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Principe d'invariance individuel pour une diffusion dans un environnement périodique.

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ABSTRACT

We prove here, using stochastic analysis methods, the invariance principle for a $\mathbb{R}^d$- diffusions $d \geq 2$, in a periodic potential beyond uniform boundedness assumptions of potential. The potential is not assumed to have any regularity. So the stochastic calculus theory for processes associated to Dirichlet forms is used to justify the existence of a continuous Markov process starting from almost all $x \in \mathbb{R}^d$ and denoted by $(X_t, t > 0)$ (cf chapter 1). In chapter 2, we prove a new Sobolev inequality with different weights, which allows us to deduce the existence and boundedness of the density of probability transition associated to the continuous Markov process. This inequality, Theorem 2.1.1 in chapter 2, which is the principal key of this work, is proved by using some materials in harmonic analysis. In chapter 3, we prove the main result (Theorem 1) of this work, the invariance principle. Our strategy for proving Theorem 1 follows some classical steps: we rely on the construction of the so-called *corrector*: this is a periodic function $v : \mathbb{R}^d \to \mathbb{R}^d$ such that the process $t \to X_t + v(X_t)$ is a Martingale with stationary increments under P_x. It then follows that the process $X^{(\epsilon)} + \epsilon v(\frac{1}{\epsilon}X^{(\epsilon)})$ satisfies the invariance principle (result of I.S. Helland in 1982, see [2]), and the key step of the proof of the Theorem 1 consists in showing that the corrector part $\epsilon v(\frac{1}{\epsilon}X^{(\epsilon)})$ tends to 0.

In order to control the corrector, and actually also in order to show its existence, we rely on the Sobolev inequality: Theorem 2.1.1 of chapter 2.
After proving this inequality, we construct a so-called *time changed process* denoted by $(\widetilde{X}_t, t \geq 0)$ from the initial process $(X_t, t \geq 0)$, using an additive functional. We show the invariance principle for $(\widetilde{X}_t, t \geq 0)$ before deducing the invariance principle for $(X_t, t \geq 0)$ by the Ergodic theorem.

All the work is done under the following hypothesis: the potential V is periodic and satisfies $e^V + e^{-V} \in L^1_{loc}(\mathbb{R}^d; dx)$ where dx is the Lebesgue measure.

RÉSUMÉ DES TRAVAUX

Nous montrons ici, en utilisant les méthodes de l'analyse stochastique, le principe d'invari-

ance pour des diffusion sur $\mathbb{R}^d$, $d \geq 2$, en milieu périodique au delà des hypothèses d'uniforme ellipticité et au delà des hypothèses de régularité sur le potentiel.
La théorie du calcul stochastique pour les processus associés aux formes de Dirichlet est largement utilisée pour justifier l'existence du processus de Markov à temps continus, défini pour presque tout point de départ sur $\mathbb{R}^d$. Pour éviter l'hypothèse de régularité, on considère le générateur associé au processus, mais de manière informelle avant de considérer la forme de Dirichlet associée à ce dernier (cf chapiter 1). Puis dans le chapitre 2, nous montrons une nouvelle inégalité de type Sobolev avec des poids différents, qui nous permet de déduire l'existence et la bornitude d'une densité de la probabilité de transition associée au processus de Markov. Cette inégalité, Theorem 2.1.1 du chapitre 2, est l'outil principal de ce travail. La preuve fera appel à des techniques d'analyse harmonique (voir [12]).
Enfin, le chapitre 3 contient le résultat principal du travail de la thèse: le principe d'invariance qui veut dire que la suite de processus $(\epsilon X_{t\epsilon^{-2}})$ converge en loi quand ϵ tend vers zéro vers un mouvement Brownien. Notre stratégie pour prouver ce thórème qu'est notée Théorème 1 dans cette thèse suit quelques étapes classiques: nous nous appuyons sur la construction de ce qu'on appelle ici correcteur. C'est une fonction périodique v sur $\mathbb{R}^d$ à valeurs dans $\mathbb{R}^d$ telle que le processus $t \to X_t + v(X_t)$ soit une Martingale à accroissements stationnaires sous P_x. Il s'en suit d'après le résultat de I. S Helland en 1982 (voir [2]), que la suite de processus $X^{(\epsilon)} + \epsilon v(\frac{1}{\epsilon}X^{(\epsilon)})$ satifait un principe d'invariance et l'étape clef dans la preuve du Theorem 1 est de montrer que la partie correcteur $\epsilon v(\frac{1}{\epsilon}X^{(\epsilon)})$ tends vers zéro quand ϵ tend vers zéro. Afin de contrôler le correcteur, et aussi pour montrer son existence, nous nous appuyons sur l'inégalité de Sobolev, Théorème 2.1.1 du chapitre 2. Nous construisons à la suite cette inégalité de type-Sobolev, un processus appelé processus changé de temps noté $(\widetilde{X}_t, t \geq 0)$ à partir du processus initial $(X_t, t \geq 0)$ en utilisant une fonctionnelle additive. La construction de cette fonctionnelle additive utilise évidamment cette inégalité. Nous obtenons un processus $(\widetilde{X}_t, t \geq 0)$ dont la mesure invariante est non seulement connue mais satisfait l'inégalité du Théorème 2.1.1. De là , Nous montrons le principe d'invariance pour $(\widetilde{X}_t, t \geq 0)$ avant de déduire le principe d'invariance pour le processus $(X_t, t \geq 0)$ en passant par le theorème ergodique.

En résumé, une inégalité de Sobolev de type Théorème 2.1.1 avec certaines hypothèses sur la mesure de référence, implique le principe d'invariance.
Tout le travail effectué dans les trois chapitres utilise seulement les hypothèses suivantes: le potentiel V est périodique et $e^V + e^{-V}$ est localement dans $L^1(\mathbb{R}^d; dx)$, ou dx est la mesure de Lebesgue.

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Introduction générale

Comme le titre de cette thèse l'indique, nous nous intéressons à l'homogénéisation des diffusions dans des milieux périodiques.

L'homogénéisation est la théorie qui consiste à remplacer lorsque celà est possible, un milieu décrit de facon microscopique par une approximation à une échelle macroscopique. Autrement dit, cela revient à étudier le remplacement d'un milieu fortement hétérogène par un milieu homogène, mais dont les propriétés caractéristiques sont données par une moyennisation des hétérogénéités.

Les enjeux sont bien entendu très importants, car de nombreux milieux sont donnés par leurs propriétés microscopiques (sous-sol, coeur de réacteur nucléaire, matériaux composites, polymères,...), et des simulations numériques réalisées à des échelles sont souvent quasi impossibles. Par exemple, pour l'étude de la diffusion du pétrole dans un milieux poreux, l'échelle des pores est de l'ordre de millimètre, alors que les réservoires ont des tailles qui sont de l'ordre du kilomètre. De plus, l'utilisation des maillages adaptatifs est souvent nécessaire pour faire face aux brusques variations des hétérogénéités, ce qui augmente d'autant plus la puissance de calcul et la mémoire requises.

Cette théorie s'est considérablement développée ces dernières années et constitue une discipline à part entière. Comme précurseur de ces genre de problème nous pouvons citer Luc Tartar. Nous pouvons trouver dans cette théorie un appercu de la diversité des problèmes et des modes de résolutions par exemple dans les ouvrages [7] and [17]. C'est surtout au cours des quatres derniéres décennies que la théorie de l'homogénéisation ou *" averaging"*' des équations aux dérivées partielles a pris forme comme une discipline mathématique distincte. Cette théorie a d'importantes applications en mécanique des matériaux composites et des matériaux perforés, de la filtration, des milieux dispersés, et dans plusieurs autres branches de la physique, de la mécanique et de la technologie moderne. Le terme *" averaging"*' a souvent été associé aux méthodes mécaniques non-linéaires et aux équations aux derrivées partielles developpées dans les travaux de Poincaré, Van Der Pol, Kryslov, Bogoliubov, etc. Et c'est vers les annnées 90' que des disciplines en mécaniques et en physiques ont stimulé l'apparition du concept du milieu microscopiquement non-homogéne de type général et ont encouragé un developpement intensif des methodes de milieu effectif, champs moyennée (*averaged field*), etc., qui sont en accord avec la théorie d'homogénéisation moderne.
Dans les modèles mathématiques des milieux microscopiquement non-homogènes, les diverses caractéristiques locales sont souvent décrites par une fonction $V(\epsilon^{-1}x)$, où ϵ est un paramètre assez petit. La fonction V peut être périodique, presque périodique ou souvent une réalisation d'un environnement stationaire; il peut aussi appartenir à d'autres classes spécifiques de fonctions. Evaluer les caractéristiques d'un milieu non-homogène est un travail extrémement difficile, puisque les coefficients d'une équation différentielle corre-

spondante sont des fonctions qui oscillent très rapidement. Cependant, il est nécessaire d'appliquer l'analyse asymptotique aux problèmes de milieu non-homogène qui, immédiatement correspond au concept d'homogénéisation.
L'homogénéisation, du point de vue analytique peut être expliquée comme suit. Si on note par $L = e^{V(x)}div(e^{-V(x)}\nabla)$, un opérateur sous forme divergence sur $L^2(\mathbb{R}^d; e^{-V(x)}dx)$ montrer un résultat d'homogénéisation pour la suite $L^\epsilon = e^{V(x/\epsilon)}div(e^{-V(x/\epsilon)}\nabla)$ signifie: pour toute fonction continue et bornée f sur $L^2(\mathbb{R}^d; e^{V(\frac{x}{\epsilon})}dx)$ les solutions u^ϵ de l'équation parabolique

$$\left\{ \begin{array}{l} \frac{\partial u^\epsilon(t,x)}{\partial t} = L^\epsilon u^\epsilon(t,x), \\ u^\epsilon(t,0) = f(x) \end{array} \right. \tag{0.0.1}$$

converges simplement quand ϵ tend vers zero, vers u solution de

$$\left\{ \begin{array}{l} \frac{\partial u(t,x)}{\partial t} = \bar{L}u(t,x), \\ u(t,0) = f(x) \end{array} \right. \tag{0.0.2}$$

où $\bar{L}$ est le Laplacien.

Par ailleurs, dans la théorie des probabilités cet opérateur sous forme divergence peut engendrer un processus de diffusion $(X_t, t \geq 0; P_x)$ sur $\mathbb{R}^d$ dès que la fonction V est assez régulière. Dire que la suite de processus $(\epsilon X_{t\epsilon^{-2}})_{\epsilon \geq 0}$ converge, quand ϵ tend vers zéro, vers un mouvement Brownien sous P_x implique un résultat d'homogénéisation pour la suite L^ϵ.

Donc la question à laquelle nous nous intéressons ici est de savoir le comportement assymptotique en temps longs de $(X_t)_{t\geq 0}$, le processus de diffusion associé à l'opérateur sous forme divergence
$L = e^V div(e^{-V}\nabla)$ où V est mesurable et périodique. Plus précisement, est ce que la suite de processus $(X.^\epsilon)_{\epsilon\geq 0} := \left(\epsilon X_{./\epsilon^2}\right)_{\epsilon \geq 0}$ converges en loi vers un mouvement Brownien. Aussi, une autre question est de savoir à quel point le potentiel V influence effectivement le comportement assymptotique de cette diffusion. Ces questions qui semblent simples à première vue nécessite beaucoup de travail et d'indulgence pour y répondre très clairement.

Précisons exatement les types de diffusions qui nous intéssent:

$$dX_t = dB_t - \nabla V(X_t)dt, \tag{0.0.3}$$

où B_t est un mouvement Brownien sur $\mathbb{R}^d$ et V une fonction mesurable sur $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$. Cette équation modélise le mouvement d'une particule soumis à deux forces: une diffusion représentée par le mouvement Brownien et une force venant du milieu V, représentée par son gradient. Puisque la solution de cette équation différentielle stochastique est un processus de Markov, un résultat bien connu dans le théorie des Probabilités, on peut alors écrire son générateur sous forme divergence, donnée par :

$$L = e^{V(x)}div(e^{-V(x)}\nabla). \tag{0.0.4}$$

Cependant, deux cas sont généralement à étudier dans ces genres de problème: Les cas où V est PÉRIODIQUE et le cas où V est une réalisation de l'environnement:ALÉATOIRE. Dans le dernier cas, l'équation (0.0.3) s'écrit: pour tout environnement ω,

$$dX_t^\omega = dB_t - \nabla V(X_t,\omega)dt \tag{0.0.5}$$

and

$$L^\omega = e^{V(x,\omega)} div(e^{-V(x,\omega)}\nabla) \tag{0.0.6}$$

où l'ensemble des environnements est considéré comme un espace de probabilités $(\Omega, \mathcal{F}, \mathcal{Q})$ sur le quel nous faisons agir l'ensemble $\mathbb{R}^d$ avec des propriétés d'ergodicité et de stationarité.

Dans cette thèse, nous nous intéressons uniquement au cas ou le potentiel est périodique: le cas où la fonction V satisfait $V(x+z) = V(x)$ pour tout $x \in \mathbb{R}^d$ et pour tout $z \in \mathbb{Z}^d$. Nous nous rendons compte, même si on ne l'a pas précisé, que l'équation (**??**) a un sens que si le potentiel V est suffisamment régulièr. Dans ce cas, l'équivalence (0.0.3) et (0.0.4) devient évidente, cours de calcul stochastique de master. Donc ce que nous allons voir en premièr lieu dans cette thèse est comment construire un processus de diffusion à partir de l'opérateur sous forme divergence. La théorie des formes de Dirichlet développée dans [4] permet à partir d'un operateur sous forme divergence construit un processus de Markov qui lui est associé dès que la forme de Dirichlet est régulière et locale, deux propriétés que nous verrons dans le chapitre 1. En gros, pour éviter l'hyppothèse de régularité de l'opérateur sous forme divergence, on considère sa forme bilinéaire associée.
Dans le cas qui nous intéresse, la fonction V n'est pas régulière donc nous regardons l'opérateur ainsi: pour tout f, g dans l'ensemble

$$\mathcal{H}(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d; e^{-V(x)}dx) : \forall i = 1, ..., d \qquad \partial_i f \in L^2(\mathbb{R}^d; e^{-V(x)}dx) \right\}$$

, où $\partial_i f$ est la derrivée faible de f dans la direction i. Nous notons par $\nabla f = (\partial_i f)_{i=1,\dots,d}$.

$$\int_{\mathbb{R}^d} Lf(x)g(x)e^{-V(x)}dx = -\int_{\mathbb{R}^d} \nabla f(x).\nabla g(x)e^{-V(x)}dx. \tag{0.0.7}$$

$\nabla f(x).\nabla g(x)$ signifie le produit scalaire dans $\mathbb{R}^d$ de $\nabla f(x)$ and $\nabla g(x)$ i.e $\nabla f(x).\nabla g(x) = \sum_{i=1}^d \partial_i f(x)\partial_i g(x)$.
Nous vérifions que

$$\xi(f,g) := \int_{\mathbb{R}^d} -Lf(x)g(x)e^{-V(x)}dx,$$

pour tout $f, g \in \mathcal{H}(\mathbb{R}^d)$ est une forme bilinéaire et donc intuitivement candidat pour être la forme de Dirichlet associée à L avec une domaine à déterminer.

Il y a deux manière de regarder le problème. Premièrement, nous pouvons considerer l'opérateur sur $\mathcal{H}(\mathbb{R}^d)$, définir la diffusion associé en utilisant les formes de Dirichlet et ensuite projetter la diffusion sur le tore unité sachant qu'elle gardera toujours sa propritée Markovienne. Oubien, nous pouvons définir la diffusion sur le tore, en considérant tout au début la forme bilinéaire associée à l'opérateur sous forme divergence $\dot{L} := e^{V(\dot{x})} div(e^{-V(\dot{x})}\nabla)$, sur

$$\mathcal{H}^1(I_0; e^{-V}) := \left\{ f \in L^2(I_0; e^{-V(\dot{x})} d\dot{x}) : \forall i = 1, ...d \qquad \partial_i f \in L^2(I_0; e^{-V(\dot{x})} d\dot{x}) \right\}.$$

Nous faisons par la suite un relèvement sur $\mathbb{R}^d$ de la trajectoire obtenue. Ce qui va nous donner un processus markovien dont la projection sur le tore correspond a celui engendré par $\dot{L}$. Ce processus nouvellement défini sur $\mathbb{R}^d$ est l'unique processus tel que sa projection sur le tore est celui associé à $\dot{L}$.
Pour la première démarche, la difficulté pourrait être d'identifier la forme de Dirichlet associée au processus sur le tore. La forme de Dirichlet sur le tore constitue un outil de base, dans notre démarche pour montrer le principe d'invariance de la diffusion sur $\mathbb{R}^d$. Donc cette démarche nous posera plus de difficultés si nous ne disposons pas de la forme de Dirichlet du processus sur le tore. La propriété de Feller de la fonction de transition est restée une véritable question pour nous.
Pour la deuxième démarche, celle que nous optons à priorie, on va définir la diffusion sur $\mathbb{R}^d$, en relèvant la trajectoire de la diffusion qui est sur le tore, donnée par la forme bilinéaire sur $\mathcal{H}^1(I_0; e^{-V})$.
En gros, on part d'un opérateur sous forme divergence sur $L^2(I_0; e^{-V})$, définir sa diffusion associée sur la tore à l'aide des formes de Dirichlet avant de définir la diffusion sur $\mathbb{R}^d$ par relèvement de la trajectoire. Mais, aussi bien sur tore I_0 que sur $\mathbb{R}^d$; il est important de se rappeler que la diffusion sera définie pour presque tout point de départ. Une autre question s'intérroge. Peut-on la définir pour tout point de départ? Cette question n'est pas traitée ici. Cependant, des techniques comme les inégalités de type-Harnack, permettraient de montrer la continuité du semigroupe Markovien et donc nous obtenons une loi markovienne qui peut être définie pour tout point de départ de $\mathbb{R}^d$ grace à la propriété de Feller.

Après que le processus soit bien défini pour presque tout point de départ dans $\mathbb{R}^d$, l'objectif reste à montrer que la suite de processus $(X_\cdot^\epsilon)_{\epsilon>0}$ converges en loi vers un mouvement Brownien. En d'autres thermes, nous allons montrer la convergence en loi fini-dimensionnelle de cette suite de processus vers un Gaussien et montrer la tension sur l'espace des fonctions continues sur $[0, T]$ à valeurs dans $\mathbb{R}^d$. Pour la preuve, la démarche est assez connue: construction d'un correcteur et théorème central limite fonctionnel des Martingales à temps continus.

Beaucoup de résultats sur ce problème ont été connus. Le plus rescent est de celui Antoine Lejay en 2000, où il ne considère que les hypothèses de mesurabilité et de bornitude sur le potentiel V (hypotèses d'uniforme ellipticité). Si V est bornée et régulière les travaux de Bensoussian, Lions et Papanicolau, cf [7] en 1978 montrent le principe d'invariance. La régularité de V est très importante car celà permet d'avoir le processus sous forme d'une semi-martingale solution de l'équation (0.0.3) et dès lors, le calcul stochastique d'Ito intervient.

La nouveauté de ce travail est qu'au delà de la non régularité de V, nous ne sonsidérons que e^V and e^{-V} localement integrables par rapport à la mesure de Lebesgue sur $\mathbb{R}^d$.

Nous divisons la thèse en trois chapitres. Le premier fera une construction du processus associé à l'opérateur L en utilisant la théorie de la forme de Dirichlet. Dans le second chapitre, nous montrons une nouvelle inégalité de type-Sobolev avec des poids différents en utilisant les techniques d'analyse harmonique (la théorie sur les sensembles de Muckenhoupt, les fonctions maximale de Hardy-Littlewood, etc. developpée dans [12]). Ce qui va nous permettre dans le troisième chapitre de montrer le principe d'invariance (la convergence en loi-fini dimensionnelle et la tension) avec une démarche assez connue dans ces genres de problème: Construction et convergence du correcteur, principe d'invariance d'une suite de Martingales à temps continus (cf [2]) et un ingrédient qu'est la notion de processus changé de temps.

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Chapter 1

On the divergence-form operators

1.1 Introduction

The goal in this chapter is to recall the main results on divergence-form operators and their associated symmetric Markov processes. This association is possible because the divergence-form operator defined on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$, where $I_0 = \mathbb{R}^d/\mathbb{Z}^d$ is the unit torus of $\mathbb{R}^d$, can be associated with one Dirichlet form defined on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. The link connecting the Dirichlet form with Markov processes is: the Markovian nature of a closed symmetric form on the space $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ is equivalent to the Markovian properties of the associated semigroup and resolvent on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. The local property of Dirichlet forms ensures the continuity of the Markov processes's path. The reader can be refered to [4]. This book which is an extention of [18] published in 1980 from Kodansha and North Holland, combined with Oshima's Lecture note "*Lectures on Dirichlet space*" delivred at Universitat Erlangen-Nurenberg in 1988. Most ingredients in [18] are kept as the skeleton of [4].
Part I of this chapter contains an introductory and comprehensive account of the theory of (symmetric),Dirichlet forms. An axiomatic extension of the classical Dirichlet integrals in the direction of Markovian semigroups. In part II, this analytic theory is unified with the probabilistic potential theory based on symmetric Markov processes and developes further in conjonction with the stochastic analysis based on the additive functionals.

We recall that in this thesis, I_0 means the unit torus of $\mathbb{R}^d$: $\mathbb{R}^d/\mathbb{Z}^d$, $d \geq 2$. Let us consider the Hilbert space $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. As said in the part of Introduction, we consider the divergence-form operator, defined in (0.0.4)

$$\dot{L} = e^{V(\dot{x})} div(e^{-V(\dot{x})}\nabla). \tag{1.1.1}$$

on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. Since V is not assumed to be regular, this definition can seen formally. In other words, we define $\dot{L} = e^{V(\dot{x})} div(e^{-V(\dot{x})}\nabla)$ the operator which satisfies

$$\int_{I_0} \dot{L}f(\dot{x})g(\dot{x})e^{-V(\dot{x})}d\dot{x} = -\int_{I_0} \nabla g(\dot{x})\nabla f(\dot{x})e^{-V(\dot{x})}dx$$

in the space of functions f in $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ such that all the weak derivatives denoted by $\frac{\partial f}{\partial_i}$, $i = 1, ...d$ are also in $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. This space will be denoted later, in chapter 3 by $\mathcal{H}^1(I_0; e^{-V})$. We assume that V is $1-$periodic and $e^{-V}+e^V$ is $L^1(I_0; d\dot{x})$ the measure $d\dot{x}$ is the Lebesgue measure on the torus.

Remark 1.1.1. *It is also important to recall that the theory which will be given in this chapter, holds also in $L^2(X; m)$ as soon as*
$(X, \mathcal{B}, m)$ is a locally compact separable metric space.
m is a positive Radon measure on X such that $supp[m] = X$.
i.e m is a non-negative Borel measure on X finite on compact sets and strictly positive on non-empty open set (see [4], page 5).

1.2 Basic theory and Dirichlet forms

1.2.1 Basic notions and definitions

We consider the inner product on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$:

$$(u, v) = \int_{I_0} u(\dot{x})v(\dot{x})e^{-V(\dot{x})}d\dot{x}.$$

A non-negative definite symmetric bilinear form densely defined on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ is henceforth called simply a *symmetric form* on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. To be precise, ξ is called a symmetric form on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ if the following conditions are satisfied:
$(\xi.1)$ $\dot{\xi}$ is defined on $\mathcal{D} \times \mathcal{D}$ with values in $\mathbb{R}^1$, $\mathcal{D}$ being a dense linear subspace of $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$,

$(\xi.2)$ $\dot{\xi}(u, v) = \dot{\xi}(v, u), \dot{\xi}(u + v, w) = \dot{\xi}(u, w) + \dot{\xi}(v, w)$,
$a\, \dot{\xi}(u, v) = \dot{\xi}(au, v)$,$\dot{\xi}(u, u) \geq 0, u, v, w \in \mathcal{D}$, $a \in \mathbb{R}^1$.

$\mathcal{D}$ is called the domain of $\dot{\xi}$.
The inner product $(,)$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ is a specific symmetric form defined on the whole space $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. Given a symmetric $\dot{\xi}$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$,

$$\dot{\xi}_\alpha(u, v) = \dot{\xi}(u, v) + \alpha(u, v), u, v \in \mathcal{D}$$
$$\mathcal{D}_\alpha = \mathcal{D}$$

defines a new symmetric form on $\mathcal{D}$ for each $\alpha > 0$. Note that the space $\mathcal{D}$ is then a pre-Hilbert space with inner product $\dot{\xi}_\alpha$. Furthermore $\dot{\xi}_\alpha$ and $\dot{\xi}_\beta$ determine equivalent metric on $\mathcal{D}$ for different $\alpha, \beta > 0$.

Definition 1.2.1.

$\dot{\xi}$ is said to be *closed* if $\mathcal{D}$ is complete with respect to the metric $\dot{\xi}_\alpha$, for an $\alpha > 0$. In other word, a symmetric form $\dot{\xi}$ is said to be closed if:
$(\xi.3)$ $\quad u_n \in \mathcal{D}, \dot{\xi}_1(u_n - u_m, u_n - u_m) \longrightarrow 0, n, m \uparrow \infty,$
$\Rightarrow \exists u \in \mathcal{D}, \dot{\xi}_1(u_n - u, u_n - u) \longrightarrow 0, n \uparrow \infty.$

Definition 1.2.2.

We say that a symmetric form $\dot{\xi}$ is *closable* if the following condition is fulfilled:
$u_n \in \mathcal{D}, \xi(u_n - u_m, u_n - u_m) \longrightarrow 0, n, m \uparrow \infty,$
$(u_n, u_n) \to 0, n \uparrow \infty \Rightarrow \xi(u_n, u_n) \longrightarrow 0, n \uparrow \infty.$

Cleary $\mathcal{D}$ is then a real Hilbert space with the inner product $\dot{\xi}_\alpha$ for each $\alpha > 0$. Given two symmetric forms $\dot{\xi}^1$ and $\dot{\xi}^2$, $\dot{\xi}^2$ is said to be an extension of ξ^1 if
$\mathcal{D}_1 \subseteq \mathcal{D}_2$ and $\dot{\xi}^1 = \dot{\xi}^2$ on $\mathcal{D}_1 \times \mathcal{D}_1$ where $\mathcal{D}_1$ is the domain of $\dot{\xi}_1$. Let us give some examples to get a closable form.

Remark 1.2.3.

The following condition is sufficient for a symmetric form $\dot{\xi}$ to be closable:
$u_n \in \mathcal{D}, (u_n, u_n) \to 0, n \uparrow \infty \Rightarrow \dot{\xi}(u_n, v) \to 0, n \uparrow \infty, \forall v \in \mathcal{D}$

Definition 1.2.4.

We say that a symmetric form $\dot{\xi}$ on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$ is Markovian if there exists:

$(\xi.4)$ for each $\epsilon > 0$, a real function $\phi_\epsilon(t)$, $t \in \mathbb{R}$ such that:

$$\begin{cases} \phi_\epsilon(t) = t & \text{for all } t \in [0,1] \\ 0 \le \phi_\epsilon(t) - \phi_\epsilon(k) \le t - k & \text{for all } t \ge k \\ -\epsilon \le \phi_\epsilon(t) \le 1 + \epsilon, & \end{cases}$$

$u \in Dom(\dot{\xi}) \Rightarrow \phi_\epsilon(u) \in Dom(\dot{\xi}), \dot{\xi}(\phi_\epsilon(u), \phi_\epsilon(u)) \le \dot{\xi}(u, u).$

Definition 1.2.5.

A Dirichlet form $\dot{\xi}$ on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$ is a symmetric form on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$ which is **closed** and **Markovian**.

We now state two other conditions which look stronger but simpler than the Markovian condition $(\xi.4)$. Given a symmetric form $\dot{\xi}$ on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$, we say that *the unit contraction* (resp. *every normal contraction*) operates on $\dot{\xi}$ if the following $(\xi.5)$ (resp. $(\xi.6)$) is satisfied:

$(\xi.5)$ $\quad u \in \mathcal{D}, v = (0 \vee u) \wedge 1 \Rightarrow v \in \mathcal{D}, \dot{\xi}(v, v) \le \dot{\xi}(u, u).$

$(\xi.6)$ $\quad u \in \mathcal{D}$, v is a normal contraction of $u \Rightarrow v \in \mathcal{D}, \dot{\xi}(v,v) \leq \dot{\xi}(u,u)$.

Here a function v is called a normal contraction of a function u if:

$$|v(\dot{x}) - v(\dot{y})| \leq |u(\dot{x}) - u(\dot{y})|, \forall \dot{x}, \dot{y} \in I_0, |v(\dot{x})| \leq |u(\dot{x})|, \forall \dot{x} \in I_0.$$

We call $v \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ a normal contraction of $u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ if some Borel version of v is a normal contraction of some Borel version of u.
Notice the obvious implication: $(\xi.5) \Rightarrow (\xi.4) \Rightarrow (\xi.3)$. The three conditions are equivalent if $\dot{\xi}$ is closed.
Therefore, it is both practically and theorically important to consider Markovian symmetric forms which are closable but not necessarily closed. They don't satisfy $\xi.4$ and $\xi.5$ in general.

Definition 1.2.6.

A *core* of a Dirichlet form $\dot{\xi}$ is by definition a subset $\mathcal{C}$ of $\mathcal{D}\bigcap C(I_0)$ such that $\mathcal{C}$ is dense in $\mathcal{D}$ with $\dot{\xi}_1$-norm and dense in $C(I_0)$ with uniform norm, where $C(I_0)$ means the continuous functons on I_0. $\dot{\xi}$ is *regular* if:
$(\xi.6)$ $\dot{\xi}$ posseses a core.

Definition 1.2.7.

Let u a continuous function on I_0, then $supp[u]$ is the closure of $\{\dot{x} \in I_0 : u(\dot{x}) \neq 0\}$.
We say that a symmetric form $\dot{\xi}$ posseses a *local property* or simply is *local* if:
$(\xi.7)$ $\quad u, v \in \mathcal{D}$, $supp[u]$ and $supp[v]$ are disjoint compact sets $\Rightarrow \dot{\xi}(u,v) = 0$.

