a nonnegative sequence $(\alpha_k, k \geq 1)$ summing to 1 such that

$$\lim_{t\to\infty} \mathbb{P}_x(Z_t = j \mid Z_t > 0) = \alpha_j, \qquad \forall x \in \mathbb{N}, j \geq 1. \tag{1.3.1}$$

The sequence $(\alpha_k, k \geq 1)$ is called the Yaglom limit of the process Z. If we define

$$g(s) = \sum_{k\geq 1} \alpha_k s^k, \qquad s \in [0,1],$$

then (1.3.1) deduces

$$g(s) = \lim_{t\to\infty} \mathbb{E}_x(s^{Z_t} \mid Z_t > 0) = \lim_{t\to\infty} \frac{\psi_t(s) - \psi_t(0)}{1 - \psi_t(0)}, \qquad s \in [0,1].$$

We have the result:

Proposition 1.3.1. *In the subcritical case when $\mathbb{E}_1(Z_1 \log(Z_1)) < \infty$, we have for any $s \in [0,1]$,*

$$\lim_{t\to\infty} \mathbb{E}_x(Z_t s^{Z_t - 1} \mid Z_t > 0) = g'(s) \leq g'(1) < \infty. \tag{1.3.2}$$

The proof of Proposition 1.3.1 can be found in [1], chapter IV (page 170). Under more restrictive hypothesis that $\mathbb{E}_1(Z_1^2) < \infty$, we can give a very elementary and interesting proof of (1.3.2), which is provided by two following lemmas.

Lemma 1.3.2. *For $t \geq 0$, let $\epsilon_t(s)$ be the function defined by*

$$\frac{1-\psi_t(s)}{1-s} = \psi_t^{'}(1) - \epsilon_t(s), \qquad s \in [0,1). \tag{1.3.3}$$

Then $\epsilon_t(s)$ is monotone decreasing, tend to zero when s tend to one.

Proof. It follows from the fact that, for each t, $\psi_t(s)$ is increasing, convex, and $\psi_t(1) = 1$. □

The equality (1.3.3) is equivalent to

$$\frac{1-\psi_t(s)}{(1-s)\psi_t^{'}(1)} = 1 - \frac{\epsilon_t(s)}{\psi_t^{'}(1)}. \tag{1.3.4}$$

Replacing s by $\psi_h(s)$ in (1.3.4) we obtain

$$\frac{1-\psi_t(\psi_h(s))}{(1-\psi_h(s))\psi_t^{'}(1)} = 1 - \frac{\epsilon_t(\psi_h(s))}{\psi_t^{'}(1)} \leq 1, \qquad t, h > 0.$$

Note that $\psi_{t+h}(s) = \psi_t(\psi_h(s))$, and $\psi_{t+h}^{'}(1) = \psi_t^{'}(1)\psi_h^{'}(1)$, then

$$\frac{1-\psi_{t+h}(s)}{(1-s)\psi_{t+h}^{'}(1)} = \frac{1-\psi_t(\psi_h(s))}{(1-\psi_h(s))\psi_t^{'}(1)} \frac{1-\psi_h(s)}{(1-s)\psi_h^{'}(1)} \leq \frac{1-\psi_h(s)}{(1-s)\psi_h^{'}(1)}, \qquad t, h > 0.$$

This implies that the sequence $(1-\psi_t(s))/((1-s)\psi_t^{'}(1))$ is monotone decreasing in t and thus converges to a function $\chi(s)$. Letting $s = 0$ we have

$$\chi(0) = \lim_{t\to\infty} \frac{\mathbb{P}_1(Z_t > 0)}{\psi_t^{'}(1)} \geq 0.$$

Lemma 1.3.3. *$\chi(0)$ is positive and for all $x \in \mathbb{N}$*

$$\lim_{t\to\infty} \mathbb{E}_x(Z_t \mid Z_t > 0) = g^{'}(1) = \frac{1}{\chi(0)}.$$

Proof. We will follow the proof idea of Joffe as given in [6]. Note that

$$\chi(0) = \lim_{t\to\infty} \frac{1-\psi_t(0)}{\psi_t^{'}(1)} = \lim_{n\to\infty, n\in\mathbb{N}} \frac{1-\psi_n(0)}{\psi_n^{'}(1)}$$
$$= \lim_{n\to\infty} \prod_{k=0}^{n-1} \big[1 - \frac{\epsilon_1(\psi_k(0))}{\psi_1^{'}(1)}\big].$$

Hence it follows that $\chi(0) > 0$ if and only if the series $\sum_{k=0}^{\infty} \epsilon_1(\psi_k(0))$ converges. Since $\epsilon_t(s) \geq 0$ we get

$$\frac{1-\psi_t(s)}{1-s} \leq \psi_t^{'}(1), \qquad t \geq 0, s \in [0,1).$$

Letting $s = 0$ we obtain

$$\psi_t(0) \geq 1 - \psi_t^{'}(1), \qquad t \geq 0$$
$$\epsilon_1(\psi_k(0)) \leq \epsilon_1(1 - \psi_k^{'}(1)), \qquad k \geq 0. \tag{1.3.5}$$

On the other hand, $\mathbb{E}_1(Z_1^2) < \infty$ implies that $\psi_1^{''}(1) < \infty$, then there exists a constant $C > 0$ such that

$$\epsilon_1(s) < C(1-s), \qquad s \in [0,1). \tag{1.3.6}$$

From (1.3.5) and (1.3.6) we deduce that the series $\sum_{k=0}^{\infty} \epsilon_1(\psi_k(0))$ converges, so that $\chi(0) > 0$. This implies that $\psi_t(0) \to 1$ as $t \to \infty$. Therefore

$$\begin{aligned}
g(\psi_t(0)) &= \lim_{h\to\infty} \frac{\psi_{t+h}(0) - \psi_h(0)}{1 - \psi_h(0)} \\
&= \lim_{h\to\infty} \frac{-(1-\psi_{t+h}(0)) + (1-\psi_h(0))}{1-\psi_h(0)} \\
&= \frac{-\psi_{t+h}^{'}(1) + \psi_h^{'}(1)}{\psi_h^{'}(1)} \\
&= -\psi_t^{'}(1) + 1.
\end{aligned}$$

Thus

$$g^{'}(1) = \lim_{t\to\infty} \frac{g(\psi_t(0)) - 1}{\psi_t(0) - 1} = \lim_{t\to\infty} \frac{-\psi_t^{'}(1)}{\psi_t(0) - 1} = \frac{1}{\chi(0)}.$$

□

Denote by $\tilde{Z}$ the limiting value of Z_t conditioned on $\{Z_t \geq 2\}$ as $t \to \infty$. We have

Theorem 1.3.1. *In the subcritical case when* $\mathbb{E}_1(Z_1 \log(Z_1)) < \infty$, *the quasi-stationary distribution* $\mathbb{P}^{qs}$ *of* T *and* $\tilde{Z}$ *is defined by*

$$\mathbb{P}^{qs}(\tilde{Z} = p, T \in dh) = \lim_{t\to\infty} \mathbb{P}_x^{(t)}(Z_t = p, T \in dh \mid Z_t \geq 2), \qquad p \geq 2, h > 0.$$

Then $\mathbb{P}^{qs}$ *defines a probability distribution which does not depend on* x *and satisfies*

$$\mathbb{E}^{qs}(\tilde{Z}(\tilde{Z}-1)s^{\tilde{Z}-2}, T \le h) = \frac{g^{'}(s)}{1-g^{'}(0)}\frac{\psi_h^{''}(s)}{\psi_h^{'}(s)}.$$

In particular,

$$\mathbb{P}^{qs}(T \le h) = \frac{1}{1-g^{'}(0)}\int_0^1 ds(1-s)\frac{\psi_h^{''}(s)}{\psi_h^{'}(s)}g^{'}(s).$$

Proof. See the proof of Theorem 2 in [7]. □

1.4 Multivariate coalescence

Assume that the current generation contains at least $n+1$ individuals, $n \ge 1$. We will present the distribution of coalescence times, when $n+1$ individuals are sampled uniformly and independently at the current time t. For $k = 1, 2, ..., n$, we denote by T_k the coalescence time of the first individual and the $(k+1)$-th individual, and by T_k^* the k-th coalescence time. We have

Theorem 1.4.1. *For any* $0 < t_1 < t_2 < ... < t_n \le t$*, the joint distribution of coalescence times* T_k *is given by*

$$\mathbb{E}_x^{(t)}(Z_t(Z_t-1)...(Z_t-n)s^{Z_t-n-1}, T_1 \in dt_1, ..., T_n \in dt_n)/dt_1...dt_n =$$
$$x\psi_t^{'}(s)\psi_t(s)^{x-1}\prod_{i=1}^{n}\psi_{t_i}^{'}(s)\big[\sum_{k\ge 2}k(k-1)\mu(k)\psi_{t_i}(s)^{k-2}\big], \quad s \in [0,1).$$

Proof. We will prove this theorem by induction since the formula holds when $n = 1$ by Theorem 1.2.1. We first condition on $\{Z_{t-t_n} = y\}$. We apply the second formula of Theorem 1.2.1 to the last coalescence time T_n,

$$\mathbb{P}^{(t)}(Z_t = p, T_1 \in dt_1, ..., T_n \in dt_n \mid Z_{t-t_n} = y)/dt_n = y\sum_{k\ge 2}k\mu(k)\times$$
$$\mathbb{E}\Big(\frac{Z_{t_n}^{(1)}(1)Z_{t_n}^{(2)}(k-1)..(Z_{t_n}^{(2)}(k-1)-n+1)}{p(p-1)...(p-n)}, Z_{t_n}^{(0)}(y-1)+Z_{t_n}^{(1)}(1)+Z_{t_n}^{(2)}(k-1) = p,$$
$$T_i \in dt_i, i \le n-1\Big),$$

where the interpretation is as for $n = 1$ (see Remark 1.2.2): y corresponds to the choice of the common ancestor of all individuals in generation $t - t_n$, k is the number of offspring this ancestor had instantaneously at time $t - T_n$ and corresponds to the choice of the ancestor of the last individual within this offspring. The n remaining individuals have to be found in the descendance of the $k-1$ remaining offspring. Then

$$\mathbb{E}^{(t)}(Z_t(Z_t-1)...(Z_t-n)s^{Z_t-n-1}, T_1 \in dt_1, ..., T_n \in dt_n \mid Z_{t-t_n} = y)/dt_n = y\sum_{k\geq 2} k\mu(k)\times$$

$$\mathbb{E}\Big(Z_{t_n}^{(1)}(1)Z_{t_n}^{(2)}(k-1)...(Z_{t_n}^{(2)}(k-1)-n+1)s^{Z_{t_n}^{(0)}(y-1)+Z_{t_n}^{(1)}(1)+Z_{t_n}^{(2)}(k-1)-n-1}, T_i \in dt_i, i \leq n-1\Big)$$

$$= y\sum_{k\geq 2} k\mu(k)\mathbb{E}\big(s^{Z_{t_n}^{(0)}(y-1)}\big)\mathbb{E}\big(Z_{t_n}^{(1)}(1)s^{Z_{t_n}^{(1)}(1)-1}\big)\times$$

$$\mathbb{E}\big(Z_{t_n}^{(2)}(k-1)...(Z_{t_n}^{(2)}(k-1)-n+1)s^{Z_{t_n}^{(2)}(k-1)-n}, T_i \in dt_i, i \leq n-1\big)$$

$$= y\psi_{t_n}(s)^{y-1}\psi_{t_n}^{'}(s)\sum_{k\geq 2} k\mu(k)\times$$

$$\mathbb{E}\big(Z_{t_n}^{(2)}(k-1)...(Z_{t_n}^{(2)}(k-1)-n+1)s^{Z_{t_n}^{(2)}(k-1)-n}, T_i \in dt_i, i \leq n-1\big).$$

By the induction hypothesis, the last expression equals

$$y\psi_{t_n}(s)^{y-1}\psi_{t_n}^{'}(s)\sum_{k\geq 2} k\mu(k)\times$$

$$(k-1)\psi_{t_n}^{'}(s)\psi_{t_n}(s)^{k-2}\prod_{i=1}^{n-1}\psi_{t_i}^{'}(s)\big[\sum_{j\geq 2} j(j-1)\mu(j)\psi_{t_i}(s)^{j-2}\big]dt_1...dt_{n-1}$$

$$= y\psi_{t_n}(s)^{y-1}\psi_{t_n}^{'}(s)\prod_{i=1}^{n}\psi_{t_i}^{'}(s)\big[\sum_{k\geq 2} k(k-1)\mu(k)\psi_{t_i}(s)^{k-2}\big]dt_1...dt_{n-1}.$$

Hence the result follows by integrating w.r.t. to the distribution of Z_{t-t_n} conditional on $\{Z_0 = x\}$. □

Theorem 1.4.2. *For any $0 < t_1 < t_2 < ... < t_n \leq t$, the joint distribution of coalescence times T_k^* is given by*

$$\mathbb{E}_x^{(t)}(Z_t(Z_t-1)...(Z_t-n)s^{Z_t-n-1}, T_1^* \in dt_1, ..., T_n^* \in dt_n)/dt_1...dt_n =$$

$$\frac{n!(n+1)!}{2^n}x\psi_t^{'}(s)\psi_t(s)^{x-1}\prod_{i=1}^{n}\psi_{t_i}^{'}(s)\big[\sum_{k\geq 2} k(k-1)\mu(k)\psi_{t_i}(s)^{k-2}\big], \quad s \in [0,1).$$

Proof. The proof is similar to that of Theorem 1.4.1 above. We reason by induction since the formula holds when $n = 1$ by Theorem 1.2.1. We first condition on $\{Z_{t-t_n} = y\}$ and apply the second formula of Theorem 1.2.1 to the last coalescence time T_n^*,

$$\mathbb{P}^{(t)}(Z_t = p, T_1^* \in dt_1, ..., T_n^* \in dt_n \mid Z_{t-t_n} = y)/dt_n = \frac{1}{2} y \sum_{k\geq 2} k\mu(k) \sum_{i=1}^{n} \binom{n+1}{i} \sum_{1\leq j_1 <..< j_{i-1} \leq n-1}$$
$$\mathbb{E}\Big(\frac{Z_{t_n}^{(1)}(1)..(Z_{t_n}^{(1)}(1) - i + 1) Z_{t_n}^{(2)}(k-1)..(Z_{t_n}^{(2)}(k-1) - n + i)}{p(p-1)...(p-n)}, Z_{t_n}^{(0)}(y-1) + Z_{t_n}^{(1)}(1) + Z_{t_n}^{(2)}(k-1)$$
$$= p, T_h^*(i) \in dt_h \text{ for } h \in \{j_1, .., j_{i-1}\} \text{ and } T_h^*(n+1-i) \in dt_h \text{ for } h \notin \{j_1, .., j_{i-1}\}, h \leq n-1\Big),$$

where the interpretation is as follows: y corresponds to the choice of the common ancestor of all individuals in generation $t - t_n$, k is the number of offspring this ancestor had instantaneously at time $t - T_n^*$ and corresponds to the choice of the ancestor of the last i individuals within this offspring (there are $\binom{n+1}{i}$ possible choices for the last i individuals). The $n+1-i$ remaining individuals have to be found in the descendance of the $k-1$ remaining offspring. For $m = 1, .., i-1$, $T_{j_m}(i)$ is the m-th coalescence time of the last i individuals, and for $h \notin \{j_1, .., j_{i-1}\}, h \leq n-1$, $T_h^*(n+1-i)$ is a coalescence time of the $n+1-i$ remaining individuals. And we have to divide the expression by 2 because each sample has been counted twice. We then have

$$\mathbb{E}^{(t)}(Z_t(Z_t - 1)...(Z_t - n)s^{Z_t - n - 1}, T_1^* \in dt_1, ..., T_n^* \in dt_n \mid Z_{t-t_n} = y)/dt_n$$
$$= \frac{1}{2} y \sum_{k\geq 2} k\mu(k) \sum_{i=1}^{n} \binom{n+1}{i} \sum_{1\leq j_1 <..< j_{i-1} \leq n-1}$$
$$\mathbb{E}\Big(Z_{t_n}^{(1)}(1)..(Z_{t_n}^{(1)}(1) - i + 1) Z_{t_n}^{(2)}(k-1)..(Z_{t_n}^{(2)}(k-1) - n + i) s^{Z_{t_n}^{(0)}(y-1) + Z_{t_n}^{(1)}(1) + Z_{t_n}^{(2)}(k-1) - n - 1},$$
$$T_h^*(i) \in dt_h \text{ for } h \in \{j_1, .., j_{i-1}\} \text{ and } T_h^*(n+1-i) \in dt_h \text{ for } h \notin \{j_1, .., j_{i-1}\}, h \leq n-1\Big)$$
$$= \frac{1}{2} y \sum_{k\geq 2} k\mu(k) \sum_{i=1}^{n} \binom{n+1}{i} \sum_{1\leq j_1 <..< j_{i-1} \leq n-1} \mathbb{E}\big(s^{Z_{t_n}^{(0)}(y-1)}\big) \times$$
$$\mathbb{E}\big(Z_{t_n}^{(1)}(1)..(Z_{t_n}^{(1)}(1) - i + 1) s^{Z_{t_n}^{(1)}(1) - i}, T_h^*(i) \in dt_h \text{ for } h \in \{j_1, .., j_{i-1}\}\big) \times$$
$$\mathbb{E}\big(Z_{t_n}^{(2)}(k-1)..(Z_{t_n}^{(2)}(k-1) - n + i) s^{Z_{t_n}^{(2)}(k-1) - n + i - 1}, T_h^*(n+1-i) \in dt_h$$
$$\text{for } h \notin \{j_1, .., j_{i-1}\}, h \leq n-1\big).$$

By the induction hypothesis, the last expression equals

$$\frac{1}{2}y\sum_{k\geq 2}k\mu(k)\sum_{i=1}^{n}\binom{n+1}{i}\sum_{1\leq j_1<..<j_{i-1}\leq n-1}\psi_{t_n}(s)^{y-1}\times$$
$$\frac{(i-1)!i!}{2^{i-1}}\psi^{'}_{t_n}(s)\prod_{h\in\{j_1,..,j_{i-1}\}}\psi^{'}_{t_h}(s)\big[\sum_{j\geq 2}j(j-1)\mu(j)\psi_{t_h}(s)^{j-2}\big]\times$$
$$\frac{(n-i)!(n-i+1)!}{2^{n-i}}(k-1)\psi^{'}_{t_n}(s)\psi_{t_n}(s)^{k-2}\prod_{1\leq h\leq n-1, h\notin\{j_1,..,j_{i-1}\}}\psi^{'}_{t_h}(s)\big[\sum_{j\geq 2}j(j-1)\mu(j)\psi_{t_h}(s)^{j-2}\big]$$
$$dt_1dt_2..dt_{n-1}$$
$$=\frac{1}{2}y\sum_{k\geq 2}k\mu(k)\sum_{i=1}^{n}\binom{n+1}{i}\sum_{1\leq j_1<..<j_{i-1}\leq n-1}\frac{(i-1)!i!(n-i)!(n-i+1)!}{2^{n-1}}\psi^{'}_{t_n}(s)\psi_{t_n}(s)^{y-1}\times$$
$$(k-1)\psi^{'}_{t_n}(s)\psi_{t_n}(s)^{k-2}\prod_{h=1}^{n-1}\psi^{'}_{t_h}(s)\big[\sum_{j\geq 2}j(j-1)\mu(j)\psi_{t_h}(s)^{j-2}\big]dt_1dt_2..dt_{n-1}$$
$$=\frac{1}{2}y\sum_{k\geq 2}k\mu(k)\sum_{i=1}^{n}\binom{n+1}{i}\binom{n-1}{i-1}\frac{(i-1)!i!(n-i)!(n-i+1)!}{2^{n-1}}\psi^{'}_{t_n}(s)\psi_{t_n}(s)^{y-1}\times$$
$$(k-1)\psi^{'}_{t_n}(s)\psi_{t_n}(s)^{k-2}\prod_{h=1}^{n-1}\psi^{'}_{t_h}(s)\big[\sum_{j\geq 2}j(j-1)\mu(j)\psi_{t_h}(s)^{j-2}\big]dt_1dt_2..dt_{n-1}$$
$$\frac{n!(n+1)!}{2^n}y\psi^{'}_{t_n}(s)\psi_{t_n}(s)^{y-1}\prod_{h=1}^{n}\psi^{'}_{t_h}(s)\big[\sum_{j\geq 2}j(j-1)\mu(j)\psi_{t_h}(s)^{j-2}\big]dt_1dt_2..dt_{n-1}.$$

Hence the result follows by integrating w.r.t. to the distribution of Z_{t-t_n} conditional on $\{Z_0=x\}$. □

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Chapter 2

"TREES UNDER ATTACK": A RAY-KNIGHT REPRESENTATION OF FELLER'S BRANCHING DIFFUSION WITH LOGISTIC GROWTH

2.1 Introduction

Feller's branching diffusion with logistic growth is governed by the SDE

$$dZ_t = \sigma\sqrt{Z_t}\,dW_t + (\theta Z_t - \gamma Z_t^2)\,dt, \qquad Z_0 = x > 0, \tag{2.1.1}$$

with positive constants σ, θ and γ. It has been studied in detail by Lambert [9], and models the evolution of the size of a large population with competition. The diffusion term in (2.1.1) incorporates the individual offspring variance, and the drift term includes a super–criticality in the branching that is counteracted by a killing with a rate proportional to the "number of pairs of individuals".

For $\theta = \gamma = 0$, equation (2.1.1) is the SDE of *Feller's critical branching diffusion with variance parameter* σ^2. In this case, a celebrated theorem due to Ray and Knight says that Z has a representation in terms of the local times of reflected Brownian motion. To be specific, let $H = (H_s)_{s\geq 0}$ be a Brownian motion on $\mathbb{R}_+$ with variance parameter $4/\sigma^2$, reflected at the origin, and for $s, t \geq 0$ let $L_s(t, H)$ be the (semi–martingale) local time accumulated by H at level t up to time s. Define

$$S_x := \inf\{s > 0 : (\sigma^2/4)L_s(0, H) > x\}. \tag{2.1.2}$$

Then $(\sigma^2/4)L_{S_x}(t, H)$, $t \geq 0$, is a weak solution of (2.1.1) with $\theta = \gamma = 0$, and is called the *Ray-Knight representation* of Feller's critical branching diffusion.

The Ray-Knight representation has a beautiful interpretation in an individual-based picture. Reflected Brownian motion $H = (H_s)_{s\geq 0}$ arises as a concatenation of excursions, and each of these excursions codes a *continuum random tree*, the genealogical tree of the progeny of an individual that was present at time $t = 0$. The size of this progeny at time $t > 0$ is $\sigma^2/4$ times the (total) local time spent by this excursion at level t. Starting with mass x at time $t = 0$ amounts to collecting a local time $(4/\sigma^2)x$ of H at level 0. The local time of H at level t then arises as a sum over the local time of the excursions, just as the state at time t of Feller's branching diffusion, Z_t, arises as a sum of the masses of countably many families, each of which belongs to the progeny of one ancestor that lived at time $t = 0$. The path $(H_s)_{0\leq s\leq S_x}$ can be viewed as the *exploration path of the genealogical forest* arising from the ancestral mass x. We will briefly illustrate this in Section 2.2 along a discrete mass – continuous time approximation. For a more detailed explanation and some historical background we refer to the survey [13].

The motivation of the present paper was the question whether a similar picture is true also for (2.1.1) with strictly positive θ and γ, and whether also in this case a Ray-Knight representation is available for a suitably re-defined dynamics of an exploration process H. At first sight this seems prohibiting since the nonlinear term in (2.1.1) destroys the independence in the reproduction. However, it turns out that introducing an order among the individuals helps to overcome this hurdle. We will think of the individuals as being arranged "from left to right", and decree that the pairwise fights are always won by the individual "to the left", and lethal for the individual "to the right". In this way we arrive at a population dynamics which leaves the evolution (2.1.1) of the total mass unchanged, see again the explanation in Section 2.2. The death rate coming from the pairwise fights leads in the exploration process of the genealogical forest to a downward drift which is proportional to $L_s(H_s, H)$, that is, proportional to the amount of mass seen to the left of the individual encountered at exploration time s (and living at real time H_s) – more rigorously, $L_r(t, H)$ denotes the local time accumulated by the semimartingale H up to time r at level t. For the reader's convenience, we recall a possible definition (borrowed from [16], Theorem

VI.1.2) of that quantity:

$$L_s(t) := 2(H_s - t)^+ - 2\int_0^s \mathbf{1}_{\{H_r > t\}} dH_r.$$

As a consequence, those excursions of H which come later in the exploration tend to be smaller - the trees to the right are "under attack from those to the left".

In quantitative terms, we will consider the stochastic differential equation

$$H_s = \frac{2}{\sigma} B_s + \frac{1}{2} L_s(0, H) + \frac{2\theta}{\sigma^2} s - \gamma \int_0^s L_r(H_r, H) dr, \qquad s \geq 0, \tag{2.1.3}$$

where B is a standard Brownian motion. The last two terms are the above described components of the drift in the exploration process, and the term $L_s(0, H)/2$ takes care of the reflection of H at the origin. The following result is proved in [14] using Girsanovs theorem.

Proposition 2.1.1. *The SDE* (2.1.3) *has a unique weak solution.*

Our main result is the

Theorem 2.1.1. *Assume that H solves the SDE* (2.1.3)*, and let, for $x > 0$, S_x be defined as in* (2.1.2)*. Then $(\sigma^2/4)L_{S_x}(t, H)$, $t \geq 0$, solves* (2.1.1)*.*

We will prove Proposition 2.1.1 by a Girsanov argument, and Theorem 2.1.1 along a discrete mass–continuous time approximation that is presented in Section 2.2. In section 2.3 we take the limit in the total population process along the discrete mass approximation. An important step in the proof of Theorem 2.1.1, and interesting in its own right, is Theorem 2.4.1 in Section 2.4, in which we obtain a convergence in distribution, in the case $\theta = \gamma = 0$, of processes that approximate reflected Brownian motion, together with their local times (considered as random fields indexed by their two parameters, time and level). A similar convergence result was proved in [15] for piecewise linear interpolation of discrete time random walks and their local time. The proof of Theorem 2.1.1 is completed in Section 2.5, using again Girsanov's theorem, this time for Poisson point processes. In that section we will also prove Theorem 2.5.1, which says that in the case $\theta, \gamma > 0$, the exploration process *together* with its local time (now evaluated at a certain random time, while the parameter for the various levels is varying) converges along the discrete mass approximation. Finally, for the convenience of the reader, we collect in an Appendix, at the end of this paper, several results from the literature, concerning tightness and weak convergence in the space D, the Doléans exponential and "goodness", and the two versions of Girsanov's theorem which we need : the one for Brownian motion and the one for Poisson point processes.

When this work was already completed, our attention was drawn by Jean–François Le Gall on the article [12] by J. Norris, L.C.G. Rogers and D. Willams, who proved a Ray–Knight theorem for a Brownian motion with a "local time drift", using tools from stochastic analysis, in particular the "excursion filtration". With a similar methodology we were recently able to establish another, shorter but less intuitive, proof of the main result of this paper [14].

2.2 A discrete mass approximation

The aim of this section is to set up a "discrete mass - continuous time" approximation of (2.1.1) and (2.1.3). This will explain the intuition behind Theorem 2.1.1, and also will prepare for its proof.

For $x > 0$ and $N \in \mathbb{N}$ the approximation of (2.1.1) will be given by the total mass $Z^{N,x}$ of a population of individuals, each of which has mass $1/N$. The initial mass is $Z_0^{N,x} = \lfloor Nx \rfloor / N$, and $Z^{N,x}$ follows a Markovian jump dynamics: from its current state k/N,

$$Z^{N,x} \text{ jumps to } \begin{cases} (k+1)/N \text{ at rate } kN\sigma^2/2 + k\theta \\ (k-1)/N \text{ at rate } kN\sigma^2/2 + k(k-1)\gamma/N. \end{cases} \tag{2.2.1}$$

For $\gamma = 0$, this is (up to the mass factor $1/N$) as a Galton-Watson process in continuous time: each individual independently spawns a child at rate $N\sigma^2/2 + \theta$, and dies (childless) at rate $N\sigma^2/2$. For $\gamma \neq 0$, the additional quadratic death rate destroys the independence, and hence also the branching property. However, when viewing the individuals alive at time t as being arranged "from left to right", and by decreeing that each of the pairwise fights (which happen at rate 2γ and always end lethal for one of the two involved individuals) is won by the individual to the left, we arrive at the additional death rate $2\gamma\mathcal{L}_i(t)/N$ for individual i, where $\mathcal{L}_i(t)$ denotes the number of individuals living at time t to the left of individual i.

The just described reproduction dynamics gives rise to a *forest* $F^{N,x}$ of *trees of descent*, drawn into the plane as sketched in Figure 2.1. At any branch point, we imagine the "new" branch being placed to the right of the mother branch. Because of the asymmetric killing, the trees further to the right have a tendency to stay smaller: they are "under attack" by the trees to their left. Note also that, with the above described construction, the $F^{N,x}$, $x > 0$, are coupled: when x is increased by $1/N$, a new tree is added to the right. We denote the union of the $F^{N,x}$, $x > 0$, by F^N.

From F^N we read off a continuous and piecewise linear $\mathbb{R}_+$-valued path $H^N = (H^N_s)$ (called the *exploration path* of F^N) in the following way:

Starting from the root of the leftmost tree, one goes upwards at speed $2N$ until one hits the top of the first mother branch (this is the leaf marked with **L** in Figure 2.1). There one turns and goes downwards, again at speed $2N$, until arriving at the next branch point (which is **B** in Figure 2.1). From there one goes upwards into the (yet unexplored) next branch, and proceeds in a similar fashion until being back at height 0, which means that the exploration of the leftmost tree is completed. Then explore the next tree, and so on.