1.2.2 Closed forms and semigroups

All the following of this chapter, we denote by $\mathcal{D}(\dot{\xi})$ the domain of a symmetric form $\dot{\xi}$ instead $\mathcal{D}$ above.
In this section we consider only the real space $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ with the inner product $(,)$ defined above.
Consider a family $\{T_t, t > 0\}$ of linear operators on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ satisfying the following conditions:

$(T_t.1)$ each T_t is a symmetric operator with domain $\mathcal{D}(T_t) = L^2(I_0; e^{-V(\dot{x})}d\dot{x})$.

$(T_t.2)$ semigroup property $T_tT_s = T_{s+t}$, $t, s > 0$.

$(T_t.3)$ contraction property: $(T_tu, T_tu) \leq (u,u), t > 0, u \in L^2(I_0; e^{V(\dot{x})}d\dot{x})$.

Then $\{T_t, t > 0\}$ is called a semigroup (of symmetric operators) on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. It is called strongly continuous if in addition

$(T_t.4)$ $(T_t u - u, T_t u - u) \longrightarrow 0, t \downarrow 0, u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$.

A resolvent on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ is by definition a family $\{G_\alpha, \alpha > 0\}$ of linear operators on $L^2(I_0; e^{V(\dot{x})}d\dot{x})$ satisfying the following conditions.

$(G_\alpha.1)$ each G_α is a symmetric operator with domain $\mathcal{D}(G_\alpha) = L^2(I_0; e^{V(\dot{x})}d\dot{x})$.

$(G_\alpha.2)$ resolvent equation $(\alpha - \beta)G_\alpha G_\beta = -G_\alpha + G_\beta$, $\alpha, \beta > 0$.

$(G_\alpha.3)$ contraction property: $(\alpha G_\alpha u, \alpha G_\alpha u) \leq (u, u), \alpha > 0, u \in L^2(I_0; e^{V(\dot{x})}d\dot{x})$.
If in addition

$(G_\alpha.4)$ $(\alpha G_\alpha u - u, \alpha G_\alpha u - u) \longrightarrow 0, \alpha \uparrow \infty, u \in L^2(I_0; e^{V(\dot{x})}d\dot{x})$ is satified, the resolvent is said to be strongly continuous. The following proposition is well known.

Proposition 1.2.8. *Given a strongly continuous semigroup $\{T_t, t > 0\}$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. The strong limit of the Riemann sum*

$$G_\alpha u = \int_0^\infty e^{-\alpha t}(T_t u)dt \tag{1.2.1}$$

determines a strongly continuous resolvent $\{G_\alpha, \alpha > 0\}$. This is called the resolvent of the given semigroup. The generator A with domain denoted by $\mathcal{D}(A)$, of the semigroup $\{T_t, t > 0\}$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ is defined by

$$\begin{cases} Au = \lim_{t\downarrow 0} \frac{T_t u - u}{t} \\ \mathcal{D}(A) = \left\{u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x}) : \textit{ Au exists as a strong limit } \right\}. \end{cases}$$

Given a strongly continuous resolvent $\{G_\alpha, \alpha > 0\}$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ let us assume that $G_\alpha u = 0$. Then $G_\beta u = 0 \; \forall \beta > 0$, from $(G_\alpha.2)$, and $u = \lim_{\beta \longrightarrow 0} \beta G_\beta u = 0$ from $(G_\alpha.4)$. Hence G_α is invertible and we can set

$$\begin{cases} Au = \alpha u - G_\alpha^{-1} u. \\ \mathcal{D}(A) = G_\alpha \left(L^2(I_0; e^{-V(\dot{x})}d\dot{x})\right). \end{cases}$$

This operator is easily seen to be independent of $\alpha > 0$ and is called the generator of a given resolvent $\{G_\alpha, \alpha > 0\}$.

Lemma 1.2.9. *i)The generator of a strongly continuous resolvent is a non positive definite self-adjoint operator.*
ii)The generator of a strongly continuous semigroup on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ coincides with the generator of the resolvent.

By this lemma, we are led from semigroups to self adjoint operators. In the following we make full use of the spectral calculus relevant to self-adjoint operator.

Lemma 1.2.10. *Let $-A$ be a non negative definite self-adoint operator on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$.*
(i) $\left\{T_t = e^{tA}, t > 0\right\}$ and $\{G_\alpha = (\alpha - A)^{-1}, \alpha > 0\}$ are a strongly continuous semigroup and strongly continuous resolvent on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ respectively.
(ii) The generator of T_t of (i) coincides with A. The strongly continuous semigroup possessing A as its generator unique. The same statement holds for the resolvent.

The above two lemmas tell us that there are one to one correspondances among the family of self-adjoint operator on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$, the family of strongly continuous semigroups, and the family of strongly continuous resolvents.

The following proposition corresponds to the problem (1.3.2) of [4]

Proposition 1.2.11. *For strongly resolvent $\{G_\alpha, \alpha > 0\}$, the associated semigroup $\{T_t, t > 0\}$ is given by:*

$$T_t u = \lim_{\beta\to\infty} e^{-t\beta} \sum_{n=0}^{\infty} \frac{(t\beta)^n}{n!} (\beta G_\beta)^n u, \qquad u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x}). \tag{1.2.2}$$

We now state the main theorem in this section.

Theorem 1.2.12. *There is one to one correspondance between the family of closed symmetric forms $\dot{\xi}$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ and the family of non-positive definite self-adjoint operator A on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. The correspondance is determined by:*

$$\begin{cases} \mathcal{D}(\dot{\xi}) = \mathcal{D}(\sqrt{-A}) \\ \dot{\xi}(u, v) = (\sqrt{-A}u, \sqrt{-A}u). \end{cases} \tag{1.2.3}$$

Proof. We start by recalling some definitions about the spectral theory associated with the self-adjoint operator.
Definition A symmetric S satisfying $\mathcal{D}(S) = L^2(I_0; e^{-V(\dot{x})}d\dot{x})$, $S^2 = S$ is called a projection operator.
A family $\{S_\lambda, -\infty < \lambda < \infty\}$ of projection operators on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ is called a spectral family if:

$S_\lambda S_\mu = S_\lambda, \lambda \leq \mu$

$\lim_{\lambda' \to \lambda} S_{\lambda'} u = S_\lambda u \qquad \forall u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$

$\lim_{\lambda \to -\infty} S_\lambda u = 0$

$\lim_{\lambda \to \infty} S_\lambda u = u \qquad \forall u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$

Definition An operator A defined on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ is said to be closed if its domain is

closed with respect to $L^2(I_0; e^{\dot{x}} d\dot{x})$-norm.
Theorem For a given self-adjoint operator A on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$ there exists a unique spectral family $\{E_\lambda, -\infty < \lambda < \infty\}$ such that $A = \int_{-\infty}^{+\infty} \lambda dE_\lambda$. This is called the spectral representation of A.
If A is non-negative, then the corresponding spectral representation satisfies $E_\lambda = 0, \lambda < 0$.

Let us prove Theorem 1.2.12. Since $-A$ is non-negative definite self-adjoint operator on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$ then so is $\sqrt{-A}$. Hence the operator $\sqrt{-A}$ is closed, which means that the symmetric form $\dot{\xi}$ on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$ defined by 1.2.3 is closed. Indeed, if $u_n \in \mathcal{D}(\sqrt{-A})$ satisfies $u_n \longrightarrow v$, $\sqrt{-A} u_n \longrightarrow w \in L^2(I_0; e^{-V(\dot{x})} d\dot{x})$ then $v \in \mathcal{D}(\sqrt{-A})$ and $w = \sqrt{-A} u$ which is nothing but the closedness of $\dot{\xi}$.
Let us observe the following relation. The strongly continuous resolvent $\{G_\alpha, \alpha > 0\}$ generated by A satisfies

$$(1.2.4) \qquad \begin{cases} G_\alpha \left(L^2(I_0; e^{-V(\dot{x})} d\dot{x}) \right) \subset \mathcal{D}(\dot{\xi}) \\ \dot{\xi}(G_\alpha u, v) = (u, v), u \in L^2(I_0; e^{-V(\dot{x})} d\dot{x}) \text{ and } v \in \mathcal{D}(\xi) \end{cases}$$

This follows easily from the expression

$$(1.2.5) \qquad \begin{cases} \mathcal{D}(\dot{\xi}) = \left\{ u \in L^2(I_0; e^{-V(\dot{x})} d\dot{x}) : \int_{[0,\infty[} \lambda d(E_\lambda u, u) < \infty \right\} \\ \dot{\xi}(u, v) = \int_{[0,\infty[} \lambda d(E_\lambda u, v) \end{cases}$$

$\{E_\lambda\}$ being the spectral family associated with $-A$.
Conversely, given a symmetric form $\dot{\xi}$ on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$, there exists by the Riesz representation theorem a unique element $G_\alpha u \in \mathcal{D}(\dot{\xi})$ such that

$$(1.2.6) \qquad \dot{\xi}_\alpha(G_\alpha u, v) = (u, v) \qquad \forall v \in \mathcal{D}(\dot{\xi})$$

for each $\alpha > 0$ and $u \in L^2(I_0; e^{-V(\dot{x})} d\dot{x})$. We easily see that the family of $\{G_\alpha, \alpha > 0\}$ defined this way is a strongly continuous resolvent. For instance, the contraction property follows from $\alpha(G_\alpha u, G_\alpha u) \leq \xi_\alpha(G_\alpha u, G_\alpha u) = (u, G_\alpha u)$ and Schwarz inequality. To see the strong continuity, it suffices to show the strong convergence $\beta G_\beta u \to u \qquad \beta \to \infty$ only for $u \in \mathcal{D}(\dot{\xi})$ because $\mathcal{D}(\dot{\xi})$ is dense in $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$ and βG_β is contractive. For $u \in \mathcal{D}(\dot{\xi})$, $\beta(\beta G_\beta u - u, \beta G_\beta u - u) \leq \dot{\xi}_\beta(\beta G_\beta u - u, \beta G_\beta u - u) = \beta^2 (G_\beta u, u) - \beta(u, u) + \dot{\xi}(u, u) \leq \dot{\xi}(u, u)$, which implies the the desired convergence.
Denote by A the generator of this resolvent $\{G_\alpha, \alpha > 0\}$. Since $-A$ is non-negative definite self-adjoint, we may associate with A a closed symmetric form $\dot{\xi}'$ by the formula (1.2.3). We claim that $\dot{\xi} = \dot{\xi}'$. From (1.2.4) $G_\alpha \left(L^2(I_0; e^{-V(\dot{x})} d\dot{x}) \right) \subset \mathcal{D}(\dot{\xi}')$ and $\dot{\xi}'_\alpha(G_\alpha u, G_\alpha v) = (G_\alpha u, v)$ which also equals $\xi_\alpha(G_\alpha u, G_\alpha v)$ by (1.2.6). Thus $\dot{\xi} = \dot{\xi}'$ on $G_\alpha \left(L^2(I_0; e^{-V(\dot{x})} d\dot{x}) \right) \times G_\alpha \left(L^2(I_0; e^{-V(\dot{x})} d\dot{x}) \right)$. But the same equations (1.2.4) and (1.2.6) imply that $G_\alpha(H)$ is dense in $\mathcal{D}(\dot{\xi}')$ and $\mathcal{D}(\dot{\xi})$ proving $\dot{\xi}' = \dot{\xi}$.
For a given $\dot{\xi}$, A satisfying (1.2.3) is unique, because the resolvent $\{G_\alpha, \alpha > 0\}$ generated by a such A satisfies (1.2.4) which in turn means that $\{G_\alpha, \alpha > 0\}$ and A are uniquely determined by $\dot{\xi}$. □

In the above proof, we actually showed that the correspondance between $\dot{\xi}$ and the resolvent of A is caracterized by the equation (1.2.4). A restatement of this is the following.

Corollary 1.2.13. *The correspondance in Theorem 1.2.12 can be caracterized by*

$$\begin{cases} \mathcal{D}(A) \subset \mathcal{D}(\dot{\xi}) \\ \dot{\xi}(u,v) = (-Au, v), u \in \mathcal{D}(A), v \in \mathcal{D}(\dot{\xi}) \end{cases} \tag{1.2.7}$$

We state two lemmas for later use.

Lemma 1.2.14. *Let a closed form $\dot{\xi}$ and a non-negative definite self-adjoint operator $-A$ correspond to each other by (1.2.7). Let $\{T_t, t > 0\}$ and $\{G_\alpha, \alpha > 0\}$be the strongly semigroup and resolvent corresponding to A. Then*
(i) $T_t\left(L^2(I_0; e^{-V(\dot{x})}d\dot{x})\right) \subset \mathcal{D}(\dot{\xi}), \dot{\xi}(T_tu, T_tu) \leq (1/2t)\{(u,u) - (T_tu, T_tu)\} \leq \dot{\xi}(u,u), u \in \mathcal{D}(\dot{\xi})$.
(ii) $G_\alpha\left(L^2(I_0; e^{-V(\dot{x})}d\dot{x})\right) \subset \mathcal{D}(\dot{\xi}), \dot{\xi}_\alpha(G_\alpha u, v) = (u,v), u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x}), v \in \mathcal{D}(\dot{\xi})$.
(iii) The following convergence takes place strongly in $\mathcal{D}(\dot{\xi})$ for any $u \in \mathcal{D}(\dot{\xi})$:
$T_tu \longrightarrow u, t \downarrow 0.$
$(1/t)(G_1u - e^{-t}G_1T_tu) = (1/t)(G_1u - e^{-t}T_tG_1u) \longrightarrow u, t \downarrow 0.$
$\alpha G_\alpha u \longrightarrow u, \alpha \longrightarrow \infty.$

Proof. (ii) has been proved using (1.2.4). The other assertions can be proved in the same way. Integrating $\lambda e^{-2t\lambda} \leq (1/2t)(1 - e^{-2t\lambda})$ with respect to $d(E_\lambda u, u)$, we get (i).
For $w_t = (1/t)(G_1u - e^{-t}G_1T_tu)$, we have

$$\xi_1(w_t - u, w_t - u) = \int_{[0,\infty)} ((1 - e^{-t(\lambda+1)})/t(\lambda+1) - 1)^2(\lambda+1)d(E_\lambda u, u) \longrightarrow 0, t \downarrow 0, u \in \mathcal{D}(\xi).$$

□

For a semigroup $\{T_t, t > 0\}$ and a resolvent $\{G_\alpha, \alpha > 0\}$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$, we define the symmetric form ξ^t and ξ^β on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ by

$$\dot{\xi}^t(u,u) = \frac{1}{t}(u - T_tu, v), u, v \in L^2(I_0; e^{-V(\dot{x})}d\dot{x}) \tag{1.2.8}$$

$$\dot{\xi}^\beta(u,u) = \beta(u - \beta G_\beta u, v), u, v \in L^2(I_0; e^{-V(\dot{x})}d\dot{x}). \tag{1.2.9}$$

The next lemma justifies our saying that $\dot{\xi}^t$ and $\dot{\xi}^\beta$ are approximating forms determined by T_t and G_β respectively.

Lemma 1.2.15. *Consider $\dot{\xi}$, $-A$, T_t, G_α of lemma 1.2.14 and let $\dot{\xi}^t$ and $\dot{\xi}^\beta$ be the approximating forms determined by T_t and G_α respectively.*
(i) For $u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$, $\dot{\xi}^t(u,u)$ is non-decreasing as $t \downarrow 0$ and

$$(1.2.10) \qquad \begin{cases} \mathcal{D}(\xi) = \left\{ u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x}) : \lim_{t\to 0} \dot{\xi}^t(u,u) < \infty \right\} \\ \dot{\xi}(u,v) = \lim_{t\to 0} \dot{\xi}^t(u,v), u, v \in \mathcal{D}(\dot{\xi}). \end{cases}$$

(ii) For any $u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$, $\dot{\xi}^\beta(u,u)$ is non-decreasing as $\beta \uparrow \infty$ and

$$(1.2.11) \qquad \begin{cases} \mathcal{D}(\dot{\xi}) = \left\{ u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x}) : \lim_{\beta\uparrow\infty} \xi^\beta(u,u) < \infty \right\} \\ \dot{\xi}(u,v) = \lim_{\beta\uparrow\infty} \dot{\xi}^\beta(u,v), u, v \in \mathcal{D}(\dot{\xi}). \end{cases}$$

The lemma can be proved using the spectral family as in the proof of the preceding lemma. This lemma is very useful in that it provides us with a simple direct description of ξ in terms of T_t and G_α. In partcular, a direct correspondance between the family of closed symmetric forms and the family of strongly continuous resolvents is given by (1.2.11) and (1.2.4).
Such a correspondance can be extended to the relationship between resolvent which are not necessarily strongly continuous and closed forms whose domains are not necessarily dense in $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. We call ξ a symmetric form on $L^2(I_0 e^{-V(\dot{x})}d\dot{x})$ in the wide sense if $\dot{\xi}$ satisfies all conditions of the symmetric form except for the denseness of $\mathcal{D}(\dot{\xi})$ in $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$.

1.2.3 Dirichlet forms and Markovian semigroups

Definition 1.2.16. $\longrightarrow$ *A linear operator S on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ with $\mathcal{D}(S) = L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ is called Markovian if $0 \le Su \le 1$ almost everywhere whenever $u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$, $0 \le u \le 1$ almost everywhere. We abreviate in the following "almost everywhere" by a.e. Here almost everywhere is almost everywhere with respect to the measure $e^{-V(\dot{x})}d\dot{x}$ called the reference measure.*
$\longrightarrow$ *We say that S is positivity preserving if $Su \ge 0$ a.e whenever $u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ and $u \ge 0$.*

Theorem 1.2.17. *Let $\dot{\xi}$ a closed symmetric form on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. Let $\{T_t, t > 0\}$ and $\{G_\alpha, \alpha > 0\}$ be the strongly continuous semigroup and resolvent on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ which are associated with $\dot{\xi}$ in the manner of the preceding section. Then the next five conditions are equivalent to each other: (a) T_t is Markovian for each $t > 0$.*

(b) *αG_α is Markovian for each $\alpha > 0$.*

(c) *ξ is Markovian.*

(d) *The unit contraction operates on* $\dot{\xi}$.

(e) *Every normal contraction operates on* $\dot{\xi}$.

A semigroup (resp. a resolvent) on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ satisfying condition (a) (resp. (b)) is called a Markovian semigroup (resp.markovian resolvent). In particular, 1.2.17 means the family of all Dirichlet forms on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ and the family of strongly continuous markovian semigroup on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ stand in one to one correspondance. The implications $(a) \implies (b)$ and $(b) \implies (a)$ are evident from proposition 1.2.8 and proposition 1.2.11 respectively. $(e) \implies (d) \implies (c)$ is trivial. Hence it suffices to prove the relations $(c) \implies (b)$ and $(b) \implies (e)$.

Proof. $(c) \implies (b)$. Fix $\alpha > 0$ and $u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ such that $0 \leq u \leq 1$ a.e. Introduce the quadratic form ψ on $\mathcal{D}(\dot{\xi})$ by

$$\psi(v) = \dot{\xi}(v,v) + \alpha\left(v - \frac{u}{\alpha}, v - \frac{u}{\alpha}\right), \qquad v \in \mathcal{D}(\dot{\xi}). \tag{1.2.12}$$

Then by virtue of (1.2.4), we have

$$\psi(G_\alpha u) + \dot{\xi}_\alpha(G_\alpha u - v, G_\alpha u - v) = \psi(v), v \in \mathcal{D}(\dot{\xi}). \tag{1.2.13}$$

In other words, $G_\alpha u$ is the unique element of $\mathcal{D}(\dot{\xi})$ minimizing ψ. Now suppose that $\dot{\xi}$ is Markovian, i.e., there exists for each ϵ a function $\phi_\epsilon(t)$ satisfying the Markovian condition $(\xi.4)$. Put $\widetilde{\phi_\epsilon}(t) = (1/\alpha)\phi_{\alpha\epsilon}(\alpha t)$ and $w = \widetilde{\phi_\epsilon}(G_\alpha u)$. Then

$$w \in \mathcal{D}(\dot{\xi}), \qquad \dot{\xi}(w,w) \leq \dot{\xi}(G_\alpha u, G_\alpha u). \tag{1.2.14}$$

On the other hand, $\left|\widetilde{\phi_\epsilon}(t) - s\right| \leq |t - s|$ for $s \in [0, 1/\alpha]$ and $t \in \mathbb{R}$ and so $|w(\dot{x}) - u(\dot{x})/\alpha| \leq |G_\alpha u(\dot{x}) - u(\dot{x})/\alpha|$ a.e. Hence $(w - u/\alpha, w - u/\alpha) \leq (G_\alpha u - u\alpha, G_\alpha u - u\alpha)$. Combining this with (1.2.14), we get $\psi(w) \leq \psi(G_\alpha u)$ which implies that $w = G_\alpha u$. In particular $-\epsilon \leq G_\alpha u \leq 1/\alpha + \epsilon$. Since ϵ is arbitrary, the Markovian property of αG_α is proven.

The implication $(b) \implies (e)$ uses the following lemma.

Lemma
(i) S is a positive symmetric linear operator on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$, then there exists a unique positive symmetric Radon measure σ on the product space $I_0 \times I_0$ satisfying the following property: for any Borel functions $u, v \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$

$$(u, Sv) = \int_{I_0 \times I_0} u(\dot{x})v(\dot{y})\sigma(d\dot{x}, d\dot{y}). \tag{1.2.15}$$

(ii) If in addition S is Markovian,

$$\sigma(I_0 \times I_0) \leq \int_E e^{-V(\dot{x})} d\dot{x}, \qquad \forall E \in \mathcal{B}(I_0). \tag{1.2.16}$$

$\mathcal{B}(I_0)$ means the set of Borelians.

Proof of lemma

(ii) follows from (i). To prove (i), let us consider a function $f(\dot{x},\dot{y}) = \sum_{i=1}^{l} u_i(\dot{x})v_i(\dot{y}), u_i, v_i \in C(I_0)$, on $I_0 \times I_0$. $C(I_0)$ means the set of continuous functions with compact support in I_0. Let

$$I(f) = \sum_{i=1} l(u_i, Sv_i). \tag{1.2.17}$$

Assuming that $f(\dot{x},\dot{y}) \geq 0, \dot{x},\dot{y} \in I_0$, we show $I(f) \geq 0$. Let $K = \cup_{i=1}^{l} supp[u_i]$. Since each u_i is uniformly continuous on the compactum K, we can choose, for any $\epsilon > 0$, a finite decomposition $K = \sum_{k=1}^{l} E_k, E_k \in \mathcal{B}(I_0)$, and points $\dot{\xi}_k \in E_k$, $1 \leq k \leq p$ such that $\sup_{\dot{x}\in K} |u_i(\dot{x}) - \widetilde{u}_i(\dot{x})| < \epsilon$ where $\widetilde{u}_i = \sum_{k=1}^{p} u_i(\dot{\xi}_k) 1_{E_k}(\dot{x})$, $1 \leq i \leq l$. Then $\left| I(f) - \sum_{i=1}^{l} (\widetilde{u}_i, Sv_i) \right| \leq \epsilon \sum_{i=1}^{l} (1_K, |Sv_i|)$.

On the other hand, $\sum_{i=1}^{l} (\widetilde{u}_i, Sv_i) = \sum_{k=1}^{p} (1_{E_k}, Sf_{\dot{\xi}_k})$ with $f_{\xi_k}(\dot{y}) = f(\dot{\xi}_k, y)$. The last sum is non-negative because $f \geq 0$ a.e. We have shown $I(f) \geq 0$.

By the above observation, we conclude that (1.2.17) defines a positive linear functional on the space $\widetilde{C}_0(I_0 \times I_0) = \left\{ f(\dot{x},\dot{y}) = \sum_{i=1}^{l} u_i(\dot{x})v_i(\dot{x}) : u_i, v_i \in C(I_0), l \geq 1 \right\}$. The value $I(f)$ for $f \in \widetilde{C}_0(I_0 \times I_0)$ does not depend on the manner of the expression of f. So I can be extended to a positive linear functional on $C(I_0)$.

Proof of $(b) \Longrightarrow (e)$. Assume that αG_α is Markovian. Then by precedent lemma there is a positive Radon measure σ_α on $I_0 \times I_0$ such that

$$\alpha(u, G_\alpha v) = \int_{I_0 \times I_0} u(\dot{x})v(\dot{y})\sigma_\alpha(d\dot{x}, d\dot{y}) \tag{1.2.18}$$

for any Borel function $u, v \in L^2(I_0, e^{-V(\dot{x})}d\dot{x})$. Using this measure, the approximating form (1.2.9) can be rewritten as

$$\begin{aligned} \dot{\xi}^{\beta}(u,u) &= \frac{1}{2}\beta \int_{I_0 \times I_0} (\widetilde{u}(\dot{x}) - \widetilde{u}(\dot{y}))^2 \sigma_\beta(d\dot{x}, d\dot{y}) \\ &+ \int_{I_0} \widetilde{u}(\dot{x})^2 (1 - s_\beta(\dot{x})) e^{-V(\dot{x})} d\dot{x}, u \in L^2(I_0, e^{-V(\dot{x})}d\dot{x}) \end{aligned} \tag{1.2.19}$$

where $s_\alpha(\dot{y}) = \sigma_\alpha(I_0, d\dot{y})/(e^{-V(\dot{y})}d\dot{y})$ and $\widetilde{u}$ is any borel modification of u. In view of precedent lemma,

$$0 \leq s_\alpha \leq 1 \text{ a.e} \tag{1.2.20}$$

It is obvious from the expression (1.2.19) and lemma 1.2.15 that every normal contraction operates on $\dot{\xi}$.

We recall that a Dirichlet form which is a closed , and Markovian form. Thus,the proof of (a) and (e) of theorem 1.2.17 are obvious. We collect below some important properties of the Dirichlet related to the property (e). □

Theorem 1.2.18. *A Dirichlet form $\dot{\xi}$ on $L^2(I_0, e^{-V(\dot{x})}d\dot{x})$ possesses the following properties: (i) $u, v \in \mathcal{D}(\dot{\xi}) \Rightarrow u \wedge v, u \vee v, u \wedge 1 \in \mathcal{D}(\dot{\xi})$.*
(ii) $u, v \in \mathcal{D}(\dot{\xi}) \cap L^\infty(I_0, e^{-V(\dot{x})}d\dot{x}) \Rightarrow u.v \in \mathcal{D}(\dot{\xi})$ and $\sqrt{\dot{\xi}(u.v, u.v)} \leq \|u\|_\infty \sqrt{\dot{\xi}(v,v)} + \|v\|_\infty \sqrt{\dot{\xi}(u,u)}$
(iii) $u \in \mathcal{D}(\dot{\xi}), u_n = ((-n) \vee u) \wedge n \Rightarrow u_n \in \mathcal{D}(\dot{\xi})$ and $u_n \to u, n \to \infty$ with respect to $\dot{\xi}_1$-metric.
(iv) $u \in \mathcal{D}(\dot{\xi}), u^\epsilon = u - ((-\epsilon) \vee u) \wedge \epsilon \Rightarrow u^\epsilon \in \mathcal{D}(\dot{\xi})$ and $u^\epsilon \to u, \epsilon \to 0$ with respect to $\dot{\xi}_1$-metric.
(v) $u_n, u \in \mathcal{D}(\dot{\xi}), u_n \to u, n \to \infty$ with respect to $\dot{\xi}_1$ metric and $\phi(t)$ is real function such that $\phi(0) = 0, |\phi(t) - \phi(t')| \leq |t - t'|, t, t' \in \mathbb{R} \Rightarrow \phi(u_n), \phi(u) \in \mathcal{D}(\dot{\xi})$ and $\phi(u_n) \to \phi(u), n \to \infty$, weakly with respect to $\dot{\xi}_1$. If in addition, $\phi(u) = u$, then the convergence is strong with respect to $\dot{\xi}_1$.

Proof. (i) By Theorem 1.2.17 (e), $u \in \mathcal{D}(\dot{\xi})$ implies that $|u| \in \mathcal{D}(\dot{\xi})$ and $u \wedge 1 \in \mathcal{D}(\dot{\xi})$. Then it suffices to note that $u \vee v = 1/2\{(u+v) + |u-v|\}$ and $u \wedge v = 1/2\{(u+v) - |u-v|\}$.
(ii) By making use of formula (1.2.19), we can prove the following which is even stronger than (e):: if $u_1, u_2 \in \mathcal{D}(\dot{\xi}), w \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ satisfy $|\widetilde{w}(\dot{x}) - \widetilde{w}(\dot{y})| \leq |\widetilde{u}_1(\dot{x}) - \widetilde{u}_1(\dot{y})| + |\widetilde{u}_2(\dot{x}) - \widetilde{u}_2(\dot{y})|$, $|\widetilde{w}(\dot{x})| \leq |\widetilde{u}_1(\dot{x}) - \widetilde{u}_2(\dot{x})|$, $\dot{x}, \dot{y} \in I_0$, for some Borel modification $\widetilde{u}_1, \widetilde{u}_2$ and $\widetilde{w}$, then $w \in \mathcal{D}(\dot{\xi})$ and $\sqrt{\dot{\xi}(w,w)} \leq \sqrt{\dot{\xi}(u_1,u_1)} + \sqrt{\dot{\xi}(u_2,u_2)}$. Assertion (ii) is now obtained by setting $w = u.v, u_1 = \|u\|_\infty .v$ and $u_2 = \|v|_\infty .u$.
(iii) Since u_n is a normal contraction of u, $\dot{\xi}_1(u_n, u_n)$ is uniformly bounded by $\dot{\xi}_1(u, u$. Moreover, $\dot{\xi}_1(u_n, G_1 v) = (u_n, v) \to (u, v) = \dot{\xi}_1(u, G_1 v), n \to \infty, v \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ by formula (1.2.4). Since $G_(L^2(I_0; e^{-V(\dot{x})}d\dot{x}))$ is dense in $\mathcal{D}(\dot{\xi})$ with metric $\dot{\xi}_1$, u_n weakly converges to u with respect to $\dot{\xi}_1$: $\dot{\xi}_1(u_n, w) \to \dot{\xi}_1(u, w), n \to \infty, \forall w \in \mathcal{D}(\dot{\xi})$.
But then $\dot{\xi}_1(u_n - u, u_n - u) \leq 2\dot{\xi}_1(u,u) - 2\dot{\xi}_1(u_n, u_n) \to 0, n \to \infty$.
(iv) The proof is the same as above because u^ϵ is a normal contraction of u.
(v) The proof is also similar to the above since $\phi(u_n)$ is a normal contraction of u_n □

1.2.4 Closability and the smallest closed extension

We show that the Markovian nature and the local property of a closable form are preserved under the operation of taking the smallest closed form.

Theorem 1.2.19. *Let $(\dot{\xi}, \mathcal{D}(\dot{\xi})$ a closable form on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. Then, the smallest closed extension $\bar{\dot{\xi}}$ is again Markovian and hence a Dirichlet form.*

Proof. Let $\left\{\alpha \dot{G}_\alpha, \alpha\right\}$ be a strongly continuous resolvent associated with the closed form $\bar{\dot{\xi}}$. By vertue of Theorem 1.2.17, it suffices to show that $\alpha \dot{G}_\alpha$ is Markovian.
Take $u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ such that $0 \leq 1$ almost everywhere, and consider the quadratic functional ψ defined by (1.2.12). By virtue of (1.2.13), we can see that $\alpha \dot{G}_\alpha u$ is the unique element minimizing ψ on $\mathcal{D}(\bar{\dot{\xi}})$ and $v_n \in \mathcal{D}(\bar{\dot{\xi}})$ is $\bar{\dot{\xi}}_1$ convergent to $\dot{G}_\alpha u$ if and only if $\lim_{n\to\infty} \psi(v_n) = \psi(\dot{G}_\alpha u)$.
We can select a such sequence $\{v_n\}$ from $\mathcal{D}(\dot{\xi})$ because $\mathcal{D}(\dot{\xi})$ is $\bar{\dot{\xi}}_1$ dense in $\mathcal{D}(\bar{\dot{\xi}})$. For any $\epsilon > 0$, let $\phi_\epsilon(t)$ be a real function as given in the Markovian condition $(\xi.4)$ above and put $\tilde{\phi}_\epsilon(t) = (1/\alpha)\phi_{\alpha\epsilon}(\alpha t)$ and $w_n = \tilde{\phi}_\epsilon(v_n)$. In the same way as in the proof of the implication $(e) \Rightarrow (b)$ of Theorem 1.2.17, we have $\psi(w_n) \leq \psi(v_n)$. Therefore, $\lim_{n\to\infty} \psi(w_n) = \psi(\dot{G}_\alpha u)$ and w_n is $\bar{\dot{\xi}}_1$ convergent to $\dot{G}_\alpha u$. In particular, a subsequence of w_n converges to $\alpha \dot{G}_\alpha u$ almost everywhere with respect to $e^{-V(\dot{x})}dx$. Since $-\epsilon \leq w_n \leq 1/\alpha + \alpha + \epsilon$ almost evrywhere with respect to $e^{-V(\dot{x})}dx$, $-\epsilon \leq \dot{G}_\alpha \leq 1/\alpha + \alpha + \epsilon$ almost everywhere with respect to $e^{-V(\dot{x})}dx$ for arbitrary ϵ, we get the Markovian nature of $\alpha \dot{G}_\alpha$.
□

The theory given up here is a general theory of Dirichlet forms, what is true, as stated at the beginning, for any space $L^2(X;m)$ where X is a locally compact separable metric space and m a positive Radon measure on X such that $supp[m] = X$. In the following, we fix our divergence-form operator $\dot{L} = e^{V(\dot{x})}div(e^{-V(\dot{x})}\nabla)$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ where I_0 is the unit torus of $\mathbb{R}^d$, which is a compact metric space. The hypothesis on V, in this thesis; will be used: V is measurable and $e^V + e^{-V} \in L^1(I_0; d\dot{x})$. The next objective is to construct a continuous and conservative Markov process $\left(\Omega, \mathcal{F}, P_{\dot{x}}, \dot{X}_t\right)$ on I_0 associated with $\dot{L}$ by using the preceeding general theory.

1.3 Construction of continuous Markov process

For the divergence-form operator $\dot{L}$ defined in (1.1.1), we associate a bilinear form $\dot{\xi}$, a family of resolvent $\left\{\dot{G}_\alpha, \alpha > 0\right\}$ and semigroup $\left\{\dot{P}_t, t > 0\right\}$ in as in Section 1. Our problem now is to construct a stochastic process from the divergence-form operator using the theory of Dirichlet forms. All the following definitions we give here, come from [4].

Hunt process A process $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ is called a Hunt diffusion process if it satisfies the following conditions:
1) $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ is strong Markov Process on I_0 with respect to the filtration $\mathcal{F}$.
2) Let Δ be an adjoined isolation point of I_0 (recall that I_0 is compact). We have:
(i) $\dot{X}_\infty(\omega) = \Delta, \forall \omega \in \Omega$,
(ii) $\dot{X}_t(\omega) = \Delta, \forall t \geq \tau(\omega)$ where $\tau(\omega) = \inf\left\{t > 0 : \dot{X}_t = \Delta\right\}$
(iii) for each $t \in [0, \infty]$, there exists a map θ_t from Ω to Ω such that

$\dot{X}_s o\theta_t = \dot{X}_{t+s}, s > 0$
(iv) for almost all $\omega \in \Omega$, the sample path $t \longmapsto \dot{X}_t(\omega)$ is continuous $P_{\dot{x}}-$ a.e on $[0, \tau(\omega)]$.
If the transition function associated with $(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}})$ is symmetric with respect to $e^{-V(\dot{x})}d\dot{x}$ in the sense of precedent section, we say hat the Hunt process is $e^{-V(\dot{x})}d\dot{x}-$ symmetric.

In the remainder of this section, we examine the case when strongly continuous Markovian semigroups and resolvents on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ are determined by Markovian transition functions and resolvent kernels respectively.

Let us consider the measurable space $(I_0, \mathcal{B}(I_0))$, I_0 is the unit torus and $\mathcal{B}(I_0))$ the family of all Borel measurable subsets of I_0.

Definition 1.3.1. ***Some definitions and construction of a strongly continuous semigroup***

A non-negative function $k(\dot{x}, A)$, $\dot{x} \in I_0, A \in \mathcal{B}(I_0)$, is called a kernel on $(I_0, \mathcal{B}(I_0))$ if $k(\dot{x}, .)$ is a positive measure on $\mathcal{B}(I_0)$ for each $\dot{x}$ in I_0 and if $k(., A)$ is a measurable function on I_0 for each $A \in \mathcal{B}(I_0)$. If in addition $k(\dot{x}, I_0) \leq 1, \dot{x} \in I_0$ is imposed, then k is called a Markovian kernel. We write $ku(\dot{x}) = \int_{I_0} u(\dot{y})k(\dot{x}, d\dot{y})$ whenever the integral makes sense.

A family $\{\dot{p}_t, t > 0\}$ of Markovian kernel on $(I_0, \mathcal{B}(I_0))$ is said to be Markovian transition function if:

$$\dot{p}_t\dot{p}_s u = \dot{p}_{t+s}u \qquad t, s > 0, u \in \mathcal{B}(I_0) \text{ and bounded.} \tag{1.3.1}$$

A family $\left\{\dot{R}_\alpha, \alpha > 0\right\}$ is said to be a Markovian resolvent kernel if $\left\{\alpha\dot{R}_\alpha, \alpha > 0\right\}$ is a family of Markovian kernel on $(I_0, \mathcal{B}(I_0))$ and :

$$\dot{R}_\alpha u - \dot{R}_\beta u + (\alpha - \beta)\dot{R}_\alpha\dot{R}_\beta u = 0, \qquad \alpha, \beta > 0, \tag{1.3.2}$$

for all measurable and bounded function u. We denote by $\mathcal{B}_b(I_0)$ the set of all measurable and bounded functions.
A kernel $\dot{p}_t$ on $(I_0, \mathcal{B}(I_0))$ is called symmetric with respect to $e^{-V(\dot{x})}d\dot{x}$ if

$$(u(\dot{x}), \dot{p}_t v(\dot{x})) = (v, \dot{p}_t u(\dot{x})) \tag{1.3.3}$$

for all non-negative measurable and bounded function u and v. Besides, we know that if $\dot{p}_t$ is a $e^{-V(\dot{x})}d\dot{x}-$ symmetric Markovian kernel. Then

$$\int_{I_0} (\dot{p}_t(u(\dot{x}))^2 e^{-V(\dot{x})}dx \leq \int_{I_0} u(\dot{x})^2 e^{-V(\dot{x})}d\dot{x}, \forall u \in \mathcal{B}_b \cap L^2(I_0; e^{-V(\dot{x})}d\dot{x}) \tag{1.3.4}$$

because by Shwarz inequality, $(\dot{p}_t u(\dot{x}))^2 \leq \dot{p}_t(\dot{x}).\dot{p}_t u(\dot{x})^2$, which leads us to (1.3.4) because of the symmetry of p_t. Equation (1.3.4) means that p_t can be extended uniquely to a symmetric contractive operator on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. We can say now that a family of $e^{-V(\dot{x})}d\dot{x}-$ symmetric Markovian transition function determines a Markovian semigroup $\dot{P}_t$ not necessary strongly continuous. $\dot{P}_t$ becomes strongly continuous if $\lim_{t\to 0}\dot{p}_t u(x) = u(\dot{x})\forall u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ or for all u in a subset of $\mathcal{B}_b(I_0) \cap L^1(I_0; e^{-V(\dot{x})}d\dot{x})$, dense in $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$.

Theorem 1.3.2. ***Theorem 7.2.2 of [4]***
Let $(\xi, \mathcal{D}(\xi))$ a ***regular and local Dirichlet form*** *on $L^2(I_0; m)$. Then there exists a $m-$symmetric Hunt diffusion process, $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ defined for almost all $\dot{x} \in I_0$ and whose Dirichlet form is given by $(\xi, \mathcal{D}(\xi))$*

We consider the divergence form operator $\dot{L} = e^{V(\dot{x})}div(e^{-V(\dot{x})}\nabla)$ on $L^2(I_0, e^{-V(\dot{x})}d\dot{x})$ where I_0 is the unit torus of $\mathbb{R}^d$, which is a compact. Let us consider $\mathcal{H}^1(I_0, e^{-V})$ the set of all functions $f \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ with all derivatives $\partial_i f$ belongs to $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. By Definition of $\dot{L}$ we have for all $f, g \in \mathcal{H}^1(I_0, e^{-V})$,

$$\dot{\xi}(f,g) := \int_{I_0} -\dot{L}f(\dot{x})g(\dot{x})e^{-V(\dot{x})}d\dot{x} = \int_{I_0} \nabla f(\dot{x}).\nabla g(\dot{x})e^{-V(\dot{x})}d\dot{x}, \tag{1.3.5}$$

is a symmetric form on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. It is very easy to verify (ξ.1) and (ξ.2).

Proposition 1.3.3. *Assume that e^V is integrable on I_0 with respect to Lebesgue. The symmectric form $\dot{\xi}$ on $\mathcal{H}^1(I_0; e^{-V})$ is a local Dirichlet form.*

Proof. The Markovian property is proved in [9] page 36, lemma 3.2. The local property is obvious from the definition.
We prove the closable property: let (f_n) be a sequence in $\mathcal{H}^1(I_0; e^{-V})$ which goes to zero in $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ and such that (f_n) is $\dot{\xi}$-Cauchy.
Since (f_n) is $\dot{\xi}$-Cauchy, we see that ∇f_n is Cauchy in $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. Therefore ∇f_n converges to some limit h in $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$.
Since $\int_{I_0} e^{V(\dot{x})}d\dot{x} < \infty$, then, for all $g \in C^\infty(I_0)$, we have

$$c := \left(\int_{I_0} \left(|g(\dot{x})|^2 + |\nabla g(\dot{x})|^2\right) e^{V(\dot{x})}d\dot{x}\right)^{\frac{1}{2}} < \infty,$$

and, using the Cauchy-Schwarz inequality:

$$\begin{aligned}\left|\int_{I_0} g(\dot{x})\nabla f_n(\dot{x})d\dot{x} - \int_{I_0} g(\dot{x})h(\dot{x})d\dot{x}\right| &\leq \int_{I_0} |g(\dot{x})|\,|\nabla f_n(\dot{x}) - h(\dot{x})|\, e^{\frac{1}{2}V(\dot{x})-\frac{1}{2}V(\dot{x})}d\dot{x} \\ &\leq c\left(\int_{I_0} |\nabla f_n(\dot{x}) - h(\dot{x})|^2\, e^{-V(\dot{x})}d\dot{x}\right)^{\frac{1}{2}} \to 0\end{aligned}$$

when $n \to \infty$.

As (f_n) converges to 0 in $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$, we also have:

$$\left|\int_{I_0} g(\dot{x})\nabla f_n(\dot{x})d\dot{x}\right| = \left|\int_{I_0} \nabla g(\dot{x}) f_n(\dot{x}) e^{\frac{1}{2}V(\dot{x})} e^{-\frac{1}{2}V(\dot{x})} d\dot{x}\right|$$
$$\leq c(\int_{I_0} |f_n(\dot{x})|^2 e^{-V(\dot{x})} d\dot{x})^{\frac{1}{2}} \to 0 \text{ when } n \to \infty.$$

As a consequence of these two facts, we see that $\int_{I_0} g(\dot{x})h(\dot{x})d\dot{x} = 0$ for all $g \in C^\infty(I_0)$. Therefore $h = 0$ almost everywhere and

$$\dot{\xi}(f_n, f_n) \to 0 \text{ when } n \to \infty.$$

Thus we have proved that $\dot{\xi}$ is closable. □

Let us set $H^1(I_0; e^{-V}) := \overline{C^\infty(I_0)}^{\dot{\xi}_1}$ be the completion of $C^\infty(I_0)$ with respect to the norm $\dot{\xi}_1$. Since $C^\infty(I_0)$ is a subset of $\mathcal{H}^1(I_0; e^{-V})$ then $\dot{\xi}$ is also a closable form on $C^\infty(I_0)$. Thus, $(\dot{\xi}, H^1(I_0; e^{-V}))$ is a regular and local Dirichlet form because by definition of core in (1.2.6), $C^\infty(I_0)$ is obviously a core of $\dot{\xi}$.

Let us consider the $e^{-V(\dot{x})}d\dot{x}-$symmetric Hunt diffusion process $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ on $(I_0, \mathcal{B}(I_0))$ associated with the regular and local Dirichlet form $(\dot{\xi}, H^1(I_0; e^{-V}))$. The Hunt diffusion process is also a Markov process, then the transition function defined by $\dot{p}_t(x, E) = P_{\dot{x}}(\dot{X}_t \in E), \dot{x} \in I_0, E \in \mathcal{B}(I_0)$ is a Markovian symmetric transition function, and the resolvent Markovian kernel of the process is $\dot{R}_\alpha(\dot{x}, E) = \int_{I_0} e^{-\alpha t}\dot{p}_t(\dot{x}, E)dt$. They determine a strongly continuous semigroup $\dot{P}_t$ and resolvent $\dot{G}_\alpha$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$.

In the next paragraph, we prove easily that our diffusion process $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ is conservative. In other words, we prove that the death time defined by:

$\tau(\omega) = \inf\left\{t > 0 : \dot{X}_t(\omega) = \Delta\right\}$ where Δ is an adjoined isolation point of I_0 satisfies:

$$P_{\dot{x}}(\tau(\omega) < +\infty) = 0.$$

1.3.1 Conservativeness of the diffusion process $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$

The proof of this property uses the equivalence beetwen conservativeness of Dirichlet form and conservativeness of the associated process.
We consider our regular and local Dirichlet form $(\dot{\xi}, H^1(I_0; e^{-V}))$ associated with the diffusion process $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$.

Definition

We say that the Dirichlet form is conservative if the associated semigroup $\dot{P}_t$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ satisfies:

$$\dot{P}_t 1(\dot{x}) = 1, \text{ for almost all } \dot{x} \in I_0, \forall t > 0. \tag{1.3.6}$$

The following Theorem is proved in Problem 4.5.1 of [4].

Theorem 1.3.4. *$(\dot{\xi}, H^1(I_0; e^{-V}))$ is conservative $\Leftrightarrow$ $P_{\dot{x}}(\tau < \infty) = 0$ for almost all $\dot{x} \in I_0$.*

By this Theorem, we deduce:

Proposition 1.3.5. *The diffusion process $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ is conservative.*

Proof. proving the conservativeness property for $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ is not very difficult, because of compactness property of the torus. Indeed, since the function 1 is in the domain of $\dot{\xi}$ we have:

$$\forall f \in H^1(I_0; e^{-V}) \qquad \dot{\xi}(f, 1) = 0.$$

Taking $f = \dot{P_t}1$ we get:

$$\begin{aligned} 0 &= -\int_{I_0} \dot{L} P_t 1(\dot{x}) 1(\dot{x}) e^{-V(\dot{x})} d\dot{x} \\ &= -\int_{I_0} \frac{\partial}{\partial t} P_t 1(\dot{x}) 1(\dot{x}) e^{-V(\dot{x})} d\dot{x} \\ &= -\frac{\partial}{\partial t} \int_{I_0} P_t 1(\dot{x}) 1(\dot{x}) e^{-V(\dot{x})} d\dot{x}. \end{aligned}$$

This implies that $\int_{I_0} P_t 1(\dot{x}) 1(\dot{x}) e^{-V(\dot{x})} d\dot{x} = c < \infty, \forall t \geq 0$. Thus $c = \int_{I_0} e^{-V(\dot{x})} d\dot{x}$ by taking $t = 0$. We have now

$\int_{I_0} \dot{P}_t 1(\dot{x}) e^{-V(\dot{x})} d\dot{x} = \int_{I_0} e^{-V(\dot{x})} d\dot{x}$.

By consequence, $\dot{P}_t 1 = 1$ a.e, $\forall t > 0$

□

Therefore we will give for readers, in chapter Appendix; a sufficient condition to prove the conservativeness of diffusions processes on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$ or in general on $(X, \mathcal{B}(X), m)$ satisfying the condition in Remark 1.1.1. For what, we refer to the last part of chapter 5 of [4] entitled "Forward and backward martingale additive functionals": see part 1 of Appendix.

We return our diffusion on the torus $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ whose Dirichlet form is $(\dot{\xi}, H^1(I_0; e^{-V}))$. The space $(\dot{\xi}, H^1(I_0; e^{-V}))$ is called a Dirichlet space associated with $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$. In the next section, we talk about the extended Dirichlet spaces and their properties. More theory can be found in section 1.5 of [4].

1.3.2 The extended Dirichlet space

We consider $(\dot{\xi}, H^1(I_0; e^{-V}))$ the Dirichlet from associated to $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ as a closed symmetric and Markovian form on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$. We already know that $H^1(I_0; e^{-V})$ is a Hilbert space with respect the norm $\dot{\xi}_\alpha)$ for each $\alpha > 0$, in particular if $\alpha = 1$. But $H^1(I_0; e^{-V})$ is not a pre-Hilbert space with respect to $\dot{\xi}$ in general. However $\dot{\xi}$ is not a norm.
We define the *extended domain* denoted here by $H^1_e(I_0; e^{-V})$: this is the set of measurable functions u on I_0, such that $|f| < \infty$ a.e and there exists a $\dot{\xi}$-Cauchy sequence (u_n) in $H^1(I_0; e^{-V})$ such that $\lim_{n\to\infty} u_n = u$ a.e.
We call u_n above the approximating sequence for $u \in H^1(I_0; e^{-V})$. $H^1_e(I_0; e^{-V})$ is a linear space containing $H^1(I_0; e^{-V})$.
In general, the Dirichlet space can be extended to the function space $H^1_e(I_0; e^{-V})$ on which $\dot{\xi}$ can be well defined.

It will be seen in chapter 4 (Appendix) that in general a extended Dirichlet space $(\xi, \mathcal{D}_e)$ is a Hilbert space if and only if $(\xi, \mathcal{D})$ is transient. The notion of transient will be defined on chapter 4.

Let's see the following proposition.

Proposition 1.3.6. *Let $(\dot{\xi}, H^1(I_0; e^{-V}))$ be a Dirichlet space on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$.*
(i) For any $u \in H^1_e(I_0; e^{-V})$ and its approximating sequence $\{u_n\}$, the limit

$$\dot{\xi}(u, u) = \lim_{n\to\infty} \dot{\xi}(u_n, u_n) \text{ exists}$$

and does not depend on the approximating sequence $\{u_n\}$ for u.
(ii) Extend the expression (1.2.19) of the approximating form associated with $(\dot{\xi}, H^1(I_0; e^{-V}))$ to any Borel function $u \in I_0$:

$$\xi^\beta(u, u) = \frac{\beta}{2} \int_{I_0\times I_0} (u(\dot{x}) - u(\dot{y}))^2 \sigma_\beta(d\dot{x}, d\dot{y}) + \beta \int_{I_0} u(\dot{x})^2 (1 - s_\beta(\dot{x})) e^{-V(\dot{x})} d\dot{x}.$$

If $u \in H^1(I_0; e^{-V})$ *then for all Borel modification* $\widetilde{u}$ *of* u, $\dot{\xi}^\beta(\widetilde{u}, \widetilde{u})$ *increases to* $\dot{\xi}(u,u)$ *as* $\beta \to \infty$.
(iii) $H^1(I_0; e^{-V}) = H^1_e(I_0; e^{-V}) \cap L^2(I_0; e^{-V(\dot{x})} d\dot{x})$.

Proof. Let $u \in H^1_e(I_0; e^{-V})$ and $\{u_n\}$ its approximating sequence. The limit in (i) exists by the triangle inequality.
We may assume that some Borel modifications $\widetilde{u}_n$ and $\widetilde{u}$ satisfy $\lim_{n\to\infty} \widetilde{u}_n(\dot{x}) = \widetilde{u}(\dot{x})$ for any $\dot{x} \in I_0$. Then by Fatou's lemma

$$\begin{aligned}\dot{\xi}^\beta(\widetilde{u}_n - \widetilde{u}, \widetilde{u}_n - \widetilde{u}) &\le \liminf_{m\to\infty} \dot{\xi}^\beta(\widetilde{u}_n - \widetilde{u}_m, \widetilde{u}_n - \widetilde{u}_m)\\ &\le \liminf_{m\to\infty} \dot{\xi}^\beta(u_n - u_m, u_n - u_m)\end{aligned}$$

which can be may arbitrary small for sufficiently large n. In particular $\lim_{n\to\infty} \dot{\xi}^\beta(\widetilde{u}_n, \widetilde{u}_n) = \dot{\xi}(\widetilde{u}, \widetilde{u})$. Hence $\dot{\xi}^\beta(\widetilde{u}, \widetilde{u})$ is non-decreasing with β.
Furthermore, the inequality

$$\begin{aligned}\left|\dot{\xi}(u,u)^{1/2} - \dot{\xi}(\widetilde{u}, \widetilde{u})^{1/2}\right| &\le \left|\dot{\xi}(u,u)^{1/2} - \dot{\xi}(u_n, u_n)^{1/2}\right| + \left|\dot{\xi}(u_n, u_n) - \dot{\xi}^\beta(\widetilde{u}_n, \widetilde{u}_n)\right|\\ &+ \dot{\xi}^\beta(\widetilde{u}_n - \widetilde{u}, \widetilde{u}_n - \widetilde{u})^{1/2}\end{aligned}$$

implies that $\lim_{\beta\to\infty} \dot{\xi}^\beta(\widetilde{u}, \widetilde{u}) = \dot{\xi}(u,u)$. We have shown (ii). In particular, if $u = 0$ a.e, then $\dot{\xi}^\beta(\widetilde{u}, \widetilde{u}) = 0$ and $\dot{\xi}(u,u) = 0$. This means the second statement in (i).
(iii) is an immediate consequence of (ii).

□

By Proposition 1.3.6, $\dot{\xi}$ can be extended to $H^1_e(I_0; e^{-V})$ as a non-negative definite symmetric bilinear form. We call $(\dot{\xi}, H^1_e(I_0; e^{-V}))$ the extended Dirichlet space of $(\dot{\xi}, H^1(I_0; e^{-V}))$. As we have seen in Lemma 1.2.14, the Markovian semigroup $\left\{\dot{P}_t, t > 0\right\}$ on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$ has the properties

$$\dot{P}_t(H^1(I_0; e^{-V})) \subset H^1(I_0; e^{-V}), \dot{\xi}(\dot{P}_t u, \dot{P}_t u) \le \dot{\xi}(u,u)$$

$$\lim_{t\to 0} \dot{\xi}(\dot{P}_t u - u, \dot{P}_t u - u) = 0$$

This two properties can be extended to the extended Dirichlet space $(\dot{\xi}, H^1_e(I_0; e^{-V}))$:

Lemma 1.3.7. *$\dot{P}_t$ can be extended to a linear operator on $(\dot{\xi}, H^1_e(I_0; e^{-V}))$ such that $u - \dot{P}_t u \in H^1(I_0; e^{-V})$ for $u \in H^1_e(I_0; e^{-V})$ and*

$$\text{(1.3.7)}\quad \dot{\xi}(\dot{P}_t u, \dot{P}_t u) \le \dot{\xi}(u,u) \qquad u \in H^1_e(I_0; e^{-V}); \qquad \lim_{t\to 0} \dot{\xi}(\dot{P}_t u - u, \dot{P}_t u - u) = 0.$$

Proof. For $u \in H_e^1(I_0; e^{-V})$ take $\{u_n\}$ the approximating sequence for it. (1.3.7) applied to $\{u_n\}$ yields that $\left\{\dot{P}_t u_n\right\}$ is $\dot{\xi}$-Cauchy. We recall that

$$(u, v) = \int_{I_0} u(\dot{x}) v(\dot{x}) e^{-V(\dot{x})} dx \text{ for all } u, v \in L^2(I_0; e^{-V(\dot{x})} d\dot{x}).$$

Since $(\dot{P}_t w, \dot{P}_t w) \leq (w, w)$ for $w \in L^2(I_0; e^{-V(\dot{x})} d\dot{x})$, we get

$$\begin{aligned}
\frac{1}{t} \left\| (u_n - u_m) - \dot{P}_t(u_n - u_m) \right\|_{L^2(I_0; e^{-V(\dot{x})} d\dot{x})} \\
&= \frac{1}{t}((u_n - u_m) - \dot{P}_t(u_n - u_m), u_n - u_m) \\
&- \frac{1}{t}((u_n - u_m) - \dot{P}_t(u_n - u_m), \dot{P}_t(u_n - u_m)) \\
&\leq \frac{1}{t}((u_n - u_m) - \dot{P}_t(u_n - u_m), u_n - u_m) \\
&\leq \dot{\xi}(u_n - u_m, u_n - u_m).
\end{aligned}$$

□

Consequently, $\left\{u_n - \dot{P}_t u_n\right\}$ is Cauchy in $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$. In particular, a subsequence $\left\{\dot{P}_t u_{n_k}\right\}$ converges a.e to a function v, which can be easily to be independant of the choice of the approximating sequence $\{u_n\}$. We let $\dot{P}_t u = v$. Then $\dot{P}_t u \in H_e^1(I_0; e^{-V})$ and $\dot{P}_t u_{n_k}$ is its approximating sequence and $u - \dot{P}_t u \in H_e^1(I_0; e^{-V}) \cap L^2(I_0; e^{-V(\dot{x})} d\dot{x}) = H^1(I_0; e^{-V})$. Clearly (1.3.7) is extended to $u \in H_e^1(I_0; e^{-V})$. The last assertion follows from

$$\begin{aligned}
\dot{\xi}(\dot{P}_t u - u, \dot{P}_t u - u) &\leq \dot{\xi}(\dot{P}_t u_n - u_n, \dot{P}_t u_n - u_n) + \dot{\xi}(\dot{P}_t(u - u_n), \dot{P}_t(u - u_n)) \\
&+ \dot{\xi}(u - u_n, u - u_n) \\
&\leq \dot{\xi}(\dot{P}_t u_n - u_n, \dot{P}_t u_n - u_n) + 2\dot{\xi}(u - u_n)
\end{aligned}$$

if we let first $t \to 0$ and then $n \to \infty$.

1.4 Some results about Potential theory for Dirichlet forms

We already know from definition 1.2.5 that a Dirichlet form is regular if it admits a core . This section is devoted to a presentation of potential theory for regaular Dirichlet form mainly due to Beurling and Deny.

Fundamental notions in the potential theory: definition of capacity, sets of capacity zero, quasi continuity of functions, equilibrium potentials etc. will be developped in this part. We will give also theirs interpretations in probability.

1.4.1 Capacity and notion of quasi-continuity

Denote by $\mathcal{O}$ the set of all open subsets of I_0. For $A \in \mathcal{O}$ we define

$$\mathcal{L}_A = \left\{ u \in H^1(I_0; e^{-V}) : u \geq 1 \text{ a.e. on } A \right\} \tag{1.4.1}$$

$$Cap(A) = \begin{cases} \inf\limits_{u \in \mathcal{L}_A} \dot{\xi}_1(u,u), & \mathcal{L}_A \neq \emptyset \\ \infty, & \mathcal{L}_A = \emptyset \end{cases} \tag{1.4.2}$$

and for any set $A \subset I_0$ we let

$$Cap(A) = \inf_{B \in \mathcal{O}, A \subset B} Cap(B) \tag{1.4.3}$$

We call this quantity, the $1-capacity$ of A or simply the $capacity$ of A. We can simillarly defined the $\alpha - capacity$ for all $\alpha > 0$ by replacing $\dot{\xi}_1$ in (1.4.2) by $\dot{\xi}_\alpha$.
Let

$$\mathcal{O}_0 = \{A \in \mathcal{O}; \mathcal{L}_A \neq \emptyset\}.$$

Proposition 1.4.1. *(i) For each $A \in \mathcal{O}_0$, there exists a unique element $e_A \in \mathcal{L}_A$ such that*

$$\dot{\xi}_1(e_A, e_A) = Cap(A). \tag{1.4.4}$$

Proof. Clarly $\mathcal{L}_A$ is a convex closed subset of $(\dot{\xi}, H^1(I_0; e^{-V}))$. By making use of the equality

$$\begin{aligned} &\dot{\xi}_1\left(\frac{u-v}{2}, \frac{u-v}{2}\right) + \dot{\xi}_1\left(\frac{u+v}{2}, \frac{u+v}{2}\right) \\ &= \frac{1}{2}\dot{\xi}(u,u) + \frac{1}{2}\xi(v,v) \end{aligned} \tag{1.4.5}$$

we can see that all minimizing sequence ($\lim_{n\to\infty} \xi_1(u_n, u_n) = Cap(A)$) is ξ_1 convergent to an element $e_A \in \mathcal{L}_A$ satisfying (1.4.4) and that such an e_A is unique. □

This proposition is the lemma 2.1.1 of chapter 2 of [4]. In this same lemma also we can found the following remark

Remark 1.4.2. (i) $0 \leq e_A \leq 1$ $a.e$ *and* $e_A = 1$ $a.e$ *on* A.
(ii) e_A *is a unique element of* $H^1(I_0; e^{-V})$ *satisfying* $e_A = 1$ $a.e$ *on* A *and* $\xi_1(e_A, v) \geq 0$, $\forall v \in H^1(I_0; e^{-V}), v \geq 0$ $a.e$ *on* A.
(iii) $v \in H^1(I_0; e^{-V}), v = 1$ $a.e$ *on* $A \Rightarrow \xi_1(e_A, v) = Cap(A)$.
(iv) $A, B \in \mathcal{O}_0, A \subset B \Rightarrow e_A \leq e_B$.