For $x > 0$ we denote by S^N_x the time at which the exploration of the forest $F^{N,x}$ is completed. Obviously, for each $t \geq 0$, the number of branches in $F^{N,x}$ that are alive at time t equals half the number of t-crossings of the exploration path of F^N stopped

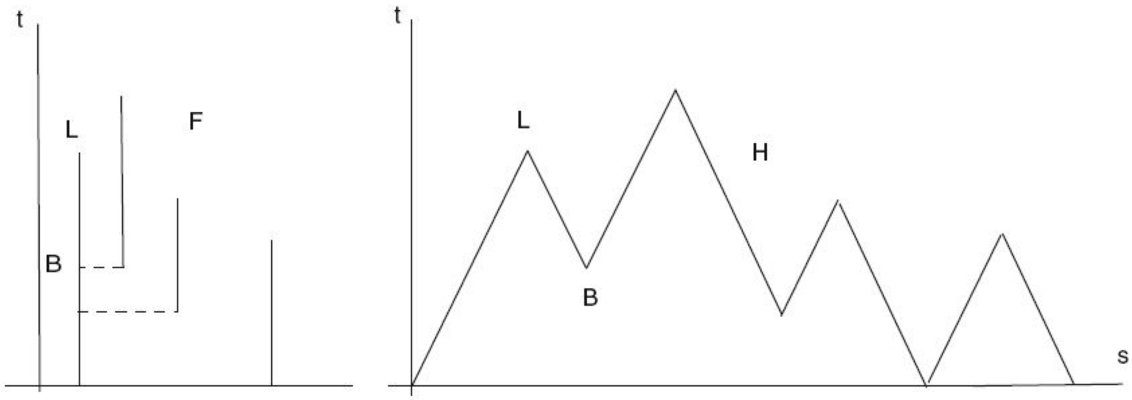

Figure 2.1: A realization of (the first two trees of) F^N and (the first two excursions of) its exploration path H^N. The t-axis is *real time* as well as *exploration height*, the s-axis is *exploration time*.

at S^N_x. Recalling that the slope of H^N is $\pm 2N$, we define

$$\Lambda^N_s(t) := \frac{1}{2N}\# \text{ of } t\text{-crossings of } H^N \text{ between exploration times } 0 \text{ and } s, \qquad (2.2.2)$$

where we count a local minimum of H^N at t as two t-crossings, and a local maximum as none. Note that by our convention both $s \mapsto \Lambda^N_s(t)$ and $t \mapsto \Lambda^N_s(t)$ are right continuous, and in particular $\Lambda^N_0(0) = 0$. We call $\Lambda^N_s(t)$ the *(unscaled) local time* of H^N accumulated at height t up to time s. This name is justified also by the following *occupation times formula*, valid for all measurable $f : \mathbb{R}_+ \to \mathbb{R}_+$

$$\int_0^s f(H^N_r)\,dr = \int_0^\infty f(t)\,\Lambda^N_s(t)\,dt, \qquad s \geq 0. \qquad (2.2.3)$$

The exploration time S^N_x which it takes to traverse all of the $\lfloor Nx \rfloor$ trees in $F^{N,x}$ can be expressed as

$$S^N_x = \inf\{s : \Lambda^N_s(0) \geq \lfloor Nx \rfloor / N\}. \qquad (2.2.4)$$

Proposition 2.2.1. *The exploration path $s \mapsto H^N_s$ obeys the following stochastic dynamics:*

- *At time $s = 0$, H^N starts at height 0 and with slope $2N$.*

- *While H^N moves upwards, its slope jumps from $2N$ to $-2N$ at rate $N^2\sigma^2+4\gamma N\ell$, where $\ell = \Lambda_s^N(H_s^N)$ is the local time accumulated by H^N at the current height H_s^N up to the current exploration time s.*

- *While H^N moves downwards, its slope jumps from $-2N$ to $2N$ at rate $N^2\sigma^2 + 2N\theta$.*

- *Whenever H^N reaches height 0, it is reflected above 0.*

Proof. We give here an informal proof which contains the essential ideas. (A more formal proof can be carried out along the arguments of the proof of Theorem 2.4 in [1].)

Recall that the death rate of an individual i living at real time t is $N\sigma^2/2 + 2\gamma\mathcal{L}_i(t)/N$, where $\mathcal{L}_i(t)$ is the number of individuals living at time t to the left of individual i. Assume the individual i living at time t is explored first at time s, hence $H_s = t$, and H has slope $2N$ at time s. Because of (2.2.2), while H_s^N goes upward, we have $\mathcal{L}_i(t) = N\Lambda_s(H_s^N)$. The rate in t is the rate in s multiplied by the factor $2N$ which is the absolute value of the slope. This gives the claimed jump rate $2N(N\sigma^2/2 + 2\gamma\Lambda_s(H_s^N))$ from slope $2N$ to slope $-2N$, which can be seen as the rate at which the "death clock" rings (and leads to a downward jump of the slope) along the rising pieces of the exploration path H^N. On the other hand, the "birth clock" rings along the falling pieces of H^N, its rate being $N\sigma^2/2 + \theta$ in real time and $2N(N\sigma^2/2 + \theta)$ in exploration time, as claimed in the proposition. Note that the process of birth times along an individual's lifetime is a homogeneous Poisson process which (in distribution) can as well be run backwards from the individual's death time. Also note that, due to the "depth-first-search"-construction of H^N, along falling pieces of H^N always yet unexplored parts of the forest are visited as far as the birth points are concerned. □

The next statement is a discrete version of Theorem 2.1.1, and will later be used for the proof of Theorem 2.1.1 by taking $N \to \infty$.

Corollary 2.2.2. *Let H^N be a stochastic process following the dynamics specified in Proposition 2.2.1, and Λ^N be its local time as defined by* (2.2.2)*. For $x > 0$, let S_x^N be the stopping time defined by* (2.2.4)*. Then $t \mapsto \Lambda^N_{S_x^N}(t)$ follows the jump dynamics* (2.2.1)*.*

Proof. By Proposition 2.2.1, H^N is equal in distribution to the exploration path of the random forest F^N. Hence $\Lambda_{S_x^N}(t)$ is equal in distribution to $Z_t^{N,x}$, where $NZ_t^{N,x}$ is the number of branches alive in $F^{N,x}$ at time t. Since $Z^{N,x}$ follows the dynamics (2.2.1), so does $\Lambda^N_{S_x^N}$. □

The next lemma will also be important in the proof of Theorem 2.1.1.

Lemma 2.2.3. *Let H^N and S_x^N be as in Corollary 2.2.2. Then $S_x^N \to \infty$ a.s. as $x \to \infty$.*

Proof. Consider $x = a/N$ for $a = 1, 2 \ldots$. Applying (2.2.3) with $s = S_x^N$ and $f \equiv 1$ we obtain the equality $S_x^N = \int_0^\infty \Lambda^N_{S_x^N}(t)\, dt$.

According to Corollary 2.2.2, $\Lambda^N_{S_x^N}(t)$ follows the jump dynamics (2.2.1), with initial condition $\Lambda_{S_x^N}(t) = a/N$. By coupling $\Lambda^N_{S_x^N}(t)$ with a “pure death process” K^N that starts in a/N and jumps from k/N to $(k-1)/N$ at rate $k(k-1)(N\sigma^2/2 + \gamma/N)$, we see that $\int_0^\infty \Lambda^N_{S_x^N}(t)\, dt$ is stochastically bounded from below by $\int_0^{T_2} K_t^N dt$, where T_2 is the first time at which K^N comes down to $2/N$. The latter integral equals a sum of independent exponentially distributed random variables with parameters $(j-1)(N^2\sigma^2/2 + \gamma))$, $j = 2, \ldots, a$. This sum diverges as $a \to \infty$. □

2.3 Convergence of the mass processes $Z^{N,x}$ as $N \to \infty$

The process $\{Z_t^{N,x},\ t \geq 0\}$ with dynamics (2.2.1) is a Markov process with values in the set $E_N := \{k/N,\ k \geq 1\}$, starting from $\lfloor Nx \rfloor / N$, with generator A^N given by

$$A^N f(z) = Nz \left(N\frac{\sigma^2}{2} + \theta \right) \left[f\left(z + \frac{1}{N} \right) - f(z) \right] \qquad (2.3.1)$$
$$+ Nz \left(N\frac{\sigma^2}{2} + \gamma \left(z - \frac{1}{N} \right) \right) \left[f\left(z - \frac{1}{N} \right) - f(z) \right],$$

for any $f : E_N \to \mathbb{R}$, $z \in E_N$. (Note that the distinction between symmetric and ordered killing is irrelevant here.) Applying successively the above formula to the cases $f(z) = z$ and $f(z) = z^2$, we get that

$$Z_t^{N,x} = Z_0^{N,x} + \int_0^t \left[\theta Z_r^{N,x} - \gamma Z_r^{N,x} \left(Z_r^{N,x} - \frac{1}{N} \right) \right] dr + M_t^{(1)}, \qquad (2.3.2)$$

$$\left(Z_t^{N,x} \right)^2 = \left(Z_0^{N,x} \right)^2 + 2 \int_0^t Z_r^{N,x} \left[\theta Z_r^{N,x} - \gamma Z_r^{N,x} \left(Z_r^{N,x} - \frac{1}{N} \right) \right] dr$$
$$+ \int_0^t \left[\sigma^2 Z_r^{N,x} + \frac{\theta}{N} Z_r^{N,x} + \frac{\gamma}{N} \left(Z_r^{N,x} - \frac{1}{N} \right) Z_r^{N,x} \right] dr + M_t^{(2)}, \qquad (2.3.3)$$

where $\{M_t^{(1)},\ t \geq 0\}$ and $\{M_t^{(2)},\ t \geq 0\}$ are local martingales. It follows from (2.3.2) and (2.3.3) that

$$\langle M^{(1)} \rangle_t = \int_0^t \left[\sigma^2 Z_r^{N,x} + \frac{\theta}{N} Z_r^{N,x} + \frac{\gamma}{N} \left(Z_r^{N,x} - \frac{1}{N} \right) Z_r^{N,x} \right] dr. \qquad (2.3.4)$$

We now prove

Lemma 2.3.1. *For any $T > 0$,*

$$\sup_{N \geq 1} \sup_{0 \leq t \leq T} \mathbb{E} \left[\left(Z_t^{N,x} \right)^4 \right] < \infty.$$

An immediate Corollary of this Lemma is that $\{M_t^{(1)}\}$ and $\{M_t^{(2)}\}$ are in fact martingales.

Proof. The same computation as above, but now with $f(z) = z^4$, gives

$$\left(Z_t^{N,x} \right)^4 = \left(Z_0^{N,x} \right)^4 + \int_0^t \Phi_N \left(Z_r^{N,x} \right) dr + M_t^{(4)}, \qquad (2.3.5)$$

where $\{M_t^{(4)},\ t \geq 0\}$ is a local martingale and for some $c > 0$ independent of N,

$$\Phi_N(z) \leq c(1 + z^4). \tag{2.3.6}$$

We note that $NZ_t^{N,x}$ is bounded by the value at time t of a Yule process (which arises when suppressing the deaths), which is a finite sum of mutually independent geometric random variables, hence has finite moments of any order. Hence $M^{(4)}$ is in fact a martingale. We then can take the expectation in (2.3.5), and deduce from (2.3.6) and Gronwall's Lemma that for $0 \leq t \leq T$,

$$\mathbb{E}\left[\left(Z_t^{N,x}\right)^4\right] \leq \left[\left(Z_0^{N,x}\right)^4 + cT\right] e^{cT},$$

which implies the result. □

We shall also need below the

Lemma 2.3.2. *For any $T > 0$,*

$$\sup_{N \geq 1} \mathbb{E}\left[\sup_{0 \leq t \leq T} \left(Z_t^{N,x}\right)^2\right] < \infty.$$

Proof. Since from (2.3.2), $Z_t^{N,x} \leq Z_0^{N,x} + \theta \int_0^t Z_r^{N,x} dr + M_t^{(1)}$,

$$\sup_{r \leq t} |Z_r^{N,x}|^2 \leq 3|Z_0^{N,x}|^2 + 3t\theta^2 \int_0^t |Z_r^{N,x}|^2 dr + 3 \sup_{r \leq t} |M_r^{(1)}|^2.$$

This together with (2.3.4), Doob's L^2-inequality for martingales and Lemma 2.3.1 implies the result. □

Remark 2.6.8 in the Appendix combined with (2.3.2), (2.3.4) and Lemma 2.3.1 guarantees that the tightness of $\{Z_0^N\}_{n \geq 1}$ implies that of $\{Z^N\}_{N \geq 1}$ in $D([0, +\infty))$.

Standard arguments exploiting (2.3.2) and (2.3.3) now allow us to deduce the convergence of the mass processes (for a detailed proof, see e.g. Theorem 18 in [11]).

Proposition 2.3.3. *As $N \to \infty$, $Z^{N,x} \Rightarrow Z^x$, where Z^x is the unique solution of the SDE* (2.1.1) *and thus is a Feller diffusion with logistic growth.*

2.4 *Convergence of the exploration path in the case* $\theta = \gamma = 0$*.*

Let H^N be a stochastic process as in Proposition 2.2.1 with $\theta = \gamma = 0$. The aim of this section is to provide a version of the joint convergence (as $N \to \infty$) of H^N and its local time which is suitable for the change of measure that will be carried through in Section 2.5. This is achieved in Theorem 2.4.1 and its Corollary 2.4.1. The proof of Theorem 2.4.1 is carried out in two major parts. The first part (Proposition 2.4.3) provides a refined version of the joint convergence of H^N and its local time at level 0, the second part (starting from Lemma 2.4.5) extends this to the other levels as well.

We define the *(scaled) local time accumulated by* H^N *at level* t *up to time* s as

$$L_s^N(t) := \frac{4}{\sigma^2} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t < H_u^N < t+\varepsilon\}} du$$

Note that this process is neither right– nor left–continuous as a function of s. However since the jumps are of size $O(1/N)$, the limit of L^N as $N \to \infty$ will turn out to be continuous. In fact, we will show that L^N converges as $N \to \infty$ to the semi–martingale local time of the limiting process H, hence the scaling factor $4/\sigma^2$. It is readily checked that $L_{0+}^N(0) = \frac{2}{N\sigma^2}$, and

$$L_s^N(t) = \frac{4}{\sigma^2} \Lambda_s^N(t), \quad \forall s \geq 0, t \geq 0, \tag{2.4.1}$$

where Λ^N was defined in (2.2.2). Then with S_x^N defined in (2.2.4), we may rewrite

$$S_x^N = \inf\{s > 0 : L_s^N(0) \geq \frac{4}{\sigma^2} \lfloor Nx \rfloor / N\}. \tag{2.4.2}$$

From (2.4.1) and Corollary 2.2.2 we see that

$$Z_t^{N,x} := \frac{\sigma^2}{4} L_{S_x^N}^N(t)$$

follows the jump dynamics (2.2.1) in the case $\theta = \gamma = 0$.

Let $\{V_s^N, s \geq 0\}$ be the càdlàg $\{-1, 1\}$-valued process which is such that for a. a. $s > 0$,

$$\frac{dH_s^N}{ds} = 2NV_s^N.$$

We can express L^N in terms of H^N and V^N as

$$L^N_s(t) = \frac{4}{\sigma^2}\frac{1}{2N}\left[\sum_{0\le r<s}\mathbf{1}_{\{H^N_r=t\}}\left(1+\frac{1}{2}(V^N_r - V^N_{r-})\right) + \mathbf{1}_{\{H^N_s=t\}}\mathbf{1}_{\{V^N_{s-}=-1\}}\right]$$

where we put $V_{0-}=+1$. Note that any $r<s$ at which t is a local minimum of H^N counts twice in the sum of the last line, while any $r<s$ at which t is a local maximum of H^N is not counted in the sum.

We have

$$\begin{aligned} H^N_s &= 2N\int_0^s V^N_r dr\,, \\ V^N_s &= 1 + 2\int_0^s \mathbf{1}_{\{V^N_{r-}=-1\}}dP^N_r - 2\int_0^s \mathbf{1}_{\{V^N_{r-}=1\}}dP^N_r + \frac{N\sigma^2}{2}(L^N_s(0) - L^N_{0+}(0)), \end{aligned} \tag{2.4.3}$$

where $\{P^N_s, s\ge 0\}$ is a Poisson process with intensity $N^2\sigma^2$. Note that $M^N_s = P^N_s - N^2\sigma^2 s$, $s\ge 0$, is a martingale. The second equation is prescribed by the statement of Proposition 2.2.1 (in fact its simplified version in case $\theta=\gamma=0$) : the initial velocity is positive, whenever $V^N=1$, it jumps to -1 (i. e. makes a jump of size -2) at rate $N^2\sigma^2$, whenever $V^N=-1$, it jumps to +1 (i. e. makes a jump of size +2) at rate $N^2\sigma^2$, and in addition it makes a jump of size 2 whenever H^N hits 0. Writing V^N_r in the first line of (2.4.3) as

$$\mathbf{1}_{\{V^N_{r-}=+1\}} - \mathbf{1}_{\{V^N_{r-}=-1\}}$$

and denoting by M^N_s the martingale $P^N_s - N^2\sigma^2 s$, we deduce from (2.4.3)

$$H^N_s + \frac{V^N_s}{N\sigma^2} = M^{1,N}_s - M^{2,N}_s + \frac{1}{2}L^N_s(0) - \frac{1}{2}L^N_{0+}(0), \tag{2.4.4}$$

where

$$M^{1,N}_s = \frac{2}{N\sigma^2}\int_0^s \mathbf{1}_{\{V^N_{r-}=-1\}}dM^N_r \quad \text{and} \quad M^{2,N}_s = \frac{2}{N\sigma^2}\int_0^s \mathbf{1}_{\{V^N_{r-}=1\}}dM^N_r \tag{2.4.5}$$

are two mutually orthogonal martingales. Thanks to an averaging property of the V^N (see step 2 in the proof of Proposition 2.4.3 below) these two martingales will converge as $N \to \infty$ to two independent Brownian motions with variance parameter $2/\sigma^2$ each. Together with the appropriate convergence of $L^N(0)$, (2.4.4) then gives the required convergence of H^N, see Proposition 2.4.3. We are now ready to state the main result of this section.

Theorem 2.4.1. *For any $x > 0$, as $N \to \infty$,*

$$\begin{aligned}&(\{H_s^N, M_s^{1,N}, M_s^{2,N},\ s \geq 0\}, \{L_s^N(t), s, t \geq 0\}, S_x^N)\\ &\quad \Rightarrow \left(\left\{H_s, \frac{\sqrt{2}}{\sigma}B_s^1, \frac{\sqrt{2}}{\sigma}B_s^2,\ s \geq 0\right\}, \{L_s(t), s, t \geq 0\}, S_x\right)\end{aligned}$$

for the topology of locally uniform convergence in s and t. B^1 and B^2 are two mutually independent standard Brownian motions, H solves the SDE

$$H_s = \frac{2}{\sigma}B_s + \frac{1}{2}L_s(0),$$

whose driving Brownian motion B is given as

$$B_s = \frac{1}{\sqrt{2}}(B_s^1 - B_s^2),$$

L is the semi–martingale local time of H, and S_x has been defined in (2.1.2)*.*

An immediate consequence of this result is

Corollary 2.4.1. *For any $x > 0$, as $N \to \infty$,*

$$(\{H_s^N, M_s^{1,N}, M_s^{2,N},\ s \geq 0\}, \{L_{S_x^N}^N(t), t \geq 0\}) \Rightarrow (\{H_s, \frac{\sqrt{2}}{\sigma}B_s^1, \frac{\sqrt{2}}{\sigma}B_s^2,\ s \geq 0\}, \{L_{S_x}(t), t \geq 0\})$$

in $C([0,\infty)) \times (D([0,\infty)))^3$.

Recall (see Lemma 2.6.1 in the Appendix) that convergence in $D([0,\infty))$ is equivalent to locally uniform convergence, provided the limit is continuous. Also note that in the absence of reflection, the weak convergence of H^N to Brownian motion would be a consequence of Theorem 7.1.4 in [5], and would be a variant of "Rayleigh's random flight model", see Corollary 3.3.25 in [17].

A first preparation for the proof of Theorem 2.4.1 is

Lemma 2.4.2. *The sequence $\{H^N\}$ is tight in $C([0,\infty))$.*

Proof. To get rid of the local time term in (2.4.4), we consider a process R^N of which H^N is the absolute value. More explicitly, let (R^N, W^N) be the $\mathbb{R} \times \{-1,1\}$–valued process that solves the system (which is exactly (2.4.3) without reflection)

$$R_s^N = 2N \int_0^s W_r^N dr\,,$$
$$W_s^N = 1 + 2\int_0^s \mathbf{1}_{\{W_{r-}^N = -1\}} dP_r^N - 2\int_0^s \mathbf{1}_{\{W_{r-}^N = +1\}} dP_r^N\,.$$

We observe that

$$(H^N, V^N) \equiv (|R^N|, \operatorname{sgn}(R^N) W^N)\,.$$

Clearly tightness of $\{R^N\}$ will imply that of $\{H^N\}$, since $|H_s^N - H_t^N| \le |R_s^N - R_t^N|$ for all $s, t \ge 0$. Now we have

$$R_s^N + \frac{W_s^N}{N\sigma^2} = \frac{1}{N\sigma^2} - \frac{2}{N\sigma^2}\int_0^s W_{r-}^N\, dM_r^N\,.$$

By Proposition 2.6.6 in the Appendix, the sequence $\{R^N\}_{N\ge1}$ is tight, and so is $\{H^N\}_{N\ge1}$. □

Proposition 2.4.3. *Fix $x > 0$. As $N \to \infty$,*

$$\big(H^N, M^{1,N}, M^{2,N}, L^N(0), S_x^N\big) \Rightarrow \Big(H, \frac{\sqrt{2}}{\sigma}B^1, \frac{\sqrt{2}}{\sigma}B^2, L(0), S_x\Big)$$
$$\text{in} \quad C([0,\infty)) \times \big(D([0,\infty))\big)^3 \times [0,\infty),$$

where B^1 and B^2 are two mutually independent standard Brownian motions, and H solves the SDE

$$H_s = \frac{2}{\sigma}B_s + \frac{1}{2}L_s(0), \quad s \ge 0, \tag{2.4.6}$$

with $B_s := (1/\sqrt{2})(B_s^1 - B_s^2)$, and $L(0)$ denoting the local time at level 0 of H. (Note that B is again a standard Brownian motion.)

Proof. The proof is organized as follows. Step 1 establishes the weak convergence of $(H^N, M^{1,N}, M^{2,N}, L^N(0))$ along a subsequence. Step 2 and step 3 together characterize the law of the limiting two–dimensional martingale, step 4 identifies the limit of the local time term. In step 5 we note that the entire sequence converges. Finally step 6 takes the limit in the quintuple (including S^N_x).

Step 1. Note that

i) from Lemma 2.4.2, the sequence $\{H^N, N \geq 1\}$ is tight in $C([0,\infty])$;

ii) $\sup_{s\geq 0} \frac{|V^N_s|}{N\sigma^2} \to 0$ in probability as $N \to \infty$;

iii) from Proposition 2.6.6, $\{M^{1,N}, N \geq 1\}$ and $\{M^{2,N}, N \geq 1\}$ are tight in $D([0,\infty])$, any limiting martingales M^1 and M^2 being continuous;

iv) it follows from the first 3 items, (2.4.4) and Proposition 2.6.5 that $\{L^N_s(0), N \geq 1\}$ is tight in $D([0,\infty])$, the limit K of any converging subsequence being continuous and increasing.

Working along a diagonal subsequence we can extract a subsequence, still denoted as an abuse like the original sequence, such that along that subsequence

$$\big(H^N, M^{1,N}, M^{2,N}, L^N(0)\big) \Rightarrow \big(H, M^1, M^2, K\big).$$

Step 2. We claim that for any $s > 0$,

$$\int_0^s \mathbf{1}_{\{V^N_r=1\}} dr \to \frac{s}{2}, \quad \int_0^s \mathbf{1}_{\{V^N_r=-1\}} dr \to \frac{s}{2}$$

in probability, as $N \to \infty$. This follows by taking the limit in the sum and the difference of the two following identities :

$$\int_0^s \mathbf{1}_{\{V^N_r=1\}} dr + \int_0^s \mathbf{1}_{\{V^N_r=-1\}} dr = s,$$
$$\int_0^s \mathbf{1}_{\{V^N_r=1\}} dr - \int_0^s \mathbf{1}_{\{V^N_r=-1\}} dr = (2N)^{-1} H^N_s,$$

since $H_s^N/N \to 0$ in probability, as $N \to \infty$, thanks to Lemma 2.4.2.

STEP 3. By Step 1 iii), $M_s = \begin{pmatrix} M_s^1 \\ M_s^2 \end{pmatrix}$, the weak limit of $\begin{pmatrix} M_s^{1,N} \\ M_s^{2,N} \end{pmatrix}$ along the chosen subsequence, is a 2–dimensional continuous martingale. In order to identify it, we first introduce some useful notation. We write $M_s^{\otimes 2}$ for the 2×2 matrix whose (i,j)–entry equals $M_s^i \times M_s^j$, and $\langle\langle M \rangle\rangle_s$ for the 2×2 matrix–valued predictable increasing process which is such that

$$M_s^{\otimes 2} - \langle\langle M \rangle\rangle_s$$

is a martingale, and note that the (i,j)–entry of the matrix $\langle\langle M \rangle\rangle_s$ equals $\langle M^i, M^j \rangle_s$. We adopt similar notations for the pair $M^{1,N}$, $M^{2,N}$.

From Step 2 we deduce that, as $N \to \infty$,

$$\begin{aligned} \langle\langle \begin{pmatrix} M^{1,N} \\ M^{2,N} \end{pmatrix} \rangle\rangle_s &= \frac{4}{\sigma^2} \int_0^s \begin{pmatrix} \mathbf{1}_{\{V_r^N = -1\}} & 0 \\ 0 & \mathbf{1}_{\{V_r^N = 1\}} \end{pmatrix} dr \\ &\to \frac{2}{\sigma^2} sI \end{aligned}$$

in probability, locally uniformly in s, where I denotes the 2×2 identity matrix. Consequently

$$\begin{pmatrix} M_s^{1,N} \\ M_s^{2,N} \end{pmatrix}^{\otimes 2} - \langle\langle \begin{pmatrix} M^{1,N} \\ M^{2,N} \end{pmatrix} \rangle\rangle_s \Rightarrow \begin{pmatrix} M_s^1 \\ M_s^2 \end{pmatrix}^{\otimes 2} - \frac{2}{\sigma^2} sI$$

in $D([0,\infty);\mathbb{R}^4)$ as $N \to \infty$, and since the weak limit of martingales is a local martingale, there exist two mutually independent standard Brownian motions B^1 and B^2 such that

$$M_s^1 = \frac{\sqrt{2}}{\sigma} B_s^1, \; M_s^2 = \frac{\sqrt{2}}{\sigma} B_s^2, \;\; s \geq 0.$$

Taking the weak limit in (2.4.4) we deduce that

$$\begin{aligned} H_s &= \frac{\sqrt{2}}{\sigma}(B_s^1 - B_s^2) + \frac{1}{2} K_s \\ &= \frac{\sqrt{2}}{\sigma} B_s + \frac{1}{2} K_s, \end{aligned}$$

where $B_s = (B^1_s - B^2_s)/\sqrt{2}$ is also a standard Brownian motion.

STEP 4. For each $\ell \geq 1$, we define the function $f_\ell : \mathbb{R}_+ \to [0,1]$ by $f_\ell(x) = (1-\ell x)^+$. We have that for each $N, \ell \geq 1, s > 0$, since $L^N(0)$ increases only when $H^N = 0$,

$$\mathbb{E}\Big(\int_0^s f_\ell(H^N_r)dL^N_r(0) - L^N_s(0)\Big) \geq 0.$$

Thanks to Lemma 2.6.3 in the Appendix we can take the limit in this last inequality as $N \to \infty$, yielding

$$\mathbb{E}\Big(\int_0^s f_\ell(H_r)dK_r - K_s\Big) \geq 0.$$

Then taking the limit as $\ell \to \infty$ yields

$$\mathbb{E}\Big(\int_0^s \mathbf{1}_{\{H_r=0\}}dK_r - K_s\Big) \geq 0.$$

But the random variable under the expectation is clearly nonpositive, hence it is zero a.s., in other words

$$K_s = \int_0^s \mathbf{1}_{\{H_r=0\}}dK_r, \quad \forall s \geq 0,$$

which means that the process K increases only when $H_r = 0$.

From the occupation times formula

$$\frac{4}{\sigma^2}\int_0^s g(H_r)dr = \int_0^\infty g(t)L_s(t)dt$$

applied to the function $g(h) = \mathbf{1}_{\{h=0\}}$, we deduce that the time spent by the process H at 0 has a.s. zero Lebesgue measure. Consequently

$$\int_0^s \mathbf{1}_{\{H_r=0\}}dB_r \equiv 0 \quad a.s.$$

hence a.s.

$$B_s = \int_0^s \mathbf{1}_{\{H_r>0\}}dB_r \quad \forall s \geq 0.$$

It then follows from Tanaka's formula applied to the process H and the function $h \to h^+$ that $K = L(0)$.

STEP 5. We have proved so far that $Q^N \Rightarrow Q$ along some subsequence, where $Q^N = (H^N, M^{1,N}, M^{2,N}, L^N(0))$, $Q = (H, \frac{\sqrt{2}}{\sigma}B^1, \frac{\sqrt{2}}{\sigma}B^2, L(0))$. Note that not only subsequences but the entire sequence $Q^1, Q^2, Q^3, \ldots$ converges, since the limit law is uniquely characterized.