We recall the following properties

Proposition 1.4.3. *The following properties are obvious. (cf lemma 2.1.2 of [4]).*
(i) $A, B \in \mathcal{O}, A \subset B \Rightarrow Cap(A) \leq Cap(B)$.
(ii) $Cap(A \cup B) + Cap(A \cap B) \leq Cap(A) + Cap(B), A, B \in \mathcal{O}$.
(iii) $A_n \in \mathcal{O}, A_n \uparrow \Rightarrow Cap(\bigcup_n A_n) = \sup_n Cap(A_n)$.
(iv) $A_n \in \mathcal{O}, A_n \downarrow \Rightarrow Cap(\bigcap_n A_n) = \inf_n Cap(A_n)$.

It holds also that

$$Cap(A) = \sup_{K compact, K \subset A} Cap(K). \tag{1.4.6}$$

The present notion of capacity allows us to deduce that all set A of zero capacity is zero $e^{-V(\dot{x})}d\dot{x}$ measure and also zero Lebesgue measure. Indeed, the inequality $\int_A e^{-V(\dot{x})}d\dot{x} \leq Cap(A, A)$ for all $A \in \mathcal{O}$ follows from definition (1.4.2).
Let's recall some definitions

Definition 1.4.4. ***(quasi-everywhere)*** *Let A a subset of I_0. A statement depending on $\dot{x} \in A$ is said to be hold $quasi - everywhere$, in abreviation $q.e$ if there exists a subset $N \subset A$ of zero capacity and such that the statement is true for every $\dot{x} \in A - N$.*

A property "q.e" implies a property "a.e" (almost everywhere) with respect to $d\mu$ for all measure μ on I_0 charging no set of zero capacity. Then we get almost everywhere with respect to Lebesgue measure. In this Thesis , "a.e" means almost everywhere with respect to Lebesgue measure.

Definition 1.4.5. *For some measure μ finite on I_0, we denote*

$$P_\mu(.) = \int_{I_0} P_{\dot{x}}(.) d\mu(\dot{x}).$$

Nearly Borel set
A subset A of I_0 is said a nearly Borel set if there exists two Borelian sets A_1 and A_2 such that $A_1 \subset A \subset A_2$ and $P_\lambda(\exists t \geq 0, \dot{X}_t \in A_2 - A_1) = 0$, where λ means Lebesgue measure on I_0.
Polar set
A subset A of I_0 is said polar if it is contained in nearly Borel set A_1 such that for almost all $\dot{x} \in I_0$ $P_{\dot{x}}(\sigma_{A_1} < \infty) = 0$, where $\sigma_{A_1} = \inf \left\{ t > 0 : \dot{X}_t \in A_1 \right\}$.
Exceptional set
A subset A of I_0 is said exceptional if it is contained in a nearly Borel set A_1 such that for almost all $\dot{x} \in I_0$ $P_\lambda(\sigma_{A_1} < \infty) = 0$, where λ is the Lebesgue measure.
Consequence: every Polar set is exceptional. Every exceptional set can be including in a properly exceptional set: a subset M of I_0 with zero Lebesgue measure and such that $I_0 - M$ satisfies

$$P_{\dot{x}}[\left\{ \omega \in \Omega : \exists t \geq 0, \dot{X}_t(\omega) \in I_0 - M \right\}] = 1 \textit{ for almost all } \dot{x} \in I_0.$$

We deduce easily the following equivalence :

$$\textit{Polar} \iff \textit{Exceptional} \iff \textit{Properly exceptional} .$$

We see easily also that any exceptional set is negligible set with respect to $d\mu$, for all μ absolutly continuous with respect to Lebesque measure. Indeed, for a Borel exceptional set N we have $\dot{p}_t(\dot{x}, N) = 0$ $\mu-$ $a.e$ $\dot{x} \in I_0$ and consequently $\mu(N) = 0$ because $0 = \lim_{t\to 0} \int_{I_0} \dot{p}_t(\dot{x}, N)d\mu(\dot{x}) = \lim_{t\to 0} \int_N \dot{P}_t 1(\dot{x})d\mu(\dot{x}) \geq \mu(N)$.

Remark 1.4.6. *On I_0 a subset of zero capacity is a negligible subset with respect to μ for all μ strictly positive and integrable measure.*

Definition 1.4.7. ***(quasi-continuity)*** *Let u an extended real value function defined $q.e$ on I_0. We call u $quasi-continuous$ if there exists for any $\epsilon > 0$ an open set $G \subset I_0$ such that $Cap(G) < \epsilon$ and $u_{|I_0-G}$ is finite and continuous. Here $u_{|I_0-G}$ denotes the restriction of u to $I_0 - G$.*

A sequence $\{F_k\}$of closed sets such that $F_k \uparrow$ and $cap(I_0 - F_k) \downarrow 0, k \uparrow \infty$, is called a $nest$ on I_0.

Definition 1.4.8. *Given two functions u and v, v is said to be a quasi-continuous modification of u if v is quasi-continuous and $v = u$ a.e. In this case we designate v by $\widetilde{u}$.*

We now state theorem based on property of regularity of $\dot{\xi}$.
We can see in theorem 2.1.3 of [4] that each element in the Dirichlet space $(\dot{\xi}, H^1(I_0; e^{-V}))$ admits a quasi-continuous modification (with respect to Cap). Let us turn back to a general extended Dirichlet space $(\dot{\xi}, H^1(I_0; e^{-V}))$ on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ and consider its extended Dirichlet space $(\dot{\xi}, H^1_e(I_0))$. The proof is the same as in Theorem 2.1.3 of [4]

Proposition 1.4.9. *Any $u \in H^1_e(I_0; e^{-V})$, the extended domain of the Dirichlet form $\dot{\xi}$; admits a quasi-continuous modification $\widetilde{u}$.*

Proof. First we prove the theorem with $u \in H^1(I_0; e^{-V})$. We establish the inequality:

$$Cap\left(\{\dot{x} \in I_0 : |u(\dot{x})| > \lambda\}\right) \leq \frac{1}{\lambda^2}. \tag{1.4.7}$$

Since $G = \{\dot{x} \in I_0 : |u(\dot{x})| > \lambda\} \in \mathcal{O}$ and $|u|/\lambda \in \mathcal{L}_G$ for any $\lambda > 0$ and $u \in H^1(I_0; e^{-V})$, we have $Cap(G) \leq (1/\lambda^2)\dot{\xi}_1(|u|, |u|) \leq (1/\lambda^2)\dot{\xi}_1(u, u)$, proving (1.4.7).

In view of the regularity of $\dot{\xi}$, any $u \in H^1(I_0, e^{-V})$ can be approximated with respect to the $\dot{\xi}_1$-metric by some $u_n \in H^1(I_0, e^{-V}) \cap C(I_0)$. We may assume $\dot{\xi}_1(u_{n_{k+1}} - u_{n_k}) < 2^{-3k}$ by selecting a subsequence if necessary. Then $Cap(G_k) \leq 2^{-k}$ by (1.4.7), where $G_k = \left\{\dot{x} \in I_0 : |u_{k+1}(\dot{x}) - u_k(\dot{x})| > 2^{-k}\right\}$. Let $F_k = \bigcap_{l=k}^{\infty} G_l^c$. Cleary $\{F_k\}$ is a nesk and $|u_m(\dot{x}) - u_n(\dot{x})| \leq \sum_{v=N+1}^{\infty} |u_{v+1}(\dot{x}) - u_v(\dot{x})| \leq 2^{-N}$ for any $\dot{x} \in F_k$ and any

$n, m > N \geq k$. This means, for each k, $u_n|_{F_k}$ are uniformly convergent as $n \to \infty$. Let $\widetilde{u}(\dot{x}) = \lim_{n\to\infty} u_n(\dot{x})$, $\dot{x} \in \bigcup_{k=1}^{\infty} F_k$; then $\widetilde{u} \in C(\{F_k\})$ and $u = \widetilde{u}$ a.e.

□

The consequence of quasi-continuous functions is

Proposition 1.4.10. *Let f be a quasi-continuous function defined on I_0. Let $\left(\Omega, \dot{X}_t, \mathcal{F}_t, P_{\dot{x}}\right\}$ be a continuous and conservative Markov process on I_0. Then*

$$t \longrightarrow f(\dot{X}_t)$$

is continuous in $[0, +\infty[$ $P_{\dot{x}}-$almost surely, for almost all $\dot{x} \in I_0$.

1.4.2 Mesures of finite energy integrals

A Radon measure μ on I_0 is said to be of finite energy integral if

$$\int_{I_0} |v(\dot{x})| \, \mu(d\dot{x}) \leq C\sqrt{\dot{\xi}_1(v, v)} \qquad \forall v \in H^1(I_0; e^{-V}) \cap C(I_0), \tag{1.4.8}$$

for some positive constante C. A positive Radon measure μ on I_0 is of finite energy integral if and only if there exists for each $\alpha > 0$ a unique function $U_\alpha \mu \in H^1(I_0; e^{-V})$ such that

$$\xi_\alpha(U_\alpha \mu, v) = \int_{I_0} |v(\dot{x})| \, \mu(d\dot{x}), \qquad \forall v \in H^1(I_0; e^{-V}) \cap C(I_0). \tag{1.4.9}$$

We call $U_\alpha \mu$ an $\alpha-$**potential**. Let S_0 be a set of measures of finite energy integral, and denote by S_{00} the subset of S_0 such that

$$\|U_1 \mu\|_\infty < \infty.$$

Lemma 1.4.11. *Let μ be in S_0. Then μ charges no set of zero capacity.*

Proof. It suffices to prove for $\mu \in S_0$

$$\mu(A) \leq \sqrt{\dot{\xi}_1(U_1\mu, U_1\mu)}.\sqrt{Cap(A)}, \qquad \forall A \in \mathcal{O}_0. \tag{1.4.10}$$

Let $g_n = n(U_1\mu - n\dot{G}_{n+1}(U_1\mu))$. This follows from Lemma 2.2.2 of [4] $g_n.e^{-V}$ converges vaguely to μ and $\dot{G}_1 g_n$ converges $\dot{\xi}_1-$weakly to $U_1\mu$. We get then,

$$\begin{aligned}
\mu(A) &\leq \lim_{n\to\infty} \int_A g_n(\dot{x}) e^{-V(\dot{x})} d\dot{x} \\
&= \lim_{n\to\infty} \int_A e_A(\dot{x}) g_n(\dot{x}) e^{-V(\dot{x})} d\dot{x} \\
&\leq \lim_{n\to\infty} \int_{I_0} e_A(\dot{x}) g_n(\dot{x}) e^{-V(\dot{x})} d\dot{x} \\
&= \lim_{n\to\infty} \dot{\xi}_1(\dot{G}_1 g_n, e_A) = \dot{\xi}_1(U_1\mu, e_A) \\
&\leq \sqrt{\dot{\xi}_1(U_1\mu, U_1\mu)}\sqrt{\dot{\xi}_1(e_A, e_A)}
\end{aligned}$$

which equals the right hand side of (1.4.10) in view of (1.4.4). □

Definition 1.4.12. *A positive Radon measure μ is said* ***smooth*** *if:*
(S_1) *μ charges no set of zero capacity.*
(S_2) *There exists a sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets such that:*

$$\mu(F_n) < \infty \text{ and } \lim_{n\to\infty} Cap(I_0 - F_n)) = 0.$$

Such a sequence is called $nest$ associated with the smooth measure μ.
Since smooth measures contain all positive Radon measures charging no set of zero capacity, then it contains S_0 in view of lemma 1.4.11.
Let $\left\{\dot{P}_t, t > 0\right\}$ be the Markovian semigroup on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ associated with Dirichlet form $\dot{\xi}$. A function $u \in L^2(I_0; e^{-V(\dot{x})}d\dot{x})$ is called $\alpha-$**excessif** (with respect to $\left\{\dot{P}_t, t > 0\right\}$) if

$$u \geq 0, e^{\alpha t}\dot{P}_t u \leq u \text{ a.e}, \forall t > 0. \tag{1.4.11}$$

Proposition 1.4.13. *For all $u \in H^1(I_0; e^{-V})$ the following conditions are equivalent to each other. (i) u is a $\alpha-$**potential**. (ii) u is a $\alpha-$**excessif**.*

Proof. The see this equivalence, refer to Theorem 2.2.1 and Lemma 2.2.1 of [4]. □

Let A be a subset of $\mathcal{O}$, let σ_A be

$$\sigma_A = \inf\left\{t > 0 : \dot{X}_t \in A\right\}.$$

σ_A is obviously a stopping time with respect to $\mathcal{F}$. The link beetwen the capacity theory and the Markov process is: if we defined $e_A(\dot{x}) := E_{\dot{x}}[e^{-\sigma_A}]\ a.e$, we have

$$\dot{\xi}_1(e_A, e_A) = Cap(A). \tag{1.4.12}$$

Remark 1.4.14. ***α-capacity***

We study reduced functions of potentials. Fixing $\alpha > 0$, we consider an $\alpha-$potential $f \in H^1(I_0; e^{-V})$ and an arbitrary set $B \subset I_0$. We put

$$\mathcal{L}_{f,B} = \left\{w \in H^1(I_0; e^{-V}) : \widetilde{w} \geq \widetilde{f} \text{ q.e on } B\right\}. \tag{1.4.13}$$

In the same way as in the proof of Proposition 1.4.1, we see that $\mathcal{L}_{f,B}$ admits a unique element, say $f_B \in \mathcal{L}_{f,B}$, minimizing $\dot{\xi}_\alpha(w, w)$ on $\mathcal{L}_{f,B}$. The function f_B is called the $\alpha-$reduced function of f on B. Cleary by Remark 1.4.2 (ii),

$$\dot{\xi}_\alpha(f_B, v) \geq 0 \qquad \forall v \in H^1(I_0; e^{-V}), \widetilde{v} \geq 0 \text{ q.e on } B, \tag{1.4.14}$$

where $\widetilde{v}$ is the quasi-continuous modification of v.
We observe the following assertion: The $\alpha-$reduced function f_B on B of an $\alpha-$potential $f \in H^1(I_0; e^{-V})$ is an unique element satisfying (1.4.14).
If we let $e_A^\alpha(\dot{x}) := E_{\dot{x}}(e^{-\alpha\sigma_A})$, then (1.4.12) can be rewritten by

$$\dot{\xi}_\alpha(e_A^\alpha, e_A^\alpha) = \inf_{w \in \mathcal{L}_A} \dot{\xi}_\alpha(w, w) \tag{1.4.15}$$

as in (1.4.2).

1.5 Stochastic analysis by additive functionals

We assume troughout this part that we are given a symmetric continuous Markov process on the unit torus of $\mathbb{R}^d$, $d \geq 2$, denoted by I_0. The continuous Markov Process is denoted by $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ defined for almost all starting point $\dot{x} \in I_0$ and whose Dirichlet form $(\dot{\xi}, H^1(I_0; e^{-V}))$ is regular and local. And by vertue of Theorem 1.4.9, every function u in the extended domain denoted here by $H_e^1(I_0; e^{-V})$ has a quasi-continuous modification. Throughout this section every function in the space $H_e^1(I_0; e^{-V})$ is considered to be quasi-continuous already.
We will talk about additive functionals, energy of continuous additive functional and in particular those that are zero energy. We will see that the AF $u(\dot{X}_t) - u(\dot{X}_0)$ generated by $u \in H_e^1(I_0; e^{-V})$ admits a unique decomposition

$$u(\dot{X}_t) - u(\dot{X}_0) = M_t + N_t$$

where M_t is Martingale additive functonal with finite energy and N_t a continuous additive functional (CAF) of zero energy. This decomposition will be more discussed in Part (1.5.2) of this section.

1.5.1 PCAF and smooth measures

Let $(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}})$ be a stochastic process on I_0, Let a subset $T \subset [0; \infty]$. We define $\mathcal{F}_\infty^0 = \sigma\left\{\dot{X}_s, s \in T, s < \infty\right\}$. For all $\Lambda \in \mathcal{F}_\infty^0$ and μ a finite measure on I_0 we define

$$P_\mu(\Lambda) = \int_{I_0} P_{\dot{x}}(\Lambda) d\mu(\dot{x})$$

a probability measure P_μ on $(\Omega, \mathcal{F}_\infty^0)$. We denote by $\mathcal{F}_\infty^\mu$ the completion of $\mathcal{F}_\infty^0$ with respect to P_μ. We also set $\mathcal{F}_\infty = \bigcap_{\mu:\mu(I_0)<\infty} \mathcal{F}_\infty^\mu$.

Definition 1.5.1.

An extended real valued function $A_t(\omega), t \geq 0, \omega \in \Omega$ is called and additive functional (AF in abreviation) if it satisfies the following conditions:

$(A.1)$ A_t is $\mathcal{F}_t$-measurable for each $t \geq 0$.

$(A.2$ There exists $\Lambda \in \mathcal{F}_\infty$ with $P_{\dot{x}}(\Lambda) = 1$, for almost all $\dot{x} \in I_0, \theta_t\Lambda \subset \Lambda, \forall t > 0$, and, for each $\omega \in \Lambda$, $A_.(\omega)$ is right continuous and has the left limit on $[0, \infty]$, $A_0(\omega) = 0, |A_t(\omega)| < \infty, \forall t < \infty$ and

$$A_t(\omega) = A_s(\omega) + A_t(\theta_s\omega), \forall t, s \geq 0.$$

The set Λ in the above is called a defining set for A. An additive functional is said to be finite (respec. continuous) if $|A_t(\omega)| < \infty, \forall t \in [0, \infty]$ (respec. $A_t(\omega)$ is continuous in $t \in [0, , \infty]$). A $[0, \infty]$-valued continuous is called a positive continuous additive funtional (PCAF in abreviation). We note that a CAF is not necessarily defined for all starting point of the process but for almost all starting point. In case where it is defined for all starting point, we say an AF in strict sense. But this case doesn't interest us here.

Example 1.5.2. *Let f a quasi-continuous function in the domain of the regular and local Dirichlet form $\dot{\xi}$. Let $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ a continuous Markov process associated. Then $f(\dot{X}_t) - f(\dot{X}_0)$ is CAF.*

Example 1.5.3. *Let $f \in L^1(I_0; e^{-V(\dot{x})}d\dot{x})$ and $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$ a continuous Markov process. Then $A_t = \int_0^t f(\dot{X}_s)ds$ is CAF;*

Theorem 1.5.4. *For any $\mu \in S_0$, there exists a finite PCAF A such that $E_{\dot{x}}(\int_0^\infty e^{-t}dA_t)$ is a quasi-continuous version of $U_1\mu$. And more this association is unique.*

Proof. This proof of the existence can be found in [4] and coincide with Theorem 5.1.1 of [4], the uniqueness is the Theorem 5.1.2 of [4]. □

This association can be enlarged on S_{00} also. The Theorem 1.5.4 allows us to caracterise any PCAF by a measure of finite energy integral. A version of this Theorem and which is more general is, for all $\mu \in S_0$, for all bounded measurable function, $E_{\dot{x}}(\int_0^\infty e^{-\alpha t}f(\dot{X}_t)dA_t)$ is quasi-continuous version of $U_\alpha(f.\mu)$. The following Theorem correspond to Theorem 0.8 of [8]
Before formulating the main assertion of this section, we need the definition of the smooth measure given above in definition 1.4.12

Proposition 1.5.5. *Let us denote by $dm(\dot{x}) = e^{-V(\dot{x})}d\dot{x}$ et let h be a $\alpha-$excessif function for all $\alpha > 0$. Then, all PCAF A can be associated with a smooth measures μ by the relation*

$$\lim_{t \to 0} \frac{1}{t} E_{h.m}\left[\int_0^t f(\dot{X}_s)dA_s\right] = \int_{I_0} h(\dot{x})f(\dot{x})d\mu(\dot{x}), \tag{1.5.1}$$

for all positive measurable bounded function f. This association is unique.

In this case, we call μ the Revuz measure of PCAF A and the association is called in the following the Revuz association beetwen a PCAF A and the smooth measure μ. Lemma 5.1.7 and Lemma 5.1.8 of [4] say that any A PCAF admits its Revuz measure μ a smooth measure and any smooth measure μ admits an A PCAF whose Revuz measure is μ.
As a consequance of this Proposition, we have: for all $f \in L^1(I_0; e^{-V(\dot{x})}d\dot{x})$ the Revuz measure associated to PCAF $A_t = \int_0^t f(\dot{X}_s)ds$ defined in Example 1.5.3, is $f(\dot{x})e^{-V(\dot{x})}d\dot{x}$ by using simply Theorem 1.5.5.

1.5.1.1 Energy of CAF

Let A be a CAF. Then we let

$$e(A) = \lim_{t\to 0} \frac{1}{2t} E_m[A_t^2], \tag{1.5.2}$$

when this limit exists. This value $e(A)$ is called energy of A. First of all we shall exibit three important classes of $AF's$ of finite energy.

(I) AF generated by functions
Suppose that a function u on I_0 posses a version $\widetilde{u}$ $(u = \widetilde{u})$ a.e such that $\widetilde{u}$ is quasi-continuous. Then

$$\widetilde{u}(\dot{X}_t) - \widetilde{u}(\dot{X}_0), \qquad t > 0 \tag{1.5.3}$$

defines a finite AF in our sense. Thus, $\widetilde{u}(\dot{X}_t) - \widetilde{u}(\dot{X}_0)$ is well defined whenever $u \in H_e^1(I_0; e^{-V})$ because we can take $\widetilde{u} \in H_e^1(I_0; e^{-V})$ see Theorem 1.4.9. Moreover, $A^{[u]} := \widetilde{u}(\dot{X}_t) - \widetilde{u}(\dot{X}_0)$ has finite energy and

$$e(A^{[u]}) = \dot{\xi}(u, u). \tag{1.5.4}$$

(II) Martingale AF's of finite energy
Consider the family

$$\mathcal{M} = \left\{ M : {\text{is a finite CAF such that for each } t>0 \atop E_x(M_t^2)<\infty \text{ and } E_{\dot{x}}(M_t)=0 \text{ a.e } \dot{x}\in I_0} \right\}$$

Since $E_m(M_t^2)$ is subadditive in t, $e(M)$ is well defined and

$$e(M) = \sup_{t>0} \frac{1}{2t} E_m(M_t^2) (\leq \infty) \tag{1.5.5}$$

for any $M \in \mathcal{M}$. We set

$$\tilde{\mathcal{M}} = \{M \in \mathcal{M} : e(M) < \infty\} \tag{1.5.6}$$

The process $M \in \tilde{\mathcal{M}}$ is a square integrable AF with mean zero. In other word, M_t is square integrable $martingale$ additive functional. We conclude from the following Theorem A.3.18 of [4] that there exists a unique quadratic variation denoted by $< M >$ which is also a PCAF and verifies

$$E_{\dot{x}}(M_t^2) = E_{\dot{x}}(< M >_t) \text{ a.e } \dot{x} \in I_0, t > 0. \tag{1.5.7}$$

We call $M \in \mathcal{M}$ a Martingale additive functional and $< M >$ a quadratic variation.
Let $\mu_{<M>}$ the Revuz measure assiciated to $< M >$ called also the energy measure of the Martingale additive functional M. We have from (1.5.1), (1.5.2) and (1.5.7),

$$\mu_{<M>}(I_0) = 2e(M), \qquad M \in \mathcal{M} \tag{1.5.8}$$

We extend the definition of energy for two CAF M and N by:

$$e(M, N) = \lim_{t \to 0} \frac{1}{2t} E_m(M_t N_t).$$

We note this following theorem

Theorem 1.5.6. *The space $\tilde{\mathcal{M}}$ is a Hilbert space with respect to e.*

(III) CAF's of zero energy

$$\mathcal{N} = \left\{ N : \substack{\text{is finite } CAF \text{ such that for each } t>0 \\ E_{\dot{x}}(|N_t|)<\infty \text{ a.e and } e(N)=0.} \right\}$$

The quadratic variation of $N \in \mathcal{N}$ vanishes in the following sense:

$$\sum_{k=1}^{[nT]} (N_{(k+1)/n} - N_{k/n})^2 \longrightarrow 0, n \to \infty, \text{ in } L^1(P_m) \tag{1.5.9}$$

Because the expectation of the left hand side equals

$$\sum_{k=1}^{[nT]} E_m(E_{X_{k/n}}(N_{1/n})^2) \leq nT.E_m(N_{k/n})^2 \longrightarrow 0, n \to \infty.$$

Example 1.5.7. *An example of $N \in \mathcal{N}$ is given by*

$$N_t = \int_0^t f(\dot{X}_s) ds \tag{1.5.10}$$

for a nearly Borel function $f \in L^2(I_0; e^{-V(\dot{x})} d\dot{x})$.

Indeed, N is PCAF by Example 1.5.3. Besides, $E_{\dot{x}} |N_t| \leq E_{\dot{x}}(\int_0^t \left| f(\dot{X}_s) \right| ds) \leq e^t G_1 |f| (\dot{x}) < \infty$ a.e. Furthermore, if we recall that (f, g) means the integral with respect to the measure m defined in Theorem 1.5.5, we have

$$
\begin{aligned}
E_m(N_t^2) &= 2E_m(\int_0^t f(\dot{X}_s)\int_s^t f(\dot{X}_v)dvds) \\
&= 2E_m(\int_0^t\int_0^{t-s} f(\dot{X}_s)p_v f(\dot{X}_s)dvds) = 2\int_0^t\int_0^{t-s}(p_s 1, f.p_v f)dvds.
\end{aligned}
$$

Hence

$$
\frac{1}{2t}E_m(N_t^2) = \frac{1}{t}\int_0^t (p_{t-s}1, f.S_s f)fs, \qquad S_s f(\dot{x}) = \int_0^s p_v f(\dot{x})dv. \tag{1.5.11}
$$

Since the right hande side of (1.5.11) is dominated by $\frac{1}{s}\int_0^t sds.(f,f)$, N is of zero energy. $N \in \mathcal{N}$.

Example 1.5.8. *A CAF with a Revuz measure $\mu \in S_0$ is zero energy.*

Indeed, since $\mu \in S_0$, then $E_{\dot{x}}(A_t) \leq e^t\widetilde{U_1\mu}(\dot{x}) < \infty$ a.e and

$$
\begin{aligned}
E_m(A_t^2) &= 2E_m(\int_0^t (A_t - A_s)dA_s) \\
&= 2\lim_{n\to\infty} E_m\left(\sum_{k=1}^n (A_t - A_{kt/n})(A_{kt/n} - A_{(k-1)t/n})\right) \\
&= 2\lim_{n\to\infty} E_m\left(\sum_{k=1}^n (E_{X_{kt/n}}(A_{t-kt/n}))(A_{kt/n} - A_{(k-1)t/n})\right) \\
&\leq 2\lim_{n\to\infty} E_m\left(\sum_{k=1}^n (E_{X_{kt/n}}(A_t))(A_{kt/n} - A_{(k-1)t/n})\right) \\
&= 2E_m\left(\int_0^t E_{X_s}(A_t)dA_s\right) \leq 2t\int_{I_0} E_x(A_t)d\mu(\dot{x})
\end{aligned}
$$

by the fact that for all $\alpha-$excessif function h ($\alpha > 0$) and for all positive measurable bounded function f, if we denote by $(fA)_t = \int_0^t f(\dot{X}_s)dA_s$; we have

$$
E_{h.m}\left((fA)_t\right) = \int_0^t(\int_{I_0} f(\dot{x})p_s h(\dot{x})d\mu(\dot{x}))ds. \text{Theorem 5.1.3 of [4]},
$$

and further

$$
\int_{I_0} E_{\dot{x}}(A_t)d\mu(\dot{x}) \leq e^{-t}\int_{I_0}(U_A^1 1(\dot{x}) - e^{-t}p_t U_A^1 1(\dot{x}))d\mu(\dot{x}) \to 0, t \to 0,
$$

where $U_A^\alpha f(\dot{x}) = E_{\dot{x}}\left(\int_0^\infty e^{-\alpha t}f(\dot{X}_t)dA_t\right)$ for all measurable bounded function f.

We are particularly interested in the sum of classes (II) and (III):

$$\mathcal{A} = \tilde{\mathcal{M}} \oplus \mathcal{N} \tag{1.5.12}$$

namely, $\mathcal{A}$ conists of AF's A such that

$$A_t = M_t + N_t, \qquad M \in \tilde{\mathcal{M}}, N \in \mathcal{N} \tag{1.5.13}$$

Clearly $\mathcal{A}$ is a linear space of AF's of finite energy. Morever, the expression (1.5.13) of $A \in \mathcal{A}$ is unique because $\tilde{\mathcal{M}} \cap \mathcal{N} = \{0\}$ where 0 is the additive functional identically zero. In fact if $A \in \tilde{\mathcal{M}}$ is of zero energy, the Revuz mesure associated with $< A >$ vanishes by (1.5.8) and so does $< A >$. Hence $A = 0$ by (1.5.7). The following theorem is proved in Theorem 5.2.2 of [4] and constitutes an important result of Dirichlet forms theory.

Theorem 1.5.9. *For all $u \in H_e^1(I_0; e^{-V})$, the AF $\widetilde{u}(\dot{X}_t) - \widetilde{u}(\dot{X}_0)$ denoted by $A_t^{[u]}$ can be expressed uniquely as:*

$$A_t^{[u]} = M_t^{[u]} + N_t^{[u]} \textit{ where } M^{[u]} \in \tilde{\mathcal{M}}, N^{[u]} \in \mathcal{N}. \tag{1.5.14}$$

As consequence of this Theorem, we have: for all $u \in H_e^1(I_0; e^{-V})$, since by (1.5.4) $e(A^{[u]}) = \dot{\xi}(u,u)$, and by (1.5.8) and (1.5.14) $e(A^{[u]}) = e(M^{[u]}) = (1/2)\mu_{<M^{[u]}>}(I_0)$; we get:

$$(1/2)\mu_{<M^{[u]}>}(I_0) = \dot{\xi}(u,u) \text{ for all } u \in H_e^1(I_0; e^{-V}). \tag{1.5.15}$$

1.5.2 Martingale additive functionals

In this subsection we give some discussions in the decomposition called Ito-Fukushima decomposition.

$$\widetilde{u}(\dot{X}_t) - \widetilde{u}(\dot{X}_0) = M_t^{[u]} + N_t^{[u]}, M_t^{[u]} \in \tilde{\mathcal{M}}, N_t^{[u]} \in \mathcal{N}, u \in H^1(I_0; e^{-V}), \tag{1.5.16}$$

in (1.5.13). This decomposition may be regarded as a generalization of a Doob-Meyer decomposition of supermatingales and Ito's formula of semimartingales. The question in this subsection is: in what conditions $N_t^{[u]} = 0 \; a.e$? The answer is the following theorem.

Definition 1.5.10. *A function $u \in H^1(I_0; e^{-V})$ is said to be $\dot{\xi}-$ harmonic if*

$$\dot{\xi}(u,v) = 0 \qquad \forall v \in \mathcal{C}, \textit{ where } \mathcal{C} \textit{ is some core of } \dot{\xi}. \tag{1.5.17}$$

Theorem 1.5.11. *Let $u \in H^1(I_0; e^{-V})$ be a $\dot{\xi}-$ harmonic. Then*

$$P_{\dot{x}}(N_t^{[u]} = 0) = 1 \textit{ for almost all } \dot{x} \in I_0. \tag{1.5.18}$$

Proof. Let us consider the following definitions:
Spectrum
Let $(\dot{\xi}, H^1(I_0; e^{-V})$ be a regular and local Dirichlet form and let $u \in H^1(I_0; e^{-V})$. A subset $F \subseteq I_0$ is called a spectrum of u denoted by $\sigma(u)$ if it is the complement of the largest set G such that $\dot{\xi}(u, v)$ vanishes for any $v \in H^1(I_0; e^{-V})$ with $support[v] \subset G$.
Thorem 5.4.1 of [4]
For any $u \in H^1(I_0; e^{-V})$, the AF $N_t^{[u]}$ vanishes on the complement of the spectrum $F = \sigma(u)$ of u in the following sense

$$P_{\dot{x}}(N_t^{[u]} = 0, t < \sigma_F) = 1 \text{ q.e } \dot{x} \in I_0, \tag{1.5.19}$$

where $\sigma_F = \inf \left\{ t > 0 : \dot{X}_t \in F \right\}$ and if $F = \emptyset$ then $\sigma_F = +\infty$.

By Theorem 5.4.1 of [4] to deduce Theorem 1.5.11, it suffices to show that for all $u \in H^1(I_0; e^{-V})$, u $\dot{\xi}-$ harmonic, then $F = \sigma(u)$ is empty. And we get Theorem 1.5.11. It is not difficult because if $\dot{\xi}(u, v) = 0$ for any $v \in H^1(I_0; e^{-V})$ then $G = I_0$ and $F = \sigma(u) = \emptyset$ □

Consequence: now, we can say more about the formulae (1.5.16). In fact for all $u \in H^1(I_0; e^{-V})$, u $\dot{\xi}-$ harmonic, $\widetilde{u}(\dot{X}_t) - \widetilde{u}(\dot{X}_0) = M_t^{[u]}$, where $M_t^{[u]}$ is a Martingale additive functional, square integrable and of finite energy.

1.6 Construction of time-changed process by additive functional

Let $(\dot{\xi}, H^1(I_0; e^{-V}))$ the regular and local Dirichlet form associated with the divergence-form operator $\dot{L}$, let $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}\right)$ be the $e^{-V(\dot{x})}d\dot{x}$ symmetric continuous Markov process associated and defined for almost all $\dot{x} \in I_0$. Let $H_e^1(I_0; e^{-V})$ be the extended domaine of $\dot{\xi}$. Throughout this section we assume that all functions in $H_e^1(I_0; e^{-V})$ are quasi-continuous.

We fix an arbitrary measure μ on I_0 charging no set of zero capacity and let w be a strictly positive and integrable function such that $d\mu(\dot{x}) = w(\dot{x})d\dot{x}$. Let A be the PCAF associated with μ by Proposition 1.5.5. We assume that μ is positive then support of μ which is the smallest closed set outside of which μ vanishes, is I_0. The support of A is defined by:
let

$$R(\omega) = \inf \{t > 0 : A_t(\omega) > 0\} \text{ and}$$
$$supp[A] = \{\dot{x} \in I_0 : P_{\dot{x}}(R(\omega) = 0) = 1\}.$$

If $A_t(\omega)$ is positive almost surely, then by denoting by N the subset of I_0 such that for all $\dot{x} \in N$ the Markovian law $P_{\dot{x}}$ is well defined, N is also the support of A. By consequence

the complementary of the support of A in the support of μ is negligible with respect to Lebesgue measure. Since w is positive, the complementary of the support of A in the support of μ is also negligible with respect to $d\mu$.
We define the hitting function:

$$H_N u(\dot{x}) = E_{\dot{x}}(u(\dot{X}_{\sigma_N}))\dot{x} \in N, \tag{1.6.1}$$

where $\sigma_N = \inf\left\{t > 0 : \dot{X}_t \in N\right\}$.
In fact, $\sigma_N = 0$ a.e. Thus $H_N u(\dot{x}) = u(\dot{x})$ a.e. The space:

$$\mathcal{H}_N = \left\{H_N u : u \in H^1_e(I_0; e^{-V})\right\}. \tag{1.6.2}$$

which is the space of projections, coincides with $H^1_e(I_0; e^{-V})$. We define a symmetric form $(\widetilde{\dot{\xi}}, \widetilde{H}^1(I_0; w))$ on $L^2(I_0; w)$ by
(1.6.3)

$$\begin{cases} \widetilde{H}^1(I_0; w) = \left\{\phi \in L^2(I_0; \mu) : \phi = u \ - \text{a.e on } I_0 \text{ for some } u \in H^1_e(I_0; e^{-V})\right\} \\ \widetilde{\dot{\xi}}(\phi, \phi) = \dot{\xi}(u, u)\ \phi \in H^1(I_0; w), \phi = u \ \text{-a.e on } I_0, u \in H^1_e(I_0; e^{-V}) \end{cases}$$

We may call $(\widetilde{\dot{\xi}}, \widetilde{H}^1(I_0; w))$ the trace of $(\dot{\xi}, H^1(I_0; e^V))$ on I_0 relative to μ. Let us also define the time changed process $\left(\Omega, \widetilde{\mathcal{F}}, \widetilde{\dot{X}}_t, P_{\dot{x}}, \dot{x} \in N\right)$ with respect to the PACF A by

$$\widetilde{\dot{X}}_t = \dot{X}_{A_t^{-1}}, \qquad A_t^{-1}(\omega) = \inf\left\{s > 0; A_s(\omega) > t\right\} \tag{1.6.4}$$

Remark 1.6.1. *$A_t^{-1}(\omega)$ is $\mathcal{F}$ stopping time for all $t \geq 0$.*

Indeed, $\left\{A_t^{-1}(\omega) < s\right\} = \cup_{n \geq 1}\left\{A_{s-1/n}(\omega) > t\right\} \in \mathcal{F}_s$

Proposition 1.6.2. *$\left(\Omega, \widetilde{\mathcal{F}}, \widetilde{\dot{X}}_t, P_{\dot{x}}, \dot{x} \in N\right)$ is continuous Markov process on I_0 called time changes process, where $\widetilde{\mathcal{F}} = (\widetilde{\mathcal{F}}_t)_{t>0}$, $\widetilde{\mathcal{F}}_t = \mathcal{F}_{A_t^{-1}}$. $\widetilde{\mathcal{F}}$ is continuous and strictly increassing family of $\sigma-$field. The Markovian transition function and the Markovian resolvent kernel are:*

$$\widetilde{\dot{P}}_t \phi(\dot{x}) = E_{\dot{x}}\left(\phi(\dot{X}_{A_t^{-1}})\right), \dot{x} \in N, \tag{1.6.5}$$

$$\widetilde{\dot{R}}_\alpha \phi(\dot{x}) = E_{\dot{x}}\left(\int_0^\infty e^{-\alpha t}\phi(\dot{X}_{A_t^{-1}})\right) = E_{\dot{x}}\left(\int_0^\infty e^{-\alpha A_t}\phi(\dot{X}_t)dA_t\right). \tag{1.6.6}$$

In view of Theorem A 2.12 of [4].

Thus if we denote by $U^0_{\alpha,A}\phi = \widetilde{\dot{R}}_\alpha \phi$ on N, since $\mu(I_0 - N) = 0$ and $U^0_{\alpha,A}$ is $\mu-$symmetric; by Corollary 6.1.1 of [4], $\widetilde{\dot{R}}_\alpha$ and $\widetilde{\dot{P}}_t$ are $\mu-$symmetric. Furthermore $\lim_{t\downarrow 0} \widetilde{\dot{P}}_t \phi(\dot{x}) = \phi(\dot{x}), \forall \dot{x} \in N, \forall \phi \in C(I_0)$. On account of section 1.3, Definition 1.3.1; $\widetilde{\dot{p}}_t$ gives a strongly continuous semigroup denoted by $\widetilde{\dot{P}}_t$ on $L^2(I_0, \mu)$. The main theorem of this section is the following.

Theorem 1.6.3. *Let $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}\right)$ be a symmetric Markov process associated with a regular Dirichlet space $(\dot{\xi}, H^1(I_0; e^{-V}))$ relative to $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$.*
(i) The Dirichlet space on $L^2(I_0; \mu)$ associated with $(\widetilde{\dot{P}}_t)_{t>0}$ coincides with $(\widetilde{\dot{\xi}}, \widetilde{H}^1(I_0; w))$ of (1.6.3).
(ii) $(\widetilde{\dot{\xi}}, \widetilde{H}^1(I_0; w))$ is regular. Furthermore, if $\mathcal{C}$ is a core of $(\dot{\xi}, H^1(I_0; e^{-V}))$ then $\mathcal{C}$ is also a core of $(\widetilde{\dot{\xi}}, \widetilde{H}^1(I_0; w))$

The following corollary prove that the extended domain of the process $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}\right)$ denoted by $H^1_e(I_0; e^{-V(\dot{x})}d\dot{x})$ contains the extended domain of $\left(\Omega, \mathcal{F}, \widetilde{\dot{X}}_t, P_{\dot{x}}\right)$ denoted by $H^1_e(I_0; w)$.

Corollary 1.6.4. $H^1_e(I_0; e^{-V}) = H^1_e(I_0; w)$.

Proof. The implication $\widetilde{H}^1(I_0; w) \subset H^1_e(I_0; e^{-V})$ is obvious by definition of $\widetilde{H}^1(I_0; w)$.

And then $H^1_e(I_0; w) \subseteq H^1_e(I_0; e^{-V})$. Indeed, Let $\varphi \in H^1_e(I_0; w)$ then there exists a sequence $(u_n) \subseteq \widetilde{H}(I_0; w) \subseteq H^1_e(I_0; e^{-V})$ such that

$$\begin{cases} (u_n) \text{ is-}\dot{\xi}\text{-Cauchy} \\ u_n \to \varphi \text{ a.e.} \end{cases}$$

In the other hand, for all n there exists a subsequence $\left(v_n^k\right)_{k\geq 0} \subseteq H^1(I_0, e^{-V})$ such that $v_n^k \to v_n$ when $k \to \infty$. The almost sure limit of $\left(v_n^k\right)$ when $k \to \infty, n \to \infty$ is φ. Recall that $\left(v_n^k\right)_{k\geq 0}$ is also $\dot{\xi}$-Cauchy then $\varphi \in H^1_e(I_0, e^{-V})$.
To prove that $H^1_e(I_0; e^{-V}) \subseteq H^1_e(I_0; w)$, it suffices to recall that as we have consctruct $(\widetilde{\dot{\xi}}, \widetilde{H}^1(I_0; w))$ from $(\dot{\xi}, H^1(I_0; e^{-V})$ to deduce that $H^1_e(I_0; w) \subseteq H^1_e(I_0; e^{-V})$, we can similarly do the same consruction to deduce that $H^1_e(I_0; e^{-V}) \subseteq H^1_e(I_0; w)$.

□

The following Lemma is very important and will be used in chapter 3, in the proof of convergence of corrector.

Lemma 1.6.5. *Let w be a strictly positive and integrable function on I_0 and let μ be, the measure defined by $d\mu(\dot{x}) = w(\dot{x})d\dot{x}$. Recall that $P_w(.) := \int_{I_0} w(\dot{x})P_{\dot{x}}(.)d\dot{x}$. Then we have, for any $\eta > 0$ and any f in the extended domain of $\dot{\xi}$,*

$$\limsup_{\epsilon\downarrow 0} P_w(\sup_{0\leq t\leq \epsilon^{-2}} \left|\epsilon f(\widetilde{\dot{X}}_t)\right| > \eta) \leq \frac{e^1\sqrt{\dot{\xi}(f,f)}}{\eta},$$

where $\dot{\xi}$ is the Dirichlet form associated to the process $\dot{X}$.

Proof. We refer to first part of chapter 5 of [4].
For a nearly Borel set A in I_0, let $\sigma_A = \inf\left\{t > 0 : \dot{X}_t \in A\right\}$ and $p^\epsilon_A(\dot{x}) = E_{\dot{x}}(e^{-\epsilon^2\sigma_A})$.
Let

$$\mathcal{L}_A = \left\{u \in H^1_e(I_0) : u \geq 1 \text{ q.e on A}\right\}.$$

(q.e. means 'quasi everywhere'.) By (1.4.15) of Part 1.4.2 of chapter 1, $p^\epsilon_A(x)$ is the unique element of $\mathcal{L}_A$ minimizing $\dot{\xi}_{\epsilon^2}(u,u)$ on $\mathcal{L}_A$.

Since $\mu \in S_0$, the set defined in Section 1.4.2, we let $U_{\epsilon^2}\mu$ the unique $\epsilon-$potential associated to μ. It satisfies:

$$\begin{aligned}\int_{I_0} E_{\dot{x}}(e^{-\epsilon^2\sigma_A})d\mu(\dot{x})d\dot{x} &= \dot{\xi}_{\epsilon^2}(p^\epsilon_A, U_\epsilon\mu)\\ &\leq \sqrt{\dot{\xi}_{\epsilon^2}(U_{\epsilon^2}\mu, U_{\epsilon^2}\mu)}\sqrt{\dot{\xi}_{\epsilon^2}(p^\epsilon_A, p^\epsilon_A)}\end{aligned} \tag{1.6.7}$$

by Cauchy-Schwarz inequality.

Apply this inequality to $A = \{\dot{x} \in I_0 : |f(\dot{x})| > \eta\}$. We note that since $f \in H^1_e(I_0)$, then $\frac{f}{\eta} \geq 1$ q.e. on A.
Thus, $\frac{f}{\eta} \in \mathcal{L}_A$ and we obtain that

$$\dot{\xi}_{\epsilon^2}(p^\epsilon_A, p^\epsilon_A) \leq \eta^{-2}\dot{\xi}_{\epsilon^2}(f,f).$$

Moreover, we can write:

$$\begin{aligned}P_w(\sup_{0\leq t\leq \epsilon^{-2}} \left|f(\widetilde{\dot{X}}_t)\right| > \eta) &\leq P_w(\frac{1}{\epsilon^2} \geq \sigma_A)\\ &= \int_{I_0} P_{\dot{x}}(\frac{1}{\epsilon^2} \geq \sigma_A)w(d\dot{x})\\ &= \int_{I_0} P_{\dot{x}}(e^{1-\epsilon^2\sigma_A} \geq 1)w(d\dot{x})\\ &\leq e^1 \int_{I_0} E_{\dot{x}}(e^{-\epsilon^2\sigma_A})w(d\dot{x})\end{aligned}$$

We deduce from inequality (1.6.7) above that:

$$P_w(\sup_{0\leq t\leq \epsilon^{-2}} \left|f(\widetilde{\dot{X}}_t)\right| > \eta) \leq \frac{e^1}{\eta}\sqrt{\dot{\xi}_{\epsilon^2}(U_\epsilon\mu, U_\epsilon\mu)}\sqrt{\dot{\xi}_{\epsilon^2}(f,f)}.$$

We obviously have $\dot{\xi}_{\epsilon^2}(U_{\epsilon^2}\mu, U_{\epsilon^2}\mu) = \int_{I_0} U_{\epsilon^2}\mu(\dot{x})d\mu(\dot{x}) < \infty$ because $U_{\epsilon^2}\mu$ is continuous, taking its quasi-continuous version. And therefore

$$P_w(\sup_{0\leq t\leq \epsilon^{-2}} \left|f(\widetilde{\dot{X}}_t)\right| > \eta) \leq \frac{e^1}{\eta}\frac{1}{\epsilon}\sqrt{\dot{\xi}_{\epsilon^2}(f,f)}.$$

Replacing η by $\frac{\eta}{\epsilon}$, we obtain

$$P_w(\sup_{0\leq t\leq \epsilon^{-2}} \left|\epsilon f(\widetilde{\dot{X}}_t)\right| > \eta) \leq \frac{e^1}{\eta}\sqrt{\dot{\xi}_{\epsilon^2}(f,f)} = \frac{e^1}{\eta}\sqrt{\dot{\xi}(f,f) + \epsilon^2\left\|f\right\|^2_{L^2(I_0;w)}},$$

and Lemma 1.6.5 is proved letting ϵ tend to 0. □

1.7 Conclusion

In this chapter by only the hypothesis: V measurable, e^V and e^{-V} integrable on I_0 with respect to Lebesgue measure, there exists a continuous and conservative Markov process on I_0 $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$, defined for almost all starting point $\dot{x} \in I_0$, associated to the divergence-form operator $\dot{L} = e^{V(\dot{x})} div(e^{-V(\dot{x})}\nabla)$ defined on $L^2(I_0; e^{-V(\dot{x})}d\dot{x})$. The Chapter 2, which is independant of this one; prove one inequality of type-Sobolev, with different weights and allows us to prove in chapter 3 that the rescalled process $\left(X^{\epsilon}_{\cdot} = \epsilon X_{\cdot/\epsilon^2}\right)$ satisfies a Invariance principle, where $\left(\Omega, \mathcal{F}, X_t, P_x, x \in \mathbb{R}^d\right)$ is the lifting process on $\mathbb{R}^d$ of $\left(\Omega, \mathcal{F}, \dot{X}_t, P_{\dot{x}}, \dot{x} \in I_0\right)$.

Chapter 2

A Sobolev inequality

2.1 Introduction

This chapter deals essentially with real-variable methods in harmonic analysis, developed by Alberto Torschinsky in [12]. However, it is very important to note that the theory given in this book is done largely on the space $\mathbb{R}^d$. Our first goal is to recall the results on the unit torus of $\mathbb{R}^d$ denoted in this thesis by I_0 and which is $\mathbb{R}^d/\mathbb{Z}^d$. The results that we recall here are Muckenhoupt's theory of A_p weights, the results of Coifman-Rochberg and Sawyer's two weight maximal theorem. Thus, we define the fractional integration of Riesz with some properties. We give the theorem of Welland which link the maximal function of Hardly-Littlewood and the fractional integral of Riesz. The goal in this chapter, is to prove one Sobolev inequality on the unit torus, with only the hypothesis $e^V + e^{-V} \in L^1(I_0; d\dot{x})$.
The chapter will be divided in two main sections. First, we develope the Muckenhoupt theory of A_p weights as principle keys of the proofs in this chapter. In the second section; we give some sufficient conditions to get a Sobolev inequality with different weights. A first new result will be proved at the end of this chapter. It is the following Theorem:

Theorem 2.1.1. *With the hypothesis $e^V + e^{-V} \in L^1(I_0; d\dot{x})$; there exists a function w strictly positive and integrable on I_0, there exists $r^* > 2$ and one constant c such that:*

$$\left(\int_{I_0} |f(\dot{x})|^{r^*} w(\dot{x}) d\dot{x} \right)^{\frac{2}{r^*}} \leq c \int_{I_0} |\nabla f(\dot{x})|^2 e^{-V(\dot{x})} d\dot{x}.$$

for all function f defined on I_0, centered and C^1. The function w will be explicitly given in the proof of the theorem.

We start recalling the results we shall need from from real harmonic analysis. We refer to the book of A. Torchinsky [12] where all the material below can be found.

We recall that I_0 is the unit torus $\mathbb{R}^d/\mathbb{Z}^d$; $d\dot{x}$ is the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of a measurable subset $I \subseteq I_0$.

2.2 Muckenhoupt's theory for maximal functions and weights

2.2.1 The Hardy-Littlewood maximal theorem for regular measures

Let μ be a non negative Borel measure in I_0, finite on bounded sets. We define for all $\dot{x} \in I_0$ and f in $L^1(I_0;\mu)$:

$$M_\mu(f)(\dot{x}) = \sup_{I \subseteq I_0 : \dot{x} \in I} \frac{1}{|\mu(I)|} \int_I |f(\dot{y})| \, d\mu(\dot{y}), \tag{2.2.1}$$

where the I's are open cubes containing $\dot{x}$. Observe that the function $M_\mu(f)$ is non-negative and measurable. M_μ is a sublinear operator.
Before discussing the properties of M_μ we recall some general definitions for sublinear operator.
Definition
We say that a sublinear operator T defined on $L^p(I_0; d\nu)$ is weak-type from $L^p(I_0;\nu)$ to $L^q(I_0, d\mu)$ if:

$$\lambda^q \mu \{Tf > \lambda\} \leq c \left(\|f\|_{L^p(I_0;d\nu)} \right)^q \text{ for some sonstant } c. \tag{2.2.2}$$

We say simply weak-type (p, q) for $p, q > 0$

We ask the question: is $f \longmapsto M_\mu f$ of weak-type $(1, 1)$ for the measure μ? In other words, we can go about answering this as follows: let $\mathcal{O}_\lambda = \{M_\mu f > \lambda\}$. M_μ is weak-type $(1, 1)$ means

$$\lambda \mu(\mathcal{O}_\lambda) \leq c \|f\|_{L^1(I_0;\mu)} . \tag{2.2.3}$$

Before stating the principal theorem of this part, we recall some definitions with respect to Borel measure.

2.2.2 Definition

Definition 1: let μ be a non negative Borel measure in I_0, finite on bounded sets. μ is said to be regular if for all $\mathcal{U}$, μ-measurable, then

$$\mu(\mathcal{U}) = \sup_{\mathcal{K} \subseteq \mathcal{U}, \mathcal{K} \text{ compact}} \mu(\mathcal{K}). \tag{2.2.4}$$

Definition 2: μ is said a doubling measure, if for all open cube I of I_0:

$$\mu(2I) \leq c\mu(I), \text{where c is independant of } I.$$

Theorem 2.2.1. *Let μ be a non negative Borel measure in I_0, finite on bounded sets which in addition is doubling and regular. Then the map $f \to M_\mu f$ is of weak-type (1, 1).*

Proof. We give the principal part of the proof of this theorem. Indeed, $\mathcal{O}_\lambda = \{M_\mu f > \lambda\}$ is open since to each $\dot{x} \in \mathcal{O}_\lambda$ there corresponds an open cube $I_{\dot{x}}$ containing $\dot{x}$ such that

$$\frac{1}{|\mu(I_{\dot{x}})|} \int_{I_{\dot{x}}} |f(\dot{y})| \, d\mu(\dot{y}) > \lambda \tag{2.2.5}$$

and consequently $I_{\dot{x}} \subseteq \mathcal{O}_\lambda$. In fact,

$$\{M_\mu f > \lambda\} = \mathcal{O}_\lambda = \bigcup_{\dot{x} \in \mathcal{O}_\lambda} I_{\dot{x}}. \tag{2.2.6}$$

We want to estimate $\mu(\mathcal{O}_\lambda)$ in terms of the μ-measure of the set on the right-hand side of (2.2.6); it is apparent that we need some control over this set. So, since, in addition μ is regular; if $\mathcal{K}$ is a compact subset of $\mathcal{O}_\lambda$ now, there are finitely many $I'_{\dot{x}}s$, $I_{\dot{x}_1}, I_{\dot{x}_2}, ..., I_{\dot{x}_m}$ say, so that $\mathcal{K} \subseteq \bigcup_{j=1}^{m} I_{x_j}$. We may assume that $I'_{\dot{x}}s$ satisfy: every $I_{\dot{x}_k}$ is such that $I_{\dot{x}_k} \neq \bigcup_{j=1}^{k-1} I_{\dot{x}_j}$. Since we are dealing with a finite number of cubes, there is one of largest sidelength (if there is more than one just pick any); separate it and rename I_1. Now, if any of the remaning cubes, say I, intersects I_1, since sidelength $I \leq$ sidelength I_1, it follows that $I \subseteq 3I_1$, the cube concentric with I_1 with sidelength three times that of I_1; all these cubes I can be discarded as well. We are thus left with a finite collection of open cubes, each one disjoint with I_1. Repeat for this family the procedure used to select I_1, that is select a cube with largest sidelength, call it I_2, and discard all other cubes which intersect it. After a finite number of steps we are left with a collection $I_1, I_2, ..., I_k$ of disjoint open cubes so that $\mathcal{K} \subseteq \bigcup_{j=1}^{k} 3I_j$. Thus

$$\mu(\mathcal{K}) \leq \sum_{j=1}^{k} \mu(3I_j)$$

Besides, μ is a doubling measure. Then we replace $\mu(3I)$ by one constant multiplied by $\mu(I)$ we obtain by consequence

$$\mu(\mathcal{K}) \leq c^2 \sum_{j=1}^{k} \mu(I_j) \tag{2.2.7}$$

where c is the doubling constant.
But all $I'_j s$ satisfy (2.2.5). Whence combining (2.2.5) and (2.2.7), and since the $I'_j s$ are pairwise disjoint, we get

$$\lambda \mu(\mathcal{K}) \leq c^2 \sum_{j=1}^{k} \int_{I_j} |f(x)| \, d\mu(y) = c^2 \int_{\bigcup I_j} |f(y)| \, d\mu(y) \tag{2.2.8}$$

Finally (2.2.2) and the definition (2.2.4) give

$$\lambda \mu(\mathcal{O}_\lambda) \leq c^2 \|f\|_{L^1(I_0;\mu)}. \tag{2.2.9}$$

□

Simple examples show, for instance, that μ can not have atoms, but in fact much more is true.

In the next paragraph, we discuss an important result which will allow us to show the consequence of this weak-type result. It is:

Theorem 2.2.2. ***(The Marcinkiewicz interpolation theorem)***
Assume that a sublinear operator T is defined in $L^{p_0}+L^{p_1}$ and is simultaneously of weak-type (p_0, p_0) with norm $\leq c_0$ and of type (p_1, p_1) with norm $\leq c_1$, for $1 \leq p_0 < p_1 \leq \infty$. If now $p_0 < p < p_1$ and $\frac{1}{p} = (1-\eta)/p_0 + \eta/p_1$, $0 < \eta < 1$, then T is also of type (p,p) with norm $\leq c\left(1/(p-p_0)^{(1-\eta)/p_0}\right) c_0^{1-\eta} c_1^{\eta_1}$. Here, c is an absolute constant $\leq 8e^{\frac{1}{e}}$ independant of the mapping T.

As a consequence of this theorem we have:

Corollary 2.2.3. *Let μ be as in theorem 2.2.1, then there is a constant $c = c_p$ independant of f such that:*

$$\|M_\mu f\|_{L^p_\mu} \leq c \, \|f\|_{L^p_\mu}, 1 < p < \infty. \tag{2.2.10}$$

Proof. M_μ is weak-type $(1,1)$, and is bounded in $L^\infty(I_0)$. Thus the Marcinkiewicz interpolation theorem applies. □

2.2.3 Relation between A_p weights and the Hardly-Littlewood maximal function

The Hardly-Littlewood maximal function has important application in the study of weighted norm inequalities. Many results which will be used here, have been proved for the maximal function. For example, Theorem 2.2.1 above. Other results have been proved by using the maximal function. We can cite the results of Mukenhoupt and the results of Coifman-Rochberg on which we will give some discussions in the following.

2.2.3.1 Definition

Let f be a measurable function on I_0. We assume that f is a non zero function, $f \in L^1(I_0; d\dot{x})$ and not bounded.
We define the Hardy-Littlewood maximal function by:

$$\forall \dot{x} \in I_0; M(f)(\dot{x}) = \sup_{I \subseteq I_0 : \dot{x} \in I} \frac{1}{|I|} \int_I |f(\dot{y})| \, d\dot{y}, \tag{2.2.11}$$

where the I's are open cubes containing $\dot{x}$. It is well known that the function $M(f)$ is non-negative and measurable but not integrable (see exemple given in 4.2.1, Part(B) of chaper 4). The definition is the same as in (2.2.1) except the measure μ is replaced by the Lebesgue measure and by consequence since the Lebesgue measure is regular and has a

doubling property; Theorem 2.2.1 holds for the maximal function defined in (2.2.11).
In the same way we define also the so-called local maximal function, for f measurable and in $L^1(I_0; d\dot{x})$.

$$M_\eta f(\dot{x}) = \sup_{I \subseteq I_0 : \dot{x} \in I} \frac{1}{|I|^{1-\eta}} \int_I |f(\dot{y})| \, d\dot{y}, \qquad \eta < 1. \tag{2.2.12}$$

We define the two Muckenhoupt's conditions as follows:
1) A_p **condition:** let $p > 1$. We say that a non-negative function $w \in L^1(I_0; d\dot{x})$ verifies the $A_p(I_0)$ condition, and we write $w \in A_p(I_0)$, if there exists a constant c such that for all cube $I \subseteq I_0$:

$$\left(\frac{1}{|I|} \int_I w(\dot{y}) d\dot{y} \right) \left(\frac{1}{|I|} \int_I w(\dot{y})^{\frac{-1}{p-1}} d\dot{y} \right)^{p-1} \leq c, \qquad if \qquad 1 < p < \infty.$$

$$\frac{1}{|I|} \int_I w(\dot{y}) d\dot{y} \leq c \, (ess \inf_I w) \qquad if \qquad p = 1.$$

The two constants above are called A_p constant (respect. A_1 constant).
2) A_∞ **condition:** we say that w verifies the $A_\infty(I_0)$ condition and we write $w \in A_\infty(I_0)$, if for each $0 < \epsilon < 1$ there corresponds $0 < \delta < 1$ so that for all measurable subset E of I we have $\int_E w(\dot{y}) d\dot{y} < \epsilon \int_I w(\dot{y}) d\dot{y}$ whenever $|E| < \delta \, |I|$. And we prove the following theorem which caracterises the A_∞ condition and which will play a very important role in the proof of theorem 2.1.1.