STEP 6. It remains to check that for any $x > 0$, as $N \to \infty$,

$$(Q^N, S_x^N) \Rightarrow (Q, S_x) \quad \text{in } C([0,\infty]) \times \big(D([0,\infty])\big)^3 \times [0,\infty].$$

To this end, let us define the function Φ from $\mathbb{R}_+ \times C_\uparrow(\mathbb{R}_+)$ into $\mathbb{R}_+$ by

$$\Phi(x,y) = \inf\{s > 0 : y(s) > \frac{4}{\sigma^2}x\}.$$

For any x fixed, the function $\Phi(x,.)$ is continuous in the neighborhood of a function y which is strictly increasing at the time when it first reaches the value x. Clearly $S_x = \Phi(x, L_\cdot(0))$. Define

$$S_x'^N := \Phi(x, L_\cdot^N(0)).$$

We note that for any $x > 0, s \mapsto L_s(0)$ is a.s. strictly increasing at time S_x, which is a stopping time. This fact follows from the strong Markov property, the fact that $H_{S_x} = 0$, and $L_\varepsilon(0) > 0$, for all $\varepsilon > 0$. Consequently S_x is a.s. a continuous function of the trajectory $L_\cdot(0)$, then also of Q, and

$$(Q^N, S_x'^N) \Rightarrow (Q, S_x).$$

It remains to prove that $S_x'^N - S_x^N \to 0$ in probability. For any $y < x$ and N large enough

$$0 \le S_x'^N - S_x^N \le S_x'^N - S_y'^N.$$

Clearly $S_x'^N - S_y'^N \Rightarrow S_x - S_y$, hence for any $\varepsilon > 0$,

$$0 \le \limsup_N \mathbb{P}(S_x'^N - S_x^N \ge \varepsilon) \le \limsup_N \mathbb{P}(S_x'^N - S_y'^N \ge \varepsilon) \le \mathbb{P}(S_x - S_y \ge \varepsilon).$$

The result follows, since $S_y \to S_{x-}$ as $y \to x, y < x$, and $S_{x-} = S_x$ a.s. $\square$

For the proof of Theorem 2.4.1 we will need the following lemma:

Lemma 2.4.4. *For any $s > 0$, $t > 0$, the following identities hold a.s.*

$$(H_s^N - t)^+ = 2N \int_0^s V_r^N \mathbf{1}_{\{H_r^N > t\}} dr,$$
$$V_s^N \mathbf{1}_{\{H_s^N > t\}} = \frac{\sigma^2 N}{2} L_s^N(t) + \int_0^s \mathbf{1}_{\{H_r^N > t\}} dV_r^N.$$

Proof. The first identity is elementary, and is true along any piecewise linear, continuous trajectory $\{H_r^N\}$ satisfying $dH_s^N/ds = 2NV_s^N$ for almost all s, with $V_s^N \in \{-1, 1\}$. The other identities which we will state in this proof are true a.s. In these identities we exclude the trajectories of H^N which have a local maximum or minimum at the level t. This implies that the two processes $s \to V_s^N$ and $s \to \mathbf{1}_{\{H_s^N > t\}}$ do not jump at the same time. Hence from

$$\begin{aligned}\mathbf{1}_{\{H_s^N > t\}} &= \sum_{0<r<s} \mathbf{1}_{\{H_r^N = t\}} V_r^N - \mathbf{1}_{\{V_s^N = -1\}} \mathbf{1}_{\{H_s^N = t\}} \\ &= \frac{\sigma^2 N}{2} \int_0^s V_r^N dL_r^N(t),\end{aligned}$$

we deduce by differentiating the product that

$$V_s^N \mathbf{1}_{\{H_s^N > t\}} = \frac{\sigma^2 N}{2} \int_0^s (V_r^N)^2 dL_r^N(t) + \int_0^s \mathbf{1}_{\{H_r^N > t\}} dV_r^N.$$

Since $(V_r^N)^2 = 1$, this is the second identity in the lemma. □

Lemma 2.4.5. *Denote by $L_s(t)$ the local time at level t up to time s of H. Then with probability one $(s,t) \mapsto L_s(t)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}$.*

Proof. This is Theorem VI.1.7 page 225 of Revuz, Yor [16]. □

Proposition 2.4.6. *For each $d \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_d$,*

$$\Big\{\big(H_s^N, L_s^N(t_1), L_s^N(t_2), .., L_s^N(t_d)\big), s \geq 0\Big\} \Rightarrow \Big\{\big(H_s, L_s(t_1), L_s(t_2), .., L_s(t_d)\big), s \geq 0\Big\}$$
$$\textit{in } C([0,\infty)) \times (D([0,\infty)))^d.$$

Proof. We prove the result in case $d = 1$ only, the proof of the general case being very similar. From (2.4.3) and Lemma 2.4.4 we deduce that for any $t \geq 0$, a. s.

$$L_s^N(t) = 2(H_s^N - t)^+ + \frac{2}{N\sigma^2} V_s^N \mathbf{1}_{\{H_s^N > t\}} - 2\int_0^s \mathbf{1}_{\{H_r^N > t\}}(dM_r^{1,N} - dM_r^{2,N}),\ s \geq 0. \tag{2.4.7}$$

Let

$$U_s^N = \int_0^s \mathbf{1}_{\{H_r^N > t\}}(dM_r^{1,N} - dM_r^{2,N}).$$

By Proposition 2.6.6 we have that $\{U^N\}_{N\geq 1}$ is tight in $D([0,\infty))$. Moreover,

$$\langle M^{1,N} - M^{2,N}\rangle_s = \frac{4}{\sigma^2} s \tag{2.4.8}$$

$$\langle U^N\rangle_s = \langle U^N, M^{1,N} - M^{2,N}\rangle_s = \frac{4}{\sigma^2}\int_0^s \mathbf{1}_{\{H_r^N > t\}} dr. \tag{2.4.9}$$

From the occupation times formula

$$\int_0^s \mathbf{1}_{\{H_r = t\}} dr = \frac{\sigma^2}{4}\int_0^\infty \mathbf{1}_{\{r=t\}} L_s(r) dr = 0 \quad a.s.$$

Then by Lemma 2.6.2 from the Appendix we deduce that along an appropriate sequence

$$\Big\{\int_0^s \mathbf{1}_{\{H_r^N > t\}} dr, s \geq 0\Big\} \Rightarrow \Big\{\int_0^s \mathbf{1}_{\{H_r > t\}} dr, s \geq 0\Big\} \tag{2.4.10}$$

From (2.4.8), (2.4.9) and (2.4.10), we have again along an appropriate subsequence

$$(U_s^N, M_s^{1,N} - M_s^{2,N}) \Rightarrow \big(\frac{2}{\sigma}\int_0^s \mathbf{1}_{\{H_r > t\}} dB_r, \frac{2}{\sigma} B_s\big) \quad \text{in} \quad \big(D([0,\infty))\big)^2.$$

Moreover, arguments similar to that used in the proof of Proposition 2.4.3 establish that

$$(H_s^N, U_s^N) \Rightarrow \big(H_s, \frac{2}{\sigma}\int_0^s \mathbf{1}_{\{H_r > t\}} dB_r\big) \quad \text{in} \quad \big(D([0,\infty))\big)^2.$$

Now from any subsequence, we can extract a subsequence along which we can take the weak limit in (2.4.7). But Tanaka's formula gives us the identity

$$L_s(t) = 2(H_s - t)^+ - \frac{4}{\sigma}\int_0^s \mathbf{1}_{\{H_r > t\}} dB_r,$$

which characterizes the limit of L^N as the local time of H. Since the law of H is uniquely characterized, the whole sequence converges. □

Proposition 2.4.7. *For each $s \geq 0$ fixed, $\{L_s^N(t), t \geq 0\}_{N\geq 1}$ is tight in $D([0,\infty))$.*

Proof. We have

$$L_s^N(t) = 2(H_s^N - t)^+ + \frac{2}{N\sigma^2} V_s^N \mathbf{1}_{\{H_s^N > t\}} + \frac{4}{N\sigma^2}\int_0^s V_{r-}^N \mathbf{1}_{\{H_r^N > t\}} dM_r^N \tag{2.4.11}$$
$$= K_t^N + G_t^N,$$

where

$$K_t^N = 2(H_s^N - t)^+ + \frac{2}{N\sigma^2} V_s^N \mathbf{1}_{\{H_s^N > t\}},$$
$$G_t^N = \frac{4}{N\sigma^2}\int_0^s V_{r-}^N \mathbf{1}_{\{H_r^N > t\}} dM_r^N.$$

From

$$\{K_0^N = 2H_s^N + \frac{2}{N\sigma^2} V_s^N \mathbf{1}_{\{H_s^N > 0\}},\ N \geq 1\} \quad \text{is tight and}$$
$$\limsup_{N\to\infty} |K_t^N - K_{t'}^N| \leq 2|t - t'|,$$

it follows from Theorem 15.1 in [3] that the sequence $\{K_\cdot^N\}_{N\geq 1}$ is tight, and any limit of a converging subsequence is a. s. continuous.

We next show that the sequence $\{G_\cdot^N\}_{N\geq 1}$ satisfies the conditions of Proposition 2.6.4.

Condition (1) follows easily from the fact that $\mathbb{E}\big(|G_t^N|^2\big) \leq 16s/\sigma^2$. In order to verify condition (2), we will show that for any $T > 0$, there exists $C > 0$ such that for any $0 < t < T, \varepsilon > 0$,

$$\mathbb{E}\big[(G_{t+\varepsilon}^N - G_t^N)^2 (G_t^N - G_{t-\varepsilon}^N)^2\big] \leq C(\varepsilon^{3/2} + \varepsilon^2).$$

In order to simplify the notations below we let

$$\varphi_r^N := V_{r-}^N \mathbf{1}_{\{t-\varepsilon < H_r^N \leq t\}},$$
$$\psi_r^N := V_{r-}^N \mathbf{1}_{\{t < H_r^N \leq t+\varepsilon\}}.$$

An essential property, which will be crucial below, is that $\varphi_r^N \psi_r^N = 0$. Also $(\varphi_r^N)^2 = |\varphi_r^N|$, and similarly for ψ^N, since those functions take their values in the set $\{-1, 0, 1\}$. The quantity we want to compute equals up to a fixed multiplicative constant

$$N^{-4}\mathbb{E}\Big[(\int_0^s \varphi_r^N dM_r^N)^2(\int_0^s \psi_r^N dM_r^N)^2\Big].$$

We note that we have the identity

$$(\int_0^s \varphi_r^N dM_r^N)^2 = 2\int_0^s \int_0^{r-} \varphi_u^N dM_u^N \varphi_r^N dM_r^N + \int_0^s |\varphi_r^N| dM_r^N + \sigma^2 N^2 \int_0^s |\varphi_r^N| dr,$$

and similarly with φ^N replace by ψ^N. Because $\varphi_r^N \psi_r^N = 0$, the expectation of the product of

$$\int_0^s \int_0^{r-} \varphi_u^N dM_u^N \varphi_r^N dM_r^N \quad or \quad \int_0^s |\varphi_r^N| dM_r^N$$

with

$$\int_0^s \int_0^{r-} \psi_u^N dM_u^N \psi_r^N dM_r^N \quad or \quad \int_0^s |\psi_r^N| dM_r^N$$

vanishes. We only need to estimate the expectations

$$\mathbb{E}\Big(\int_0^s \int_0^{r-} \varphi_u^N dM_u^N \varphi_r^N dM_r^N \int_0^s |\psi_r^N| dr\Big), \mathbb{E}\Big(\int_0^s |\varphi_r^N| dM_r^N \int_0^s |\psi_r^N| dr\Big),$$
$$and \quad \mathbb{E}\Big(\int_0^s |\varphi_r^N| dr \int_0^s |\psi_r^N| dr\Big),$$

together with similar quantities with φ^N and ψ^N interchanged. The estimates of the first two expectations are very similar. We estimate the second one as follows, using the Cauchy-Schwarz inequality, and Lemma 2.4.8 below :

$$\mathbb{E}\Big(\int_0^s |\varphi_r^N| dM_r^N \int_0^s |\psi_r^N| dr\Big) \leq \sqrt{\mathbb{E}\int_0^s |\varphi_r^N| d\langle M^N\rangle_r} \sqrt{\mathbb{E}\Big[\Big(\int_0^s |\psi_r^N| dr\Big)^2\Big]}$$
$$\leq CN\varepsilon^{3/2}$$

Finally, again from Lemma 2.4.8,

$$\mathbb{E}\Big(\int_0^s |\varphi_r^N| dr \int_0^s |\psi_r^N| dr\Big) \leq C\varepsilon^2.$$

The first quantity should be multiplied by N^2, and the second by N^4, and then both should be divided by N^4. The proposition now follows from Proposition 2.6.5. □

Lemma 2.4.8. *Let $s, \epsilon, T > 0$. Then there exists a constant C such that for all $N \geq 1$ and $0 < t, t' < T$,*

$$\mathbb{E}\Big(\int_0^s \mathbf{1}_{\{t-\varepsilon < H_r^N \leq t\}} dr\Big) \leq C\varepsilon,$$
$$\mathbb{E}\Big(\int_0^s \mathbf{1}_{\{t-\varepsilon < H_r^N \leq t\}} dr \int_0^s \mathbf{1}_{\{t'-\varepsilon < H_r^N \leq t'\}} dr\Big) \leq C\varepsilon^2.$$

Proof. We will prove the second inequality, the first one follows from the second one with $t = t'$ and the Cauchy-Schwarz inequality.

For $s, t > 0$ define $F_s^N(t) := \int_0^s \mathbf{1}_{\{0 \leq H_r^N \leq t\}} dr$. It follows readily from the definition of L^N that

$$\frac{\partial F_s^N}{\partial t}(t) = \frac{\sigma^2}{4} L_s^N(t).$$

Hence

$$\begin{aligned}
\mathbb{E}\Big(\int_0^s \mathbf{1}_{\{t-\varepsilon < H_r^N \leq t\}} dr \int_0^s \mathbf{1}_{\{t'-\varepsilon < H_r^N \leq t'\}} dr\Big) &= \frac{\sigma^4}{16}\mathbb{E}\Big(\int_{t-\varepsilon}^t L_s^N(r)dr \int_{t'-\varepsilon}^{t'} L_s^N(u)du\Big) \\
&= \frac{\sigma^4}{16}\mathbb{E}\Big(\int_{t-\varepsilon}^t \int_{t'-\varepsilon}^{t'} L_s^N(r) L_s^N(u) dr du\Big) \\
&= \frac{\sigma^4}{16}\int_{t-\varepsilon}^t \int_{t'-\varepsilon}^{t'} \mathbb{E}\big(L_s^N(r) L_s^N(u)\big) dr du \\
&\leq \frac{\sigma^4}{16}\varepsilon^2 \sup_{0 \leq r, u \leq T} \mathbb{E}\big(L_s^N(r) L_s^N(u)\big) \\
&= \frac{\sigma^4}{16}\varepsilon^2 \sup_{0 \leq r \leq T} \mathbb{E}\big((L_s^N(r))^2\big).
\end{aligned}$$

On the other hand, since by Itô's formula there exists a martingale $\bar{M}_s^N$ such that

$$(H_s^N)^2 + \frac{2}{N\sigma^2} H_s^N V_s^N = \frac{4}{\sigma^2}s + \bar{M}_s^N.$$

we conclude that

$$\sup_{N \geq 1} \mathbb{E}\big((H_s^N)^2\big) < \infty.$$

The second inequality now follows from (2.4.11). □

Proposition 2.4.9. *For all $d > 1$, $0 \leq s_1 < s_2 < \cdots < s_d$,*

$$(H^N, L_{s_1}^N, \ldots, L_{s_d}^N) \Rightarrow (H, L_{s_1}, \ldots, L_{s_d}) \quad in \quad C([0,\infty)) \times (D([0,\infty)))^d.$$

Proof. We prove the result in the case $d = 1$ only, the proof in the general case being very similar. From Proposition 2.4.6 there follows in particular that for all $k \geq 1, 0 \leq t_1 < t_2 < \cdots < t_k$, we have

$$\big(H^N, L_s^N(t_1), L_s^N(t_2), .., L_s^N(t_k)\big) \Rightarrow \big(H, L_s(t_1), L_s(t_2), .., L_s(t_k)\big)$$

That is, $\{L_s^N\}$ converges in finite-dimensional distributions to $\{L_s\}$, jointly with H^N. By Proposition 2.4.7, $\{L_s^N(t), t \geq 0\}_{N\geq 1}$ is tight. The result follows. $\square$

We are now prepared to complete the

Proof of Theorem 2.4.1: The main task is to combine the assertions of Propositions 2.4.6 and 2.4.9, which means to turn the “partial” convergences asserted for L^N in these propositions into a convergence that is joint in s and t. We will also combine this result with Proposition 2.4.3 in order to get joint convergence of all our processes. To facilitate the reading, we will divide the proof into several steps.

Step 1. Let $\{s_n, n \geq 1\}$ denote a countable dense subset of $\mathbb{R}_+$. Our first claim is that for all $n \in \mathbb{N}$,

$$(H^N, M^{1,N}, M^{2,N}, L_{s_1}^N, \ldots, L_{s_n}^N, S_x^N) \Rightarrow (H, \frac{2}{\sigma}B^1, \frac{2}{\sigma}B^2, L_{s_1}, \ldots, L_{s_n}, S_x) \quad \text{in } C(\mathbb{R}_+) \times D(\mathbb{R}_+)^{n+2} \times \mathbb{R}_+. \tag{2.4.12}$$

To make the core of the argument clear, let us write just for the moment

$$Y^N := (M^{1,N}, M^{2,N}, S_x^N),\ Y := (\frac{2}{\sigma}B^1, \frac{2}{\sigma}B^2, S_x),\ \Lambda^N := (L_{s_1}^N, \ldots, L_{s_n}^N),\ \Lambda := (L_{s_1}, \ldots, L_{s_n}).$$

Then (2.4.12) translates into

$$(H^N, Y^N, \Lambda^N) \Rightarrow (H, Y, \Lambda). \tag{2.4.13}$$

By Proposition 2.4.3, $(H^N, Y^N) \Rightarrow (H, Y)$, and by Proposition 2.4.9, $(H^N, \Lambda^N) \Rightarrow (H, \Lambda)$. Because in our situation Λ is a.s. a function of H, these two convergences imply (2.4.13). (More generally, this implication would be true if Y and Λ would be conditionally independent given H.)

STEP 2. Now having established (2.4.12), it follows from a well known theorem due to Skorohod that all the processes appearing there can be constructed on a joint probability space, such that there exists an event $\mathcal{N}$ with $\mathbb{P}(\mathcal{N}) = 0$ and for all $\omega \notin \mathcal{N}$,

$$S_x^N(\omega) \to S_x(\omega), \tag{2.4.14}$$

$$(H_s^N(\omega), M_s^{1,N}(\omega), M_s^{2,N}(\omega)) \to (H_s(\omega), \frac{2}{\sigma}B_s^1(\omega), \frac{2}{\sigma}B_s^2(\omega)) \quad \text{locally uniformly in } s \geq 0, \tag{2.4.15}$$

and for all $n \geq 1$,

$$L_{s_n}^N(t)(\omega) \to L_{s_n}(t)(\omega) \quad \text{locally uniformly in } t \geq 0, \tag{2.4.16}$$

as $N \to \infty$. Here we have made use of Lemma 2.4.5, which allows us to assume that $(s,t) \mapsto L_s(t)(\omega)$ is continuous from $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}$ for all $\omega \notin \mathcal{N}$, possibly at the price of enlarging the null set $\mathcal{N}$, and of Lemma 2.6.1 from the Appendix.

STEP 3. We claim that in the situation described in the previous step one even has for all $C, T > 0, \omega \notin \mathcal{N}$,

$$\sup_{0 \leq s \leq C,\, 0 \leq t \leq T} |L_s^N(t,\omega) - L_s(t,\omega)| \to 0, \tag{2.4.17}$$

as $N \to \infty$. In other words, in Skorokhod's construction there is a.s. convergence of $L_s^N(t)$ to $L_s(t)$, locally uniformly in s and t. To prove (2.4.17), we will make use of the fact that for any $\omega \notin \mathcal{N}$, and all N, t, the mapping $s \mapsto L_s^N(t)(\omega)$ is increasing and the mapping $s \mapsto L_s(t)(\omega)$ is continuous and increasing. Moreover, since the mapping $(s,t) \mapsto L_s(t,\omega)$ is continuous from the compact set $[0,C] \times [0,T]$ into $\mathbb{R}_+$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 \leq s < s' \leq C, 0 \leq t \leq T$ and $s' - s \leq \delta$ implies that

$$L_{s'}(t,\omega) - L_s(t,\omega) \leq \varepsilon.$$

Hence there exists $k \geq 1$ and $0 =: s_0 < r_1 < \cdots < r_k := C$ such that $\{r_i, 0 \leq i < k\} \subset \{s_n, n \geq 1\}$ and moreover, $r_i - r_{i-1} \leq \delta$ for all $1 \leq i \leq k$. We have

$$\sup_{0 \leq s \leq C, 0 \leq t \leq T} |L_s^N(t,\omega) - L_s(t,\omega)| \leq \sup_{1 \leq i \leq k} [A_{N,i} + B_{N,i}]$$

where

$$A_{N,i} = \sup_{r_{i-1} \leq s \leq r_i, 0 \leq t \leq T} (L_s^N(t,\omega) - L_s(t,\omega))^+$$

$$B_{N,i} = \sup_{r_{i-1} \leq s \leq r_i, 0 \leq t \leq T} (L_s^N(t,\omega) - L_s(t,\omega))^-.$$

For $r_{i-1} \leq s \leq r_i$,

$$\begin{aligned}(L_s^N(t,\omega) - L_s(t,\omega))^+ &\leq (L_{r_i}^N(t,\omega) - L_s(t,\omega))^+ \\ &\leq (L_{r_i}^N(t,\omega) - L_{r_i}(t,\omega))^+ + \varepsilon, \\ (L_s^N(t,\omega) - L_s(t,\omega))^- &\leq (L_{r_{i-1}}^N(t,\omega) - L_s(t,\omega))^- \\ &\leq (L_{r_{i-1}}^N(t,\omega) - L_{r_{i-1}}(t,\omega))^- + \varepsilon.\end{aligned}$$

Finally,

$$\sup_{0 \leq s \leq C, 0 \leq t \leq T} |L_s^N(t,\omega) - L_s(t,\omega)| \leq 2 \sup_{0 \leq i \leq k} \sup_{0 \leq t \leq T} |L_{r_i}^N(t,\omega) - L_{r_i}(t,\omega)| + 2\varepsilon,$$

while from (2.4.16),

$$\limsup_{N \to \infty} \sup_{0 \leq s \leq C, 0 \leq t \leq T} |L_s^N(t,\omega) - L_s(t,\omega)| \leq 2\varepsilon.$$

This implies (2.4.17), since $\varepsilon > 0$ is arbitrary. The assertion of Theorem 2.4.1 is now immediate by combining (2.4.14), (2.4.15) and (2.4.17).

□

2.5 Change of measure and proof of Theorem 2.1.1

As in the previous section, let, for fixed $N \in \mathbb{N}$, H^N be a process that follows the dynamics described in Proposition 2.2.1 for $\theta = \gamma = 0$. We denote the underlying probability measure by $\mathbb{P}$, and the filtration by $\mathcal{F} = (\mathcal{F}_s)$. Our first aim is to construct, by a Girsanov reweighting of the restrictions $\mathbb{P}|_{\mathcal{F}_s}$, a measure $\tilde{\mathbb{P}}^N$ under which H^N follows the dynamics from Proposition 2.2.1 for a prescribed $\theta \geq 0$ and $\gamma > 0$.

Here, a crucial role is played by the point process P^N of the successive local maxima and minima of H^N, excluding the minima at height 0. Under $\mathbb{P}$, this is a Poisson process with intensity $\sigma^2 N^2$. More precisely, the process $Q^{1,N}$ which counts the successive local minima of H^N (except those at height 0) is a point process with predictable intensity $\lambda_s^{1,N} := N^2\sigma^2 \mathbf{1}_{\{V_{s-}^N=-1\}}$, and the process $Q^{2,N}$ which counts the successive local maxima of H^N is a point process with predictable intensity $\lambda_s^{2,N} := N^2\sigma^2 \mathbf{1}_{\{V_{s-}^N=+1\}}$. (Recall that the process V^N is the (càdlàg) sign of the slope of H^N.)

For the rest of this section we fix $\theta \geq 0$ and $\gamma > 0$. In view of Proposition 2.2.1 we want to change the rate $\lambda_s^{1,N}$ to $\lambda_s^{1,N}(1 + \frac{2\theta}{N\sigma^2})$ and the rate $\lambda_s^{2,N}$ to $\lambda_s^{2,N}(1 + \frac{4\gamma\Lambda_s^N(H_s)}{N\sigma^2})$.

As in Section 2.4 we will use the process $M_s^N = P_s^N - N\sigma^2 s$, $s \geq 0$, which is a martingale under $\mathbb{P}$. Taking the route designed by Proposition 2.6.13 in the Appendix, we consider the local martingales

$$X_s^{N,1} := \int_0^s \frac{2\theta}{N\sigma^2} \mathbf{1}_{\{V_{r-}^N=-1\}} dM_r^N, \quad X_s^{N,2} := \int_0^s \frac{\gamma L_r^N(H_r^N)}{N} \mathbf{1}_{\{V_{r-}^N=1\}} dM_r^N, \quad X^N := X^{N,1} + X^{N,2}.$$

Let $Y^N := \mathcal{E}(X^N)$ denote the Doléans exponential of X^N. Proposition 2.6.9 in the Appendix recalls this concept and the fact that Y^N is the solution of

$$Y_s^N = 1 + \int_0^s Y_{r-}^N \left(\frac{2\theta}{N\sigma^2} \mathbf{1}_{\{V_{r-}^N=-1\}} + \frac{\gamma L_r^N(H_r^N)}{N} \mathbf{1}_{\{V_{r-}^N=1\}} \right) dM_r^N, \quad s \geq 0. \qquad (2.5.1)$$

We will show that Y^N is a martingale, which from Proposition 2.6.13 will directly render the required change of measure.

Proposition 2.5.1. *Y^N is a $(\mathcal{F}, \mathbb{P})$-martingale.*

Proof. Under $\mathbb{P}$, Y^N is a positive super–martingale and a local martingale. It is a martingale if and only if

$$\mathbb{E}[Y_s^N] = Y_0^N = 1, \qquad (2.5.2)$$

which we will show. A key idea is to work along the excursions of H^N, that is, along the sequence of stopping times $\tau_a^{N,s} := S_{a/N}^N \wedge s$, $a = 0, 1, 2, \ldots$ Since N and s are fixed, we will suppress the superscripts N and s for brevity and write τ_a instead of $\tau_a^{N,s}$.

STEP 1. We want to show that

$$\mathbb{E}[Y_{\tau_a}^N] = 1, \quad a = 1, 2, \ldots \tag{2.5.3}$$

Between τ_{a-1} and τ_a, the solution of (2.5.1) is bounded above by the solution of the same equation with M^N replaced by P^N, which takes the form $d\tilde{Y}_r^N = \tilde{Y}_{r-}^N a_r^N dP_r^N$. If we denote by $\{T_k,\ k \geq 1\}$ the successive jump times of the Poisson process P^N, we have or each k $\tilde{Y}_{T_k}^N = \tilde{Y}_{T_k-}^N(1 + a_{T_k}^N)$. Consequently for all $\tau_{a-1} \leq r \leq \tau_a$,

$$\frac{Y_r^N}{Y_{\tau_{a-1}}^N} \leq \prod_{k\geq 1:\tau_{a-1}\leq T_k\leq\tau_a} \left(1 + \frac{2\theta}{N\sigma^2}\mathbf{1}_{\{V_{T_k-}^N=-1\}} + \frac{\gamma L_{T_k}^N(H_{T_k}^N)}{N}\mathbf{1}_{\{V_{T_k-}^N=1\}}\right).$$

Within the excursion of H^N between the times τ_{a-1} and τ_a, these jump times coincide with the times of the local maxima and minima of H^N in the time interval (τ_{a-1}, τ_a). Since for $a > 1$ there are reflections of X^N at 0 in the time interval $(0, \tau_{a-1})$, the parity of those k for which $V_{T_k-}^N = 1$, $\tau_{a-1} \leq T_k \leq \tau_a$, depends on a. However, noting that

$$L_{T_k}^N(H_{T_k}^N) \leq \frac{4}{N\sigma^2}k, \tag{2.5.4}$$

we infer the existence of a constant $c > 0$ such that

$$\frac{Y_r^N}{Y_{\tau_{a-1}}^N} \leq c^{P_{\tau_a}^N} \times (P_{\tau_a}^N + 1)!!\,, \quad \tau_{a-1} \leq r \leq \tau_a, \tag{2.5.5}$$

where we define for $k \in \mathbb{N}$, $k!! = 1 \cdot 3 \cdot 5 \cdots k$ if k is odd, and $k!! = 1 \cdot 3 \cdot 5 \cdots (k-1)$ if k is even.

Now

$$Y_{\tau_a}^N = Y_{\tau_{a-1}}^N \left(1 + \int_{\tau_{a-1}}^{\tau_a} \frac{Y_{r-}^N}{Y_{\tau_{a-1}}^N}[p(r) + L_{r-}^N(H_r^N)q(r)]dM_r^N\right), \tag{2.5.6}$$

where $0 \le p(r) \le \frac{2}{\theta}N^2$, $0 \le q(r) \le \frac{\gamma}{N}$, p and q are predictable. The claimed equalities $\mathbb{E}[Y^N_{\tau_a}] = 1$, $a = 1, 2, \ldots$, follow by induction on a from (2.5.6), provided the process

$$\mathcal{M}^N_s = \int_0^s \mathbf{1}_{]\tau_{a-1},\tau_a]}(r) \frac{Y^N_{r-}}{Y^N_{\tau_{a-1}}} [p(r) + L^N_{r-}(H^N_r) q(r)] dM^N_r, \quad s \ge 0,$$

is a martingale. From Theorem T8 in Brémaud [4] page 27, this is a consequence of the fact that

$$\mathbb{E} \int_0^s \mathbf{1}_{]\tau_{a-1},\tau_a]}(r) \frac{Y^N_{r-}}{Y^N_{\tau_{a-1}}} [p(r) + L^N_{r-}(H^N_r) q(r)] dr < \infty.$$

In order to verify the latter inequality, we compute

$$\begin{aligned}
\mathbb{E}\left[\int_{\tau_{a-1}}^{\tau_a} \frac{Y^N_{r-}}{Y^N_{\tau_{a-1}}} [p(r) + L^N_{r-}(H^N_r) q(r)] dr\right] &\le C_N s \mathbb{E}\left[c^{P^N_{\tau_a}} \times (P^N_{\tau_a} + 1)!!(1 + P^N_{\tau_a})\right] \\
&\le C_N s \mathbb{E}\left[c^{P^N_s} \times (P^N_s + 1)!!(1 + P^N_s)\right] \\
&\le C_N s C_{N,s},
\end{aligned}$$

where we have used (2.5.5), (2.5.4) and $\tau_a \le s$, and where C_N and $C_{N,s}$ are constants which depend only on N and (N, s), respectively. The fact that $C_{N,s} < \infty$ follows from

$$\mathbb{E}[c_2^{P^N_s} (P^N_s + 1)!! P^N_s] = \exp\{-N^2\sigma^2 s\} \sum_{k=0}^{\infty} c_2^k (k+1)!! k \frac{(N^2\sigma^2 s)^k}{k!}$$

Since

$$\begin{aligned}
\frac{(k+1)!!k}{k!} = \frac{k(k+1)}{2.4\ldots\big(2[\frac{k}{2}]\big)} &= \frac{k(k+1)}{2^{[\frac{k}{2}]}} . \frac{1}{[\frac{k}{2}]!} \\
&< \frac{1}{[\frac{k}{2}]!} \qquad \forall k \ge 20,
\end{aligned}$$

we deduce that $\mathbb{E}[c_2^{P^N_s} (P^N_s + 1)!! P^N_s] < \infty$. This completes the proof of (2.5.3).