Theorem 2.2.4.

$$A_\infty = \bigcup_{p \geq 1} A_p, \tag{2.2.13}$$

Proof.

Proposition 2.2.5. ***(Reverse Holder)***
Suppose that $w \in A_1(I_0)$, then there is a positive number η so that

$$\left(\frac{1}{|I|} \int_I w(\dot{x})^{1+\eta} d\dot{x} \right)^{\frac{1}{1+\eta}} \leq c \frac{1}{|I|} \int_I w(\dot{x}) d\dot{x}, \forall I \subseteq I_0 \tag{2.2.14}$$

where c is independent of I, but not of course of η. We say $w \in RH_{1+\eta}$.

This proposition corresponds to Theorem 3.5 chapter IX of [12]. The detailed proof can be seen in that book. As consequence, we have also the following proposition.

Proposition 2.2.6. ***(Reverse doubling)***
Suppose that $w \in A_1(I_0)$ and let be the measure μ such that $d\mu(\dot{x}) = w(\dot{x})d\dot{x}$. Then there exists $\gamma > 0$ such that for all open cube $I \subseteq I_0$ and measurable subset E of I

$$\frac{\mu(E)}{\mu(I)} \leq c \left(\frac{|E|}{|I|} \right)^\gamma \tag{2.2.15}$$

where c is independent of E, I.

Proof. By proposition (2.2.5) $w \in RH_{1+\eta}$. Then Hölder inequality give,

$$\mu(E) \leq \left(\int_E w(\dot{x})^{1+\eta} d\dot{x}\right)^{\frac{1}{1+\eta}} |E|^{\frac{\eta}{1+\eta}} \leq c\,|I|^{\frac{1}{1+\eta}} \left(\frac{1}{|I|}\int_I w(\dot{x})d\dot{x}\right) |E|^{\frac{\eta}{1+\eta}} = c\left(\frac{|E|}{|I|}\right)^{\frac{\eta}{1+\eta}} \mu(I).$$

□

Now, let us start to prove the (2.2.13) above. The first inclusion $\bigcup_{p\geq 1} A_p \subseteq A_\infty$ is obvious because if $w \in A_1$, the proposition (2.2.6) is applied.
We show that the converse is true. If $w \in A_\infty$, we have to find one $p > 1$ such that $w \in A_p$. Let us use the following lemma.

Lemma 2.2.7. *Assume that a non negative function w verifies* $\left|\left\{\dot{y} \in I; w(\dot{y}) < \frac{w_I}{B^k}\right\}\right| \leq c\eta^k |I|$ *all I, where c is independant of I,* $w_I = \frac{1}{|I|}\int_I w(\dot{y})d\dot{y}$, *for some constant η and B such that $0 < \eta < 1 < B < \infty$. Then, there is a $p > 1$ such that $w \in A_p$.*

Proof. Note that:

$$\begin{aligned}
\int_I w(\dot{y})^{\frac{-1}{p-1}} d\dot{y} &\leq \int_{[0,\infty[} \left|\left\{\frac{1}{w} > \lambda\right\}\right| d(\lambda^{\frac{1}{p-1}}) \\
&= \left(\int_{[0,B/w_I[} + \sum_{k=1}^{\infty} \int_{[B^k/w_I, B^{k-1}/w_I[}\right) \left|\left\{w < \frac{1}{\lambda}\right\}\right| d(\lambda^{\frac{1}{p-1}}) \\
&\leq c\left(\frac{B}{w_I}\right)^{\frac{1}{p-1}} |I| + c\sum_{k=1}^{\infty} (\eta B^{\frac{1}{p-1}})^k (w_I)^{\frac{-1}{p-1}} |I|,
\end{aligned}$$

and choose p so large that $\eta B^{\frac{1}{p-1}} < 1$. □

Before continuing the proof of the second inclusion, we recall the following definition.
Definition
We say that a point $\dot{x} \in I_0$ is a Lebesgue point of $f \in L^1(I_0; d\dot{x})$, where $d\dot{x}$ is the Lebesgue measure, if

$$\lim_{r\to 0^+} \frac{1}{|B(\dot{x}, r)|} \int_{B(\dot{x},r)} |f(t) - f(\dot{x})|\, dt = 0.$$

And the Lebesgue's differentiation theorem allows us to say that if $f \in L^1(I_0; d\dot{x})$ then almost all $\dot{x} \in I_0$ is a Lebesgue point of f. In other words, the set of $\dot{x} \in I_0$ which are not Lebesgue point is negligible.

Let us prove that $A_\infty \subseteq \bigcup_{p>1} A_p$.
We fixe I and k, we define $w_I = \frac{1}{|I|}\int_I w(\dot{y})d\dot{y}$. Our objective is to find the appropriate η which verifies the hypothesis of lemma 2.2.7.

We observe that since $w \in A_\infty$, for $\epsilon = 1/2$ we let δ, $0 < \delta < 1$ its corresponding, so that for all measurable subset E of I we have $\int_E w(\dot{y})d\dot{y} < \epsilon \int_I w(\dot{y})d\dot{y}$ whenever $|E| < \delta\,|I|$.

We set $E = \left\{\dot{x} \in I : w(\dot{x}) < \frac{w_I}{8^k}\right\}$. Observe that $\mu(E) < \frac{\mu(I)}{8^k} < \frac{\mu(I)}{2}$ implies $|E| < (1-\delta)\,|I|$. Now, since almost every $\dot{x} \in E$ is a Lebesgue point of χ_E and Lebesgue measure is regular we may assume that E is compact and each point of E is Lebesgue point of χ_E. To each $\dot{x} \in E$ we may assign an open interval $I_{\dot{x}}$ centered at $\dot{x}$ such that $|I_{\dot{x}} \cap I \cap E| = (1-\delta)\,|I_{\dot{x}} \cap I|$ (this is possible since for $I_{\dot{x}}$ large, $I_{\dot{x}}$ contains I and $|E| = (1-\delta)\,|I|$ and for $I_{\dot{x}}$ small $I_{\dot{x}} \subset I$ and $|I_{\dot{x}} \cap I| / |I_{\dot{x}}| \to 1$), $I_{\dot{x}} \subseteq cI$, c independant of $\dot{x}$ and I. Let $S = \cup_{\dot{x}\in E} I_{\dot{x}}$, since E is compact we may assume that S is finite and choose I_1 as an $I_{\dot{x}}$ in S of largest length. Then after $I_1, I_2, \ldots I_k$ have been chosen let S_k be the family of the remaining $I'_{\dot{x}}s$ so that $\dot{x} \neq \cup_{j=1}^{k} I_j$ and let I_{k+1} be largest interval in S_k. Observe that each $y \in \bigcup I_j$ belongs to, at most, two of the $I'_j s$ and put $E_1 = \bigcup_j (I_j \cap I) \subseteq I$. Then $\mu(E_1) \leq \sum_j \int_{I_j \cap I} d\mu(\dot{y}) \leq 2\sum_j \int_{E\cap I_j \cap I} d\mu(\dot{y})$ (since $|E \cap I_j \cap I| = (1-\delta)\,|I_{\dot{x}} \cap I|$ implies $\mu(I_j \cap I) \leq 2\mu(E \cap I_j \cap I)$) $\leq 4\int_{E\cup I_j \cup I} d\mu(\dot{y})$ (since each $\dot{y}$ belongs to at most two of the I_j) $\leq \int_E d\mu(\dot{y}) \leq 4\mu(I)/8^k$.
In the other hand, the Lebesgue measure of E_1 satisfies:

$$|E_1| = |E| + |\cup_j (I_j \cap I \cap (I \setminus E))| \geq |E| + \frac{1}{2}\sum_j |I_j \cap I \cap (I \setminus E)|$$

$$\geq |E| + \frac{\delta}{2}\sum_j |I_j \cap I| \geq |E| + (\frac{\delta}{2})\,|E_1|$$

or $|E_1| \geq \frac{|E|}{\eta}$. Now, if $k \geq 2$, it is possible to start with $\mu(E_1) < \mu(E)/8^{k-1}$ and repeat the above argument with E replaced by E_1. This give $E_2 \subset I$, $\mu(E_2) \leq \frac{\mu(I)}{8^{k-2}}$ and $|E_2| > \frac{|E|}{\eta^2}$; repeating the process k times we are done. This result is Muckenhoupt's [1974] and ensure that $A_\infty = \bigcup_{p\geq 1} A_p$ □

Proposition 2.2.8. *Assume μ is a nonnegative Borel measure, finite on bounded sets defined by $d\mu(\dot{x}) = w(\dot{x})d\dot{x}$ and assume that for some $1 < p < \infty$,*

(2.2.16) $$\lambda^p \mu(\{Mf > \lambda\}) \leq k^p \|f\|_{L^p(I_0;\mu)}\,, \textit{ for all } f \in L^p(I_0; d\mu),$$

where k is some positive constant. Then, the A_p condition holds.

Proof. Fix an open cube I and consider for $f \in L^p(I_0; d\mu)$ the quantity $(1/|I|)\int_I |f(\dot{y})|\,d\dot{y} = |f|_I$ which we may assume >0. Since

(2.2.17) $$\inf_{\dot{x}\in I} M(f1_I)(\dot{x}) \geq |f|_I\,,$$

$|f|_I$ must be finite for each I, for otherwise (2.2.16) can not hold unless μ is the zero measure. Thus, if we put $\mathcal{O} = \{M(f1_I) > |f|_I/2\}$, by (2.2.16) and by (2.2.17) it follows that $\mu(I) \leq \mu(\mathcal{O}) \leq k^p(1/|f|_I)^p \|f\|_{L^p(I_0;d\mu)}$ which is equivalent to

(2.2.18) $$(1/|I|)\int_I |f(\dot{y})|\,d\dot{y} \leq k^p \left(1/\mu(I)\int_I |f(\dot{y})|^p\,d\mu(\dot{y})\right)^{\frac{1}{p}}$$

Since we have no a priori reason why w can not vanish on a set of positive measure, we introduce the measure $d\nu(\dot{y}) = d\mu(\dot{y}) + \epsilon d\dot{x}, \epsilon > 0$, to avoid unnecessary technical difficulties. Clearly, ν is also absolutely continuous with respect to Lebesgue measure, $d\nu(y) = v(\dot{y})d\dot{y}, v > 0$, and more importantly, (2.2.16) holds for μ replaced by ν with constant independent of ϵ. Assume $p > 1$ first. In order to estimate $\int_I v(\dot{y})^{-1/(p-1)}d\dot{y}$ we note that it equals $\|1/v\|_{L^{p'}(I_0;d\nu)}$, $p' = p/(p-1)$, which, by Hölder's inequality may be estimated by

$$\sup_{\|f\|_{L^p(I_0;d\nu)}\leq 1} \left| \int_I \frac{f(\dot{y})}{v(\dot{y})} v(\dot{y})dy \right|^{p'} = \sup_{\|f\|_{L^p(I_0;d\nu)}\leq 1} \left| \int_I f(\dot{y})d\dot{y} \right|^{p'}.$$

Now by (2.2.18), which also holds for ν, it follows that for all $f \in L^p(I_0; d\mu)$

$$\left| \int_I f(\dot{y})d\dot{y} \right| \leq c.k\,|I| \left(\frac{1}{\nu(I)} \int_I |f(\dot{y})|^p\, d\nu \right)^{1/p} \leq \frac{c.k\,|I|}{\nu(I)^{1/p}}$$

and consequently

$$\frac{1}{|I|} \int_I v(\dot{y})^{-1/(p-1)} d\dot{y} \leq c.k \frac{1}{|I|} \left(\frac{|I|}{\nu(I)^{1/p}} \right)^{p'}.$$

The last inequality gives A_p condition for $p > 1$. □

2.2.3.2 A_1 weights

As we have seen in Theorem 2.2.8, A_1 is a necessary condition for the Hardy-Littlewood maximal operator M to map $L^1(I_0; d\mu)$ into weak-$L^1(I_0; d\mu)$. It is also a sufficient condition. This is the following first theorem of Muckenhoupt.

Proposition 2.2.9. *Let μ be the measure defined by $d\mu(\dot{x}) = w(\dot{x})d\dot{x}$ and suppose $w \in A_1$. Then M maps $L^1(I_0; d\mu)$ into weak-$L^1(I_0; \mu)$, with norm independant in A_1.*

Proof. First note that if $w \in A_1$, then μ is doubling with doubling constant $\leq c$ (c is the A_1 constant of w); Indeed, since $(\frac{1}{|2I|}) \int_{2I} d\mu(\dot{y}) \leq c\; ess\inf_I w \leq c(\frac{1}{|I|}) \int_I d\mu(\dot{y})$, it follows that $\mu(2I) \leq c\mu(I)$, $c \leq 2^d(A_1$ constant of $w)$. Moreover, since

$$\begin{aligned}\frac{1}{|I|} \int_I |f(\dot{y})|\, dy &= \frac{\mu(I)}{|I|} \frac{1}{\mu(I)} \int_I |f(\dot{y})|\, d\dot{y} \\ &\leq c \frac{1}{\mu(I)} (ess \inf_I w) \int_I |f(\dot{y})|\, d\dot{y} \\ &\leq c \frac{1}{\mu(I)} \int_I |f(\dot{y})|\, d\mu(\dot{y}).\end{aligned}$$

We also have that $Mf(\dot{x}) \leq cM_\mu f(\dot{x})$ where c is the A_1 constant of w. Thus $\{Mf > \lambda\} \subseteq \left\{M_\mu f > \frac{\lambda}{c}\right\}$ and by theorem (2.2.1) $\lambda\mu(\{Mf > \lambda\}) \leq c(\frac{\lambda}{c})\mu(\left\{M_\mu f > \frac{\lambda}{c}\right\}) \leq c^3 \|f\|_{L_\mu}$ □

Remark 2.2.10. *Somes observations concerning A_1 are obvious: for instance, A_1 is the limiting A_p condition as $p \to 1^+$ and an equivalent way of stating A_1 is*

$$Mw(\dot{x}) \leq c.w(\dot{x}) \text{ a.e} \tag{2.2.19}$$

What are the A_1 weights? Can we give some examples or even characterize them? the lecteur can find the answer in section (B) of the Appendix. Also, another important result of this theory is the following proposition.

Proposition 2.2.11. ***(Coifman and Rochberg)***
Let μ be a non-negative Borel measure so that $M_\mu(\dot{x})$, defined by

$$M_\mu(\dot{x}) = \sup_{I \subseteq I_0 : \dot{x} \in I} \frac{1}{|I|} \int_I \mu(d\dot{y});$$

is not identically ∞. Then for each $0 \leq \epsilon < 1$, $M_\mu(\dot{x})^\epsilon \in A_1(I_0)$.

Proof. Recall that $M_\mu(\dot{x}) = \sup_{\dot{x} \in I}(\frac{1}{|I|})\mu(I)$. For a fixed open cube I
we estimate $(\frac{1}{|I|})\int_I M_\mu(\dot{x})^\epsilon d\dot{x}$ by $A^\epsilon = (\inf_I M_\mu(\dot{x}))^\epsilon$ as follows: for each $\dot{x}$ in I we divide those open cubes $\mathcal{Q}$ containing x into two families by setting $\mathcal{J}_1 = \{\mathcal{Q} : |\mathcal{Q}| \leq 2\,|I|\}$ and $\mathcal{J}_2 = \{\mathcal{Q} : |\mathcal{Q}| \geq 2\,|I|\}$. Thus

$$\begin{aligned} M_\mu(\dot{x}) &\leq \sup_{\mathcal{Q} \in \mathcal{J}_1} \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} d\mu(\dot{y}) + \sup_{\mathcal{Q} \in \mathcal{J}_2} \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} d\mu(\dot{y}) \\ &= A(\dot{x}) + B(\dot{x}), \text{say.} \end{aligned} \tag{2.2.20}$$

The estimate of $B(\dot{x})$ is readily obtained; since for $\mathcal{Q} \in \mathcal{J}_2$, we have $3\,|\mathcal{Q}| \supseteq I$, it follows that

$$\frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} d\mu(\dot{y}) \leq \frac{c}{|3\mathcal{Q}|} \int_{3\mathcal{Q}} d\mu(\dot{y}) \leq c \inf_{3\mathcal{Q}} M_\mu \leq cA$$

and,

$$B(\dot{x}) \leq cA. \tag{2.2.21}$$

where c is independent of μ. As for $A(\dot{x})$, let μ_1 denote the restriction of μ to $6I$ i.e $d\mu_1(\dot{y}) = \chi_{6I}(y)d\mu(\dot{y})$, and note that:

$$A(\dot{x}) \leq M\mu_1(\dot{x}). \tag{2.2.22}$$

Thus on account of (2.2.20), (2.2.21) and (2.2.22) we get that

$$\frac{1}{|I|} \int_I M\mu(\dot{x})^\epsilon d\dot{x} \leq \frac{1}{|I|} \int_I M\mu_1(\dot{x})^\epsilon d\dot{x} + cA^\epsilon$$

and it suffices to prove the desired estimate with $M\mu$ replaced by $M\mu_1$. But by Theorem 7.5 of chapter (iv) of [12], we readily see that:

$$\begin{aligned} \frac{1}{|I|} \int_I M\mu_1(\dot{y})^\epsilon dy &\leq \frac{1}{|I|} c(\text{weak-L norm of } M\mu_1)^\epsilon\, |I|^{1-\epsilon} \\ &\leq c \left(\frac{1}{|I|} \int_{6I} d\mu \right)^\epsilon \leq cA^\epsilon \end{aligned}$$

where c is a constant depending only on ϵ, and we have finished. □

2.2.3.3 A_p weights, p>1

Also, in theorem 2.2.8 A_p condition is necessary for the Hardy-Littlewood maximale function M to map $L^p(I_0; d\mu)$ into $L^p(I_0; d\mu)$. The following second theorem of Muckenhoupt shows that it is also sufficient. The argument is similar to theorem 2.2.9. However, a strong result holds.

Proposition 2.2.12. *Let w a strictly positive and integrable function on I_0 with respect to Lebesgue measure. Suppose $w \in A_p$, $p > 1$ and we set $d\mu(\dot{x}) = w(\dot{x})d\dot{x}$. Then M maps $L^p(I_0; d\mu)$ continuously into $L^p(I_0; d\mu)$.*

Proof. The proof is similar to proposition (2.2.9). Let I_0 be fixed, let $I \subseteq I_0$ such that $\dot{x} \in I$.

$$\begin{aligned}\frac{1}{|I|}\int_I |f(\dot{y})|\,d\dot{y} &\leq \frac{1}{|I|}\left(\int_I |f(\dot{y})|^p\,d\mu(\dot{y})\right)^{\frac{1}{p}}\left(\int_I w(\dot{y})^{\frac{-1}{p-1}}d\dot{y}\right)^{\frac{p-1}{p}}\\ &= \|f\|_{L^p(I_0;d\mu)}\,|I|^{-1}\,|I|^{\frac{p-1}{p}}\left(\frac{1}{|I|}\int_I w(\dot{y})^{\frac{-1}{p-1}}d\dot{y}\right)^{\frac{p-1}{p}}.\\ &\leq c^p\,|I|^{\frac{-1}{p}}\,\|f\|_{L^p(I_0;d\mu)}\left(\frac{1}{|I|}\int_I w(\dot{y})d\dot{y}\right)^{\frac{-1}{p}}\ \text{by definition of } A_p \text{ condition.}\end{aligned}$$

Taking the supremum over $I \subseteq I_0$ such that $\dot{x} \in I$ we get

$$\begin{aligned}Mf(\dot{x}) &\leq \|f\|_{L^p(I_0;d\mu)}\,sup_{I\subseteq I_0:\dot{x}\in I}\left[\frac{1}{|I|^{\frac{1}{p}}}\left(\frac{1}{|I|}\int_I w(\dot{y})d\dot{y}\right)^{\frac{-1}{p}}\right]\\ &= c^p\,\|f\|_{L^p(I_0;d\mu)}\,sup_{I\subseteq I_0:\dot{x}\in I}\left(\int_I w(\dot{y})d\dot{y}\right)^{\frac{-1}{p}}\end{aligned}$$

$\Rightarrow |Mf(\dot{x})|^p \leq c^p\,\|f\|^p_{L^p(I_0;d\mu)}\left(\inf_{I\subseteq I_0:\dot{x}\in I}\int_I w(\dot{y})d\dot{y}\right)^{-1}$
Taking integral on I_0 with respect to $d\mu(\dot{x})$ the result follows. □

The consequence of this theorem is the $L^p(I_0; d\mu)$ weak-type condition is necessary and sufficient for the Hardy-Littlewood maximale function to maps from $L^p(I_0; d\mu)$ to $L^p(I_0; d\mu)$.

2.3 Fractional integration

In this part, we introduce a new tol of harmonic analysis, the Riesz fractional integral. This operator denoted by I_α operates on $L^p(I_0; d\dot{x})$ and has many similar properties as the local maximal function of Hardy-Littlewood denoted here by M_η, following the hypothesis on f. The precise statement is the following: for all real $0 < \alpha < 1$, for all $f \in L^p(I_0; d\dot{x}), 1 \leq p < \frac{1}{\alpha}$, we define the Riesz fractional integral I_α by:

$$I_\alpha f(\dot{x}) = \int_{I_0}\frac{f(\dot{x}-\dot{y})}{|\dot{y}|^{1-\alpha}}d\dot{y}, \qquad \dot{x} \in I_0.$$

Let us define M_η. For all $f \in L^1(I_0; d\dot{x})$, $\eta < 1$, we define:

$$M_\eta f(\dot{x}) = \sup_{I \subseteq I_0; \dot{x} \in I} \frac{1}{|I|^{1-\eta}} \int_I |f(\dot{y})| \, d\dot{y}. \tag{2.3.1}$$

We will recall here an important result of Welland (see Theorem 2.4 of chapter of VI of [12]) , which gives the relation between the fractional integration and the local maximal function of Hardy-Littlewood. We have then the following proposition which corresponds to theorem 2.4 of [12].

Proposition 2.3.1. ***(Welland's theorem)***
Suppose that $f \in L^1(I_0, d\dot{x})$, $0 < \epsilon < \alpha < \alpha + \epsilon < 1$. Then there exists a constant c independant of f such that

$$|I_\alpha f(\dot{x})| \leq c(M_{\alpha-\epsilon} f(\dot{x}) M_{\alpha+\epsilon} f(\dot{x}))^{\frac{1}{2}} \tag{2.3.2}$$

Proof. For all ϵ satisfying $0 < \epsilon < \alpha < \alpha + \epsilon < 1$ we have:

$$\begin{aligned} M_{\alpha-\epsilon} f(\dot{x}) M_{\alpha+\epsilon} f(\dot{x}) &\geq \frac{1}{|I|^{1-(\alpha-\epsilon)+1-(\alpha+\epsilon)}} \left(\int_I |f(\dot{y})| \, d\dot{y} \right)^2 \\ &= \frac{1}{|I|^{2-2\alpha}} \left(\int_I |f(\dot{y})| \, d\dot{y} \right)^2 \\ &= \left(\frac{1}{|I|^{1-\alpha}} \int_I |f(\dot{y})| \, d\dot{y} \right)^2 \qquad \forall I \subseteq I_0 : \dot{x} \in I. \end{aligned}$$

We have then, $(M_{\alpha-\epsilon} f(\dot{x}) M_{\alpha+\epsilon} f(\dot{x}))^{\frac{1}{2}} \geq \frac{1}{|I|^{1-\alpha}} \int_I |f(\dot{y})| \, d\dot{y} \qquad \forall I \subseteq I_0 : \dot{x} \in I.$

$(M_{\alpha-\epsilon} f(\dot{x}) M_{\alpha+\epsilon} f(\dot{x}))^{\frac{1}{2}} \geq \sup_{I \subseteq I_0 : \dot{x} \in I} \frac{1}{|I|^{1-\alpha}} \int_I |f(\dot{y})| \, d\dot{y}$

In the other hand, we can see that there exists a positive constant c such that :

$$\int_{I_0} \frac{|f(\dot{y})|}{|\dot{x} - \dot{y}|^{(1-\alpha)d}} d\dot{y} \leq c \sup_{I \subseteq I_0 : \dot{x} \in I} \frac{1}{|I|^{1-\alpha}} \int_I |f(\dot{y})| \, d\dot{y}. \tag{2.3.3}$$

Indeed, $\forall I \subseteq I_0 : \dot{x} \in I$,

$$\frac{1}{|I|} \int_I \frac{1}{|\dot{x} - \dot{y}|^{(1-\alpha)d}} d\dot{y} \leq c \frac{1}{|I|^{1-\alpha}}.$$

Thus $\forall f \in L^1(I_0; dx)$,

$$\int_I |f(\dot{y})| \left(\frac{1}{|I|} \int_I \frac{1}{|\dot{x} - \dot{y}|^{(1-\alpha)d}} d\dot{y} \right) d\dot{y} \leq c \int_I \frac{|f(\dot{y})|}{|I|^{1-\alpha}} d\dot{y}$$

$$\int_I \frac{|f(\dot{y})|}{|\dot{x} - \dot{y}|^{(1-\alpha)d}} d\dot{y} \leq c \int_I \frac{|f(\dot{y})|}{|I|^{1-\alpha}} d\dot{y} \qquad \forall I \subseteq I_0 : \dot{x} \in I.$$

We get $\forall I \subseteq I_0 : \dot{x} \in I$,

$$\int_I \frac{|f(\dot{y})|}{|\dot{x}-\dot{y}|^{(1-\alpha)d}} d\dot{y} \leq c \sup_{I \subseteq I_0 : \dot{x} \in I} \frac{1}{|I|^{1-\alpha}} \int_I |f(\dot{y})| \, d\dot{y}. \tag{2.3.4}$$

The inequality (2.3.3) holds by taking $I = I_0$ in the left hand side of (2.3.4). With the inequality (2.3.3) we obtain:

$$(M_{\alpha-\epsilon} f(\dot{x}) M_{\alpha+\epsilon} f(\dot{x}))^{\frac{1}{2}} \geq \frac{1}{c} \int_{I_0} \frac{|f(\dot{y})|}{|\dot{x}-\dot{y}|^{1-\alpha}} d\dot{y} = \frac{1}{c} |I_\alpha |f(\dot{x})|| \geq \frac{1}{c} |I_\alpha f(\dot{x})| \qquad \dot{x} \in I_0.$$

□

2.4 Sobolev and Poincaré inequalities

In this part of this chapter, we show some applications of the theory of Muckenhoupt weights. First we need some observations of general interest. In what follows I_0 denotes the unit torus $\mathbb{R}^d/\mathbb{Z}^d$ and I an arbitrary open subcube of I_0. Let's recall some definitions.
Definition:
Let w be a strictly positive and bounded function on I_0. A integrable function f is said to be centered if $\int_{I_0} f(\dot{x}) w(\dot{x}) d\dot{x} = 0$. Set $d\mu(\dot{x}) = w(\dot{x}) d\dot{x}$, we have:

Proposition 2.4.1. *Suppose that f is defined on I_0, centered and C^1 there. Then there is a constant c such that:*

$$|f(\dot{x})| \leq c \int_{I_0} \frac{|\nabla f(\dot{y})|}{|\dot{x}-\dot{y}|^{d-1}} dy, \qquad \dot{x} \in I_0, c \textit{ independent of } \dot{x}.$$

Proof. $|f(\dot{x})| - |f(\dot{y})| \leq |f(\dot{x}) - f(\dot{y})|$. Taking integral with the measure $d\mu(\dot{y})$ we get:

$$|f(\dot{x})| \int_{I_0} d\mu(\dot{y}) - \int_{I_0} f(\dot{y}) d\mu(\dot{y}) \leq c \int_{I_0} |f(\dot{x}) - f(\dot{y})| \, d\dot{y} \text{ because } w \text{ is bounded.}$$

$$\leq c \int_{I_0} \frac{|\nabla f(\dot{y})|}{|\dot{x}-\dot{y}|^{d-1}} d\dot{y}, \qquad \dot{x} \in I \text{ where c is a dimensional constant.}$$

Let us prove the last inequality. It is easy to see that
$|f(\dot{x}) - f(\dot{y})| = \left| \int_{[0;1]} \nabla f(\dot{x} + t(\dot{y} - \dot{x})).(z - \dot{x}) dt \right|$. We see at once that
$\int_{I_0} |f(\dot{x}) - f(\dot{y})| \, d\dot{y} \leq \int_{[0,1]} \int_{I_0} |\nabla f(\dot{x} + t(\dot{y} - \dot{x}))| \, |\dot{y} - \dot{x}| \, d\dot{y} dt = A$, say.
To bound A, we observe the line segment joining $\dot{x}$ and $\dot{z}$ is totally contained in I_0, if we put $\dot{z} = \dot{x} + t(\dot{y} - \dot{x})$ then also z is in I_0, $d\dot{z} = t^n d\dot{y}$ and $|\dot{z} - \dot{x}| = t \, |\dot{y} - \dot{x}| \leq c.t.L$ where L denotes the sidelength of I_0, $L = 1$ and c a dimensional constant. Then

$$A \leq c \int_{I_0} |\nabla f(\dot{z})| \, |\dot{x} - \dot{z}| \int_{[\frac{|\dot{x}-\dot{z}|}{c}, \infty]} t^{-(n+1)} dt d\dot{y}$$

$$c \int_{I_0} |\nabla f(\dot{z})| \, |\dot{x} - \dot{z}| \, (\frac{1}{|\dot{x} - \dot{z}|^d}) d\dot{z}.$$

We have now

$$\int_{I_0} |f(\dot{x}) - f(\dot{y})|\, d\dot{y} \le c \int_{I_0} \frac{|\nabla f(\dot{z})|}{|\dot{x} - \dot{z}|^{d-1}} d\dot{z}. \tag{2.4.1}$$

The conclusion follows. □

We restate the estimates (2.4.1) in terms of maximal functions. We introduce the following definitions: for f defined on I_0 and $\dot{x} \in I_0$, we consider the expressions $\sup(\frac{1}{|I|}\int_I |f(\dot{y}) - f_I|\, d\dot{y})$, where $I \subseteq I_0$ and $\dot{x} \in I$, $f_I = \frac{1}{|I|}\int_I |f(\dot{x})|\, d\dot{x}$; in order to keep notations simple we still call this expression $\widetilde{M} f(\dot{x})$. With this notation we have:

Corollary 2.4.2. *Assume f is a function defined on I_0 and C^1 there and let $I \subseteq I_0$. Then,*

$$\widetilde{M} f(\dot{x}) \le c M_{\frac{1}{d}}(|\nabla f|)(\dot{x}), \qquad \dot{x} \in I_0 \tag{2.4.2}$$

Proof. Fix $\dot{x} \in I_0$ and let I contains $\dot{x}$. Then

$$\frac{1}{|I|}\int_I |f(\dot{y}) - f_I|\, d\dot{y} \le \frac{1}{|I|^2}\int_I \int_I |f(\dot{y}) - f(\dot{z})|\, d\dot{y} d\dot{z}. \tag{2.4.3}$$

We can see also, by the same proof as inequality (2.4.1) that

$$\frac{1}{|I|}\int_I |f(\dot{x}) - f(\dot{z})|\, d\dot{z} \le c \int_I \frac{|\nabla f(\dot{y})|}{|\dot{x} - \dot{y}|^{d-1}} d\dot{y}. \tag{2.4.4}$$

By inequality (2.4.4) , the right-hand side of (2.4.3) does not exceed

$$c\frac{1}{|I|}\int_I \int_I \frac{|\nabla f(\dot{y})|}{|\dot{y} - \dot{z}|^{d-1}} d\dot{y} d\dot{z}$$

But since we can easily see that:

$$\frac{1}{|I|}\int_I \frac{1}{|\dot{y} - \dot{z}|^{d-1}} d\dot{z} \le c\, |I|^{\frac{(1-d)}{d}},$$

(2.4.2) holds and we are done. □

We collect now some facts concerning the maximal functions which will be useful in the sequel.