Step 2. We can now define a consistent family of probability measures $\tilde{\mathbb{P}}^{N,s,a}$ on $\mathcal{F}_{\tau_a}$, $a = 1, 2, \ldots$ by putting

$$\frac{d\tilde{\mathbb{P}}^{N,s,a}}{d\mathbb{P}\mid_{\mathcal{F}_{\tau_a}}} = Y^N_{\tau_a}, \quad a \in \mathbb{N}.$$

We write $\tilde{\mathbb{P}}^{N,s}$ for the probability measure on the σ-field generated by union of the σ-fields $\mathcal{F}_{\tau_a}$, $a = 1, 2, \ldots$, whose restriction to $\mathcal{F}_{\tau_a}$ is $\tilde{\mathbb{P}}^{N,s,a}$ for all $a = 1, 2, \ldots$, and put

$$A := \inf\{a \in \mathbb{N} : \tau_a = s\}.$$

We will now show that

1. (i) $A < \infty \quad \tilde{\mathbb{P}}^{N,s}$- a.s. (and consequently $\tau_A = s \quad \tilde{\mathbb{P}}^{N,s}$- a.s.),

2. (ii) under $\tilde{\mathbb{P}}^{N,s}$, $(H_r^N)_{0 \le r \le \tau_A} = (H_r^N)_{0 \le r \le s}$ is a stochastic process following the dynamics specified in Proposition 2.2.1.

Indeed, applying Girsanov's theorem (Proposition 2.6.13 in the Appendix) to the 2-variate point process

$$(Q_r^{1,N}, Q_r^{2,N}) = \left(\int_0^r \mathbf{1}_{\{V_{u-}^N = -1\}} dP_u^N, \int_0^r \mathbf{1}_{\{V_{u-}^N = 1\}} dP_u^N \right), \quad 0 \le r \le \tau_a, \tag{2.5.7}$$

we have that under $\tilde{\mathbb{P}}^{N,s,a}$

$$\begin{aligned} Q_r^{1,N} &\quad \text{has intensity} \quad (N^2\sigma^2 + 2\theta N)\mathbf{1}_{\{V_{r-}^N = -1\}} dr \\ Q_r^{2,N} &\quad \text{has intensity} \quad \sigma^2[N^2 + \gamma N L_r^N(H_r^N)]\mathbf{1}_{\{V_{r-}^N = 1\}} dr. \end{aligned}$$

Thus, for all $a \in \mathbb{N}$, $(H_r^N)_{0 \le r \le \tau_a}$ is, under $\tilde{\mathbb{P}}^{N,s,a}$, a stochastic process following the dynamics from Proposition 2.2.1 up to the stopping time τ_a. Considering the sequence of excursions $(H_r^N)_{\tau_{a-1} \le r \le \tau_a}$, $a = 1, 2, \ldots$ under $\tilde{\mathbb{P}}^{N,s}$, we infer from Lemma 2.2.3 the validity of the claims (i) and (ii).

Step 3. We now prove (2.5.2). For this we observe that

$$\mathbb{E}[Y_s^N] = \sum_{a \ge 1} \mathbb{E}[Y_s^N; A = a] = \sum_{a \ge 1} \mathbb{E}[Y_{\tau_a}^N; A = a] = \sum_{a \ge 1} \tilde{\mathbb{P}}^{N,s}(A = a) = \tilde{\mathbb{P}}^{N,s}(A < \infty) = 1.$$

□

Corollary 2.5.2. *Let $\tilde{\mathbb{P}}^N$ be the probability measure on $\mathcal{F}$ whose restriction to $\mathcal{F}_s$, $s > 0$, has density Y_s^N (given by (2.5.1)) with respect to $\mathbb{P}|_{\mathcal{F}_s}$. Then under $\tilde{\mathbb{P}}^N$ the process H^N follows the dynamics from Proposition 2.2.1 for the prescribed θ and γ.*

Proof. This is immediate from Proposition 2.5.1 and the discussion preceding it, combined with Proposition 2.6.13 in the Appendix applied to the process defined in (2.5.7), now with $0 \le r < \infty$. □

Next we will analyze the behaviour of the Girsanov densities as $N \to \infty$. For this we use the two martingales $M^{1,N}$ and $M^{2,N}$ defined in (2.4.5), and note that (2.5.1) can be rewritten as

$$Y_s^N = 1 + \int_0^s Y_{r-}^N \{\theta dM_r^{1,N} + \frac{\gamma\sigma^2 L_r^N(H_r^N)}{2} dM_r^{2,N}\}, \quad s \ge 0.$$

The two (pure jump) martingales $M^{1,N}$ and $M^{2,N}$ have jump sizes $2/(N\sigma^2)$, hence the random variable under the expectation in formula (2.6.6) vanishes for suitably large N. Thus (see Definition 2.6.10 in the Appendix), the sequences $\{M^{1,N}\}_{N\ge1}$ and $\{M^{2,N}\}_{N\ge1}$ have uniformly controlled variations, and because of Proposition 2.6.11 (1) they are "good". Hence

$$X_\cdot^N \Rightarrow \int_0^\cdot \{\frac{\sqrt{2}\theta}{\sigma} dB_r^1 + \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2} dB_r^2\} := X_\cdot.$$

Moreover, by Proposition 2.6.11 (3), $\{X_s^N\}_{N\ge1}$ is also a good sequence, hence by Proposition 2.6.11 (2)

$$Y^N = \mathcal{E}(X^N) \Rightarrow \mathcal{E}(X) =: Y.$$

Combining these facts with Corollary 2.4.1, we deduce, again from Proposition 2.6.11 (3), that

$$(H^N, L_{S_x^N}^N, Y^N) \Rightarrow (H, L_{S_x}, Y). \tag{2.5.8}$$

Since B^1 and B^2 are mutually orthogonal, by Proposition 2.6.9 we have

$$\begin{aligned} Y_s &= \mathcal{E}\Big(\int_0^{\cdot}\{\frac{\sqrt{2}\theta}{\sigma}dB_r^1 + \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2}dB_r^2\}\Big)_s \\ &= \mathcal{E}\Big(\frac{\sqrt{2}\theta}{\sigma}B^1\Big)_s\,\mathcal{E}\Big(\int_0^{\cdot}\frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2}dB_r^2\Big)_s \\ &= \exp\Big\{\frac{\sqrt{2}\theta}{\sigma}B_s^1 + \int_0^s \frac{\sqrt{2}\gamma\sigma L_r(H_r)}{2}dB_r^2 - \int_0^s \big[\frac{\theta^2}{\sigma^2} + \frac{\gamma^2\sigma^2}{4}L_r(H_r)^2\big]dr\Big\} \end{aligned}$$

Let us recall two lemmas. The first one is Theorem 1.1, chapter 7 (page 152) in [6].

Lemma 2.5.3. *Assume that the quadratic variation of the continuous local martingale M is of the form $\langle M\rangle_s = \int_0^s R_r\,dr$, and that for all $s > 0$ there exist constants $a > 0$ and $c < \infty$ such that*

$$\mathbb{E}\exp(aR_r) \le c, \qquad 0 \le r \le s.$$

Then

$$\mathbb{E}\exp\left(M_s - \frac{1}{2}\langle M\rangle_s\right) = 1, \qquad s \ge 0,$$

holds.

The next lemma is proved in [14].

Lemma 2.5.4. *Let H be a Brownian motion on $\mathbb{R}_+$ reflected at the origin, with variance parameter v^2. Then for all $s > 0$ there exists $\alpha = \alpha(s, v) > 0$ and a constant $c < \infty$ such that*

$$\mathbb{E}\left(\exp(\alpha L_r(H_r)^2)\right) \le c, \qquad 0 \le r \le s.$$

Applying those two lemmas, we deduce that Y is a martingale. In particular $\mathbb{E}[Y_s] = 1$ for all $s \ge 0$. Define the probability measure $\tilde{\mathbb{P}}$ by

$$\frac{d\tilde{\mathbb{P}}\,|_{\mathcal{F}_s}}{d\mathbb{P}\,|_{\mathcal{F}_s}} = Y_s, \quad \forall s \ge 0,$$

then H, under $\tilde{\mathbb{P}}$, solves the SDE (2.1.3) with B_s there replaced by

$$\tilde{B}_s := \frac{1}{\sqrt{2}}(B_s^1 - B_s^2) - \frac{\theta}{\sigma}s + \frac{\gamma\sigma}{2}\int_0^s L_r(H_r)dr,$$

which is a standard Brownian motion under $\tilde{\mathbb{P}}$ due to Proposition 2.6.12.

The following general and elementary Lemma will allow us to conclude the required convergence under the transformed measures.

Lemma 2.5.5. *Let (ξ_N, η_N), (ξ, η) be random pairs defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with η_N, η nonnegative scalar random variables, and ξ_N, ξ taking values in some complete separable metric space $\mathcal{X}$. Assume that $\mathbb{E}[\eta_N] = \mathbb{E}[\eta] = 1$. Write $(\tilde{\xi}_N, \tilde{\eta}_N)$ for the random pair (ξ_N, η_N) defined under the probability measure $\tilde{\mathbb{P}}^N$ which has density η_N with respect to $\mathbb{P}$, and $(\tilde{\eta}, \tilde{\xi})$ for the random pair (η, ξ) defined under the probability measure $\tilde{\mathbb{P}}$ which has density η with respect to $\mathbb{P}$. Then $(\tilde{\xi}_N, \tilde{\eta}_N)$ converges in distribution to $(\tilde{\eta}, \tilde{\xi})$, provided that (ξ_N, η_N) converges in distribution to (ξ, η).*

Proof. Due to the equality $\mathbb{E}[\eta_N] = \mathbb{E}[\eta] = 1$ and a variant of Scheffé's theorem (see Thm. 16.12 in [2]), the sequence η_N is uniformly integrable. Hence for all bounded continuous $F : \mathcal{X} \times \mathbb{R}_+ \to \mathbb{R}$,

$$\mathbb{E}[F(\tilde{\xi}_N, \tilde{\eta}_N)] = \mathbb{E}[F(\xi_N, \eta_N)\eta_N] \to \mathbb{E}[F(\xi, \eta)\eta] = \mathbb{E}[F(\tilde{\xi}, \tilde{\eta})].$$

□

Combining (2.5.8) with Lemma 2.5.5 yields the

Theorem 2.5.1. *Let H^N be a stochastic process following the dynamics specified in Proposition 2.2.1, and let H be the unique weak solution of the SDE* (2.1.3)*. We have*

$$(H^N, L^N_{S^N_x}) \Rightarrow (H, L_{S_x}) \quad in \quad C([0,\infty]) \times D([0,\infty]), \tag{2.5.9}$$

where S^N_x and S_x are defined in (2.4.2) *and* (2.1.2)*.*

We can now proceed with the

Completion of the proof of Theorem 2.1.1 : Define $Z^{N,x}_t := \frac{\sigma^2}{4} L^N_{S^N_x}(t)$. By Corollary 2.2.2, $Z^{N,x}$ follows the dynamics (2.2.1). From (2.5.9), $\frac{\sigma^2}{4} L_{S_x}$ is the limit in distribution of $Z^{N,x}$ as $N \to \infty$. Hence by Proposition 2.3.3, $t \mapsto \frac{\sigma^2}{4} L_{S_x}(t)$ is a weak solution of the SDE (2.1.1), which completes the proof of Theorem 2.1.1.

□

Remark 2.5.6. Theorem 2.1.1 establishes a correspondence between the solution H of the SDE (2.1.3) and the logistic Feller process, i.e. the solution of (2.1.1). This connection can be expressed in particular through the occupation times formula for H, which states that for any Borel measurable and positive valued function f,

$$\int_0^{S_x} f(H_s)ds = \int_0^\infty f(t) Z_t^x dt.$$

This formula in the particular case $f \equiv 1$ states that

$$S_x = \int_0^\infty Z_t^x dt.$$

The quantity on the right is the area under the trajectory Z^x. It is the limit of the properly scaled total branch length of the approximating forests F^N defined in Section 2.2. We now establish another identity concerning this same quantity, with the help of a time change introduced by Lambert in [9]. Consider the additive functional

$$A_t = \int_0^t Z_r^x dr,$$

and the associated time change

$$\alpha_t = \inf\{r > 0, \; A_r > t\}.$$

As noted in [9], the process $U_t^x := Z_{\alpha_t}^x$ is an Ornstein–Uhlenbeck process, solution of the SDE

$$dU_t^x = (\theta - \gamma U_t^x)dt + \sigma dB_t, \quad U_0^x = x.$$

Of course this identification is valid only for $0 \le t \le \tau_x$, where $\tau_x := \inf\{t > 0, \; U_t = 0\}$. Let T_x be the extinction time of the logistic Feller process Z_t^x. We clearly have $\alpha_{\tau_x} = T_x$, and consequently

$$\tau_x = \int_0^\infty Z_r^x dr.$$

We have identified the time at which the local time at 0 of the exploration process H reaches x with the area under the logistic Feller trajectory starting from x, and with the time taken by the Ornstein–Uhlenbeck process U^x to reach 0. The reader may notice that in the particular case $\gamma = 0$, the identity $S_x = \tau_x$ is not a surprise, see also the discussion and the references in [13] Section 6.

2.6 Appendix

2.6.1 Skorohod's topology and tightness in $D([0,\infty))$

We denote by $D([0,\infty))$ the space of functions from $[0,\infty)$ into $\mathbb{R}$ which are right continuous and have left limits at any $t > 0$ (as usual such a function is called càdlàg). We briefly write $\mathbb{D}$ for the space of adapted, càdlàg stochastic processes. We shall always equip the space $D([0,\infty))$ with the Skorohod topology, for the definition of which we refer the reader to Billingsley [3] or Joffe, Métivier [7]. The next Lemma follows from considerations which can be found in [3], bottom of page 124.

Lemma 2.6.1. *Suppose $\{x_n, n \geq 1\} \subset D([0,\infty))$ and $x_n \to x$ in the Skorohod topology.*

(i) If x is continuous, then x_n converges to x locally uniformly.

(ii) If each x_n is continuous, then so is x, and x_n converges to x locally uniformly.

In particular, the space $C([0,\infty))$ is closed in $D([0,\infty))$ equipped with the Skorohod topology.

The following two lemmas are used in the proofs of Propositions 2.4.6 and 2.4.3:

Lemma 2.6.2. *Fix $t > 0$. Let $x_n, x \in C([0,\infty)), n \geq 1$ be such that*

1. *$x_n \to x$ locally uniformly, as $n \to \infty$.*

2. *for each* $s > 0$,

$$\int_0^s \mathbf{1}_{\{x(r)=t\}} dr = 0.$$

Then

$$\int_0^s \mathbf{1}_{\{x_n(r)>t\}} dr \to \int_0^s \mathbf{1}_{\{x(r)>t\}} dr \quad \textit{locally uniformly in } s \geq 0.$$

Proof. We prove convergence for each $s > 0$. The local uniformity is then easy. Given $\varepsilon > 0$, there exists N_0 such that

$$\sup_{0 \leq r \leq s} |x_n(r) - x(r)| < \varepsilon \quad \forall n \geq N_0.$$

Then for all $n \geq N_0$,

$$|\mathbf{1}_{\{x_n(r)>t\}} - \mathbf{1}_{\{x(r)>t\}}| \leq \mathbf{1}_{\{t-\varepsilon<x(r)<t+\varepsilon\}}$$
$$\Big| \int_0^s \mathbf{1}_{\{x_n(r)>t\}} dr - \int_0^s \mathbf{1}_{\{x(r)>t\}} dr \Big| \leq \int_0^s \mathbf{1}_{\{t-\varepsilon<x(r)<t+\varepsilon\}} dr.$$

The result follows from

$$\lim_{\varepsilon \to 0} \int_0^s \mathbf{1}_{\{t-\varepsilon<x(r)<t+\varepsilon\}} dr = \int_0^s \mathbf{1}_{\{x(r)=t\}} dr = 0.$$

□

Lemma 2.6.3. *Let* $x_n, y_n \in D([0,\infty)), n \geq 1$ *and* $x, y \in C([0,\infty))$ *be such that*

1. *for all* $n \geq 1$*, the function* $t \to y_n(t)$ *is increasing;*

2. $x_n \to x$ *and* $y_n \to y$*, both locally uniformly.*

Then y *is increasing and*

$$\int_0^t x_n(s) dy_n(s) \to \int_0^t x(s) dy(s), \quad \textit{locally uniformly in } t \geq 0.$$

Proof. We prove convergence for each $t > 0$. The local uniformity is then easy.

$$\begin{aligned}
&\Big|\int_0^t x(s)dy(s) - \int_0^t x_n(s)dy_n(s)\Big| \\
&\le \Big|\int_0^t [x(s) - x_n(s)]dy_n(s)\Big| + \Big|\int_0^t x(s)[dy(s) - dy_n(s)]\Big| \\
&\le \sup_{0\le s\le t} |x(s) - x_n(s)|y_n(t) + \int_0^t |x(s) - \xi_\varepsilon(s)|[dy(s) - dy_n(s)] + \int_0^t |\xi_\varepsilon(s)|[dy(s) - dy_n(s)],
\end{aligned}$$

where ξ_ε is a step function which is such that $\sup_{0\le s\le t} |x(s) - \xi_\varepsilon(s)| \le \varepsilon$. The first and last term of the above right hand side clearly tend to 0 as $n \to \infty$. Then

$$\begin{aligned}
\limsup_{n\to\infty} \Big|\int_0^t x(s)dy(s) - \int_0^t x_n(s)dy_n(s)\Big| &\le \varepsilon \limsup_{n\to\infty} [y_n(t) + y(t)] \\
&\le 2y(t) \times \varepsilon.
\end{aligned}$$

It remains to let $\varepsilon \to 0$. □

We first state a tightness criterion, which is Theorem 13.5 from [3] :

Proposition 2.6.4. *Let* $\{X^n_t,\ t \ge 0\}_{n\ge1}$ *be a sequence of random elements of* $D([0,\infty))$. *A sufficient condition for* $\{X^n\}$ *to be tight is that the two conditions (i) and (ii) be satisfied :*

(i) For each $t \ge 0$, *the sequence of random variables* $\{X^n_t,\ n \ge 1\}$ *is tight in* $\mathbb{R}$;

(ii) for each $T > 0$, *there exists* $\beta, C > 0$ *and* $\theta > 1$ *such that*

$$\mathbb{E}\left(\left|X^n_{t+h} - X^n_t\right|^\beta \left|X^n_t - X^n_{t-h}\right|^\beta\right) \le Ch^\theta,$$

for all $0 \le t \le T$, $0 \le h \le t$, $n \ge 1$.

Note that convergence in $D([0,\infty))$ is not additive : $x_n \to x$ and $y_n \to y$ in $D([0,\infty))$ does not imply that $x_n + y_n \to x + y$ in $D([0,\infty))$. This is due to the fact that to the sequence x_n is attached a sequence of time changes, and to the sequence y_n is attached another sequence of time changes, such that the time changed x_n and y_n

converge uniformly. But there may not exist a sequence of time changes which makes $x_n + y_n$ converge. If now $\{X_t^n,\ t \geq 0\}_{n\geq 1}$ and $\{Y_t^n,\ t \geq 0\}_{n\geq 1}$ are two tight sequences of random elements of $D([0,\infty))$, we cannot conclude that $\{X_t^n + Y_t^n,\ t \geq 0\}_{n\geq 1}$ is tight. However, if $x_n \to x$ and $y_n \to y$ in $D([0,\infty))$ and x is continuous, then we deduce easily from Lemma 2.6.1 that $x_n + y_n \to x + y$ in $D([0,\infty))$. It follows

Proposition 2.6.5. *If $\{X_t^n,\ t \geq 0\}_{n\geq 1}$ and $\{Y_t^n,\ t \geq 0\}_{n\geq 1}$ are two tight sequences of random elements of $D([0,\infty))$ such that any limit of a weakly converging subsequence of the sequence $\{X_t^n,\ t \geq 0\}_{n\geq 1}$ is a. s. continuous, then $\{X_t^n + Y_t^n,\ t \geq 0\}_{n\geq 1}$ is tight in $D([0,\infty))$.*

Consider a sequence $\{X_t^n, t \geq 0\}_{n\geq 1}$ of one-dimensional semi–martingales, which is such that for each $n \geq 1$,

$$X_t^n = X_0^n + \int_0^t \varphi_n(X_s^n)ds + M_t^n, \qquad t \geq 0;$$
$$\langle M^n \rangle_t = \int_0^t \psi_n(X_s^n)ds, \qquad t \geq 0;$$

where for each $n \geq 1, M^n$ is a locally square-integrable martingale, φ_n and ψ_n are Borel measurable functions from $\mathbb{R}$ into $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

The following result is an easy consequence of Theorems 16.10 and 13.4 from [3].

Proposition 2.6.6. *A sufficient condition for the above sequence $\{X_t^n, t \geq 0\}_{n\geq 1}$ of semi–martingales to be tight in $D([0,\infty))$ is that both*

$$\text{the sequence of r.v.'s } \{X_0^n, n \geq 1\} \text{ is tight;} \tag{2.6.1}$$

and for some $p > 1$,

$$\forall T > 0,\ \text{the sequence of r.v.'s } \Big\{ \int_0^T [\mid \varphi_n(X_t^n) \mid + \psi_n(X_t^n)]^p dt, n \geq 1 \Big\} \text{ is tight.} \tag{2.6.2}$$

Those conditions imply that both the bounded variation parts $\{V^n, n \geq 1\}$ and the martingale parts $\{M^n, n \geq 1\}$ are tight, and that the limit of any converging subsequence of $\{V^n\}$ is a.s. continuous.

If moreover, for any $T > 0$, as $n \to \infty$,

$$\sup_{0\le t\le T} \mid M_t^n - M_{t-}^n \mid \to 0 \quad \textit{in probability,}$$

then any limit X of a converging subsequence of the original sequence $\{X^n\}_{n\ge 1}$ is a.s. continuous.

Remark 2.6.7. A sufficient condition for (2.6.2) is that for all $T > 0$,

$$\{\sup_{0\le t\le T} [|\varphi_n(X_t^n)| + \psi_n(X_t^n)],\ n \ge 1\} \text{ is tight.} \tag{2.6.3}$$

Remark 2.6.8. A sufficient condition for (2.6.2) is that for all $T > 0$

$$\limsup_{n\ge 1} \sup_{0\le t\le T} \mathbb{E}[\varphi_n^2(X_t^n) + \psi_n^2(X_t^n)] < \infty. \tag{2.6.4}$$

Indeed, (2.6.4) yields

$$\limsup_n \mathbb{E} \int_0^T [|\varphi_n(X_t^n)| + \psi_n(X_t^n)]^2 dt < \infty,$$

which in turn implies (2.6.2).

2.6.2 Doléans exponential and "goodness"

For a càdlàg semi–martingale $X = (X_t, t \ge 0)$, consider the stochastic linear equation of Doléans

$$Y_t = 1 + \int_0^t Y_{r-} dX_r. \tag{2.6.5}$$

The following proposition follows from Theorem 1 and Theorem 2 in [10], page 122.

Proposition 2.6.9. (1) *Equation* (2.6.5) *has a unique solution (up to indistinguishability) within the class of semi–martingales. This solution is denoted by $\mathcal{E}(X)$ and is called the Doléans exponential of X. It has the following representation*

$$\mathcal{E}(X)_t = \exp\{X_t - X_0 - \frac{1}{2}\langle X^c\rangle_t\} \prod_{r\le t} (1 + \Delta X_r) e^{-\Delta X_r}.$$

(2) *If U and X are two semi–martingales, then*

$$\mathcal{E}(U)_t\,\mathcal{E}(X)_t = \mathcal{E}(U + X + [U, X])_t.$$

(3) *If X is a local martingale, then $\mathcal{E}(X)$ is a nonnegative local martingale and a super–martingale.*

For $\delta > 0$ we define $h_\delta : \mathbb{R}_+ \to \mathbb{R}_+$ by $h_\delta(r) = (1 - \delta/r)^+$. For $x \in D([0, \infty))$, we define $x^\delta \in D([0, \infty))$ by

$$x_t^\delta := x_t - \sum_{0<s\leq t} h_\delta(\mid \Delta x_s \mid)\Delta x_s.$$

Definition 2.6.10. (1) *Let G, G^n in $\mathbb{D}, \{G^n, n \geq 1\}$ be a sequence of semimartingales adapted to a given filtration $(\mathcal{F}_t)$ and assume $G^n \Rightarrow G$ as $n \to \infty$. The sequence (G^n) is called* good *if for any sequence $\{I^n, n \geq 1\}$ of $(\mathcal{F}_t)$–progressively measurable processes in $\mathbb{D}$ such that $(I^n, G^n) \Rightarrow (I, G)$ as $n \to \infty$, then G is a semi–martingale for a filtration with respect to which I is adapted, and $(I^n, G^n, \int I^n_{s-} dG^n_s) \Rightarrow (I, G, \int I_{s-} dG_s)$ as $n \to \infty$.*

(2) *A sequence of semi–martingales $\{G^n\}_{n\geq 1}$ is said to have* uniformly controlled variations *if there exists $\delta > 0$, and for each $\alpha > 0, n \geq 1$, there exists a semi–martingale decomposition $G^{n,\delta} = M^{n,\delta} + A^{n,\delta}$ and a stopping time $T^{n,\alpha}$ such that $\mathbb{P}(\{T^{n,\alpha} \leq \alpha\}) \leq \frac{1}{\alpha}$ and furthermore*

$$\sup_n \mathbb{E}\Big\{[M^{n,\delta}, M^{n,\delta}]_{t\wedge T^{n,\alpha}} + \int_0^{t\wedge T^{n,\alpha}} \mid dA^{n,\delta} \mid \Big\} < \infty. \tag{2.6.6}$$

It follows from pages 32 ff. in [8]

Proposition 2.6.11. *Let G, G^n in $\mathbb{D}, \{G^n, n \geq 1\}$ be a sequence of semi–martingales and assume $G^n \Rightarrow G$.*

(1) *The sequence $\{G^n\}$ is good if and only if it has uniformly controlled variations.*

(2) *If* $\{G^n\}$ *is good, then* $(G^n, \mathcal{E}(G^n)) \Rightarrow (G, \mathcal{E}(G))$.

(3) *Suppose* $(I^n, G^n) \Rightarrow (I, G)$, *and* $\{G^n\}$ *is good. Then* $J^n = \int I^n_{s-} dG^n_s$, $n = 1, 2, \ldots$, *is also a good sequence of semi–martingales. Moreover under the same conditions,* $(I^n, G^n, \mathcal{E}(G^n)) \Rightarrow (I, G, \mathcal{E}(G))$.

2.6.3 Two Girsanov theorems

We state two multivariate versions of the Girsanov theorem, one for the Brownian and one for the point process case. The second one combines Theorems T2 and T3 from [4], pages 165–166.

Proposition 2.6.12. *Let* $\{(B^{(1)}_s, \ldots, B^{(d)}_s),\ s \geq 0\}$ *be a d–dimensional standard Brownian motion defined on the filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *Moreover, let* $\phi = (\phi_1, ..., \phi_d)$ *be an* $\mathcal{F}$*-progressively measurable process with* $\int_0^s \phi_i(r)^2 dr < \infty$ *for all* $1 \leq i \leq d$ *and* $s \geq 0$. *Let* $X^{(i)}_s := \int_0^s \phi_i(r)\, dB^{(i)}_r$ *and put* $Y := \mathcal{E}(X^{(1)} + \cdots + X^{(d)})$, *or in other words*

$$Y_s = \exp\Big\{ \int_0^s \langle \phi(r), dB_r \rangle - \frac{1}{2} \int_0^s \mid \phi(r) \mid^2 dr \Big\}.$$

If $\mathbb{E}[Y_s] = 1$, $s \geq 0$, *then* $\tilde{B}_s := B_s - \int_0^s \phi(r) dr$, $s \geq 0$, *is a d-dimensional standard Brownian motion under the probability measure* $\tilde{\mathbb{P}}$ *defined by* $d\tilde{\mathbb{P}} \mid_{\mathcal{F}_s} / d\mathbb{P} \mid_{\mathcal{F}_s} = Y_s$, $s \geq 0$.

Proposition 2.6.13. *Let* $\{(Q^{(1)}_s, ..., Q^{(d)}_s), s \geq 0\}$ *be a d-variate point process adapted to some filtration* $\mathcal{F}$, *and let* $\{\lambda^{(i)}_s, s \geq 0\}$ *be the predictable* $(\mathbb{P}, \mathcal{F})$*-intensity of* $Q^{(i)}, 1 \leq i \leq d$. *Assume that none of the* $Q^{(i)}$, $Q^{(j)}$, $i \neq j$, *jump simultaneously. Let* $\{\mu^{(i)}_r, r \geq 0\}, 1 \leq i \leq d$, *be nonnegative* $\mathcal{F}$*-predictable processes such that for all* $s \geq 0$ *and all* $1 \leq i \leq d$

$$\int_0^s \mu^{(i)}_r \lambda^{(i)}_r dr < \infty \qquad \mathbb{P}\text{ -a.s.}$$

For $i = 1, \ldots, d$ *and* $s \geq 0$ *define*

$$X^{(i)}_s := \int_0^s (\mu^{(i)}_r - 1) dM^{(i)}_r, \quad Y^{(i)} := \mathcal{E}(X^{(i)}), \quad Y = \mathcal{E}(X^{(1)} + \ldots + X^{(d)}).$$

Then, with $\{T^i_k, k = 1, 2 \ldots\}$ *denoting the jump times of* $Q^{(i)}$,

$$Y^{(i)}_s = \Big(\prod_{k \geq 1 : T^i_k \leq s} \mu^{(i)}_{T^i_k} \Big) \exp \Big\{ \int_0^s (1 - \mu^{(i)}_r) \lambda^{(i)}_r dr \Big\} \quad \textit{and} \quad Y_s = \prod_{j=1}^{d} Y^{(j)}_s, \quad s \geq 0.$$

If $\mathbb{E}[Y_s] = 1$, $s \geq 0$, *then, for each* $1 \leq i \leq d$, *the process* $Q^{(i)}$ *has the* $(\tilde{\mathbb{P}}, \mathcal{F})$*-intensity* $\tilde{\lambda}^{(i)}_r = \mu^{(i)}_r \lambda^{(i)}_r$, $r \geq 0$, *where the probability measure* $\tilde{\mathbb{P}}$ *is defined by* $d\tilde{\mathbb{P}} \mid_{\mathcal{F}_s} / d\mathbb{P} \mid_{\mathcal{F}_s} = Y_s$, $s \geq 0$.

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Chapter 3

HEIGHT AND THE TOTAL MASS OF THE FOREST OF GENEALOGICAL TREES OF A LARGE POPULATION WITH GENERAL COMPETITION

3.1 Introduction

Consider a continuous time branching process, which takes values either in $\mathbb{N}$ or in $\mathbb{R}_+$ (in the second case one speaks of a continuous state branching process, and we shall consider only those such processes with continuous paths). Such processes can be used as models of population growth. However, in that context one might want to model interactions between the individuals (e.g. competition for limited resources) so that we no longer have a branching process. Such interactions can increase the number of births, or in contrary increase the number of deaths. The popular logistic competition has been considered in Le, Pardoux, Wakolbinger [10], while a much more general type of interaction appears in Ba, Pardoux [4].

We will assume that for large population size the interaction is of the type of a competition, which limits the size of the population. One may then wonder in which cases the interaction is strong enough so that the extinction time (or equivalently the height of the forest of genealogical trees) remains finite, as the number of ancestors tends to infinity, or even such that the length of the forest of genealogical trees (which in the case of continuous state is rather called its total mass) remains finite, as the population size tends to infinity.

This question has been addressed in the case of a polynomial interaction in Ba, Pardoux [3]. Here we want to generalize those results to a very general type of competition, and we will also show that whenever our condition enforces a finite extinction time (resp. total mass) for the process started with infinite mass, that random variable has some finite exponential moments.

Let us describe the two classes of models which we will consider.

We first describe the discrete state model. Consider a population evolving in continuous time with m ancestors at time $t = 0$, in which each individual, independently of the others, gives birth to one child at a constant rate λ, and dies after an exponential time with parameter μ. For each individual we superimpose additional birth and death rates due to interactions with others at a certain rate which depends upon the other individuals in the population. More precisely, given a function $f : \mathbb{R}_+ \to \mathbb{R}$ which satisfies assumption (H1) below, whenever the total size of the population is k, the total additional birth rate due to interactions is $\sum_{j=1}^{k}(f(j) - f(j-1))^+$, while the total additional death rate due to interactions is $\sum_{j=1}^{k}(f(j) - f(j-1))^-$. Let X_t^m denote the population size at time $t > 0$, originating from m ancestors at time 0. The above description is good enough for prescribing the evolution of $\{X_t^m,\ t \geq 0\}$ with one value of m. There is a natural way to couple those evolutions for different values of m which will be described in section 3.2 below, such that $m \mapsto X_t^m$ is increasing for all $t \geq 0$, a.s.

If we consider this population with $m = [Nx]$ ancestors at time $t = 0$, replace λ by $\lambda_N = 2N$, μ by $\mu_N = 2N$, f by $f_N(x) = Nf(x/N)$, and define the weighted population size process $Z_t^N = N^{-1}X_t^N$, it is shown in [4] that Z^N converges weakly to the unique solution of the SDE (see Dawson, Li [7])

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + 2\int_0^t \int_0^{Z_s^x} W(ds, du), \tag{3.1.1}$$

where W is space–time white noise on $\mathbb{R}_+ \times \mathbb{R}_+$. This SDE couples the evolution of the various $\{Z_t^x,\ t \geq 0\}$ jointly for all values of $x > 0$.

We will use the fact that for a given value of $x > 0$, there exists a standard Brownian motion W, such that

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + 2\int_0^t \sqrt{Z_s^x}dW_s. \tag{3.1.2}$$

There is a natural way of describing the genealogical tree of the discrete population. The notion of genealogical tree is discussed for the limiting continuous population as well in [10, 12], in terms of continuous random trees in the sense of Aldous [1]. Clearly one can define the height H^m and the length L^m of the discrete forest of genealogical trees, as well as the height of the continuous "forest of genealogical trees", equal to the lifetime T^x of the process Z^x, and the total mass of the same forest of trees, given by $S^x := \int_0^{T^x} Z_t^x dt$.

Our assumption concerning the function f will be

Hypothesis (H1): $f \in C(\mathbb{R}_+, \mathbb{R})$, $f(0) = 0$, and there exists $\theta \geq 0$ such that

$$f(x+y) - f(y) \leq \theta x \qquad \forall x, y \geq 0.$$

Note that the hypothesis (H1) implies that the function $\theta x - f(x)$ is increasing. In particular, we have

$$f(x) \leq \theta x \qquad \forall x \geq 0.$$

This chapter is organized as follows. Section 3.2 studies the discrete case, i.e. the case of $\mathbb{N}$–valued processes, while section 3.3 studies the continuous case, i.e. the case of $\mathbb{R}$–valued processes. Each of those two sections starts with a subsection presenting necessary preliminary material. The main results in the discrete case are Theorem 3.2.3 and 3.2.4, while the main results in the continuous case are Theorem 3.3.2, 3.3.3 and 3.3.4. Section 3.4 gives some examples to illustrate our results.

Remark 3.1.1. This remark aims at helping the reader to build his intuition about our results. Take first a locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}_+$, such that for simplicity $f(x) > 0$, for all x, and consider the ODE $\dot{x} = f(x)$. It is easily seen that the solution x explodes in finite time iff $\int_0^\infty dx/f(x) < \infty$, and in that case, denoting t_∞ the time of explosion, $\int_0^{t_\infty} x(t)dt < \infty$ iff $\int_0^\infty x dx/f(x) < \infty$.

Reversing time, we deduce that if now $f(x) < 0$ for all x (or all x sufficiently large), the same ODE has a solution which satisfies $x(t) \in \mathbb{R}$ for all $t \in (0; T]$ for some $T > 0$ and $x(t) \to +\infty$ as $t \to 0$ (i.e. in a sense $x(0) = +\infty$) iff for some $M > 0$, $\int_M^\infty dx/|f(x)| < \infty$, and that solution is locally integrable near t = 0 iff $\int_M^\infty xdx/|f(x)| < \infty$. The fact that these results can be extended to certain SDEs is essentially our argument in the continuous population case, see section 3.3 below. Once this is understood, it is clear that similar results might be expected to hold true in the finite population case, which is the content of section 3.2.

3.2 The discrete case

3.2.1 Preliminaries

We consider a continuous time $\mathbb{Z}_+$–valued population process $\{X_t^m, t \geq 0, m \geq 1\}$, which starts at time zero from the initial condition $X_0^m = m$, i.e. m is the number of ancestors of the whole population. $\{X_t^m, t \geq 0\}$ is a continuous time $\mathbb{Z}_+$–valued Markov process, which evolves as follows. If $X_t^m = 0$, then $X_s^m = 0$ for all $s \geq t$. While at state $k \geq 1$, the process

$$X_t^m \quad \text{jumps to} \begin{cases} k+1, & \text{at rate} \quad \lambda k + F^+(k) \\ k-1, & \text{at rate} \quad \mu k + F^-(k), \end{cases}$$

where f is a function satisfying (H1), λ, μ are positive constants, and

$$F^+(k) := \sum_{\ell=1}^{k} (f(\ell) - f(\ell-1))^+, \qquad F^-(k) := \sum_{\ell=1}^{k} (f(\ell) - f(\ell-1))^-.$$

We now describe a joint evolution of all $\{X_t^m, t \geq 0\}_{m\geq 1}$, or in other words of the two-parameter process $\{X_t^m, t \geq 0, m \geq 1\}$, which is consistent with the above prescriptions. Suppose that the m ancestors are arranged from left to right. The left/right order is passed on to their offsprings: the daughters are placed on the right of their mothers and if at a time t the individual i is located at the right of individual j, then all the offsprings of i after time t will be placed on the right of all the offsprings of j. Since we have excluded multiple births at any given time, this means that the forest of genealogical trees of the population is a planar forest of trees, where the ancestor of the population X_t^1 is placed on the far left, the ancestor of $X_t^2 - X_t^1$ immediately on his right, etc... Moreover, we draw the genealogical trees in such a way that distinct branches never cross. This defines in a non–ambiguous way an order from left to right within the population alive at each time t. We decree that each individual feels the interaction with the others placed on his left but not with those on his right. Precisely, at any time t, the individual i has an interaction death rate equal to $(f(\mathcal{L}_i(t)+1) - f(\mathcal{L}_i(t)))^-$ or an interaction birth rate equal to $(f(\mathcal{L}_i(t)+1) - f(\mathcal{L}_i(t)))^+$, where $\mathcal{L}_i(t)$ denotes the number of individuals alive at time t who are located on the left of i in the above planar picture. This means that the individual i is under attack by the others located at his left if $f(\mathcal{L}_i(t)+1) - f(\mathcal{L}_i(t)) < 0$ while the interaction improves his fertility if $f(\mathcal{L}_i(t)+1) - f(\mathcal{L}_i(t)) > 0$. Of course, conditionally upon $\mathcal{L}_i(\cdot)$, the occurence of a "competition death event" or an "interaction birth event" for individual i is independent of the other birth/death events and of what happens to the other individuals. In order to simplify our formulas, we suppose moreover that the first individual in the left/right order has a birth rate equal to $\lambda + f^+(1)$ and a death rate equal to $\mu + f^-(1)$.

It is plain that with this definition, $m \to \{X_t^m,\ t \geq 0\}$ is increasing, since the progeny of the $m+1$–th ancestor does not modify the fate of the progeny of the m first ancestors. Let us moreover verify that this description of the evolution of the two-parameter process $\{X_t^m, t \geq 0, m \geq 1\}$ is consistent with the jump rates from k to $k+1$ and $k-1$ which have been indicated above. This is a consequence of the fact that the rate at which X_t^m jumps from k to $k+1$ is the sum of the individual birth rates of k individuals, one having zero left neighbour, a second one having one left neighbour, etc..., the last one having $k-1$ left neighbours. A similar argument applies for the rate of jump from k to $k-1$, replacing birth rates by death rates.

Remark 3.2.1. The functions F^+ and F^- may look a bit strange. However, if f is either increasing or decreasing, which is the case in particular if f is linear, then

$F^+ = f^+$ and $F^- = f^-$.

Define the height and length of the genealogical forest of trees by

$$H^m = \inf\{t > 0, X_t^m = 0\}, \qquad L^m = \int_0^{H^m} X_t^m dt, \qquad \text{for} \quad m \geq 1.$$

Note that our coupling of the various X^m's makes H^m and L^m a.s. increasing w.r. to m. We now study the limits of H^m and L^m as $m \to \infty$. We first recall some preliminary results on birth and death processes, which can be found in [2, 6, 9].

Let Y be a birth and death process with birth rate $\lambda_n > 0$ and death rate $\mu_n > 0$ when in state $n, n \geq 1$. Let

$$A = \sum_{n \geq 1} \frac{1}{\pi_n}, \qquad S = \sum_{n \geq 1} \frac{1}{\pi_n} \sum_{k \geq n+1} \frac{\pi_k}{\lambda_k},$$

where

$$\pi_1 = 1, \qquad \pi_n = \frac{\lambda_1 ... \lambda_{n-1} \lambda_n}{\mu_2 ... \mu_n}, \quad n \geq 2.$$

We denote by T_y^m the first time the process Y hits $y \in [0, \infty)$ when starting from $Y_0 = m$.

$$T_y^m = \inf\{t > 0 : Y_t = y \mid Y_0 = m\}.$$

We say that ∞ is an entrance boundary for Y (see, for instance, Anderson [2], section 8.1) if there is $y > 0$ and a time $t > 0$ such that

$$\lim_{m \uparrow \infty} \mathbb{P}(T_y^m < t) > 0.$$

We have the following result (see [6], Proposition 7.10)

Proposition 3.2.2. *The following are equivalent:*

1) ∞ *is an entrance boundary for* Y.

2) $A = \infty, S < \infty$.

3) $\lim_{m\uparrow\infty}\mathbb{E}(T_0^m)<\infty$.

We now want to apply the above result to the process X_t^m, in which case $\lambda_n = \lambda n + F^+(n), \mu_n = \mu n + F^-(n), n \geq 1$. We will need the following lemmas.

Lemma 3.2.3. *Let f be a function satisfying (H1), $a \in \mathbb{R}$ be a constant. If there exists $a_0 > 0$ such that $f(x) \neq 0, f(x) + ax \neq 0$ for all $x \geq a_0$, then we have that*

$$\int_{a_0}^{\infty} \frac{1}{\mid f(x) \mid} dx < \infty \Leftrightarrow \int_{a_0}^{\infty} \frac{1}{\mid ax + f(x) \mid} dx < \infty,$$

and when those equivalent conditions are satisfied, we have

$$\lim_{x\to\infty} \frac{f(x)}{x} = -\infty.$$

Proof. We need only show that

$$\int_{a_0}^{\infty} \frac{1}{\mid f(x) \mid} dx < \infty \Rightarrow \int_{a_0}^{\infty} \frac{1}{\mid ax + f(x) \mid} dx < \infty.$$

Indeed, this will imply the same implication for pair $f'(x) = f(x) + ax$, $f'(x) - ax$, which is the conversed result. Because $f(x) \leq \theta x$ for all $x \geq 0$, we can easily deduce from $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx < \infty$ that

$$f(x) < 0 \qquad \forall x \geq a_0.$$

Let β be a constant such that $\beta > \theta$. We have

$$\int_{a_0}^{\infty} \frac{1}{\beta x - f(x)} dx < \int_{a_0}^{\infty} \frac{1}{-f(x)} dx < \infty.$$

It implies that

$$\lim_{x\to\infty} \int_{x}^{2x} \frac{1}{\beta u - f(u)} du = 0.$$

But since the function $x \mapsto \beta x - f(x)$ is increasing,

$$\int_{x}^{2x} \frac{1}{\beta u - f(u)} du \geq \big(2\beta - \frac{f(2x)}{x}\big)^{-1}.$$

We deduce that $\lim_{x\to\infty} \frac{f(x)}{x} = -\infty$. Hence there exists $a_1 > a_0$ such that $f(x) < -2|a|x$ for all $x \geq a_1$. The result follows from

$$\int_{a_1}^{\infty} \frac{1}{\mid ax + f(x) \mid} dx < \int_{a_1}^{\infty} \frac{2}{-f(x)} dx < \infty.$$

□

Lemma 3.2.4. *Let f be a function satisfying (H1). For all $n \geq 1$ we have the two inequalities*

$$F^+(n) \leq \theta n$$
$$-f(n) \leq F^-(n) \leq \theta n - f(n).$$

Proof. The result follows from the facts that for all $n \geq 1$

$$(f(n) - f(n-1))^+ \leq \theta$$
$$(f(n) - f(n-1))^- \geq f(n-1) - f(n)$$
$$F^-(n) - F^+(n) = -f(n).$$

□

Proposition 3.2.5. *Assume f is a function satisfying (H1) and there exists $a_0 > 0$ such that $f(x) \neq 0$ for all $x \geq a_0$. Then ∞ is an entrance boundary for X if and only if*

$$\int_{a_0}^{\infty} \frac{1}{\mid f(x) \mid} dx < \infty.$$

Proof. If $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx = \infty$, then (recall that since $\mu > 0$, $(\mu+\theta)x - f(x)$ is non–negative and increasing)

$$\int_{a_0}^{\infty} \frac{1}{(\mu+\theta)x - f(x)} dx = \infty,$$

by Lemma 3.2.3. In this case,

$$\begin{aligned} S &\geq \sum_{n \geq 1} \frac{\pi_{n+1}}{\lambda_{n+1}\pi_n} \\ &= \sum_{n \geq 1} \frac{1}{\mu_{n+1}} \\ &= \sum_{n \geq 2} \frac{1}{\mu n + F^-(n)} \\ &\geq \sum_{n \geq 2} \frac{1}{(\mu+\theta)n - f(n)} \\ &= \infty. \end{aligned}$$

Therefore, ∞ is not an entrance boundary for X, by Proposition 3.2.2. On the other hand, if $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx < \infty$, then $\lim_{x\to\infty} \frac{f(x)}{x} = -\infty$, by Lemma 3.2.3. By Lemma 3.2.4 we have

$$\lim_{n\to\infty} \frac{\pi_{n+1}}{\pi_n} = \lim_{n\to\infty} \frac{\lambda n + F^+(n)}{\mu n + F^-(n)} \leq \lim_{n\to\infty} \frac{(\lambda+\theta)n}{\mu n - f(n)} = 0,$$

so that

$$A = \sum_{n\geq 1} \frac{1}{\pi_n} = \infty.$$

Set $a_n = \lambda_n/\mu_n$, then there exists $n_0 \geq 1$ such that $a_n < 1$ for all $n \geq n_0$. The inequality of arithmetic and geometric means states that for all $m > 0$ and $x_1, x_2, ..., x_m > 0$,

$$\frac{x_1 + x_2 + ... + x_m}{m} \geq \sqrt[m]{x_1 x_2 ... x_m},$$

so that for all $k > n > 0$,

$$a_{n+1}^{k-n} + ... + a_k^{k-n} \geq (k-n) a_{n+1} ... a_k.$$

Then

$$\begin{aligned}
\sum_{n\geq n_0} \frac{1}{\pi_n} \sum_{k\geq n+1} \frac{\pi_k}{\lambda_k} &\leq \frac{1}{\lambda} \sum_{n\geq n_0} \sum_{k\geq n+1} \frac{1}{k} a_{n+1} ... a_k \\
&\leq \frac{1}{\lambda} \sum_{n\geq n_0} \sum_{k\geq n+1} \frac{1}{k(k-n)} (a_{n+1}^{k-n} + ... + a_k^{k-n}) \\
&= \frac{1}{\lambda} \sum_{k\geq n_0+1} \sum_{n=1}^{k-n_0} \frac{1}{kn} (a_{k-n+1}^{n} + ... + a_k^{n}) \\
&= \frac{1}{\lambda} \sum_{i\geq n_0+1} \sum_{n\geq 1} a_i^n \sum_{k=i}^{n-1+i} \frac{1}{kn}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\lambda} \sum_{i \geq n_0+1} \sum_{n \geq 1} \frac{a_i^n}{i} \\
&= \frac{1}{\lambda} \sum_{i \geq n_0+1} \frac{a_i}{i(1-a_i)} \\
&= \frac{1}{\lambda} \sum_{i \geq n_0+1} \frac{\lambda_i}{i(\mu_i - \lambda_i)} \\
&= \sum_{i \geq n_0+1} \frac{\lambda i + F^+(i)}{\lambda i(\mu i - \lambda i + F^-(i) - F^+(i))} \\
&\leq \frac{\lambda+\theta}{\lambda} \sum_{i \geq n_0+1} \frac{1}{\mu i - \lambda i - f(i)} \\
&< \infty,
\end{aligned}
$$

where we have used Lemma 3.2.3 to conclude. Hence $S < \infty$. The result follows from Proposition 3.2.2. $\square$

We can now prove

Theorem 3.2.1. *Assume f is a function satisfying $(H1)$ and there exists $a_0 > 0$ such that $f(x) \neq 0$ for all $x \geq a_0$. We have*

1) *If $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx = \infty$, then*

$$\sup_{m>0} T_0^m = \infty \quad a.s.$$

2) *If $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx < \infty$, then*

$$\mathbb{E}\big(\sup_{m>0} T_0^m\big) < \infty.$$

Proof. If $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx = \infty$, then by Proposition 3.2.5, ∞ is not an entrance boundary for X. It means that for all $t > 0$,

$$\lim_{m \uparrow \infty} \mathbb{P}(T_0^m < t) = 0.$$

Hence for all $t > 0$, since $m \to T_0^m$ is increasing a.s.,

$$\mathbb{P}(\sup_{m>0} T_0^m < t) = 0,$$

hence

$$\sup_{m>0} T_0^m = \infty \qquad a.s.$$

The second part of the theorem is a consequence of Proposition 3.2.5 and Proposition 3.2.2. □

Remark 3.2.6. The first part of Theorem 3.2.1 is still true when $\lambda_n = 0, n \geq 1$. In fact, in this case we have

$$T_0^m \doteq \sum_{n=1}^{m} \theta_n,$$

where $\doteq$ denotes equality in law, θ_n represents the first passage time from state n to state $n-1$,

$$\theta_n = \inf\{t > 0 : X_t = n-1 \mid X_0 = n\}.$$

Recalling the fact that θ_n is exponentially distributed with parameter $\mu n + F^-(n)$, we have (see Lemma 4.3, Chapter 7 in [11])

$$\sup_{m>0} T_0^m = \infty \quad a.s. \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{\mu n + F^-(n)} = \infty.$$

The result follows by Lemma 3.2.3 and Lemma 3.2.4.

Here a question arises: in the case $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx < \infty$, whether higher moments of $\sup_{m>0} T_0^m$ are also finite or not. We will see that the answer is Yes. Indeed, we can prove that it has some finite exponential moments.

Theorem 3.2.2. *Suppose that f is a function satisfying (H1) and there exists $a_0 > 0$ such that $f(x) \neq 0$ for all $x \geq a_0$. If $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx < \infty$ we have*

1) *For any $a > 0$, there exists $y_a \in \mathbb{Z}_+$ such that*

$$\sup_{m>y_a} \mathbb{E}\big(e^{aT_{y_a}^m}\big) < \infty.$$

2) *There exists some positive constant c such that*

$$\sup_{m>0} \mathbb{E}\big(e^{cT_0^m}\big) < \infty.$$

Proof. 1) There exists $n_a \in \mathbb{Z}_+$ large enough so that

$$\sum_{n=n_a}^{\infty} \frac{1}{\pi_n} \sum_{k \geq n+1} \frac{\pi_k}{\lambda_k} \leq \frac{1}{a}.$$

Let J be the nonnegative increasing function defined by

$$J(m) := \sum_{n=n_a}^{m-1} \frac{1}{\pi_n} \sum_{k \geq n+1} \frac{\pi_k}{\lambda_k}, \qquad m \geq n_a + 1.$$

Set now $y_a = n_a + 1$. Note that $\sup_{m > y_a} T^m_{y_a} < \infty$ a.s., then for any $m > y_a$ we have

$$J(X^m_{t \wedge T^m_{y_a}}) - J(m) - \int_0^{t \wedge T^m_{y_a}} AJ(X^m_s) ds$$

is a martingale, where A is the generator of the process X^m_t which is given by

$$Ag(n) = \lambda_n(g(n+1) - g(n)) + \mu_n(g(n-1) - g(n)), \qquad n \geq 1,$$

for any $\mathbb{R}_+$-valued, bounded function g. Therefore, by Ito's formula

$$e^{a(t \wedge T^m_{y_a})} J(X^m_{t \wedge T^m_{y_a}}) - J(m) - \int_0^{t \wedge T^m_{y_a}} e^{as}(aJ(X^m_s) + AJ(X^m_s)) ds$$

is also a martingale. It implies that

$$\mathbb{E}\Big(e^{a(t \wedge T^m_{y_a})} J(X^m_{t \wedge T^m_{y_a}})\Big) = J(m) + \mathbb{E}\Big(\int_0^{t \wedge T^m_{y_a}} e^{as}(aJ(X^m_s) + AJ(X^m_s)) ds\Big).$$

We have for $m > y_a$, $J(X^m_s) < J(\infty) \leq \frac{1}{a} \quad \forall s \leq T^m_{y_a}$, and for any $n \geq y_a$,

$$\begin{aligned}
AJ(n) &= \lambda_n(J(n+1) - J(n)) + \mu_n(J(n-1) - J(n)) \\
&= \lambda_n \frac{1}{\pi_n} \sum_{k \geq n+1} \frac{\pi_k}{\lambda_k} - \mu_n \frac{1}{\pi_{n-1}} \sum_{k \geq n} \frac{\pi_k}{\lambda_k} \\
&= \frac{\mu_2 \ldots \mu_n}{\lambda_1 \ldots \lambda_{n-1}} \sum_{k \geq n+1} \frac{\pi_k}{\lambda_k} - \frac{\mu_2 \ldots \mu_n}{\lambda_1 \ldots \lambda_{n-1}} \sum_{k \geq n} \frac{\pi_k}{\lambda_k} \\
&= -\frac{\mu_2 \ldots \mu_n}{\lambda_1 \ldots \lambda_{n-1}} \frac{\pi_n}{\lambda_n} \\
&= -1.
\end{aligned}$$

So that

$$\mathbb{E}\Big(e^{a(t\wedge T^m_{y_a})}J(X^m_{t\wedge T^m_{y_a}})\Big) \leq J(m) \qquad \forall m > y_a.$$

But J is increasing, hence for any $m > y_a$ one gets

$$0 < J(y_a) \leq J(m) < J(\infty) \leq \frac{1}{a}.$$

From this we deduce that

$$\mathbb{E}\Big(e^{a(t\wedge T^m_{y_a})}\Big) \leq \frac{1}{aJ(y_a)} \qquad \forall m > y_a.$$

Hence

$$\mathbb{E}\Big(e^{aT^m_{y_a}}\Big) \leq \frac{1}{aJ(y_a)} \qquad \forall m > y_a,$$

by the monotone convergence theorem. The result follows.

2) Using the first result of the theorem, there exists a constant $M \in \mathbb{Z}_+$ such that

$$\sup_{m>M} \mathbb{E}\Big(e^{T^m_M}\Big) < \infty,$$

or $\mathbb{E}\Big(e^{T_M}\Big) < \infty$, where $T_M := \sup_{m>M} T^m_M$.

Given any fixed $T > 0$, let p denote the probability that starting from M at time $t = 0, X$ hits zero before time T. Clearly $p > 0$. Let ζ be a geometric random variable with success probability p, which is defined as follows. Let X start from M at time 0. If X hits zero before time T, then $\zeta = 1$. If not, we look the position X_T of X at time T.

If $X_T > M$, we wait until X goes back to M. The time needed is stochastically dominated by the random variable T_M, which is the time needed for X to descend to M, when starting from ∞. If however $X_T \leq M$, we start afresh from there, since the probability to reach zero in less than T is greater than or equal to p, for all starting points in the interval $(0, M]$.

So either at time T, or at time less than $T + T_M$, we start again from a level which is less than or equal to M. If zero is reached during the next time interval of length T, then $\zeta = 2$... Repeating this procedure, we see that $\sup_{m>0} T_0^m$ is stochastically dominated by

$$\zeta T + \sum_{i=1}^{\zeta} \eta_i,$$

where the random variables η_i are i.i.d, with the same law as T_M, globally independent of ζ. We have

$$\begin{aligned}\sup_{m>0} \mathbb{E}\big(e^{cT_0^m}\big) &\leq \mathbb{E}\big(e^{c(\zeta T + \sum_{i=1}^{\zeta} \eta_i)}\big) \\ &\leq \sqrt{\mathbb{E}\big(e^{2c\zeta T}\big)}\sqrt{\mathbb{E}\big(e^{2c\sum_{i=1}^{\zeta} \eta_i}\big)}.\end{aligned}$$

Since ζ is a geometric(p) random variable, then

$$\mathbb{E}\big(e^{2c\zeta T}\big) = \frac{p}{1-p} \sum_{k=1}^{\infty} \big(e^{2cT}(1-p)\big)^k < \infty,$$

provided that $c < -\log(1-p)/2T$.

Moreover, we have

$$\begin{aligned}\mathbb{E}\big(e^{2c\sum_{i=1}^{\zeta} \eta_i}\big) &= \sum_{k=1}^{\infty} \mathbb{E}\big(e^{2c\sum_{i=1}^{k} \eta_i}\big)\mathbb{P}(\zeta = k) \\ &= \sum_{k=1}^{\infty} \big[\mathbb{E}\big(e^{2cT_M}\big)\big]^k \mathbb{P}(\zeta = k) \\ &= \frac{p}{1-p} \sum_{k=1}^{\infty} \big[\mathbb{E}\big(e^{2cT_M}\big)(1-p)\big]^k.\end{aligned}$$

Since $\mathbb{E}\big(e^{T_M}\big) < \infty$, it follows from the monotone convergence theorem that $\mathbb{E}\big(e^{2cT_M}\big) \to 1$ as $c \to 0$. Hence we can choose $0 < c < -\log(1-p)/2T$ such that

$$\mathbb{E}\big(e^{2cT_M}\big)(1-p) < 1,$$

in which case $\mathbb{E}\big(e^{2c\sum_{i=1}^{\zeta} \eta_i}\big) < \infty$.

Then $\sup_{m>0} \mathbb{E}\big(e^{cT_0^m}\big) < \infty$. The result follows.