The next proposition relates to the continuity properties of the local function $M_\eta f$.

Definition 2.4.3. *We say that two positive functions (w, v) are in $A_{p,q,\eta}(I_0)$, we write $(w, v) \in A_{p,q,\eta}(I_0)$ if*

$$\forall I \subseteq I_0, \left(\int_I w(\dot{x}) d\dot{x}\right)^{\frac{1}{q}} \left(\int_I v(\dot{x})^{\frac{-1}{p-1}} d\dot{x}\right)^{\frac{p-1}{p}} \le c\, |I|^{1-\eta}. \tag{2.4.5}$$

Proposition 2.4.4. *Suppose w, v are positive, integrable functions in I_0, and let $d\mu(\dot{x}) = w(\dot{x})d\dot{x}$, $d\nu(\dot{x}) = v(\dot{x})d\dot{x}$. Then for $p > 1$, $s > 0$, $0 < \eta < 1$, $(w, v) \in A_{p,s,\eta}(I_0)$ implies that for all $0 < r < s$ the local maximal function $M_\eta f$ verifies*

$$\|M_\eta f(\dot{x})\|_{L^r(I_0;d\mu)} \leq c\,\|f\|_{L^p(I_0;d\nu)}\,, \text{ where } c \text{ is some constant.} \tag{2.4.6}$$

Proof. I_0 being fixed, let I a cube of I_0 such that $\dot{x} \in I$.

$$\begin{aligned}\frac{1}{|I|^{1-\eta}}\int_I |f(y)|\,d\dot{y} &\leq \frac{1}{|I|^{1-\eta}}\left(\int_I |f(\dot{y})|^p\,d\nu(\dot{y})\right)^{\frac{1}{p}}\left(\int_I v(\dot{y})^{\frac{-1}{p-1}}d\dot{y}\right)^{\frac{p-1}{p}}\\ &\leq c\,\|f\|_{L^p(I_0;d\nu)}\left(\int_I w(\dot{y})d\dot{y}\right)^{\frac{-1}{s}} \text{ by condition } A_{p,s,\eta}(I_0).\end{aligned}$$

Taking the supremum over $I \subseteq I_0$ we get

$$\begin{aligned}M_\eta f(\dot{x}) &\leq c\,\|f\|_{L^p(I_0;d\nu)}\left(\inf_{I\subseteq I_0,\dot{x}\in I}\int_I w(\dot{y})d\dot{y}\right)^{\frac{-1}{s}}\\ \Rightarrow |M_\eta f(\dot{x})|^r &\leq c\,\|f\|^r_{L^p(I_0;d\nu)}\left(\inf_{I\subseteq I_0,\dot{x}\in I}\int_I w(\dot{y})d\dot{y}\right)^{\frac{-r}{s}}.\end{aligned}$$

Taking the integral on I_0 with respect to $d\mu(\dot{x})$ it follows

$$\begin{aligned}\|M_\eta f(\dot{x})\|^r_{L^r(I_0;d\mu)} &\leq c\,\|f\|^r_{L^p(I_0;d\nu)}\left(\int_{I_0} w(\dot{x})d\dot{x}\right)\left(\inf_{I\subseteq I_0,\dot{x}\in I}\int_I w(\dot{y})d\dot{y}\right)^{\frac{-r}{s}}\\ &\leq c\,\|f\|^r_{L^p(I_0;d\nu)}\end{aligned}$$

□

The following Theorem, Sobolev's embedding theorem; is the basis of the proof of the main theorem of this chapter: Theorem 2.1.1. But we will se that that a rigorous verification of hypotheses is necessary.

Theorem 4.8 of chapter X of [12]: (Sobolev's embedding theorem)
Let $1 < p < \infty$ such that $\frac{1}{p} - \frac{1}{d} \leq \frac{1}{s} < \frac{1}{p}$. Let w, v two strictly positive functions such that $w \in A_\infty(I_0)$ and $(w, v) \in A_{p,s,\frac{1}{d}}(I_0)$. Then for any q such that $p \leq q < s$, and for every u defined on I_0, centered and C^1 there,

$$\left(\int_{I_0} |u(\dot{x})|^q\,w(\dot{x})d\dot{x}\right)^{\frac{1}{q}} \leq c\left(\int_{I_0} |\nabla u(\dot{x})|^p\,v(\dot{x})d\dot{x}\right)^{\frac{1}{p}}. \tag{2.4.7}$$

Proof. We set $d\mu(\dot{x}) = w(\dot{x})d\dot{x}$ and $d\nu(\dot{x}) = v(\dot{x})d\dot{x}$. First, observe that w is in A_p for some $p \geq 1$ since w is in A_∞. By Proposition 2.2.5, there exists $r > 1$ such that:

$$\left(\frac{1}{|I|}\int_I w(\dot{x})^r d\dot{x}\right)^{\frac{1}{r}} \leq c\frac{1}{|I|}\int_I w(\dot{x})d\dot{x}, \qquad I \subseteq I_0, \tag{2.4.8}$$

and consequently $(w^r, v) \in A_{p,rs,\eta}$, where $\eta = \frac{1}{d} + \frac{1}{s}(1 - \frac{1}{r})$. That the statement holds for $0 < r < 1$ as well is an easy consequence of Hölder's inequality. Let now $p \le q < s$, and let $r > 1$ be sufficiently close to 1 so that (2.4.8) holds and the corresponding η verifies $\eta < \frac{2}{d} \le 1$.
By proposition (2.4.1) it follows that

$$\int_{I_0} |f(\dot{x})|^q \, \mu(d\dot{x}) \le c \int_{I_0} (I_{\frac{1}{d}}(|\nabla f|)(\dot{x}))^q \mu(d\dot{x}) = A,$$

say. Consequently it suffices to estimate A. Since by Welland's theorem we have that

$$I_{\frac{1}{d}}(|\nabla f|)(\dot{x}) \le c \, (M_{\eta_1}(|\nabla f|)(\dot{x}) M_{\eta}(|\nabla f|)(\dot{x}))^{\frac{1}{2}}$$

where $\eta_1 = \frac{1}{d} - \frac{1}{s}(1 - \frac{1}{r})$. Taking integral with respect to the measure μ and applying Hölder inequality with $(\frac{2r}{2r-1}, 2r)$; A may be estimated by

$$\text{(2.4.9)} \quad \left(\int_{I_0} M_{\eta_1}(|\nabla f|)(\dot{x})^{\frac{rq}{2r-1}} w(\dot{x})^{\frac{r}{2r-1}} d\dot{x} \right)^{1-\frac{1}{2r}} \left(\int_{I_0} M_{\eta}(|\nabla f|)(\dot{x})^{rq} w(x)^r d\dot{x} \right)^{\frac{1}{2r}}.$$

Moreover, since $(w^r, v) \in A_{p,rs,\eta}(I_0)$, we recall that this condition means

$$\left(\int_I w^r(\dot{x}) d\dot{x} \right)^{1/rs} \left(\int_I v(\dot{x})^{\frac{-1}{p-1}} d\dot{x} \right)^{\frac{p-1}{p}} \le c \, |I|^{1-\eta}.$$

We have then $(w(\dot{x})^{r/(2r-1)}, v) \in A_{p, \frac{rs}{2r-1}, \eta_1}$. Indeed,

$$\begin{aligned} \left(\int_I w(\dot{x})^{r/(2r-1)} d\dot{x} \right)^{\frac{2r-1}{rs}} \left(\int_I v(\dot{x})^{\frac{-1}{p-1}} d\dot{x} \right)^{\frac{p-1}{p}} &\le \left(\int_I w^r(\dot{x}) d\dot{x} \right)^{\frac{1}{rs}} \left(\int_I v(\dot{x})^{\frac{-1}{p-1}} d\dot{x} \right)^{\frac{p-1}{p}} |I|^{\frac{2r-2}{rs}} \\ &\le c \, |I|^{1-(\eta - \frac{2r-2}{rs})} = c \, |I|^{1-\eta_1}, \end{aligned}$$

by Hölder inequality with parameter $(2r-1, \frac{2r-1}{2r-2})$.

By proposition 2.4.4, since $q < s$, means $rq < rs$, these operators satisfy:

$$\left(\int_{I_0} M_{\eta}(|\nabla f|)(\dot{x})^{rq} w(\dot{x})^r d\dot{x} \right)^{\frac{1}{2r}} \le c \, \|\nabla f\|_{L^p(I_0; d\nu)}^{\frac{q}{2}},$$

and

$$\left(\int_{I_0} M_{\eta_1}(|\nabla f|)(\dot{x})^{\frac{rq}{2r-1}} w(\dot{x})^{\frac{r}{2r-1}} d\dot{x} \right)^{1-\frac{1}{2r}} \le c \, \|\nabla f\|_{L^p(I_0; d\nu)}^{\frac{q}{2}}.$$

By combining the two last inequalities we obtain,

$$\int_{I_0} |f(\dot{x})|^q \, \mu(d\dot{x}) \le c \, \|\nabla f\|_{L^p(I_0; d\nu)}^q.$$

The desired inequality follows. □

2.5 Proof of main theorem

Theorem 2.1.1
Let us now prove Theorem 2.1.1. We let

$$w(\dot{x}) = M(e^V)(\dot{x})^{-1},$$

and check that this function w satisfies all the properties of Sobolev-embedding theorem.

First observe that since $e^V \in L^1(I_0; d\dot{x})$, then $w^{-1} = M(e^V)$ belongs to the weak $L^1(I_0; d\dot{x})$ space and therefore $M(e^V) < \infty$ a.e. and $w > 0$ a.e. Also $M(e^V)$ is bounded from below by $\int_{I_0} e^{V(\dot{y})} d\dot{y}$ and therefore w is bounded by $\left(\int_{I_0} e^{V(\dot{y})} d\dot{y}\right)^{-1}$.

We shall apply Theorem 4.8 of chapter X of [12] with $v(\dot{x}) = e^{-V(\dot{x})}$ and $p = 2$. In order to do so, it is sufficient to verify that $w \in A_\infty(I_0)$ and $(w, v) \in A_{2,s,\frac{1}{d}}(I_0)$ for some $s > 2$.

We first prove that $w \in A_\infty(I_0)$: the result of Coifman and Rochberg quoted in proposition (2.2.11) above implies that $\frac{1}{\sqrt{w}} = M(e^V)^{\frac{1}{2}} \in A_1$. This implies that, for all I,

$$\frac{1}{|I|} \int_I \frac{1}{\sqrt{w(\dot{y})}} d\dot{y} \leq c \left(\inf_I \frac{1}{\sqrt{w}}\right),$$

for some constant c. Therefore

$$\begin{aligned} (\frac{1}{|I|} \int_I w(\dot{y}) d\dot{y})(\frac{1}{|I|} \int_I \frac{1}{\sqrt{w(\dot{y})}} d\dot{y})^2 &\leq c^2 \left(\frac{1}{|I|} \int_I w(\dot{y})\right) \inf_I \frac{1}{w} \\ &= c^2 \frac{1}{|I|} \int_I \frac{w(\dot{y})}{\sup_I w} d\dot{y} \leq c^2. \end{aligned}$$

Therefore $w \in A_3(I_0)$ and, using remark (2.2.4), $w \in A_\infty(I_0)$.

Let us now check that there exists $s > 2$ such that $(w, v) \in A_{2,s,\frac{1}{d}}(I_0)$.
By definition of the maximal function, we know that for all $I \subseteq I_0$ and for all $\dot{x} \in I$, then

$$w(\dot{x}) \leq |I| \left(\int_I e^{V(\dot{y})} d\dot{y}\right)^{-1}.$$

Therefore

$$\left(\int_I w\dot{y}) d\dot{y}\right)^{\frac{1}{s}} \left(\int_I e^{V(\dot{y})} d\dot{y}\right)^{\frac{1}{2}} \leq |I|^{\frac{2}{s}} \left(\int_I e^{V(\dot{y})} d\dot{y}\right)^{\frac{1}{2} - \frac{1}{s}}. \tag{2.5.1}$$

We choose $s = \frac{2d}{d-1}$ and the following verifications are easy:

$$\begin{cases} 1/2 - 1/d \leq 1/s < 1/2, \\ 1/2 - 1/s > 0, \\ 2/s = 1 - 1/d, \end{cases}$$

and it follows from (2.5.1) that

$$\left(\int_I w(\dot{y})d\dot{y}\right)^{\frac{1}{s}}\left(\int_I e^{V(\dot{y})}d\dot{y}\right)^{\frac{1}{2}} \le \left(\int_{I_0} e^{V(\dot{y})}d\dot{y}\right)^{\frac{1}{2}-\frac{1}{s}} |I|^{1-\frac{1}{d}}.$$

Thus we checked the $A_{2,s,\frac{1}{d}}(I_0)$ condition.

Now Theorem 4.8 of [12] implies Theorem 2.1.1 for any choice of r^* such that $2 < r^* < s = 2d/(d-1)$. □

As consequence of theorem 2.1.1, we have the following inequality, called Poincare inequality.

$$\int_{I_0} |f(\dot{x})|^2 w(\dot{x})d\dot{x} \le c\int_{I_0} |\nabla f(\dot{x})|^2 e^{-V(\dot{x})}d\dot{x}, \tag{2.5.2}$$

which is deduced from theorem 2.1.1 with Holder's inequality.

Remark 2.5.1. *Here an elementary proof of theorem 2.1.1 when $e^V \in L^r$ for some $r > d/2$ and $w = 1$*
Indeed, the usual Sobolev inequality states that for all $p \in [1, d[$ then,

$$\left(\int_{I_0} |f(\dot{x})|^q d\dot{x}\right)^{\frac{1}{q}} \le c\left(\int_{I_0} |\nabla f(\dot{x})|^p\right)^{\frac{1}{p}} \tag{2.5.3}$$

for every function f, defined on I_0, centered and C^1 there and $q = pd/(d-p)$. Choose $p \in [1, 2[$ and apply Holder with parameters $a = 2/p$ and $b = 2/(2-p)$, we get that,

$$\left(\int_{I_0} |f(\dot{x})|^q d\dot{x}\right)^{\frac{1}{q}} \le c\left(\int_{I_0} |\nabla f(\dot{x})|^2 e^{-V(\dot{x})}\right)^{\frac{1}{2}}\left(\int_{I_0} e^{\frac{p}{2-p}V(\dot{x})}d\dot{x}\right)^{\frac{2-p}{2p}}$$

Letting p approach $2d/(d+2)$ with $p > 2d/(d+2)$, we then get theorem 2.1.1 with constant $w = 1$ and provided that $e^V \in L^r(I_0; d\dot{x})$ for some $r > d/2$.

Although this method seems to work only if e^V belongs to $L^r(I_0; d\dot{x})$ for some $r > d/2$, it has the advantage of providing an explicit and simple expression of the constant in terms of V.

2.6 Conclusion

A new weighted Sobolev type inequality for integrable potentials is proved in Theorem2.1.1.

Chapter 3

Homogenization for diffusions in periodic potential

3.1 Introduction

We are interested here in diffusion processes on $\mathbb{R}^d$ $d \geq 2$ driven by a linear second-order divergence form operator of the type:

$$L := \frac{1}{2}e^{V(x)}div(e^{-V(x)}\nabla) \qquad \text{where } V : \mathbb{R}^d \to \mathbb{R} \text{ is measurable and periodic.}$$

When V is assumed to be regular, the diffusion process generated by L can be constructed as a solution of the stochastic differential equation:

$$dX_t = dB_t - \frac{1}{2}\nabla V(X_t)dt, \tag{3.1.1}$$

where $(B_t\,;\,t \geq 0)$ is a standard Wiener process on $\mathbb{R}^d$. The stochastic process $(X_t\,;\,t \geq 0)$ is then a semi-martingale and Itô's stochastic calculus can be applied.

To make sense of equation (3.1.1) in the more general case where V is only assumed to be measurable, we shall use Dirichlet form theory. In Section 3.2, we assume that e^V and e^{-V} are both locally integrable, and show the existence of a Markovian law on path space $C([0, +\infty); \mathbb{R}^d)$ with generator L. The stochastic calculus developed in [4] will play a key role.

Such equations as (3.1.1) model the motion of a passive tracer submitted to two effects: a diffusion movement represented by the Brownian motion B and an external force described by the potential V.

Many works in the domain of homogenization theory addressed the question of the long-time behavior of such diffusions. Two cases are generally studied: either the potential is periodic or it is a realization of a stationary random function. Clearly the first can be seen as a special case of the second. Also many results hold for similar discrete models where $\mathbb{R}^d$ is replaced by the grid $\mathbb{Z}^d$ and one studies so-called *random walks with random conductances*.

Homogenization theory states that, under appropriate restrictions on V, solutions of elliptic problems associated to the operator L on, say, a large ball, scale to solutions of similar problems where L is replaced by an *homogenized* operator with constant coefficients, say

$$\bar{L} = \frac{1}{2}\sum_{i,j}(\bar{\sigma})_{i,j}\partial_i\partial_j,$$

where $\bar{\sigma}$ is a positive symmetric matrix, the so-called *effective diffusivity*.

In probabilistic terms, proving homogenization results amounts to showing the rescaled process $(X_t^{(\epsilon)} := \epsilon X_{t/\epsilon^2}\,;\, t \geq 0)$ satisfies a *functional central limit theorem* - or *invariance principle*. Namely one shows that the distribution of the process $X^{(\epsilon)}$, on the space of continuous functions from $[0, +\infty)$ which values in $\mathbb{R}^d$, weakly converges to the law of a Brownian motion with covariance matrix $\bar{\sigma}$.

Let us now describe more precisely the different results that one finds in the literature and that are relevant here.

We let $I_0 := \mathbb{R}^d/\mathbb{Z}^d$ be the unit torus. The potential V is assumed to satisfy $V(x+z) = V(x)$ for all $x \in \mathbb{R}^d$ and $z \in \mathbb{Z}^d$. We may sometimes identify I_0 with a cube in $\mathbb{R}^d$.

We use the notation $(X_t\,;\, t \geq 0)$ to denote the canonical process on $C([0, +\infty); \mathbb{R}^d)$ and P_x to denote the law of the process generated by L with starting point $x \in \mathbb{R}^d$. Also denote with

$$P_u(.) := \int_{I_0} P_x(.)\, dx,$$

the law of the process when starting with uniform law on I_0, and more generally

$$P_w(.) := \int P_x(.)w(x)\, dx,$$

the law of the process when the initial law has density w with respect to dx.

In [7], the authors assume the function V is smooth. Observe it implies that V is bounded. They use the stochastic differential equation (3.1.1) to define the process X for any given initial point $x \in \mathbb{R}^d$ and establish the invariance principle under P_x for any $x \in \mathbb{R}^d$.

These results were later generalized in [8] to the case of a measurable and bounded potential V. Then the construction of the process is based on Dirichlet form theory. Observe however that when V is bounded, the operator L is then uniformly elliptic, so that all kind of a-priori Gaussian bounds and Hölder regularity estimates are known to hold for the fundamental solution of L. These in particular allow to define P_x for all $x \in \mathbb{R}^d$. Another consequence is that it is then sufficient to prove the invariance principle under P_u. Indeed one may combine Hölder regularity estimates and the invariance principle under P_u to deduce it under P_x for any $x \in \mathbb{R}^d$.

The singular case - when V is not assumed to be bounded anymore - is considered in [9] (as a special case of diffusions in a random environment). The authors assume that both e^V and e^{-V} are locally integrable and they use 2-scale arguments to show homogenization results and the central limit theorem under P_u: the law of $X_t/\sqrt{t}$ under P_u converges to the Gaussian distribution with covariance $\bar{\sigma}$.

An alternative approach, which also applies to random environments, was previously developed in [5]. It is based on the interpretation of the process X as an additive functional of a reversible Markovian dynamics, the so-called process of the *environment seen from the particle*. In our context, the process of the environment seen from the particle is just the projection of X on the torus I_0. Applying the general results from [5] in the periodic setting, one gets a functional central limit theorem under P_u if V is such that ∇V is integrable and $e^V + e^{-V} \in L^1(I_0; dx)$, see part 6 in [5]. It is quite possible that, at the cost of some extra work, one can remove the assumption on ∇V and then, still using the arguments in [5], obtain the invariance principle under the only assumption that $e^V + e^{-V} \in L^1(I_0; dx)$. Observe however that, as in [9], the approach in [5] can only give *averaged* results under P_u and does not tell us anything on the behavior of the process under P_x for a given starting point x.

The question which interests us in this paper is to show the *individual* invariance principle without assuming V is bounded. Namely we wish to show that, under P_x, for a given x, the process scales to Brownian motion. Note however that the approach through Dirichlet form only provides a definition of P_x for x outside a set of zero Lebesgue measure. The main result of this work is the following:

Theorem 1. *Assume that $e^V + e^{-V} \in L^1(I_0; dx)$. There exists a positive symmetric non-degenerate matrix $\bar{\sigma}$ such that for almost all x $\in \mathbb{R}^d$, under P_x, the family of processes $(X^{(\epsilon)}\,;\,\epsilon > 0)$ converges in distribution, as ϵ tends to zero, towards the law of a Brownian motion with covariance matrix $\bar{\sigma}$.*

We note that the integrability condition $e^V + e^{-V} \in L^1(I_0; dx)$ is reasonable. On the one hand, it arises naturally when one tries to prove the existence of the process through constructing its Dirichlet form, see Part 1.3. On the other hand, in the case $d = 1$, it is known that the convergence of $X^{(\epsilon)}$ towards a non-degenerate Brownian motion holds if and only if $e^V + e^{-V} \in L^1(I_0; dx)$, see [6]. It does not mean the condition $e^V + e^{-V} \in L^1(I_0; dx)$ is always necessary for the individual functional C.L.T. to hold. Indeed one might think of examples of perforated environments, where V takes the value $+\infty$ on a set of non zero measure, and nevertheless the individual functional C.L.T. may hold.

Our individual invariance principle for almost any starting point x corresponds to what is known in the more general context of random environments as a *quenched* invariance principle where one gets a functional C.L.T. for a given starting point and almost any realization of the environment.

In the context of random walks with random conductances, a lot of effort was recently made to get quenched invariance principles. In particular it was recently proved in [1] that the quenched functional C.L.T. holds for random stationary conductances satisfying some moment conditions. Observe however that the moment condition used in [1] is much more restrictive than ours. In particular it gets worse as the dimension grows.

Our strategy for proving Theorem 1 follows some classical steps: we rely on the construction of the so-called *corrector*: this is a periodic function $v : \mathbb{R}^d \to \mathbb{R}^d$ such that the process $t \to X_t + v(X_t)$ is a martingale with stationary increments under P_x. It then follows that the process $X^{(\epsilon)} + \epsilon v(\frac{1}{\epsilon} X^{(\epsilon)})$ satisfies the invariance principle, see Part 3.3.3, and the key step of the proof of the Theorem consists in showing that the corrector part $\epsilon v(\frac{1}{\epsilon} X^{(\epsilon)})$ tends to 0.

In order to control the corrector, and actually also in order to show its existence, we rely on the following Sobolev inequality:

Theorem 2. *Let V be a measurable function defined on I_0 satisfying $e^V + e^{-V} \in L^1(I_0; dx)$. Then there exists a positive and bounded function w, there exists $r^* > 2$ and there exists a constant c such that:*

$$\left(\int_{I_0} |f(x)|^{r^*} \, w(x) \, dx \right)^{2/r^*} \leq c \int_{I_0} |\nabla f(x)|^2 \, e^{-V(x)} dx. \tag{3.1.2}$$

for all function f defined on I_0, centered and C^1 there.

Theorem 2 is the main theorem of chapter 2 (see Theorem 2.1.1) of chapter 2.

Once this Sobolev-type inequality got, we may copy the strategy of [3]: we derive a first invariance principle for a time-changed version of the process X and finally prove Theorem 1 in Part 3.3.

We believe the Sobolev inequality from Theorem 2 has its own interest.

3.2 Existence of diffusion process on $\mathbb{R}^d$

We recall that I_0 stands for the unit torus: $I_0 := \mathbb{R}^d / \mathbb{Z}^d$. We denote with $d\dot{x}$ the Lebesgue measure on I_0. When we say that a function is *integrable on* I_0 without any further precision, it is understood that this function is integrable with respect to $d\dot{x}$.

In the sequel, $C([0, +\infty), I_0)$ is the space of continuous functions defined on $[0, +\infty)$ with values in I_0 and $(\dot{X}_t \, ; \, t \geq 0)$ is the canonical coordinate process on $C([0, +\infty), I_0)$.

Let $x \in \mathbb{R}^d$ whose projection on I_0 we denote with $\dot{x}$. Given a trajectory $(\dot{X}_t \, ; \, t \geq 0)$ in $C([0, +\infty), I_0)$ such that $\dot{X}_0 = \dot{x}$, we let $(X_t \, ; \, t \geq 0)$ be the $\mathbb{R}^d$-valued trajectory obtained by lifting $\dot{X}$. That is $(X_t \, ; \, t \geq 0)$ is the unique element in $C([0, +\infty), \mathbb{R}^d)$ satisfying $X_0 = x$ and whose projection on I_0 coincides with $\dot{X}_t$ for all times t.

As we have said in "Introduction générale", we shall consider the divergence-form operator $\dot{L}$ on $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$, formally defined by:

$$\dot{L} f(\dot{x}) = \frac{1}{2} e^{V(\dot{x})} div(e^{-V(\dot{x})} \nabla f(\dot{x})).$$

Ours first goal in this section is to prove that there exists a diffusion process associated with operator $\dot{L}$ when e^V and e^{-V} are both integrable on I_0. In other words, we want to prove the existence of a Markov law $(P_{\dot{x}}\,;\,\dot{x}\in I_0)$ on $C([0,+\infty);I_0)$ with generator $\dot{L}$. Once this is done, we shall define the diffusion process in $\mathbb{R}^d$ by lifting the trajectory from the torus to $\mathbb{R}^d$. We first study the Dirichlet form associated with $\dot{L}$.

Let f and g be a real-valued functions defined on I_0. For $i=1...d$, let $\partial_i f$ denote the the weak derivative of f in the i-th direction. Let f and g be such that for any $i=1...d$, then $\partial_i f$ belongs to $L^2(I_0;e^{-V(\dot{x})}d\dot{x})$. We then define the bilinear forms

$$\dot{\xi}(f,g):=\frac{1}{2}\int_{I_0}\nabla f(\dot{x})\cdot\nabla g(\dot{x})\,e^{-V(\dot{x})}d\dot{x}, \tag{3.2.1}$$

and, if f and g are further assumed to belong to $L^2(I_0;e^{-V(\dot{x})}d\dot{x})$,

$$\dot{\xi}_1(f,g):=\dot{\xi}(f,g)+\int_{I_0}f(\dot{x})g(\dot{x})\,e^{-V(\dot{x})}d\dot{x}.$$

More generally, for $\alpha>0$ and such functions f anf g, let

$$\dot{\xi}_\alpha(f,g):=\dot{\xi}(f,g)+\alpha\int_{I_0}f(\dot{x})g(\dot{x})\,e^{-V(\dot{x})}d\dot{x}.$$

Let $\mathcal{H}^1(I_0;e^{-V})$ be the set of functions in $L^2(I_0;e^{-V(\dot{x})}d\dot{x})$ with all derivatives $\partial_i f$ belonging to $L^2(I_0;e^{-V(\dot{x})}d\dot{x})$.

In view of Proposition 1.3.3 in Sectionn 1.3 $(\dot{\xi},H^1(I_0;e^{-V}))$ is a regular and local Dirichlet form associated with $\dot{L}$, where $H^1(I_0;e^{-V}):=\overline{C^\infty(I_0)}^{\dot{\xi}_1}$ be the completion of $C^\infty(I_0)$ with respect to the norm $\dot{\xi}_1$. Indeed, we have proved that $(\dot{\xi},\mathcal{H}^1(I_0;e^{-V}))$ is closable which implies that $(\dot{\xi},C(I_0))$ is also closable. By consequence, applying Theorem 1.2.19 of chapter 1, $(\dot{\xi},H^1(I_0;e^{-V}))$ is a closed form.