$\square$

3.2.2 Height and length of the genealogical forest of trees in the discrete case

The following result follows from Theorem 3.2.1 and Theorem 3.2.2

Theorem 3.2.3. *Suppose that f is a function satisfying (H1) and there exists $a_0 > 0$ such that $f(x) \neq 0$ for all $x \geq a_0$. We have*

1) *If $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx = \infty$, then*
$$\sup_{m>0} H^m = \infty \quad a.s.$$

2) *If $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx < \infty$, then*
$$\sup_{m>0} H^m < \infty \quad a.s.,$$
and moreover, there exists some positive constant c such that
$$\sup_{m>0} \mathbb{E}\big(e^{cH^m}\big) < \infty.$$

Concerning the length of the genealogical tree we have

Theorem 3.2.4. *Suppose that the function $\frac{f(x)}{x}$ satisfies (H1) and there exists $a_0 > 0$ such that $f(x) \neq 0$ for all $x \geq a_0$. We have*

1) *If $\int_{a_0}^{\infty} \frac{x}{|f(x)|} dx = \infty$, then*
$$\sup_{m>0} L^m = \infty \quad a.s.$$

2) *If $\int_{a_0}^{\infty} \frac{x}{|f(x)|} dx < \infty$, then*
$$\sup_{m>0} L^m < \infty \quad a.s.,$$
and moreover, there exists some positive constant c such that
$$\sup_{m>0} \mathbb{E}\big(e^{cL^m}\big) < \infty.$$

To prove Theorem 3.2.4 we need the following result, which is Theorem 1 in Bhaskaran [5].

Proposition 3.2.7. *Let Y^i be a birth and death process with birth rates $\{\lambda_n^{(i)}\}_{n\geq 1}$ and death rates $\{\mu_n^{(i)}\}_{n\geq 1}(i=1,2)$, where $\lambda_n^{(i)}$ and $\mu_n^{(i)}$ satisfy the condition*

$$\sum_{n\geq 1}\frac{1}{\pi_n}\sum_{k=1}^{n}\frac{\pi_k}{\lambda_k}=\infty. \tag{3.2.1}$$

Suppose that

$$\lambda_n^{(1)}\geq\lambda_n^{(2)}\quad \textit{and}\quad \mu_n^{(1)}\leq\mu_n^{(2)},\qquad n\geq 1.$$

Then one can construct two processes $\tilde{Y}^1$ and $\tilde{Y}^2$ on the same probability space such that $\{\tilde{Y}^i(k),k\geq 0\}$ and $\{Y^i(k),k\geq 0\}$ have the same law for $i=1,2$, and $\tilde{Y}^1(k)\geq\tilde{Y}^2(k)$ a.s. for all $k\geq 0$.

Remark 3.2.8. 1) Condition (3.2.1) implies that the birth and death process does not explode in finite time a.s. Note that

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{\pi_n}\sum_{k=1}^{n}\frac{\pi_k}{\lambda_k}&\geq\sum_{n\geq 1}\frac{1}{\pi_n}\times\frac{\pi_n}{\lambda_n}\\&=\sum_{n\geq 1}\frac{1}{\lambda_n}.\end{aligned}$$

Then (3.2.1) is satisfied if there exists a constant $\gamma>0$ such that

$$\lambda_n\leq\gamma n,\qquad \forall n\geq 1.$$

2) Proposition 3.2.7 is still true when $\lambda_n^2=0, n\geq 1$. In fact, the proof of Bhaskaran (as given in [5]) still works in this case.

Now we will apply Proposition 3.2.7 to prove Theorem 3.2.4. In the proof, we will not bother to check condition (3.2.1), which is obviously satisfied here.

Proof of Theorem 3.2.4

1) Let

$$f_1(n):=\frac{f(n)}{n},\qquad F_1^-(n):=\sum_{k=1}^{n}(f_1(k)-f_1(k-1))^-,\qquad n\geq 1.$$

By Lemma 3.2.9 below we have for all $n \geq 1$,

$$\begin{aligned}\mu_n = \mu n + F^-(n) &\leq \mu n + 2\theta n^2 - f(n)\\ &\leq (\mu+2\theta)n^2 - \frac{f(n)}{n}n\\ &\leq (\mu+2\theta)n^2 + F_1^-(n)n.\end{aligned}$$

Let $X^{1,m}$ be a birth and death process which starts from $X_0^{1,m} = m$, with birth rate $\lambda_n^1 = 0$ and death rate $\mu_n^1 = (\mu+2\theta)n^2 + F_1^-(n)n$ when in state $n, n \geq 1$. From Proposition 3.2.7 we deduce that for all $m \geq 1$,

$$X^m \geq X^{1,m}(\text{in dist.}), \quad H^m \geq H^{1,m}(\text{in dist.}), \quad L^m \geq L^{1,m}(\text{in dist.}),$$

and moreover, since both $m \to L^m$ and $m \to L^{1,m}$ are a.s. increasing,

$$\sup_{m>0} L^m \geq \sup_{m>0} L^{1,m}(\text{in dist.}),$$

where $H^{1,m}, L^{1,m}$ are the height and the length of the genealogical tree of the population $X^{1,m}$, respectively.

We now use a random time-change to transform the length of a forest of genealogical trees into the height of another forest of genealogical trees, so that we can apply Theorem 3.2.1. We define

$$A_t^{1,m} := \int_0^t X_r^{1,m} dr, \qquad \eta_t^{1,m} = \inf\{s > 0, A_s^{1,m} > t\},$$

and consider the process $U^{1,m} := X^{1,m} \circ \eta^{1,m}$. Let $S^{1,m}$ be the stopping time defined by

$$S^{1,m} = \inf\{r > 0, U_r^{1,m} = 0\},$$

then we have

$$S^{1,m} = \int_0^{H^{1,m}} X_r^{1,m} dr = L^{1,m} \qquad a.s.$$

The process $X^{1,m}$ can be expressed using a standard Poisson processes P, as

$$X_t^{1,m} = m - P\left(\int_0^t [(\mu+2\theta)(X_r^{1,m})^2 + F_1^-(X_r^{1,m})X_r^{1,m}]dr\right).$$

Consequently the process $U^{1,m}$ satisfies

$$U_t^{1,m} = m - P\left(\int_0^t [(\mu + 2\theta)U_r^{1,m} + F_1^-(U_r^{1,m})]dr \right).$$

Applying Theorem 3.2.1 and Remark 3.2.6 we have

$$\sup_{m>0} L^{1,m} = \sup_{m>0} S^{1,m} = \infty \qquad a.s.,$$

hence $\sup_{m>0} L^m = \infty$ a.s. The result follows.

2) For the second part of the theorem, we note that in the case $\int_{a_0}^{\infty} \frac{x}{|f(x)|}dx < \infty$, we have $\frac{f(x)}{x^2} \to -\infty$ as $x \to \infty$, by Lemma 3.2.3. Then there exists a constant $u > 0$ such that for all $n \geq u$ (using again Lemma 3.2.9),

$$\mu n + F^-(n) \geq -f(n) \geq \theta n^2 - \frac{f(n)}{2}.$$

We can choose $\varepsilon \in (0, 1)$ such that for all $1 \leq n \leq u$

$$\mu n \geq \varepsilon(\theta n^2 - \frac{f(n)}{2}).$$

It implies that for all $n \geq 1$,

$$\mu n + F^-(n) \geq \varepsilon(\theta n^2 - \frac{f(n)}{2}).$$

Let $X^{2,m}$ be a birth and death process which starts from $X_0^{2,m} = m$, with birth rate $\lambda_n^2 = (\lambda + 2\theta)n^2$ and death rate $\mu_n^2 = \varepsilon(\theta n^2 - \frac{f(n)}{2})$ when in state $n, n \geq 1$. From Lemma 3.2.9 and Proposition 3.2.7 we deduce that for all $m \geq 1$,

$$X^m \leq X^{2,m}(\text{in dist.}), \quad H^m \leq H^{2,m}(\text{in dist.}), \quad L^m \leq L^{2,m}(\text{in dist.}),$$

where $H^{2,m}, L^{2,m}$ are the height and the length of the genealogical tree of the population $X^{2,m}$, respectively. We define

$$A_t^{2,m} := \int_0^t X_r^{2,m}dr, \qquad \eta_t^{2,m} = \inf\{s > 0, A_s^{2,m} > t\},$$

and consider the process $U^{2,m} := X^{2,m} \circ \eta^{2,m}$. Let $S^{2,m}$ be the stopping time defined by

$$S^{2,m} = \inf\{r > 0, U_r^{2,m} = 0\},$$

then we have

$$S^{2,m} = \int_0^{H^{2,m}} X_r^{2,m} dr = L^{2,m} \qquad a.s.$$

Denote $f_2(x) := \frac{\varepsilon}{2}(\frac{f(x)}{x} - \theta x)$, then f_2 is a negative and decreasing function, so that for all $n \geq 1$,

$$F_2^+(n) := \sum_{k=1}^{n}(f_2(k)-f_2(k-1))^+ = 0, \qquad F_2^-(n) := \sum_{k=1}^{n}(f_2(k)-f_2(k-1))^- = -f_2(n).$$

The process $X^{2,m}$ can be expressed using two mutually independent standard Poisson processes P_1 and P_2, as

$$X_t^{2,m} = m+P_1\left(\int_0^t [(\lambda+2\theta)(X_r^{2,m})^2]dr\right)-P_2\left(\int_0^t [\frac{\varepsilon\theta}{2}(X_r^{2,m})^2+F_2^-(X_r^{2,m})X_r^{2,m}]dr\right).$$

Consequently the process $U^{2,m}$ satisfies

$$U_t^{2,m} = m+P_1\left(\int_0^t [(\lambda+2\theta)U_r^{2,m}+F_2^+(U_r^{2,m})]dr\right)-P_2\left(\int_0^t [\frac{\varepsilon\theta}{2}U_r^{2,m}+F_2^-(U_r^{2,m})]dr\right).$$

By Theorem 3.2.2, there exists some positive constant c such that

$$\sup_{m>0} \mathbb{E}\left(e^{cL^{2,m}}\right) = \sup_{m>0} \mathbb{E}\left(e^{cS^{2,m}}\right) < \infty,$$

hence

$$\sup_{m>0} \mathbb{E}\left(e^{cL^m}\right) \leq \sup_{m>0} \mathbb{E}\left(e^{cL^{2,m}}\right) < \infty.$$

The result follows.

□

It remains to prove

Lemma 3.2.9. *Suppose that the function $\frac{f(x)}{x}$ satisfies (H1). For all $n \geq 1$ we have the following inequalities*

$$F^+(n) \leq 2\theta n^2,$$
$$-f(n) \leq F^-(n) \leq 2\theta n^2 - f(n).$$

Proof. Note that for all $k \geq 1$,

$$\begin{aligned}(f(k) - f(k-1))^+ &= \Big((k-1)(\frac{f(k)}{k} - \frac{f(k-1)}{k-1}) + \frac{f(k)}{k}\Big)^+ \\ &\leq (k-1)\Big(\frac{f(k)}{k} - \frac{f(k-1)}{k-1}\Big)^+ + \Big(\frac{f(k)}{k}\Big)^+ \\ &\leq 2\theta k.\end{aligned}$$

Then

$$F^+(n) \leq \sum_{k=1}^{n} 2\theta k = \theta n(n+1) \leq 2\theta n^2.$$

The second result now follows from the fact that for all $n \geq 1$

$$(f(n) - f(n-1))^- \geq f(n-1) - f(n)$$
$$F^-(n) - F^+(n) = -f(n).$$

□

3.3 The continuous case

3.3.1 Preliminaries

We now consider the $\mathbb{R}_+$–valued two–parameter stochastic process $\{Z_t^x, t \geq 0, x \geq 0\}$ which solves the SDE (3.1.1), where the function f satisfies (H1). We note that this coupling of the $\{Z_t^x,\ t \geq 0\}$'s for various x's is consistent with that used in the discrete population case in the sense that as $N \to \infty$,

$$\{N^{-1}X_t^{\lfloor Nx \rfloor},\ t \geq 0, x > 0\} \Rightarrow \{Z_t^x,\ t \geq 0, x > 0\},$$

see [4], where the topology for which this is valid is made precise.

According again to [4], the process $\{Z^x_\cdot, x \geq 0\}$ is a Markov process with values in $C(\mathbb{R}_+, \mathbb{R}_+)$, the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$, starting from 0 at $x = 0$. Moreover, we have that whenever $0 < x \leq y, Z^y_t \geq Z^x_t$ for all $t \geq 0$ a.s. For $x > 0$, define T^x the extinction time of the process Z^x (it is also called the height of the process Z^x) by

$$T^x = \inf\{t > 0, Z^x_t = 0\}.$$

And define S^x the total mass of Z^x by

$$S^x = \int_0^{T^x} Z^x_t dt.$$

We next study the limits of T^x and S^x as $x \to \infty$. We want to show that under a specific assumption $T^x \to \infty$ (resp. $S^x \to \infty$) as $x \to \infty$, and under the complementary assumption $\sup_{x>0} \mathbb{E}(e^{cT^x}) < \infty$ for some $c > 0$ (resp. $\sup_{x>0} \mathbb{E}(e^{cS^x}) < \infty$ for some $c > 0$). Because both mappings $x \mapsto T^x$ and $x \mapsto S^x$ are a.s. increasing, the result will follow for the same result proved for any collection of r.v.'s $\{T^x, x > 0\}$ (resp. $\{S^x, x > 0\}$) which has the same monotonicity property, and has the same marginal laws as the original one. More precisely, we will consider the Z^x's solutions of (3.1.2) instead of (3.1.1), with the same W for all $x > 0$.

We first need to recall some preliminary results on a class of one–dimensional Kolmogorov diffusions (drifted Brownian motions), which can also be found in [6].

Consider a one–dimensional drifted Brownian motion with values in $[0, \infty)$ which is killed when it first hits zero

$$dX_t = q(X_t)dt + dB_t, \qquad X_0 = x > 0,$$

where q is defined and is C^1 on $(0, \infty)$, and $\{B_t, t \geq 0\}$ is a standard one- dimensional Brownian motion. In particular, q is allowed to explode at the origin. In this section, we shall assume that

Hypothesis (H2): There exists $x_0 > 0$ such that $q(x) < 0 \quad \forall x \geq x_0$, and

$$\limsup_{x \to 0^+} q(x) < \infty.$$

The condition (H2) implies that q is bounded from above by some constant. It ensures that ∞ is inaccessible, in the sense that a.s. ∞ can not be reached in finite time from $X_0 = x \in (0, \infty)$.

We denote by T_y^x the first time the process X hits $y \in [0, \infty)$ when starting from $X_0 = x$

$$T_y^x = \inf\{t > 0 : X_t = y \mid X_0 = x\}.$$

We say that ∞ is an entrance boundary for X (see, for instance, Revuz and Yor [13], page 305) if there is $y > 0$ and a time $t > 0$ such that

$$\lim_{x \uparrow \infty} \mathbb{P}(T_y^x < t) > 0.$$

Let us introduce the following condition

Hypothesis (H3):

$$\int_1^\infty e^{-Q(y)} \int_y^\infty e^{Q(z)} dz dy < \infty,$$

where $Q(y) = 2 \int_1^y q(x) dx, y \geq 1$.

Tonelli's theorem ensures that (H3) is equivalent to

$$\int_1^\infty e^{Q(y)} \int_1^y e^{-Q(z)} dz dy < \infty.$$

We have the following result which is Proposition 7.6 in [6].

Proposition 3.3.1. *The following are equivalent:*

1) ∞ *is an entrance boundary for* X.

2) *$(H3)$ holds.*

3) *For any $a > 0$, there exists $y_a > 0$ such that*

$$\sup_{x>y_a} \mathbb{E}\big(e^{aT^x_{y_a}}\big) < \infty.$$

We now state the main result of this subsection

Theorem 3.3.1. *Assume that $(H2)$ holds. We have*

1) *If $(H3)$ does not hold, then for all $y \geq 0$,*

$$\sup_{x>y} T^x_y = \infty \qquad a.s.$$

2) *If $(H3)$ holds, then for all $y \geq 0$,*

$$\sup_{x>y} T^x_y < \infty \qquad a.s.,$$

and moreover, there exists some positive constant c such that

$$\sup_{x>0} \mathbb{E}\big(e^{cT^x_0}\big) < \infty.$$

Proof. 1) If (H3) does not hold, then by Proposition 3.3.1, ∞ is not an entrance boundary for X. It means that for all $y > 0, t > 0$,

$$\lim_{x\uparrow\infty} \mathbb{P}(T^x_y < t) = 0.$$

Hence for all $t > 0$, since $x \to T^x_y$ is increasing a.s.,

$$\mathbb{P}(\sup_{x>y} T^x_y < t) = 0,$$

hence

$$\sup_{x>y} T^x_y = \infty \qquad a.s.$$

2) The result is a consequence of Proposition 3.3.1. We can prove it by using the same argument as used in the proof of Theorem 3.2.2.

□

It is not obvious when (H3) holds. But from the following result, if q satisfies some explicit conditions, we can decide whether (H3) holds or not.

Proposition 3.3.2. *Suppose that* $(H2)$ *holds. We have*

1) *If*
$$\int_{x_0}^{\infty} \frac{1}{q(x)} dx = -\infty \qquad \text{and} \qquad \limsup_{x\to\infty} \frac{q'(x)}{q(x)^2} < \infty,$$
then $(H3)$ *does not hold.*

2) *If there exists* $q_0 < 0$ *such that* $q(x) \le q_0$ *for all* $x \ge x_0$,
$$\int_{x_0}^{\infty} \frac{1}{q(x)} dx > -\infty \qquad \text{and} \qquad \liminf_{x\to\infty} \frac{q'(x)}{q(x)^2} > -2,$$
then $(H3)$ *holds.*

3) *If*
$$\int_{x_0}^{\infty} \frac{1}{q(x)} dx > -\infty \qquad \text{and} \qquad q'(x) \le 0 \qquad \forall x \ge x_0,$$
then $(H3)$ *holds.*

Proof. 1) Define $s(y) := \int_y^\infty e^{Q(z)} dz$. If $s(x_0) = \infty$, then $s(y) = \infty$ for all $y \ge x_0$, so that (H3) does not hold.

We consider the case $s(x_0) < \infty$. Integrating by parts on $\int se^{-Q} dy$ gives
$$\int_{x_0}^{\infty} se^{-Q} dy = \int_{x_0}^{\infty} \frac{s}{2q} e^{-Q} 2q dy = \frac{-s}{2q} e^{-Q}\Big|_{x_0}^{\infty} - \int_{x_0}^{\infty} \frac{1}{2q} dy - \int_{x_0}^{\infty} se^{-Q} \frac{q'}{2q^2} dy \tag{3.3.1}$$

From $\int_{x_0}^\infty \frac{1}{q(x)} dx = -\infty$ and $\frac{-s}{2q} e^{-Q}(\infty) \ge 0$, (3.3.1) implies that
$$\int_{x_0}^{\infty} se^{-Q} (1 + \frac{q'}{2q^2}) dy = \infty.$$

Since $\limsup_{x\to\infty} \frac{q^{'}(x)}{q(x)^2} < \infty$, then $\int_{x_0}^{\infty} se^{-Q}dy = \infty$. Condition (H3) does not hold.

2) We can easily deduce from $q(x) \leq q_0$ for all $x \geq x_0$ that $s(y)$ tends to zero as y tends to infinity, and $s(y)e^{-Q(y)}$ is bounded in $y \geq x_0$. Because $\int_{x_0}^{\infty} \frac{1}{q(x)}dx > -\infty$, (3.3.1) implies that $se^{-Q}(1 + \frac{q^{'}}{2q^2})$ is integrable. Then thanks to the condition $\liminf_{x\to\infty} \frac{q^{'}(x)}{q(x)^2} > -2$, we conclude that (H3) holds.

3) From $q(x) \leq q(x_0) < 0$ for all $x \geq x_0$, we can easily deduce that $Q(y) \to -\infty$ and $s(y) \to 0$ as $y \to \infty$. Applying Cauchy's mean value theorem to $s(y)$ and $q_1(y) := e^{Q(y)}$, we have for all $y \geq x_0$, there exists $\xi \in (y, \infty)$ such that

$$\frac{\int_y^{\infty} e^{Q(z)}dz}{e^{Q(y)}} = \frac{s^{'}(\xi)}{q_1^{'}(\xi)} = -\frac{1}{2q(\xi)}.$$

Because $q^{'}(x) \leq 0$ for all $x \geq x_0$, we obtain

$$s(y)e^{-Q(y)} \leq -\frac{1}{2q(y)}, \qquad \text{for all} \qquad y \geq x_0.$$

Hence

$$\int_{x_0}^{\infty} s(y)e^{-Q(y)}dy \leq -\int_{x_0}^{\infty} \frac{1}{2q(y)}dy < \infty.$$

Then (H3) holds.

□

Example 3.3.3. We are interested in the case that q is a polynomial. More precisely, we consider the function q satisfying (H2) and for all $x \geq x_0$,

$$q(x) = -x^{\alpha} \qquad \alpha > -1.$$

We have

$$\lim_{x\to\infty} \frac{q^{'}(x)}{q(x)^2} = \lim_{x\to\infty} \frac{\alpha x^{\alpha-1}}{x^{2\alpha}} = \lim_{x\to\infty} \frac{\alpha}{x^{\alpha+1}} = 0.$$

Hence condition (H3) holds if and only if

$$\int_{x_0}^{\infty} \frac{1}{-x^{\alpha}}dx > -\infty \Leftrightarrow \alpha > 1.$$

3.3.2 Height of the continuous forest of trees

We consider the process $\{Z_t^x, t \geq 0\}$ solution of (3.1.2). It follows from the Ito formula that the process $Y_t^x = \sqrt{Z_t^x}$ solves the SDE

$$dY_t^x = \frac{f((Y_t^x)^2) - 1}{2Y_t^x}dt + dW_t, \qquad Y_0^x = \sqrt{x}. \tag{3.3.2}$$

Note that the height of the process Z^x is

$$T^x = \inf\{t > 0, Z_t^x = 0\} = \inf\{t > 0, Y_t^x = 0\}.$$

We now establish the large x behaviour of T^x.

Theorem 3.3.2. *Assume that f is a function satisfying (H1) and that there exists $a_0 > 0$ such that $f(x) \neq 0$ for all $x \geq a_0$. If $\int_{a_0}^{\infty} \frac{1}{|f(x)|}dx = \infty$, then*

$$T^x \to \infty \quad a.s. \quad as \quad x \to \infty.$$

Proof. Let β be a constant such that $\beta > \theta$. By a well-known comparison theorem, $Y_t^x \geq Y_t^{1,x}$, where $Y_t^{1,x}$ solves

$$dY_t^{1,x} = -\frac{\beta(Y_t^{1,x})^2 - f((Y_t^{1,x})^2) + 1}{2Y_t^{1,x}}dt + dW_t, \qquad Y_0^{1,x} = \sqrt{x},$$

Note that the function $\beta x - f(x) + 1$ is positive and increasing, then $f_1(x) := -\frac{\beta x^2 - f(x^2)+1}{2x}$ satisfies (H2), and

$$\limsup_{x\to\infty} \frac{f_1'(x)}{f_1(x)^2} < \infty.$$

Moreover there exists $x_1 > 0$ such that $\beta x - f(x) \geq 1$ for all $x \geq x_1$, hence

$$\begin{aligned}
\int_1^{\infty} \frac{1}{f_1(x)}dx &= -\int_1^{\infty} \frac{2x}{\beta x^2 - f(x^2) + 1}dx \\
&= -\int_1^{\infty} \frac{1}{\beta x - f(x) + 1}dx \\
&\leq -\int_1^{x_1} \frac{1}{\beta x - f(x) + 1}dx - 2\int_{x_1}^{\infty} \frac{1}{\beta x - f(x)}dx \\
&= -\infty,
\end{aligned}$$

again by Lemma 3.2.3. The result now follows readily from Theorem 3.3.1 and Proposition 3.3.2. □

Theorem 3.3.3. *Assume that f is a function satisfying (H1) and that there exists $a_0 > 0$ such that $f(x) \neq 0$ for all $x \geq a_0$. If $\int_{a_0}^{\infty} \frac{1}{|f(x)|} dx < \infty$, then*

$$\sup_{x>0} T^x < \infty \qquad a.s.,$$

and moreover, there exists some positive constant c such that

$$\sup_{x>0} \mathbb{E}\left(e^{cT^x}\right) < \infty.$$

Proof. We can rewrite the SDE (3.3.2) as (with again $\beta > \theta$)

$$dY_t^x = \frac{\beta(Y_t^x)^2 - h((Y_t^x)^2)}{2Y_t^x} dt + dW_t, \qquad Y_0^x = \sqrt{x},$$

where $h(x) := \beta x - f(x) + 1$ is a positive and increasing function. By Lemma 3.2.3, we have $\int_1^{\infty} \frac{1}{h(x)} dx < \infty$ which is equivalent to $\sum_{n=1}^{\infty} \frac{1}{h(n)} < \infty$. Let

$$a_1 = h(1), \qquad a_n = \min\{h(n), 2a_{n-1}\} \quad \forall n > 1.$$

It is easy to see that for all $n > 1$,

$$a_{n-1} < a_n \leq h(n), \qquad\qquad \frac{a_n}{a_{n-1}} \leq 2.$$

We also have

$$\begin{aligned}
&\frac{1}{a_1} = \frac{1}{h(1)}\\
&\frac{1}{a_2} \leq \frac{1}{h(2)} + \frac{1}{2a_1} = \frac{1}{h(2)} + \frac{1}{2h(1)}\\
&\frac{1}{a_3} \leq \frac{1}{h(3)} + \frac{1}{2a_2} \leq \frac{1}{h(3)} + \frac{1}{2h(2)} + \frac{1}{4h(1)}\\
&\quad \ldots\ldots\ldots\ldots\\
&\frac{1}{a_n} \leq \frac{1}{h(n)} + \frac{1}{2a_{n-1}} \leq \frac{1}{h(n)} + \frac{1}{2h(n-1)} + \ldots + \frac{1}{2^{n-1}h(1)}.
\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty}\frac{1}{a_n}\le 2\sum_{n=1}^{\infty}\frac{1}{h(n)}<\infty.$$

Now, we define a continuous increasing function g as follows. We first draw a broken line which joins the points (n,a_n) and is the graph of h_1. Define the function h_2 as follows.

$$h_2(x) \quad = \quad \begin{cases} h(x), & 0\le x\le 1 \\ h_1(x), & x\ge 1. \end{cases}$$

We then smoothen all the nodal points of the graph of h_2 to obtain a smooth curve which is the graph of an increasing function g_1. Let $g(x)=\frac{1}{2}g_1(x)$. We have for all $n\ge 1$ and $x\in[n,n+1)$,

$$h(x)\ge h(n)\ge a_n\ge \frac{1}{2}a_{n+1}=g(n+1)\ge g(x).$$

By the comparison theorem, $Y_t^x\le Y_t^{2,x}$, where $Y_y^{2,x}$ solves

$$dY_t^{2,x}=\frac{\beta(Y_t^{2,x})^2-g((Y_t^{2,x})^2)}{2Y_t^{2,x}}dt+dW_t, \qquad Y_0^{2,x}=\sqrt{x}.$$

Since

$$\sum_{n=1}^{\infty}\frac{1}{g(n)}=2\sum_{n=1}^{\infty}\frac{1}{a_n}<\infty,$$

we deduce that $\int_1^\infty\frac{1}{g(x)}dx<\infty$, and $\frac{g(x)}{x}\to\infty$ as $x\to\infty$, by Lemma 3.2.3. Let $f_2(x):=\frac{\beta x^2-g(x^2)}{2x}$, then there exists $x_1>0, q_1<0$ such that $f_2(x)<q_1$ for all $x\ge x_1$, and

$$\int_{x_1}^{\infty}\frac{1}{f_2(x)}dx=\int_{x_1}^{\infty}\frac{2x}{\beta x^2-g(x^2)}dx=\int_{x_1^2}^{\infty}\frac{1}{\beta x-g(x)}dx>-\infty.$$

Moreover,

$$\liminf_{x\to\infty}\frac{f_2'(x)}{f_2(x)^2}=\liminf_{x\to\infty}\frac{-4xg'(x)}{g(x)^2}.$$

But for all $x\in[n,n+1)$,

$$\frac{g'(x)x}{g(x)^2}\le\frac{(n+1)}{g(n)^2}\max_{i\in\{n-1,n,n+1\}}\{g(i+1)-g(i)\}<\frac{(n+1)g(n+2)}{g(n)^2}\le\frac{4(n+1)}{g(n)}\to 0,$$

as $n\to\infty$. The result follows from Theorem 3.3.1 and Proposition 3.3.2.

□

3.3.3 Total mass of the continuous forest of trees

Recall that in the continuous case, the total mass of the forest of genealogical trees is given as

$$S^x = \int_0^{T^x} Z_t^x dt$$

Consider the increasing process

$$A_t^x = \int_0^t Z_s^x ds, t \geq 0,$$

and the associated time change

$$\eta^x(t) = \inf\{s > 0, A_s > t\}.$$

We now define $U_t^x = \frac{1}{2} Z^x \circ \eta^x(t), t \geq 0$. It is easily seen that the process U^x solves the SDE

$$dU_t^x = \frac{f(2U_t^x)}{4U_t^x} dt + dW_t, \qquad U_0^x = \frac{x}{2}. \tag{3.3.3}$$

Let $\tau^x := \inf\{t > 0, U_t^x = 0\}$. It follows from above that $\eta^x(\tau^x) = T^x$, hence $S^x = \tau^x$. We have

Theorem 3.3.4. *Suppose that the function $\frac{f(x)}{x}$ satisfies (H1) and there exists $a_0 > 0$ such that $f(x) \neq 0$ for all $x \geq a_0$.*

1) *If $\int_{a_0}^{\infty} \frac{x}{|f(x)|} dx = \infty$ then*

$$S^x \to \infty \quad a.s. \quad as \quad x \to \infty.$$

2) *If $\int_{a_0}^{\infty} \frac{x}{|f(x)|} dx < \infty$ then*

$$\sup_{x>0} S^x < \infty \qquad a.s.,$$

and moreover, there exists some positive constant c such that

$$\sup_{x>0} \mathbb{E}\left(e^{cS^x}\right) < \infty.$$

Proof. Note that we can rewrite the SDE (3.3.3) as

$$dU_t^x = \big(\beta U_t^x - h(U_t^x)\big)dt + dW_t, \qquad U_0^x = \frac{x}{2},$$

where $h(x) := \beta x - \frac{f(2x)}{4x}$, with again $\beta > \theta$, is a positive and increasing function.

1) By the comparison theorem, $U_t^x \geq U_t^{1,x}$, where $U_t^{1,x}$ solves

$$dU_t^{1,x} = -h(U_t^{1,x})dt + dW_t, \qquad U_0^{1,x} = \frac{x}{2}.$$

The result follows from Theorem 3.3.1, Proposition 3.3.2 and Lemma 3.2.3.

2) The result is a consequence of Theorem 3.3.1 and Proposition 3.3.2. We can prove it by using the same argument as used in the proof of Theorem 3.3.3.

□

3.4 Some examples

In this section we will discuss some special cases to illustrate our results.

Example 3.4.1. An important example is the case of a logistic interaction, where

$$f(x) := ax - bx^2, \qquad a \in \mathbb{R}, b \in \mathbb{R}_+.$$

There exists a positive constant a_0 such that $f(x) < 0$ for all $x \geq a_0$, and

$$\int_{a_0}^{\infty} \frac{1}{\mid f(x) \mid} dx = \int_{a_0}^{\infty} \frac{1}{bx^2 - ax} dx < \infty, \qquad \int_{a_0}^{\infty} \frac{x}{\mid f(x) \mid} dx = \int_{a_0}^{\infty} \frac{x}{bx^2 - ax} dx = \infty.$$

Hence in this case, there exists some positive constant c such that

$$\sup_{m>0} \mathbb{E}\big(e^{cH^m}\big) < \infty, \qquad \sup_{x>0} \mathbb{E}\big(e^{cT^x}\big) < \infty,$$

and

$$\sup_{m>0} L^m = \infty \quad a.s., \qquad \sup_{x>0} S^x = \infty \quad a.s.$$

Example 3.4.2. We consider the case where f is a function satisfying (H1) and for all $x \geq 2$,

$$f(x) = -x^{\alpha}(\log x)^{\gamma}, \qquad \alpha \geq 0, \gamma \geq 0.$$

Note that

$$\int_2^{\infty} \frac{1}{x^{\alpha}(\log x)^{\gamma}} dx \qquad \begin{cases} = \infty, & \text{if} \quad \alpha < 1 \quad \text{or} \quad \alpha = 1, \gamma \leq 1 \\ < \infty, & \text{if} \quad \alpha > 1 \quad \text{or} \quad \alpha = 1, \gamma > 1. \end{cases}$$

Hence

$$\sup_{m>0} H^m = \infty \quad a.s., \qquad \sup_{x>0} T^x = \infty \quad a.s.$$

if $\alpha < 1$ or $\alpha = 1, \gamma \leq 1$, while there exists some positive constant c such that

$$\sup_{m>0} \mathbb{E}\big(e^{cH^m}\big) < \infty, \qquad \sup_{x>0} \mathbb{E}\big(e^{cT^x}\big) < \infty$$

if $\alpha > 1$ or $\alpha = 1, \gamma > 1$. Concerning the length (resp. the total mass) of the genealogical forest of trees we have

$$\sup_{m>0} L^m = \infty \quad a.s., \qquad \sup_{x>0} S^x = \infty \quad a.s.$$

if $\alpha < 2$ or $\alpha = 2, \gamma \leq 1$, while there exists some positive constant c such that

$$\sup_{m>0} \mathbb{E}\big(e^{cL^m}\big) < \infty, \qquad \sup_{x>0} \mathbb{E}\big(e^{cS^x}\big) < \infty$$

if $\alpha > 2$ or $\alpha = 2, \gamma > 1$.

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Chapter 4

EXTINCTION TIME AND THE TOTAL MASS OF THE CONTINUOUS STATE BRANCHING PROCESSES WITH COMPETITION

4.1 Introduction

Consider a continuous state branching process (CSBP) with continuous paths. Such process can be used as model of population growth, with the notion of genealogical tree of population is described by using continuous random trees in the sense of Aldous [1]. However, in that context one might want to model interactions between the individuals (e.g. competition for limited resources) so that we no longer have a branching process. Such interactions can increase the number of births, or in contrary increase the number of deaths. The popular logistic competition has been considered in Le, Pardoux, Wakolbinger [11], while a much more general type of interaction appears in Ba, Pardoux [3].

We will assume that for large population size the interaction is of the type of a competition, which limits the size of the population. One may then wonder in which cases the interaction is strong enough so that the extinction time (or equivalently the height of the forest of genealogical trees) remains finite, as the number of ancestors tends to infinity, or even such that the total mass of the forest of genealogical trees remains finite, as the population size tends to infinity.

This question has been addressed in the case of a polynomial interaction in Ba, Pardoux [2], and in more general case of competition in Le, Pardoux [10]. Here we want to generalize those results to the case of CSBP with discontinuous paths. More precisely, suppose that $\sigma \geq 0$ is a constant, and $(r \wedge r^2)m(dr)$ is a finite measure on

$(0,\infty)$. Let ψ be a function given by

$$\psi(\lambda) = \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r)m(dr), \qquad \lambda \geq 0. \tag{4.1.1}$$

Let $W(ds, du)$ be a white noise on $(0,\infty)^2$ based on the Lebesgue measure $dsdu$, and let $N(ds, dz, du)$ be Poisson random measure on $(0,\infty)^3$ with intensity $dsm(dz)du$. Let $\tilde{N}(ds, dz, du)$ be the compensated measure of $N(ds, dz, du)$. We will consider the CSBP with competition characterised by the branching mechanism ψ, which is governed by the SDE

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + \sigma \int_0^t \int_0^{Z_s^x} W(ds, du) + \int_0^t \int_0^\infty \int_0^{Z_{s-}^x} z\tilde{N}(ds, dz, du), \tag{4.1.2}$$

where the function f satisfies the following hypothesis.

Hypothesis (H1): $f \in C(\mathbb{R}_+, \mathbb{R})$, $f(0) = 0$. There exists $\theta \geq 0$ such that

$$f(x + y) - f(x) \leq \theta y \qquad \forall x, y \geq 0.$$

The hypothesis (H1) implies that the function $\theta y - f(y)$ is increasing. In particular, we have

$$f(y) \leq \theta y \qquad \forall y \geq 0.$$

The equation (4.1.2) has a unique strong solution (see Dawson, Li [5]). This SDE couples the evolution of the various $\{Z_t^x,\ t \geq 0\}$ jointly for all values of $x > 0$.

For $x > 0$, define T^x the extinction time of the process Z^x by

$$T^x = \inf\{t > 0, Z_t^x = 0\}.$$

And define S^x the total mass of Z^x by

$$S^x = \int_0^{T^x} Z_t^x dt.$$

Note that our coupling of the various Z^x's makes T^x and S^x a.s. increasing. We next study the limits of T^x and S^x as $x \to \infty$. This chapter is organized as follows. Section

4.2 studies the extinction time, while section 4.3 studies the total mass of the CSBP with competition. The main results are Theorem 4.2.1, 4.2.2, 4.3.1 and 4.3.2. Section 4.4 gives some examples to illustrate our results and gives a counterexample to see that the results obtained in the case of CSBP with continuous paths may no longer be true for the case of CSBP with discontinuous paths.

4.2 Extinction time of the CSBP with competition

We now study the extinction time of the process Z^x. In the logistic case where $f(y) = ay - by^2, b > 0$, Lambert [8] has proved the process Z^x either remains positive, or is absorbed at 0 at finite time, depending solely on the branching mechanism, i.e. according to a criterion that does not involve a and b: absorption occurs with probability 1 if $\int^\infty d\lambda/\psi(\lambda) < \infty$, with probability 0 otherwise. In the case of Feller's branching diffusion with competition where $\psi(\lambda) = 2\lambda^2$ (the condition $\int^\infty d\lambda/\psi(\lambda) < \infty$ is satisfied in this case), it is showed in Le and Pardoux [10] that

$$\sup_{x>0} T^x < \infty \Leftrightarrow \int^\infty \frac{1}{| f(y) |} dy < \infty.$$

Hence it is intuitive to see that in the general case if

$$\int^\infty d\lambda/\psi(\lambda) < \infty, \qquad \int^\infty \frac{1}{| f(y) |} dy < \infty, \tag{4.2.1}$$

we have $\sup_{x>0} T^x < \infty$. Indeed, under the condition (4.2.1), we can prove that $\sup_{x>0} T^x$ has finite expectation.

We first need the following lemma, which is Lemma 2.3 in [10].

Lemma 4.2.1. *Let f be a function satisfying (H1), $a \in \mathbb{R}$ be a constant. If there exists $a_0 > 0$ such that $f(y) \neq 0, f(y) + ay \neq 0$ for all $y \geq a_0$, then we have that*

$$\int_{a_0}^\infty \frac{1}{| f(y) |} dy < \infty \Leftrightarrow \int_{a_0}^\infty \frac{1}{| ay + f(y) |} dy < \infty,$$

and when those equivalent conditions are satisfied, we have

$$\lim_{y\to\infty} \frac{f(y)}{y} = -\infty.$$

We now establish the main results of this section

Theorem 4.2.1. *Suppose that f is a function satisfying (H1) such that* $\lim_{y\to 0^+}\frac{f(y)}{y} > -\infty$ *and* $\int^{\infty} d\lambda/\psi(\lambda) = \infty$. *We have for all $x > 0$, $T^x = \infty$ a.s.*

Proof. From the condition (H1) and $\lim_{y\to 0^+}\frac{f(y)}{y} > -\infty$ there exists a positive constant δ such that

$$-\delta y \le f(y) \le \theta y \qquad \forall y \in [0, 2x].$$

Define $\tau_1 := \inf\{t > 0 : Z_t^x \ge 2x\}$, then $f(Z_t^x) \ge -\delta Z_t^x$ for all $t \in [0, \tau_1)$. By the comparison theorem (see Dawson, Li [5]) we have $Z_t^x \ge Z_t^{1,x}$ a.s. for all $t \in [0, \tau_1)$, where $Z_t^{1,x}$ solves

$$Z_t^{1,x} = x - \delta\int_0^t Z_s^{1,x} ds + \sigma\int_0^t\int_0^{Z_s^{1,x}} W(ds, du) + \int_0^t\int_0^{\infty}\int_0^{Z_{s-}^{1,x}} z\tilde{N}(ds, dz, du).$$

The process $Z_t^{1,x}$ is a CSBP characterised by the branching mechanism $\psi_1(\lambda) = \psi(\lambda) + \delta\lambda$. By Lemma 4.2.1 we have $\int^{\infty} d\lambda/\psi_1(\lambda) = \infty$, so that $Z_t^{1,x}$ remains positive a.s. (see Kyprianou [7], page 279). Hence Z_t^x remains positive a.s. on $[0, \tau_1)$.

Assume that we have the opposite result $\mathbb{P}(T^x < \infty) > 0$. We condition on $\{T^x < \infty\}$. We have τ_1 is finite and therefore we can define $\tau_2 := \inf\{t > \tau_1 : Z_t^x \le x\}$. Clearly τ_2 is finite and $Z_{\tau_2}^x = x$ because the process Z_t^x has no negative jumps. Define

$$\tau_{2k+1} := \inf\{t > \tau_{2k} : Z_t^x \ge 2x\}, \qquad \tau_{2k+2} := \inf\{t > \tau_{2k+1} : Z_t^x \le x\}, \quad k \ge 1,$$

then every τ_k are finite by the above argument. But it is easy to see that

$$\sum_{k=1}^{\infty}(\tau_{2k+1} - \tau_{2k}) = \infty \qquad \text{a.s.}$$

This contradicts our initial assumption. The result follows. ∎

Theorem 4.2.2. *Assume that f is a function satisfying (H1) and that there exists $a_0 > 0$ such that $f(y) \neq 0$ for all $y \ge a_0$. If the condition* (4.2.1) *is satisfied, we have*

$$\sup_{x>0}\mathbb{E}(T^x) < \infty.$$

Proof. From Lemma 4.2.1 we get

$$\lim_{y\to\infty}\frac{f(y)}{y}=-\infty,$$

then there is a constant $M>a_0$ such that $f(y)<-y$ for all $y\geq M$. Define for $x>M$,

$$T^x_M=\inf\{t>0, Z^x_t\leq M\}.$$

We have for $x>M$,

$$dZ^x_t=f(Z^x_t)dt+\sigma\int_0^{Z^x_t}W(dt,du)+\int_0^\infty\int_0^{Z^x_{t-}}z\tilde{N}(dt,dz,du),$$

$$\frac{dZ^x_t}{-f(Z^x_t)}=-dt+\sigma\int_0^{Z^x_t}\frac{1}{-f(Z^x_t)}W(dt,du)+\int_0^\infty\int_0^{Z^x_{t-}}\frac{z}{-f(Z^x_t)}\tilde{N}(dt,dz,du).$$

Hence

$$\int_0^{T^x_M\wedge t}\frac{dZ^x_s}{-f(Z^x_s)}=-(T^x_M\wedge t)+\sigma\int_0^{T^x_M\wedge t}\int_0^{Z^x_s}\frac{1}{-f(Z^x_s)}W(ds,du)$$
$$+\int_0^{T^x_M\wedge t}\int_0^\infty\int_0^{Z^x_{s-}}\frac{z}{-f(Z^x_s)}\tilde{N}(ds,dz,du). \qquad (4.2.2)$$

It is easy to show that

$$\mathbb{E}\Big[\Big|\int_0^{T^x_M\wedge t}\int_1^\infty\int_0^{Z^x_{s-}}\frac{z}{-f(Z^x_s)}\tilde{N}(ds,dz,du)\Big|\Big]\leq\mathbb{E}\Big[\int_0^{T^x_M\wedge t}\frac{Z^x_{s-}}{-f(Z^x_s)}ds\int_1^\infty zm(dz)\Big].$$

Observe also that

$$\mathbb{E}\Big[\Big|\int_0^{T^x_M\wedge t}\int_0^{Z^x_s}\frac{1}{-f(Z^x_s)}W(ds,du)\Big|^2\Big]=\mathbb{E}\Big[\int_0^{T^x_M\wedge t}\frac{Z^x_s}{f(Z^x_s)^2}ds\Big]$$

$$\mathbb{E}\Big[\Big|\int_0^{T^x_M\wedge t}\int_0^1\int_0^{Z^x_{s-}}\frac{z}{-f(Z^x_s)}\tilde{N}(ds,dz,du)\Big|^2\Big]=\mathbb{E}\Big[\int_0^{T^x_M\wedge t}\frac{Z^x_{s-}}{f(Z^x_s)^2}ds\int_0^1 z^2m(dz)\Big].$$

The above expectations are finite, then

$$t\mapsto\int_0^{T^x_M\wedge t}\int_0^{Z^x_s}\frac{1}{-f(Z^x_s)}W(ds,du)+\int_0^{T^x_M\wedge t}\int_0^\infty\int_0^{Z^x_{s-}}\frac{z}{-f(Z^x_s)}\tilde{N}(ds,dz,du)$$

is a martingale. From (4.2.2) we get

$$\mathbb{E}\Big[\int_0^{T^x_M\wedge t}\frac{dZ^x_s}{-f(Z^x_s)}\Big]=-\mathbb{E}(T^x_M\wedge t)$$

$$\mathbb{E}\Big[\int_{Z^x_{T^x_M\wedge t}}^x\frac{du}{-f(u)}\Big]=\mathbb{E}(T^x_M\wedge t).$$

We deduce that for all $x > M, t > 0$,

$$\mathbb{E}(T_M^x \wedge t) \leq \int_M^\infty \frac{du}{-f(u)}.$$

Taking the limit as $x \to \infty$ and $t \to \infty$ we have

$$\sup_{x>M} \mathbb{E}(T_M^x) < \infty,$$

or $\mathbb{E}(T_M) < \infty$, where $T_M := \sup_{x>M} T_M^x$.

We have just proved that the process Z comes down from infinity. For proving that $\sup_{x>0} \mathbb{E}(T^x) < \infty$, it remains to show that the time taken by Z to descend from M to 0 is integrable,which we now establish. By the comparison theorem we have $Z_t^M \leq Z_t^{2,M}$ a.s. for all $t \geq 0$, where $Z_t^{2,M}$ solves

$$Z_t^{2,M} = M + \theta \int_0^t Z_s^{2,M} ds + \sigma \int_0^t \int_0^{Z_s^{2,M}} W(ds, du) + \int_0^t \int_0^\infty \int_0^{Z_{s-}^{2,M}} z\tilde{N}(ds, dz, du).$$

The process $Z_t^{2,M}$ is a CSBP characterised by the branching mechanism $\psi_2(\lambda) = \psi(\lambda) - \theta\lambda$. By Lemma 4.2.1 we obtain $\int^\infty d\lambda/\psi_2(\lambda) < \infty$, so that $Z^{2,M}$ is absorbed at 0 in finite time with positive probability (see Kyprianou [7], page 279). Then there is a constant $T > 0$ such that $Z^{2,M}$ is absorbed at 0 before time T with positive probability. Let p denote the probability that starting from M at time $t = 0, Z$ hits zero before time T. Clearly $p > 0$. Let ζ be a geometric random variable with success probability p, which is defined as follows. Let Z start from M at time 0. If Z hits zero before time T, then $\zeta = 1$. If not, we look the position Z_T of Z at time T.

If $Z_T > M$, we wait until Z goes back to M. The time needed is stochastically dominated by the random variable T_M, which is the time needed for Z to descend to M, when starting from ∞. If however $Z_T \leq M$, we start afresh from there, since the probability to reach zero in less than T is greater than or equal to p, for all starting points in the interval $(0, M]$.

So either at time T, or at time less than $T + T_M$, we start again from a level which is less than or equal to M. If zero is reached during the next time interval of length

T, then $\zeta = 2$... Repeating this procedure, we see that $\sup_{x>0} T^x$ is stochastically dominated by

$$\zeta T + \sum_{i=1}^{\zeta} \eta_i,$$

where the random variables η_i are i.i.d, with the same law as T_M, globally independent of ζ. Therefore

$$\begin{aligned}\sup_{x>0} \mathbb{E}(T^x) &\leq \mathbb{E}(\zeta T + \sum_{i=1}^{\zeta} \eta_i) \\ &= \frac{T}{p} + \frac{1}{p}\mathbb{E}(T_M) \\ &< \infty.\end{aligned}$$

The result follows. □

4.3 Total mass of the CSBP with competition

In this section, we shall assume that

Hypothesis (H2): f is a function satisfying (H1) such that

$$\lim_{u \to 0^+} \frac{f(u)}{u} = \alpha,$$

for some $-\infty < \alpha \leq \theta$, and the function $f_1(u) := \frac{f(u)}{u} - \alpha$ satisfies (H1).

4.3.1 The Lamperti transform

We will study the total mass S^x of the process Z^x. In this subsection we remind the reader of a celebrated result of Lamperti [9] which relates CSBP and Lévy processes with no negative jumps. This result will allow us to give a representation of CSBP with competition in terms of spectrally positive Lévy processes with drift.

Let X be a real-valued Lévy process with no negative jumps and initial position $X_0 = x > 0$. Let T_0 be the first hitting time of zero by X. Then define

$$\rho_t = \int_0^{T_0 \wedge t} \frac{ds}{X_s} \qquad t > 0,$$

and $(C_t, t > 0)$ its right-inverse. Lamperti's result then states that if

$$Y_t = X(C_t) \qquad t > 0,$$

then Y is a CSBP with initial value $Y_0 = x$. Moreover,

$$C_t = \int_0^t Y_s ds \qquad t > 0.$$

Conversely, suppose that Y is a CSBP such that $Y_0 = x > 0$. If C is defined as above, and ρ is the right-inverse of C, then $Y \circ \rho$ is a Lévy process with no negative jumps which starts at x and is killed when it hits 0.

We now time-change the CSBP with competition Z^x in Lamperti's fashion to obtain a Lévy process with drift. Consider the increasing process

$$C_t^x = \int_0^t Z_s^x ds, t \geq 0,$$

and its right-inverse ρ_t^x. We define $U^x = Z^x \circ \rho^x$. We have

Proposition 4.3.1. *Assume that the function f satisfies $(H2)$. Then U^x is the unique strong solution of the following SDE*

$$dU_t^x = \frac{f(U_t^x)}{U_t^x} dt + dX_t, \qquad U_0^x = x. \tag{4.3.1}$$

where X is a Lévy process with Laplace exponent ψ.

Proof. The process Z^x is a càdlàg homogeneous strong Markov processes (see e.g. [5]). By standard theory of Markov processes (see e.g. [6, 12]), U^x is then a càdlàg homogeneous strong Markov process. We denote A (resp. Q, L) the infinitesimal generator of X (resp. U^x, Z^x). Using Itô 's formula one can see that Z^x solves the martingale problem associated with the infinitesimal generator L given by

$$\begin{aligned} Lg(y) &= \frac{1}{2}\sigma^2 y g^{''}(y) + f(y)g^{'}(y) + y\int_0^\infty D_z g(y)m(dz) \\ &= yAg(y) + f(y)g^{'}(y). \end{aligned}$$

Furthermore, we deduce $Qg(y) = \frac{Lg(y)}{y}$ from the fact that for any time $t > 0$, with $r = \rho_s^x$,

$$\begin{aligned}
\mathbb{E}(g(U_t^x)) &= \mathbb{E}(g(Z_{\rho_t^x}^x)) \\
&= \mathbb{E}(\int_0^{\rho_t^x} Lg(Z_r^x)dr) \\
&= \mathbb{E}(\int_0^t \frac{Lg(U_s^x)}{U_s^x} ds) \\
&= \mathbb{E}(\int_0^t Qg(U_s^x)ds).
\end{aligned}$$

Hence

$$Qg(y) = Ag(y) + \frac{f(y)}{y} g^{'}(y).$$

This shows that U^x is a solution of the SDE (4.3.1). It remains to prove the uniqueness of solution of (4.3.1). Suppose that $U_t^{1,x}$ and $U_t^{2,x}$ be two solutions of (4.3.1), we have for all $t \geq 0$,

$$U_t^{1,x} - U_t^{2,x} = \int_0^t \big(f_1(U_s^{1,x}) - f_1(U_s^{2,x})\big) ds.$$

Then by Ito's formula and Hypothesis (H1) we get

$$\begin{aligned}
(U_t^{1,x} - U_t^{2,x})^2 &= \int_0^t 2\big(U_s^{1,x} - U_s^{2,x}\big)\big(f_1(U_s^{1,x}) - f_1(U_s^{2,x})\big) ds \\
&\leq \int_0^t 2\theta (U_s^{1,x} - U_s^{2,x})^2 ds.
\end{aligned}$$

The result follows from Gronwall's inequality. □

Let $\tau^x := \inf\{t > 0, U_t^x = 0\}$. It is easy to see that $\rho^x(\tau^x) = T^x$, hence $S^x = \tau^x$. We next study the limits of S^x as $x \to \infty$. We want to show that under a specific assumption $S^x \to \infty$ a.s. as $x \to \infty$, and under the complementary assumption $\sup_{x>0} S^x < \infty$ a.s. Because the mapping $x \mapsto S^x$ is a.s. increasing, the result will follow for the same result proved for any collection of r.v.'s $\{S^x, x > 0\}$ which has the same monotonicity property, and has the same marginal laws as the original one. More precisely, we will consider the U^x's solutions of (4.3.1) with the same X for all $x > 0$.

4.3.2 About the Lévy process X

In this subsection we establish some preliminary results on Lévy processes which will be used later. Recall that X is a spectrally positive Lévy process with Laplace exponent ψ given by (4.1.1), so that for all $\lambda \geq 0$,

$$\mathbb{E}(e^{-\lambda X_t}) = e^{t\psi(\lambda)} \tag{4.3.2}$$

Because ψ is continuous and has a continuous derivative, $\psi(0) = 0$ and ψ is increasing on $\mathbb{R}_+$ so that ψ has a unique inverse ϕ which is defined and continuous on $\mathbb{R}_+$ and satisfies $\phi(0) = 0$. From (4.3.2) we get for any $t \geq 0$,

$$\mathbb{E}(X_t) = -t\psi'(0) = 0.$$

Hence for any stopping time η which is a.s. positive and integrable,

$$\mathbb{E}(X_\eta) = \mathbb{E}(\mathbb{E}(X_\eta \mid \eta)) = -\mathbb{E}(\eta)\psi'(0) = 0. \tag{4.3.3}$$

Furthermore we have (see Theorem 7.2 in [7])

$$\limsup_{t\to\infty} X_t = -\liminf_{t\to\infty} X_t = \infty \quad \text{a.s.}$$

So that if we define for $y > 0$,

$$\tau_y^+ = \inf\{t > 0, X_t > y\}, \qquad \tau_y^- = \inf\{t > 0, X_t < -y\},$$

then τ_y^+ and τ_y^- are a.s. positive and finite. We have

Proposition 4.3.2. *$\tau_y^+ \to \infty$ a.s. and $\tau_y^- \to \infty$ a.s. as $y \to \infty$, and for any $y > 0, \beta > 0$ we have*

$$\mathbb{E}\Big(\frac{1}{(\tau_y^-)^\beta}\Big) < \infty.$$

Proof. Define for $t > 0$,

$$\overline{X}_t = \sup_{0\leq s\leq t} X_s \qquad \text{and} \qquad \underline{X}_t = \inf_{0\leq s\leq t} X_s.$$

Then for all $\beta \geq 0$,

$$\mathbb{E}\big(e^{\beta \underline{X}_{e_q}}\big) = \frac{\phi(q)}{\phi(q)+\beta} \qquad \text{and} \qquad \mathbb{E}\big(e^{-\beta \overline{X}_{e_q}}\big) = \frac{q}{\phi(q)} \frac{\phi(q)-\beta}{q-\psi(\beta)}, \tag{4.3.4}$$

where e_q is an independent and exponentially distributed random variable with parameter $q > 0$ (see Kyprianou [7], page 213). Letting β tend to zero in the first expression of (4.3.4) we see that

$$\mathbb{P}(\underline{X}_{e_q} > -\infty) = 1.$$

We have

$$\begin{aligned} \mathbb{P}(\sup_{y>0} \tau_y^- < e_q) &= \lim_{y\to\infty} \mathbb{P}(\tau_y^- < e_q) \\ &= \lim_{y\to\infty} \mathbb{P}(\underline{X}_{e_q} < -y) \\ &= 0. \end{aligned}$$

Therefore for all $t > 0, q > 0$,

$$\mathbb{P}(\sup_{y>0} \tau_y^- \leq t) \leq \mathbb{P}(\sup_{y>0} \tau_y^- < e_q) + \mathbb{P}(e_q \leq t) = 1 - e^{-qt}.$$

Then taking q to zero we get $\mathbb{P}(\sup_{y>0} \tau_y^- \leq t) = 0$ for all $t > 0$, so that

$$\mathbb{P}(\sup_{y>0} \tau_y^- = \infty) = 1.$$

Hence $\tau_y^- \to \infty$ a.s. as $y \to \infty$. Similarly, from the second expression of (4.3.4) we can prove $\tau_y^+ \to \infty$ a.s. as $y \to \infty$.

For proving the last result of the Proposition, it is enough to show that

$$\mathbb{E}\big((\tau_y^-)^{-n}\big) < \infty \qquad \text{for all } n \in \mathbb{N}^*.$$

Note that (see [7], page 212) the process $\{\tau_y^-, y \geq 0\}$ is a subordinator with Laplace exponent ϕ, and

$$\mathbb{E}\big(e^{-s\tau_y^-}\big) = e^{-\phi(s)y} \qquad \text{for all } s > 0. \tag{4.3.5}$$

It is easy to see that for all $s > 0, n \in \mathbb{N}^*$

$$\mathbb{E}\big((\tau_y^-)^{-n} e^{-s\tau_y^-}\big) = F_n(s),$$

where

$$F_1(s) = \int_s^\infty e^{-\phi(u)y} du \text{ and } F_{n+1}(s) = \int_s^\infty F_n(u)du \text{ for all } n \geq 1.$$

By Lemma 4.3.3 below we have that for any $n \geq 1, F_n(s)$ is finite. Hence for all $n \geq 1$,

$$\begin{aligned}
\mathbb{E}\big((\tau_y^-)^{-n}\big) &\leq \mathbb{E}\big((\tau_y^-)^{-n}\mathbf{1}_{\{\tau_y^- \leq 1\}}\big) + 1 \\
&\leq e^s \mathbb{E}\big((\tau_y^-)^{-n} e^{-s\tau_y^-}\mathbf{1}_{\{\tau_y^- \leq 1\}}\big) + 1 \\
&\leq e^s F_n(s) + 1 \\
&< \infty.
\end{aligned}$$

The result follows. $\square$

Lemma 4.3.3. *For $n \geq 1$, there exist positive constants $m_0^n, m_1^n, .., m_n^n$ which depend upon y such that*

$$F_n(s) \leq e^{-\phi(s)y}\big(m_0^n + m_1^n\phi(s) + ... + m_n^n\phi(s)^n\big), \qquad s > 0.$$

Proof. We will prove this lemma by induction on n. It is easily seen that

$$\psi^{'}(s) \leq b_1 s + b_0, \qquad s > 0, \tag{4.3.6}$$

where

$$b_0 = \int_1^\infty rm(dr), \qquad b_1 = \sigma^2 + \int_0^1 r^2 m(dr).$$

We have for any $s > 0$, with $u = \psi(r)$,

$$\begin{aligned}
F_1(s) &= \int_s^\infty e^{-\phi(u)y} du \\
&= \int_{\phi(s)}^\infty e^{-ry}\psi^{'}(r)dr \\
&\leq \int_{\phi(s)}^\infty e^{-ry}(b_1 r + b_0)dr.
\end{aligned}$$

We deduce that the lemma holds for $n = 1$ from the fact that for all $a > 0, m \geq 1$,

$$\int_a^\infty e^{-ry} r^m dr = \frac{1}{y} e^{-ay} a^m + \frac{m}{y} \int_a^\infty e^{-ry} r^{m-1} dr. \tag{4.3.7}$$

Assume that the lemma holds for $n = k$. Hence for any $s > 0$, with $u = \psi(r)$,

$$\begin{aligned} F_{k+1}(s) &\leq \int_s^\infty e^{-\phi(u)y} \big(m_0^k + m_1^k \phi(u) + ... + m_k^k \phi(u)^k\big) du \\ &= \int_{\phi(s)}^\infty e^{-ry} (m_0^k + m_1^k r + ... + m_k^k r^k) \psi'(r) dr \\ &\leq \int_{\phi(s)}^\infty e^{-ry} (m_0^k + m_1^k r + ... + m_k^k r^k)(b_1 r + b_0) dr, \end{aligned}$$

where we have used (4.3.6) for the last inequality. From (4.3.7) we have the lemma holds for $n = k + 1$. The result follows. □

Lemma 4.3.4. *For $t \geq 0$, define $\Gamma_t = \inf\{s > 0, X_s - s < -t\}$. We have $\mathbb{E}(\Gamma_t) = t$.*

Proof. Note that $X_s - s$ is a spectrally positive Lévy process with Laplace exponent $\psi_0(\lambda) = \psi(\lambda) + \lambda$. Denote ϕ_0 the unique inverse of ψ_0. It is well known that the process $\{\Gamma_t, t \geq 0\}$ is a subordinator with Laplace exponent ϕ_0, and

$$\mathbb{E}\big(e^{-s\Gamma_t}\big) = e^{-\phi_0(s)t} \qquad \text{for all } s \geq 0, t \geq 0.$$

Therefore $\mathbb{E}(\Gamma_t) = \phi_0'(0)t$. The result follows from the fact that

$$\psi_0'(0) = 1 \qquad \text{and} \qquad \psi_0'(0)\phi_0'(0) = 1.$$

□

Lemma 4.3.5. *Assume that (H) the paths of X are of infinite variation a.s. Then for all positive constants a, b we have*

$$\mathbb{P}(a + \inf_{[0,b]} X_s \leq 0) > 0.$$

Proof. According to [4] (Corollary VII.5), assumption (H) holds iff

$$\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\lambda} = \infty. \tag{4.3.8}$$

Note that (4.3.8) happens iff at least one of the following two conditions is satisfied: $\sigma > 0$, or

$$\int_0^1 rm(dr) = \infty.$$

If $\mathbb{P}(a + \inf_{[0,b]} X_s \le 0) = 0$, we have $\tau_a^- \ge b$ a.s. Hence $\mathbb{E}\big(e^{-s\tau_a^-}\big) \le e^{-bs}$ for all $s > 0$. By (4.3.5) we get

$$e^{-\phi(s)a} \le e^{-bs}$$

$$\phi(s)a \ge bs. \tag{4.3.9}$$

Let $s = \psi(r)$ in (4.3.9) we obtain $ar \ge b\psi(r)$ for all $r > 0$. This contradicts (4.3.8), so that

$$\mathbb{P}(a + \inf_{[0,b]} X_s \le 0) > 0.$$

□

4.3.3 Main results

We now establish the main results of this section

Theorem 4.3.1. *Suppose that f is a function satisfying $(H2)$ and that there exists $a_0 > 0$ such that $f(u) \neq 0$ for all $u \ge a_0$. If $\int_{a_0}^{\infty} \frac{u}{|f(u)|} du = \infty$, then*

$$S^x \to \infty \quad a.s. \quad as \quad x \to \infty.$$

Proof. Let γ be a constant such that $f_2(u) := \gamma u - f_1(u)$ is a positive and increasing function (we can choose e.g. $\gamma > \theta$, by Hypothesis (H2)). We can rewrite the SDE (4.3.1) as

$$dU_t^x = \big(\alpha + \gamma U_t^x - f_2(U_t^x)\big)dt + dX_t, \qquad U_0^x = x,$$

Setting $V_t^x = U_t^x - X_t$, then V_t^x solves the ODE

$$\frac{dV_t^x}{dt} = \alpha + \gamma(V_t^x + X_t) - f_2(V_t^x + X_t), \qquad V_0^x = x.$$

Let $\{x_n, n \geq 1\}$ be an increasing sequence of positive real numbers such that $x_n \to \infty$ as $n \to \infty$. For any $y > 0$, there exists $n_y > 0$ such that $x_n > 2y$ for all $n \geq n_y$. Define

$$R_y^n := \inf\{t > 0, V_t^{x_n} < 2y\} \quad \text{for any} \quad y > 0, n \geq n_y.$$

For $n \geq n_y$, we have on the time interval $[0, R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2}]$,

$$\begin{aligned}
-y \leq X_t &\leq y \\
y \leq V_t^{x_n} + X_t &\leq V_t^{x_n} + y \\
\frac{dV_t^{x_n}}{dt} &\geq - \mid \alpha \mid - f_2(V_t^{x_n} + y) \\
-t &\leq \int_0^t \frac{dV_s^{x_n}}{\mid \alpha \mid + f_2(V_s^{x_n} + y)} \\
t &\geq \int_{V_t^{x_n}+y}^{x_n+y} \frac{du}{\mid \alpha \mid + f_2(u)}.
\end{aligned} \tag{4.3.10}$$

Consider now the integral $\int^\infty \frac{du}{|\alpha| + f_2(u)}$. If $\int^\infty \frac{du}{|\alpha|+f_2(u)} < \infty$, then by Lemma 4.2.1 we have

$$\lim_{u \to \infty} \frac{\mid \alpha \mid + f_2(u)}{u} = \infty.$$

We deduce that there exists a constant $a_1 > a_0$ such that

$$-\frac{f(u)}{u} \geq \alpha + \mid \alpha \mid + \gamma u \quad \text{for all} \quad u \geq a_1.$$

Therefore

$$0 < \int_{a_1}^\infty \frac{-u}{2f(u)} du \leq \int_{a_1}^\infty \frac{du}{\alpha + \mid \alpha \mid + \gamma u - \frac{f(u)}{u}} = \int_{a_1}^\infty \frac{du}{\mid \alpha \mid + f_2(u)} < \infty.$$

This contradiction shows that $\int^\infty \frac{du}{|\alpha|+f_2(u)} = \infty$. From (4.3.10) we get

$$\lim_{n \to \infty} V_t^{x_n} = \infty \text{ a.s. for all } t \in [0, R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2}].$$

Hence

$$\lim_{n\to\infty} R_y^n \geq \tau_y^- \wedge \frac{\tau_y^+}{2} \text{ a.s.}$$

Moreover, because $U_t^{x_n} = V_t^{x_n} + X_t > 0$ a.s. for all $t \in [0, R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2}]$, then

$$\begin{aligned} S^{x_n} = \tau^{x_n} &\geq R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2} \text{ a.s.} \\ \lim_{n\to\infty} S^{x_n} = \lim_{n\to\infty} \tau^{x_n} &\geq \lim_{n\to\infty} R_y^n \wedge \tau_y^- \wedge \frac{\tau_y^+}{2} \text{ a.s.} \\ &\geq \tau_y^- \wedge \frac{\tau_y^+}{2} \text{ a.s.} \end{aligned}$$

Letting y tend to infinity, the result follows from Proposition 4.3.2. □

We next consider the case $\int_{a_0}^{\infty} \frac{u}{|f(u)|} du < \infty$. We will see that in this case $\sup_{x>0} S^x < \infty$ a.s. Indeed, we can prove that it has some finite moments.

It is easy to see that $\frac{f(u)}{u} \to -\infty$ as $u \to \infty$, so that there exists a constant $a_2 > a_0$ such that $\frac{f(u)}{u} \leq -2 \mid \alpha \mid$ for all $u \geq a_2$. Hence

$$\int_{a_2}^{\infty} \frac{1}{\mid f_1(u) \mid} du = \int_{a_2}^{\infty} \frac{1}{\alpha - \frac{f(u)}{u}} du \leq \int_{a_2}^{\infty} \frac{2u}{-f(u)} du < \infty.$$

By Lemma 4.2.1 we have

$$\int_{a_2}^{\infty} \frac{1}{f_2(u)} du = \int_{a_2}^{\infty} \frac{1}{\gamma u - f_1(u)} du < \infty. \tag{4.3.11}$$

Let $g(y) := \int_y^{\infty} \frac{1}{f_2(u)} du$ for $y \geq a_0$. Then g is decreasing and $g(y) \to 0$ as $y \to \infty$. We suppose that the following hypothesis holds:

Hypothesis (H3): The function f_2 is C^1 on (a_0, ∞) and there exist some constants $d > 0, c > a_0$ such that

$$g(y) f_2^{'}(y) \geq 1 + d \quad \text{for all} \quad y \geq c.$$

Define the function $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ as follows.

$$h(y) \quad = \quad \begin{cases} \frac{1}{g(c)^d}, & 0 \le y \le c \\ \frac{1}{g(y)^d}, & y > c. \end{cases}$$

Then h is increasing and is C^2 on (c, ∞), $h(y) \to \infty$ as $y \to \infty$, and

$$h^{''}(y) = \frac{-d[g(y)f_2^{'}(y) - d - 1]}{f_2(y)^2 g(y)^{d+2}} \le 0 \quad \text{for all} \quad y > c.$$

Therefore $h^{'}(y)$ is decreasing on (c, ∞). From the fact that for $y > 2c$, there exists $\xi \in (2c, y)$ such that

$$h(y) - h(2c) = h^{'}(\xi)(y - 2c) \le h^{'}(2c)(y - 2c) \le h^{'}(2c)y,$$

we easily deduce that

$$h(y) \le h(2c) + h^{'}(2c)y \quad \text{for all} \quad y \ge 0. \tag{4.3.12}$$

We have

Lemma 4.3.6. *There exists a positive constant c_1 such that*

$$h(a + b) \le h(a) + h(b) + c_1 \quad \textit{for all} \quad a, b \ge 0.$$

Proof. For all $0 \le a, b < 2c$ we have $h(a + b) \le h(4c)$. For $a \ge 2c$, define the function $h_1 \in C((c, \infty), \mathbb{R}_+)$ by

$$h_1(y) = h(y + b) - h(y).$$

We have $h_1^{'}(y) = h^{'}(y + b) - h^{'}(y) \le 0$ for all $y > c$. Then

$$h(a + b) - h(a) = h_1(a) \le h_1(2c) = h(2c + b) - h(2c) \le h(2c + b)$$

$$h(a + b) - h(a) - h(b) \le h(2c + b) - h(b).$$

But $h(2c + b) \le h(4c)$ for $0 \le b < 2c$, and $h(2c + b) - h(b) \le h(4c) - h(2c)$ for $b \ge 2c$, by the same argument from above. The result follows by choosing $c_1 = h(4c)$. $\square$

Theorem 4.3.2. *Suppose that there exists $a_0 > 0$ such that $f(u) \neq 0$ for all $u \geq a_0$ and that $(H2), (H3)$ hold. If*

$$\int_{a_0}^{\infty} \frac{u}{\mid f(u) \mid} du < \infty \quad \textit{and} \quad \lim_{\lambda \to \infty} \frac{\psi(\lambda)}{\lambda} = \infty,$$

then

$$\mathbb{E}\big(h(\sup_{x>0} S^x)\big) < \infty.$$

Proof. From (4.3.11) and Lemma 4.2.1 we deduce that

$$\lim_{u \to \infty} \frac{f_2(u)}{u} = \infty.$$

Therefore there exists a constant $M > a_0$ such that $f_2(u) \geq 2\gamma u + 2\alpha$ for all $u \geq M$. Let $\{x_n, n \geq 1\}$ be an increasing sequence of positive real numbers such that $x_n \to \infty$ as $n \to \infty$. There exists $n_0 > 0$ such that $x_n > 2M$ for all $n \geq n_0$. Hence

$$R_M^n = \inf\{t > 0, V_t^{x_n} < 2M\} > 0 \text{ a.s. for any } n \geq n_0.$$

For $n \geq n_0$, we have on the time interval $[0, R_M^n \wedge \tau_M^-]$,

$$\begin{aligned}
-M &\leq X_t \\
M \leq \frac{1}{2} V_t^{x_n} &\leq V_t^{x_n} + X_t \\
\alpha + \gamma(V_t^{x_n} + X_t) - f_2(V_t^{x_n} + X_t) &\leq -\frac{1}{2} f_2(V_t^{x_n} + X_t) \leq -\frac{1}{2} f_2(\frac{1}{2} V_t^{x_n}) \\
\frac{dV_t^{x_n}}{dt} &\leq -\frac{1}{2} f_2(\frac{1}{2} V_t^{x_n}) \\
\int_0^t \frac{dV_s^{x_n}}{f_2(\frac{1}{2} V_s^{x_n})} &\leq -\frac{1}{2} t \\
\int_{\frac{1}{2} V_t^{x_n}}^{\frac{1}{2} x_n} \frac{du}{f_2(u)} &\geq \frac{1}{4} t \\
g(\frac{1}{2} V_t^{x_n}) &\geq \frac{1}{4} t.
\end{aligned} \tag{4.3.13}$$

Now, for proving $\mathbb{E}\big(h(\sup_{x>0} S^x)\big) < \infty$ we follow the following five steps:

Step 1. We first show that for all $n \geq n_0$, $R_M^n \wedge \tau_M^-$ is a.s. bounded above by $4g(M)$.

Indeed, if $R_M^n \wedge \tau_M^- > 4g(M)$ then

$$V_{4g(M)}^{x_n} > 2M.$$

So that

$$g\big(\frac{1}{2}V_{4g(M)}^{x_n}\big) < g(M),$$

because g is decreasing. This contradicts (4.3.13).

Step 2. We show that for all $n \geq n_0$,

$$\mathbb{E}\big(h(V_{R_M^n \wedge \tau_M^-}^{x_n})\big) < \infty.$$

Note that on the set $\{R_M^n < \tau_M^-\}$, $V_{R_M^n \wedge \tau_M^-}^{x_n} = 2M$. Consequently,

$$V_{R_M^n \wedge \tau_M^-}^{x_n} = 2M\mathbf{1}_{\{R_M^n < \tau_M^-\}} + V_{\tau_M^-}^{x_n}\mathbf{1}_{\{R_M^n \geq \tau_M^-\}}.$$

Therefore

$$\begin{aligned} h\big(V_{R_M^n \wedge \tau_M^-}^{x_n}\big) &= h(2M)\mathbf{1}_{\{R_M^n < \tau_M^-\}} + h\big(V_{\tau_M^-}^{x_n}\big)\mathbf{1}_{\{R_M^n \geq \tau_M^-\}} \\ &\leq h(2M) + 2h\big(\frac{1}{2}V_{\tau_M^-}^{x_n}\big)\mathbf{1}_{\{R_M^n \geq \tau_M^-\}} + c_1 \\ &\leq h(2M) + \frac{2^{2d+1}}{(\tau_M^-)^d} + c_1, \end{aligned}$$

where we have used Lemma 4.3.6 and (4.3.13) for the last two inequalities. Hence

$$\mathbb{E}\big(h(V_{R_M^n \wedge \tau_M^-}^{x_n})\big) \leq h(2M) + c_1 + 2^{2d+1}\mathbb{E}\big(\frac{1}{(\tau_M^-)^d}\big). \tag{4.3.14}$$

Step 2 follows from Proposition 4.3.2.

Step 3. We show that for all $n \geq n_0$,

$$\mathbb{E}\big(h(U_{R_M^n \wedge \tau_M^-}^{x_n})\big) < \infty.$$

From (4.3.3) and Step 1 we get

$$\begin{aligned} \mathbb{E}\big(X_{R_M^n \wedge \tau_M^-}\big) &= 0 \\ \mathbb{E}\big(X_{R_M^n \wedge \tau_M^-}\mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} > 0\}} + X_{R_M^n \wedge \tau_M^-}\mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} \leq 0\}}\big) &= 0 \\ \mathbb{E}\big(X_{R_M^n \wedge \tau_M^-}\mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} > 0\}} - M\big) &\leq 0 \\ \mathbb{E}\big(X_{R_M^n \wedge \tau_M^-}\mathbf{1}_{\{X_{R_M^n \wedge \tau_M^-} > 0\}}\big) &\leq M. \end{aligned} \tag{4.3.15}$$

We have

$$
\begin{aligned}
h(U^{x_n}_{R^n_M\wedge\tau^-_M}) &= h(V^{x_n}_{R^n_M\wedge\tau^-_M} + X_{R^n_M\wedge\tau^-_M}) \\
&\le h(V^{x_n}_{R^n_M\wedge\tau^-_M} + X_{R^n_M\wedge\tau^-_M}\mathbf{1}_{\{X_{R^n_M\wedge\tau^-_M}>0\}}) \\
&\le h(V^{x_n}_{R^n_M\wedge\tau^-_M}) + h(X_{R^n_M\wedge\tau^-_M}\mathbf{1}_{\{X_{R^n_M\wedge\tau^-_M}>0\}}) + c_1 \\
&\le h(V^{x_n}_{R^n_M\wedge\tau^-_M}) + h^{'}(2c)X_{R^n_M\wedge\tau^-_M}\mathbf{1}_{\{X_{R^n_M\wedge\tau^-_M}>0\}} + h(2c) + c_1,
\end{aligned}
$$

where we have used Lemma 4.3.6 and (4.3.12) for the last two inequalities. Hence by (4.3.15) and Step 2

$$
\begin{aligned}
\mathbb{E}\big(h(U^{x_n}_{R^n_M\wedge\tau^-_M})\big) &\le \mathbb{E}\big(h(V^{x_n}_{R^n_M\wedge\tau^-_M})\big) + h^{'}(2c)M + h(2c) + c_1 \\
&< \infty.
\end{aligned}
\tag{4.3.16}
$$

Step 4. For $n \ge n_0$, we denote τ^n_M the time for U^{x_n} to hit level M

$$
\tau^n_M := \inf\{t > 0, U^{x_n}_t \le M\}.
$$

We will show that $\mathbb{E}\big(h(\tau^n_M)\big) < \infty$. Note that we can choose M large enough such that

$$
\frac{f(u)}{u} \le -1 \quad \text{for all} \quad u \ge M.
$$

Conseqently, $U^{x_n}_{R^n_M\wedge\tau^-_M+r} \le Y_r$, for all $0 \le r \le \tau^n_M - R^n_M \wedge \tau^-_M$ a.s., where Y solves

$$
dY_r = -dr + dX_r, \qquad Y_0 = U^{x_n}_{R^n_M\wedge\tau^-_M}.
$$

Let $A^n_M := \inf\{r > 0, Y_r \le M\}$. Clearly $\tau^n_M \le R^n_M \wedge \tau^-_M + A^n_M \le 4g(M) + A^n_M$. Hence

$$
h(\tau^n_M) \le h(4g(M)) + h(A^n_M) + c_1, \tag{4.3.17}
$$

by Lemma 4.3.6. We now prove that $\mathbb{E}\big(h(A^n_M)\big) < \infty$, from which Step 4 will follow.

Indeed, we have for $t > 0$ (recall that $\Gamma_t = \inf\{s > 0, X_s - s < -t\}$)

$$\begin{aligned}\mathbb{P}(h(A_M^n) > t) &= \mathbb{P}(A_M^n > h^{-1}(t)) \\ &= \mathbb{P}(Y_0 + \inf_{[0,h^{-1}(t)]}(X_s - s) > M) \\ &\le \mathbb{P}(\inf_{[0,h^{-1}(t)]}(X_s - s) > -Y_0) \\ &= \mathbb{P}(\Gamma_{Y_0} > h^{-1}(t)) \\ &= \mathbb{P}(h(\Gamma_{Y_0}) > t).\end{aligned}$$

Hence

$$\mathbb{E}\big(h(A_M^n)\big) = \int_0^\infty \mathbb{P}(h(A_M^n) > t)dt \le \int_0^\infty \mathbb{P}(h(\Gamma_{Y_0}) > t) = \mathbb{E}(h(\Gamma_{Y_0})). \qquad (4.3.18)$$

Furthermore, since h is a concave function on (c, ∞), we can use Jensen's inequality and Lemma 4.3.4 to get for all $t > 0$,

$$\begin{aligned}\mathbb{E}(h(\Gamma_t \vee 2c)) &\le h(\mathbb{E}(\Gamma_t \vee 2c)) \\ &\le h(\mathbb{E}(\Gamma_t + 2c)) \\ &= h(t + 2c).\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}(h(\Gamma_{Y_0})) &\le \mathbb{E}(h(\Gamma_{Y_0} \vee 2c)) \\ &= \mathbb{E}[\mathbb{E}(h(\Gamma_{Y_0} \vee 2c)|Y_0)] \\ &\le \mathbb{E}(h(Y_0 + 2c)) \\ &\le \mathbb{E}(h(Y_0)) + h(2c) + c_1, \qquad (4.3.19)\end{aligned}$$

where we have used Lemma 4.3.6 for the last inequality. Step 4 then follows from (4.3.18), (4.3.19) and Step 3.

Step 5. From (4.3.17), (4.3.18), (4.3.19), (4.3.16) and (4.3.14) we deduce that for all $n \ge n_0$,

$$\mathbb{E}\big(h(\tau_M^n)\big) \le h(2M) + h(4g(M)) + 2h(2c) + h^{'}(2c)M + 4c_1 + 2^{2d+1}\mathbb{E}\Big(\frac{1}{(\tau_M^-)^d}\Big).$$

Hence

$$\mathbb{E}(h(\tau_M)) < \infty, \quad \text{where} \quad \tau_M := \sup_{n>n_0} \tau_M^n.$$

Let T be a positive constant. Let p denote the probability that starting from M at time $t = 0$, U hits zero before time T. There exists a constant $K > 0$ such that

$$\frac{f(u)}{u} \leq K \quad \text{for all} \quad u \geq 0.$$

We have

$$p \geq \mathbb{P}(M + KT + \inf_{[0,T]} X_t \leq 0) > 0,$$

by Lemma 4.3.5. Using the same argument used in the proof of Theorem 4.2.2 we obtain that $\sup_{x>0} \tau^x$ is stochastically dominated by

$$\zeta T + \sum_{i=1}^{\zeta} \eta_i,$$

where ζ is a geometric random variable with success probability p, the random variables η_i are i.i.d, with the same law as τ_M, globally independent of ζ. Therefore

$$\begin{aligned}
\mathbb{E}\big(h(\sup_{x>0} S^x)\big) = \mathbb{E}\big(h(\sup_{x>0} \tau^x)\big) &\leq \mathbb{E}\big(h(\zeta T + \sum_{i=1}^{\zeta} \eta_i)\big) \\
&\leq \mathbb{E}\big(\zeta h(T) + \sum_{i=1}^{\zeta} h(\eta_i) + (2\zeta - 1)c_1\big) \\
&\leq \frac{h(T)}{p} + \frac{1}{p}\mathbb{E}(h(\tau_M)) + \big(\frac{2}{p} - 1\big)c_1 \\
&< \infty,
\end{aligned}$$

where we have used Lemma 4.3.6 for the second inequality. The result follows. □

4.4 Some examples

In this section we will discuss some special cases to illustrate our results.

Example 4.4.1. An important example is the case of a logistic interaction where

$$f(u) := au - bu^2, \qquad a \in \mathbb{R}, b > 0.$$

It is easily seen that f satisfies (H2). There exists a positive constant a_0 such that $f(u) < 0$ for all $u \geq a_0$, and

$$\int_{a_0}^{\infty} \frac{1}{\mid f(u) \mid} du = \int_{a_0}^{\infty} \frac{1}{bu^2 - au} du < \infty, \qquad \int_{a_0}^{\infty} \frac{u}{\mid f(u) \mid} du = \int_{a_0}^{\infty} \frac{u}{bu^2 - au} du = \infty.$$

Hence in this case, from Theorem 4.2.1, 4.2.2 and 4.3.1 we have

$$\begin{cases} \sup_{x>0} T^x = \infty \quad \text{a.s.} & \text{if} \quad \int^{\infty} d\lambda/\psi(\lambda) = \infty \\ \mathbb{E}(\sup_{x>0} T^x) < \infty & \text{if} \quad \int^{\infty} d\lambda/\psi(\lambda) < \infty, \end{cases}$$

and

$$\sup_{x>0} S^x = \infty \quad \text{a.s.}$$

Example 4.4.2. We consider the case of a polynomial interaction where

$$f(u) := au - bu^{\beta}, \qquad a \in \mathbb{R}, b > 0, \beta > 1.$$

Then f satisfies (H2) and there exists a positive constant a_0 such that $f(u) < 0$ for all $u \geq a_0$. Since

$$\int_{a_0}^{\infty} \frac{1}{\mid f(u) \mid} du = \int_{a_0}^{\infty} \frac{1}{bu^{\beta} - au} du < \infty,$$

from Theorem 4.2.1 and 4.2.2 we have

$$\begin{cases} \sup_{x>0} T^x = \infty \quad \text{a.s.} & \text{if} \quad \int^{\infty} d\lambda/\psi(\lambda) = \infty \\ \mathbb{E}(\sup_{x>0} T^x) < \infty & \text{if} \quad \int^{\infty} d\lambda/\psi(\lambda) < \infty. \end{cases}$$

Concerning the total mass we note that

$$\int_{a_0}^{\infty} \frac{u}{\mid f(u) \mid} du = \int_{a_0}^{\infty} \frac{u}{bu^{\beta} - au} du \qquad \begin{cases} = \infty, & \text{if} \quad \beta \leq 2 \\ < \infty, & \text{if} \quad \beta > 2. \end{cases}$$

Hence $\sup_{x>0} S^x = \infty$ a.s. for $\beta \leq 2$, by Theorem 4.3.1. For $\beta > 2$, we can choose

$$f_2(u) := bu^{\beta - 1}.$$

Therefore for all $u \geq a_0$,

$$g(u) = \int_u^\infty \frac{1}{f_2(r)} dr = \frac{1}{b(\beta-2)u^{\beta-2}}.$$

Since for all $u \geq a_0$,

$$g(u) f_2'(u) = \frac{\beta-1}{\beta-2},$$

(H3) holds for $d = \frac{1}{\beta-2}$. So that for $u > a_0$,

$$h(u) = \frac{1}{g(u)^d} = (b(\beta-2))^{\beta-2} u.$$

Hence from Theorem 4.3.2, if $\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\lambda} = \infty$ we have

$$\mathbb{E}(\sup_{x>0} S^x) < \infty.$$

Example 4.4.3. According to Theorem 3.7 in [10], in the case of Feller's branching diffusion with interaction where

$$f(u) := -ue^u,$$

there exists some positive constant ℓ such that

$$\sup_{x>0} \mathbb{E}\big(e^{\ell S^x}\big) < \infty. \tag{4.4.1}$$

We will give a counterexample to see that (4.4.1) is no longer true for the case of CSBP with competition which has discontinuous paths. Let ψ be the function given by (4.1.1) such that

$$\sigma > 0 \quad \text{and} \quad \int_1^\infty r^2 m(dr) = \infty. \tag{4.4.2}$$

In this case U^x is the solution of the following SDE

$$dU_t^x = -e^{U_t^x} dt + dX_t, \qquad U_0^x = x, \tag{4.4.3}$$

where X is a Lévy process with Laplace exponent ψ. It is easily seen that an explicit formula for the unique strong solution of (4.4.3) is

$$U_t^x = x + X_t - \log(1 + \int_0^t e^{X_s} ds).$$

Recall that for $x, y > 0$,

$$\Gamma_x = \inf\{t > 0, X_t - t < -x\}, \qquad \tau_y^+ = \inf\{t > 0, X_t > y\}.$$

We have on the time interval $[0, \Gamma_x \wedge \frac{\tau_y^+}{2}]$,

$$\begin{aligned} y \geq X_t &\geq t - x \\ X_t - \log(1 + \int_0^t e^{X_s} ds) &\geq t - x - \log(1 + te^y) \\ &\geq t - x - \log(1+t) - y \\ &\geq -x - y. \end{aligned}$$

Therefore for all $x, y > 0$,

$$\tau^{x+y} = \inf\{t > 0, U_t^{x+y} = 0\} \geq \Gamma_x \wedge \frac{\tau_y^+}{2} \quad \text{a.s.}$$

By Proposition 4.3.2, we have $\tau_y^+ \to \infty$ a.s. as $y \to \infty$, hence

$$\sup_{x>0} \tau^x \geq \sup_{x>0} \Gamma_x \quad \text{a.s.}$$

We deduce from Lemma 4.4.4 below that

$$\mathbb{E}\big((\sup_{x>0} S^x)^2\big) = \mathbb{E}\big((\sup_{x>0} \tau^x)^2\big) = \infty.$$

Lemma 4.4.4. *Suppose that the condition* (4.4.2) *is satisfied. For all* $x > 0$ *we have*

$$\mathbb{E}\big((\Gamma_x)^2\big) = \infty.$$

Proof. Recall that $X_t - t$ is a spectrally positive Lévy process with Laplace exponent $\psi_0(s) = \psi(s) + s$ and ϕ_0 is the unique inverse of ψ_0. It is well known that the process $\{\Gamma_x, x \geq 0\}$ is a subordinator with Laplace exponent ϕ_0, and

$$\mathbb{E}\big(e^{-s\Gamma_x}\big) = e^{-\phi_0(s)x} \qquad \text{for all } s, x > 0.$$

Hence

$$\mathbb{E}\big(e^{-s\Gamma_x}(\Gamma_x)^2\big) = e^{-\phi_0(s)x}\big((\phi_0'(s))^2 x^2 - \phi_0''(s)x\big) \qquad \text{for all } s, x > 0. \tag{4.4.4}$$

On the other hand, we have for all $s > 0$

$$\begin{aligned} \psi_0(\phi_0(s)) &= s \\ \psi_0'(\phi_0(s))\phi_0'(s) &= 1 \\ \psi_0'(\phi_0(s))\phi_0''(s) + \psi_0''(\phi_0(s))(\phi_0'(s))^2 &= 0 \\ \phi_0''(s) + \psi_0''(\phi_0(s))(\phi_0'(s))^3 &= 0. \end{aligned} \tag{4.4.5}$$

It is easy to check that

$$\psi_0(0) = \phi_0(0) = 0, \qquad \psi_0'(0) = \phi_0'(0) = 1, \tag{4.4.6}$$

and

$$\lim_{s\to 0^+} \psi_0''(s) = \sigma^2 + \lim_{s\to 0^+} \int_0^\infty r^2 e^{-sr} m(dr) = \sigma^2 + \int_0^\infty r^2 m(dr) = \infty. \tag{4.4.7}$$

From the monotone convergence theorem and (4.4.4), (4.4.5), (4.4.6), (4.4.7) we deduce that

$$\mathbb{E}\big((\Gamma_x)^2\big) = \lim_{s\to 0^+} \mathbb{E}\big(e^{-s\Gamma_x}(\Gamma_x)^2\big) = \infty.$$

The result follows. □

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