Following [4], part 1.5 of chapter 1, we also define the *extended domain* $H^1_e(I_0;e^{-V})$: this is the set of measurable functions f on I_0, such that $|f|<\infty$ a.e and there exists a $\dot{\xi}$-Cauchy sequence (f_n) in $H^1(I_0;e^{-V})$ such that $\lim_{n\to\infty}f_n=f$ a.e.

Since $(\dot{\xi},H^1(I_0;e^{-V}))$ is a regular and local Dirichlet form, there exists a Markov law on $C([0,+\infty),I_0)$ whose Dirichlet form is $(\dot{\xi},H^1(I_0;e^{-V}))$. This law is denoted with $(P_{\dot{x}}\,;\,\dot{x}\in I_0)$. It is uniquely defined for Lebesque almost all $\dot{x}\in I_0$. The measure $e^{-V(\dot{x})}d\dot{x}$ is reversible. The process thus defined is conservative. This property is proved Subsection 1.3.1 of chapter 1. Its generator, in the L^2 sense, is given by $\dot{L}$. Let $(E_{\dot{x}}\,;\,\dot{x}\in I_0)$ denote the expectation with respect to $(P_{\dot{x}}\,;\,\dot{x}\in I_0)$.

Let $x\in\mathbb{R}^d$ and $\dot{x}$ be its projection on I_0. We denote with P_x the law of the lifting of the trajectory $(\dot{X}_t\,;\,t\geq 0)$ to $\mathbb{R}^d$ under $P_{\dot{x}}$. Then P_x is a probability on $C([0,+\infty),\mathbb{R}^d)$.

Remark 3. *One may ask whether $H^1(I_0;e^{-V})=\mathcal{H}^1(I_0;e^{-V})$. The answer is no. See counterexample in part 4.3 of chapter 4.*

In the sequel we will have to consider time-changed processes. We discuss this construction now.

3.3 Homogenization result: proof of Theorem 1

We show the invariance principle for $\widetilde{X}$ and deduce the invariance principle for X using the relation (1.6.4) of Section 1.6. Indeed, we construct the time-changed process $\left(\widetilde{\dot{X}}_t, t > 0\right)$ on the torus by the relation (1.6.4) and we denote by $\left(\widetilde{X}_t, t > 0\right)$ the lifting time of this of $\left(\widetilde{\dot{X}}_t, t > 0\right)$. We ask the question: is $\left(\widetilde{X}_t, t > 0\right)$ the time-changed process of $(X_t, t > 0)$ by the same additive function A defined in (1.6.4)? The answer is yes. We will show this answer more clearly in Subsection 3.3.3.

3.3.1 Sobolev Inequality and time-changed process

Recall that the strictly positive bounded function w which satisfies Theorem 2 is $w = M(e^V)^{-1}$ (see proof of Theorem 2.1.1 of chapter 2). Thus, $w(\dot{x})$ is integrable on I_0 and then the measure $d\mu(\dot{x}) = w(\dot{x})d\dot{x}$ has support I_0 because it is strictly positive. Obviously, It is a Radon measure and charges no set of zero capacity.
By Theorem 1.5.5 the unique PCAF denoted by A associated with μ is:

$$A_t = \int_0^t w(\dot{X}_s)e^{-V(\dot{X}_s)}ds. \tag{3.3.1}$$

The measure μ is the Revuz measure of A.
We are now in condition the apply all the theory given in Section 1.6 of chapter 1. Let us simplifly the definition of Dirichlet form in (1.6.3).
The Hitting function denoted by $H_N u(x)$ is equal to $u(x)$ $a.e$, and in view of Theorem 1.6.3, the symmetric bilinear form $\left(\widetilde{\dot{\xi}}, \widetilde{H}^1(I_0; w)\right)$ defined on $L^2(I_0; w(d\dot{x}))$ by:

$$\begin{cases} \widetilde{H}^1(I_0; w) = \{\phi \in L^2(I_0; w(d\dot{x})) : \exists f \in H^1_e(I_0; e^{-V}) : f = \phi \text{ a.e}\} \\ \widetilde{\dot{\xi}}(\phi, \phi) = \dot{\xi}(f, f). \end{cases} \tag{3.3.2}$$

is the regular and local Dirichlet form associated with $\left(\widetilde{\dot{X}}_t, t > 0\right)$ where

$$\widetilde{\dot{X}}_t = \dot{X}_{A_t^{-1}}, \qquad A_t^{-1} = \inf\{s > 0; A_s > t\}. \tag{3.3.3}$$

In view of the definition of $H^1_e(I_0; e^{-V})$, we remark that the extended domain of $\widetilde{\dot{\xi}}$ coincides with the extended domain of $\dot{\xi}$. This property is proved in corollary 1.6.4 of chapter 1. Note that $\left(\widetilde{\dot{\xi}}, \widetilde{H}^1(I_0; w)\right)$ admits $C^\infty(I_0)$ as a core (see Theorem 1.6.3).

From now on, we assume that e^V and e^{-V} are integrable on I_0. We choose the function w given by Theorem 2.1.1 of chapter 2.
Since C^1 functions are dense in the domain of $\widetilde{\dot{\xi}}$, it follows that equation (3.1.2) is true

for any function f in $\widetilde{H}^1(I_0; w)$. Let $(\widetilde{\dot{P}}_t; t \geq 0)$ be the semi-group generated by $\widetilde{\dot{X}}$. By construction, $(\widetilde{\dot{P}}_t; t \geq 0)$ is a symmetric strongly continuous semi-group acting on $L^2(I_0; w(d\dot{x}))$. It is related to the process $\widetilde{\dot{X}}$ through the formula

$$\widetilde{\dot{P}}_t f(\dot{x}) = E_{\dot{x}}[f(\widetilde{\dot{X}}_t)],$$

for almost all $\dot{x} \in I_0$, any time t and any measurable function $f \in L^2(I_0; w(d\dot{x}))$.

As a consequence of Theorem 2, we have the following

3.3.2 Sobolev inequality and, existence and boundedness of density of probability transition

Corollaire 3.3.1. *For all positive time t, for almost every $\dot{x} \in I_0$, the law of $\widetilde{\dot{X}}_t$ under $P_{\dot{x}}$ has a density with respect to the measure $w(d\dot{x})$, say $(\widetilde{\dot{p}}_t(\dot{x}, \dot{y}); \dot{y} \in I_0)$. The function $(\dot{x}, \dot{y}) \to \widetilde{\dot{p}}_t(\dot{x}, \dot{y})$ is almost everywhere bounded on $I_0 \times I_0$.*

Proof. The proof follows a classical argument that can be found in the book [13] or the papers [14] and [15] for instance.
We only sketch it here.

In the proof below, the value of the constant c may vary from line to line.

Choose r^* from Theorem 2 and let $p = r^*/2$. Equation (3.1.2) then reads: for all C^1 and centered function f, then

$$\left(\int_{I_0} |f(\dot{x})|^{2p} w(d\dot{x}) \right)^{\frac{1}{p}} \leq c \int_{I_0} |\nabla f(\dot{x})|^2 e^{-V(\dot{x})} d\dot{x}. \tag{3.3.4}$$

Using first Hölder's inequality with parameters $2p - 1$ and $(2p-1)/(2p-2)$ and then (3.3.4) we deduce that

$$\begin{aligned} \int_{I_0} f^2(\dot{x}) w(d\dot{x}) &= \int_{I_0} |f(\dot{x})|^{2p/(2p-1)} |f(\dot{x})|^{(2p-2)/(2p-1)} w(d\dot{x}) \\ &\leq \left(\int_{I_0} |f(\dot{x})|^{2p} w(d\dot{x}) \right)^{1/(2p-1)} \left(\int_{I_0} |f(\dot{x})| w(d\dot{x}) \right)^{(2p-2)/(2p-1)} \\ &\leq c \left(\int_{I_0} |\nabla f(\dot{x})|^2 e^{-V(\dot{x})} d\dot{x} \right)^{p/(2p-1)} \left(\int_{I_0} |f(\dot{x})| w(d\dot{x}) \right)^{(2p-2)/(2p-1)}. \end{aligned} \tag{3.3.5}$$

Using the density of C^1 functions, inequality (3.3.5) can be extended for all centered functions $f \in \widetilde{H}^1(I_0; w)$. Then $\int_{I_0} |\nabla f(\dot{x})|^2 e^{-V(\dot{x})} d\dot{x} = 2\widetilde{\dot{\xi}}(f, f)$.

Let $f \in L^2(I_0; w(dx))$ and set $f_t := \widetilde{\dot{P}}_t f$. Assume that f is centered. Then so is f_t for any t.
Let $v(t) := \left(\int_{I_0} |f(\dot{x})| w(d\dot{x}) \right)^{-2} \int_{I_0} f_t(\dot{x})^2 w(d\dot{x})$.

On the one hand, the function v satisfies

$$v'(t) = -2\left(\int_{I_0} |f(\dot{x})|w(d\dot{x})\right)^{-2} \tilde{\dot{\xi}}(f_t, f_t).$$

Therefore, using (3.3.5), we have

$$v(t) \leq \left(-cv^{'}(t)\right)^{\alpha} \qquad \text{with } \alpha = \frac{p}{2p-1}.$$

(We used the fact that

$$\int_{I_0} |f_t(\dot{x})|\, w(d\dot{x}) \leq \int_{I_0} \tilde{\dot{P}}_t\, |f(\dot{x})|\, w(d\dot{x}) = \int_{I_0} |f(\dot{x})|\, w(d\dot{x}).)$$

From this differential inequality, we deduce that $v(t)$ is bounded by a constant, say $c(t)$, independently of f and therefore

$$\int_{I_0} \left(\tilde{\dot{P}}_t f\right)^2 w(d\dot{x}) \leq c(t) \left(\int_{I_0} |f(\dot{x})|w(d\dot{x})\right)^2.$$

The duality property gives:

$$\left\|\tilde{\dot{P}}_t f\right\|_{L^\infty} = \sup\left\{\frac{\left|\int_{I_0} g(\tilde{\dot{P}}_t f)w(dx)\right|}{\|g\|_{L^1(I_0;w(d\dot{x}))}}; g \in L^1(I_0; w(d\dot{x})), g \neq 0\right\}.$$

Thus we have using Hölder's inequality again:

$$\left\|\tilde{\dot{P}}_t f\right\|_{L^\infty} \leq \sqrt{c(t)}\left(\int_{I_0} |f(\dot{x})|^2\, w(d\dot{x})\right)^{\frac{1}{2}} \qquad \forall t > 0.$$

As a consequence

$$\begin{aligned}
&\left\|\tilde{\dot{P}}_t f\right\|_{L^\infty} = \left\|\tilde{\dot{P}}_{t/2}(\tilde{\dot{P}}_{t/2} f\right) \|_{L^\infty} \\
&\leq \sqrt{c(t/2)}\left\|\tilde{\dot{P}}_{t/2} f\right\|_{L^2(I_0;w(d\dot{x}))} \leq \sqrt{c(t/2)}\sqrt{c(t/2)}\, \|f\|_{L^1(I_0;w(dx))}.
\end{aligned}$$

We deduce that:

$$\left\|\tilde{\dot{P}}_t f\right\|_{L^\infty(I_0)} \leq c(t/2)\, \|f\|_{L^1(I_0;w(d\dot{x}))} \qquad \forall t > 0.$$

This inequality extends to all non-negative functions f.

By taking $f = \mathbf{1}_A$, with A any Borelian contained in I_0 we deduce that the semi-group $\tilde{\dot{P}}$ is absolutely continuous with respect to the measure $w(d\dot{x})$ with a density bounded by $c(t/2)$. □

3.3.3 Sobolev inequality and, construction and convergence of corrector

We already defined the process $(X_t\,;\,t \geq 0)$ as the lifting of $(\dot{X}_t\,;\,t \geq 0)$. Recall that the process $(\widetilde{\dot{X}}_t\,;\,t \geq 0)$ is obtained from $\dot{X}$ by the time change A from equation (3.3.3). We similarly introduce the process $(\widetilde{X}_t\,;\,t \geq 0)$ as the time-change of X through the additive functional A. Note that the projection on I_0 of the trajectory of $\widetilde{X}$ is then $(\widetilde{\dot{X}}_t\,;\,t \geq 0)$.

In this section, we prove the existence of a corrector to the process $\widetilde{X}$, i.e. we construct a function v, defined on I_0, such that $\widetilde{M_t} := \widetilde{X}_t + v(\widetilde{\dot{X}}_t)$ is a continuous martingale under P_x for almost all $x \in \mathbb{R}^d$.

We use the construction of the Dirichlet form $\dot{\xi}$ from part 3.2, where the function w is the one given by Theorem 2. In particular recall that $H^1_e(I_0; e^{-V})$ is the extended domain of $\widetilde{\dot{\xi}}$. Observe that the Sobolev inequality (3.1.2) implies that functions in $H^1_e(I_0; e^{-V})$ are also in $L^{r^*}(I_0; w(d\dot{x}))$ and therefore in $L^1(I_0; w(d\dot{x}))$.
We call $H^1_{o,e}(I_0)$ the quotient space obtained by identifying functions in $H^1_e(I_0; e^{-V})$ when they differ by a constant. Equivalently $H^1_{o,e}(I_0)$ is the sub-space of centered functions in $H^1_e(I_0)$.

3.3.3.1 Construction of corrector

To start the construction of the corrector, we need the following proposition. Recall that $H^1_{o,e}(I_0)$ the quotient space obtained by identifying functions in $H^1_e(I_0; e^{-V})$ when they differ by a constant. Equivalently $H^1_{o,e}(I_0)$ is the sub-space of centered functions in $H^1_e(I_0; e^{-V})$

Proposition 3.3.2. *$(H^1_{o,e}(I_0), \widetilde{\dot{\xi}})$ is a Hilbert space.*

Proof. The proposition follows from the Poincaré inequality

$$\int_{I_0} |f(\dot{x})|^2\, w(d\dot{x}) \leq c \int_{I_0} |\nabla f(\dot{x})|^2\, e^{-V(\dot{x})} d\dot{x}, \tag{3.3.6}$$

which is itself a consequence of (3.1.2) and Hölder's inequality.

On the one hand, (3.3.6) implies that $\widetilde{\dot{\xi}}$ is a norm on $H^1_{o,e}(I_0)$ and it is equivalent to $\widetilde{\dot{\xi}}_1$. Since $\widetilde{H}^1(I_0; w)$ is complete with respect to $\widetilde{\dot{\xi}}_1$, and because the condition of being centered is closed in $\widetilde{H}^1(I_0; w)$, we get that $(H^1_{o,e}(I_0), \widetilde{\dot{\xi}})$ is complete. □

Remark 4. *Observe, as above, that functions in $H^1_e(I_0; e^{-V})$ are also in $L^2(I_0; w(dx))$. Therefore $H^1_e(I_0; e^{-V}) = \widetilde{H}^1(I_0; w)$.*

Construction of the corrector: for $i = 1...d$, consider the expression:

$$L_i : f \mapsto -\frac{1}{2}\int_{I_0} \partial_i f(x)\, e^{-V(x)} dx\,.$$

Then L_i is a continuous linear map on $(H^1_{o,e}(I_0), \widetilde{\dot{\xi}})$.
We identify $H^1_{o,e}(I_0)$ and its dual. Thus, there exists a unique v_i in $H^1_{o,e}(I_0)$ such that:

$$(3.3.7) \qquad \begin{aligned} -\frac{1}{2}\int_{I_0} \partial_i f(x)\, e^{-V(x)} dx &= \frac{1}{2}\int_{I_0} \nabla v_i \cdot \nabla f\, e^{-V(x)} dx \\ &= \widetilde{\dot{\xi}}(v_i, f), \end{aligned}$$

for all $f \in H^1_{o,e}(I_0)$.

The function v_i is called the *corrector* in the direction i. We may also consider the vector-valued corrector $v := (v_1, ..., v_d) : I_0 \mapsto \mathbb{R}^d$. We also define the function $u = (u_1, ..., u_d)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ by $u(x) = x + v(\dot{x})$ (where $\dot{x}$ is the projection of x on I_0).

Proposition 3.3.3. *The process* $(\widetilde{M}_t := u(\widetilde{X}_t) = \widetilde{X}_t + v(\widetilde{\dot{X}}_t)\,;\, t \geq 0)$ *is a continuous martingale under* P_x *for almost all* $x \in \mathbb{R}^d$ *and satisfies*

$$(3.3.8) \qquad \langle \widetilde{M} \rangle_t = \int_0^t \frac{e^{-V(\widetilde{\dot{X}}_s)}}{w(\widetilde{\dot{X}}_s)} ((\delta + \nabla v)(\delta + \nabla v))(\widetilde{\dot{X}}_t)\, dt\,,$$

where $((\delta + \nabla v)(\delta + \nabla v))(.)$ *is the matrix with* (i, j) *entry given by* $(\delta_i + \nabla v_i(.)) \cdot (\delta_j + \nabla v_j(.))$ *and* δ_i *is the unit vector in direction* i.

Proof. We recall from [4], chapter 5, that for all functions $f \in \widetilde{H}^1(I_0; w)$, the process $t \to f(\widetilde{\dot{X}}_t)$ has a unique Itô-Fukushima decomposition under $P_{\dot{x}}$, for almost every $\dot{x}$, as a sum of two terms:

$$(3.3.9) \qquad f(\widetilde{\dot{X}}_t) - f(\widetilde{\dot{X}}_0) = M^f_t + N^f_t\,,$$

where M^f is a continuous martingale additive functional and N^f is a functional of zero energy. Besides, for f and g in $\widetilde{H}^1(I_0; w)$, one has the following expression for the square bracket:

$$(3.3.10) \qquad \langle M^f, M^g \rangle_t = \int_0^t \frac{e^{-V(\widetilde{\dot{X}}_s)}}{w(\widetilde{\dot{X}}_s)} \nabla f(\widetilde{\dot{X}}_s) \cdot \nabla g(\widetilde{\dot{X}}_s)\, ds.$$

See in particular example 5.2.1 and formula (5.2.46) in [4].

These formulas do not immediately yield a decomposition for the process $\widetilde{M}$. Indeed, we could directly apply the Itô-Fukushima decomposition to the function v which belongs to $\widetilde{H}^1(I_0; w)$, but, although the process $\widetilde{X}$ is also an additive functional of $\widetilde{\dot{X}}$, it is not of the form (3.3.9). In order to deal this difficulty, we rely on a localization argument.

Let $\dot{x} \in I_0$ and choose $x \in \mathbb{R}^d$ whose projection on I_0 is $\dot{x}$. Let J_0 be a closed cube in I_0 centered at $\dot{x}$. We identify J_0 with a closed cube in $\mathbb{R}^d$ centered at x, say J_1, and let $\phi : J_0 \to J_1$ be the identification map.

We will denote with $({}^c\widetilde{\dot{X}}_t\,;\, t \geq 0)$ the process obtained by reflecting $\widetilde{\dot{X}}$ on the boundary of J_0. The construction of ${}^c\widetilde{\dot{X}}$ mimics the construction of $\widetilde{\dot{X}}$ in part 3.3.1 except that we consider the bilinear form (3.2.1) on smooth functions with support in J_0. Let ${}^c\widetilde{\dot{\xi}}$ be the Dirichlet form of the process ${}^c\widetilde{\dot{X}}$.

Let τ be the hitting time of the boundary of J_0. Note that the two processes ${}^c\widetilde{\dot{X}}_t$ and $\widetilde{\dot{X}}_t$ coincide in law until time τ. Besides, the two processes $\widetilde{X}$ and $\phi({}^c\widetilde{\dot{X}})$ also coincide until time τ. Thus we get that

$$u(\widetilde{X}_t) - u(\widetilde{X}_0) = (v+\phi)({}^c\widetilde{\dot{X}}_t) - (v+\phi)({}^c\widetilde{\dot{X}}_0)\,, \tag{3.3.11}$$

for times $t < \tau$ (in the sense that these two processes have the same law).

Now observe that the functions v and ϕ both belong to the domain of the Dirichlet form ${}^c\widetilde{\dot{\xi}}$. Thus the process $(v+\phi)({}^c\widetilde{\dot{X}})$ admits an Itô-Fukushima decomposition as

$$(v+\phi)({}^c\widetilde{\dot{X}}_t) - (v+\phi)({}^c\widetilde{\dot{X}}_0) = M_t^{(0)} + N_t^{(0)}\,.$$

On the one hand, the function $\partial_i\phi$ is constant and equals the unit vector in direction i. On the other hand, the function v satisfies equation (3.3.7). Thus we get that ${}^c\widetilde{\dot{\xi}}(f, v+\phi) = 0$ for all smooth functions f supported in the interior of J_0. In other words, the function $v+\phi$ is harmonic for the process ${}^c\widetilde{\dot{X}}$ killed at time τ. It implies that the process $(u({}^c\widetilde{\dot{X}}_t) - u({}^c\widetilde{\dot{X}}_0)\,;\, 0 \leq t < \tau)$ is a local martingale and $N_t^{(0)} = 0$ for all times $t < \tau$. Using (3.3.11), we conclude that the process $(u(\widetilde{\dot{X}}_t) - u(\widetilde{\dot{X}}_0)\,;\, 0 \leq t < \tau)$ is a local martingale.

In order to prove that $(u(\widetilde{\dot{X}}_t) - u(\widetilde{\dot{X}}_0)\,;\, 0 \leq t)$ is a local martingale for all times, one iterates this reasoning using the Markov property. The computation of the bracket follows from formula (3.3.10).

□

We show the invariance principle for $\widetilde{X}$ and deduce the invariance principle for X using the relation (3.3.3).

Let $\widetilde{X}_t^{(\epsilon)} := \epsilon \widetilde{X}_{t/\epsilon^2}$ and $\widetilde{\dot{X}}_t^{(\epsilon)} := \epsilon \widetilde{\dot{X}}_{t/\epsilon^2}$.

Proposition 3.3.4. *There exists a positive symmetric non-degenerate matrix σ such that for almost all $x \in \mathbb{R}^d$, under P_x, the family of processes $(\widetilde{X}^{(\epsilon)}\,;\, \epsilon > 0)$ converges in distribution, as ϵ tends to zero, towards the law of a Brownian motion with variance σ.*

The proof of Proposition 3.3.4 is in two steps:

First step: invariance principle for the martingale part.

We define $u_i^\epsilon(x) = \epsilon u_i(\frac{x}{\epsilon})$ and let

$$\widetilde{M}_t^{i,\epsilon} := u_i^\epsilon(\widetilde{X}_t^\epsilon) - u_i^\epsilon(\widetilde{X}_0^\epsilon),$$
$$\widetilde{M}_t^\epsilon := (\widetilde{M}_t^{1,\epsilon}, ..., \widetilde{M}_t^{d,\epsilon}).$$

Lemma 3.3.1. *There exists a positive symmetric non-degenerate matrix σ such that for almost all $x \in \mathbb{R}^d$, under P_x, the family of processes $(\widetilde{M}^\epsilon; \epsilon > 0)$ converges in distribution, as ϵ tends to zero, towards the law of a Brownian motion with covariance matrix σ.*

Proof. We will need the invariance principle for continuous martingales. For the reader's convenience, we provide here the formulation of theorem 5.1 of [2].

Theorem 5.1 of [2] (Helland 1982)
Let m^ϵ be a family of continuous real-valued martingales with quadratic variation processes $< m^\epsilon >$ satisfying the following condition:
(i) there exists a real number $a > 0$ such that for any $t > 0$, as ϵ tends to zero, then $< m^\epsilon >_t$ converges in probability to at.
Then, as ϵ tends to zero, the sequence of processes $m^\epsilon(.)$ converges in law in the uniform topology to a Brownian motion with covariance a.

Let σ be the matrix with entries given by

$$(\sigma)_{i,j} := \int_{I_0} (\delta_i + \nabla v_i(\dot{x})) \cdot (\delta_j + \nabla v_j(\dot{x})) e^{-V(\dot{x})} d\dot{x}.$$

Note that, by construction, ∇v_j belongs to $L^2(I_0; e^{-V(\dot{x})} d\dot{x})$.

In view of Proposition 3.3.3, we know that $\widetilde{M}_t^{i,\epsilon}$ is a square integrable martingale which quadratic variation

$$\int_0^t |\delta_i + \nabla v_i|^2 (\frac{\widetilde{X}_s^{(\epsilon)}}{\epsilon}) \left(\frac{e^{-V}}{w}\right) (\frac{\widetilde{X}_s^{(\epsilon)}}{\epsilon}) ds = \int_0^t |\delta_i + \nabla v_i|^2 (\frac{\dot{\widetilde{X}}_s^{(\epsilon)}}{\epsilon}) \left(\frac{e^{-V}}{w}\right) (\frac{\dot{\widetilde{X}}_s^{(\epsilon)}}{\epsilon}) ds,$$

because V is periodic, $|\delta_i + \nabla v_i|^2$ is periodic and w is also periodic.

More generally, for any vector $e \in \mathbb{R}^d$, then $e \cdot \widetilde{M}_t^\epsilon := \sum_i e_i \widetilde{M}_t^{i,\epsilon}$ is a square integrable martingale with bracket

$$\langle e \cdot \widetilde{M}^\epsilon \rangle_t = \int_0^t \left(\sum_i e_i(\delta_i + \nabla v_i) \right)^2 (\frac{\dot{\widetilde{X}}_s^{(\epsilon)}}{\epsilon}) \left(\frac{e^{-V}}{w}\right) (\frac{\dot{\widetilde{X}}_s^{(\epsilon)}}{\epsilon}) ds$$

By the ergodic Theorem for $\dot{\widetilde{X}}$, for all $t \geq 0$:

$$\langle e \cdot \widetilde{M}^\epsilon \rangle_t \longrightarrow_{\epsilon \to 0} t. \int_{I_0} \left(\sum_i e_i(\delta_i + \nabla v_i(\dot{x})) \right)^2 e^{-V(\dot{x})} d\dot{x} \text{ almost surely.}$$

Theorem 5.1 of [2], as recalled above, gives the invariance principle for the martingales $\left(e\cdot\widetilde{M}^{\epsilon}_t; t\geq 0\right)$ with asymptotic variance $e\cdot\sigma e$. Since this is true for all direction e, we deduce the invariance principle for M^ϵ itself. □

Second step:

3.3.3.2 convergence of the corrector

We have to show that the corrector part goes to zero in P_x probability for almost all $x\in\mathbb{R}^d$. For that, it suffices to prove the following equality:

$$\forall\eta>0,\ \limsup_{\epsilon\downarrow 0} P_x\left(\sup_{0\leq t\leq 1}\left|\epsilon v_i(\frac{\dot{\widetilde{X}}_t^{(\epsilon)}}{\epsilon})\right|>\eta\right)=0. \tag{3.3.12}$$

Observe that (3.3.12) implies that, for all T and for all $\eta>0$,

$$\begin{aligned}&\limsup_{\epsilon\downarrow 0} P_x\left(\sup_{0\leq t\leq T}\left|\epsilon v_i(\frac{\dot{\widetilde{X}}_t^{(\epsilon)}}{\epsilon})\right|>\eta\right)\\ =&\limsup_{\epsilon\downarrow 0} P_x\left(\sup_{0\leq t\leq 1}\left|\epsilon v_i(\frac{\dot{\widetilde{X}}_t^{(\epsilon)}}{\epsilon})\right|>\frac{\eta}{\sqrt{T}}\right)=0.\end{aligned} \tag{3.3.13}$$

We have $\widetilde{X}^{(\epsilon)}=\widetilde{M}^\epsilon - v(\dot{\widetilde{X}}^{(\epsilon)})$. Combining (3.3.13) with Lemma 3.3.1 yields Proposition 3.3.4.

□

Now, let us prove (3.3.12). We have

$$\begin{aligned}P_x\left(\sup_{0\leq t\leq 1}\left|\epsilon v_i(\frac{\dot{\widetilde{X}}_t^{(\epsilon)}}{\epsilon})\right|>\eta\right)\leq\ & P_x\left(\sup_{0\leq t\leq \epsilon^2}\left|\epsilon v_i(\frac{\dot{\widetilde{X}}_t^{(\epsilon)}}{\epsilon})\right|>\eta\right)(:=I)\\ &+P_x\left(\sup_{\epsilon^2\leq t\leq 1}\left|\epsilon v_i(\frac{\dot{\widetilde{X}}_t^{(\epsilon)}}{\epsilon})\right|>\eta\right)(:=II).\end{aligned}$$

We show that each term goes to zero.
The first term is

$$I=P_x\left(\sup_{0\leq t\leq 1}\left|v_i(\dot{\widetilde{X}}_t)\right|>\frac{\eta}{\epsilon}\right)$$

and observe that

$$P_x\left(\sup_{0\leq t\leq 1}\left|v_i(\dot{\widetilde{X}}_t)\right|>\frac{\eta}{\epsilon}\right)\to 0 \text{ when } \epsilon\longrightarrow 0$$

by continuity: the map $t \longmapsto v_i(\widetilde{\dot{X}}_t)$ is continuous because v_i is in the extended domain of $\widetilde{\dot{\xi}}$ (see Theorem 1.4.10 of chapter 1).

The second term is equal to

$$II = P_x\left(\sup_{\epsilon^2 \le t \le 1}\left|\epsilon v_i(\frac{\widetilde{\dot{X}}_t^{(\epsilon)}}{\epsilon})\right| > \eta\right) = P_x\left(\sup_{1 \le t \le \epsilon^{-2}}\left|\epsilon v_i(\widetilde{\dot{X}}_t)\right| > \eta\right).$$

By the Markov property, the existence and the boundedness of the density at $t = 1$, we get that:

$$\begin{aligned} II &= \int_{I_0} \widetilde{\dot{p}}_1(\dot{x}, \dot{y}) w(\dot{y}) P_y\left(\sup_{0 \le t \le \epsilon^{-2}-1}\left|\epsilon v_i(\widetilde{\dot{X}}_t)\right| > \eta\right) d\dot{y} \\ &\le c P_w\left(\sup_{0 \le t \le \epsilon^{-2}}\left|\epsilon v_i(\widetilde{\dot{X}}_t)\right| > \eta\right). \end{aligned}$$

We use the Lemma 1.6.5, proved in chapter 1, to show that this last term goes to zero when ϵ goes to zero.

We claim that Lemma 1.6.5 implies that, for all $\eta > 0$, then

$$P_w(\sup_{0 \le t \le \epsilon^{-2}}\left|\epsilon v_i(\widetilde{\dot{X}}_t)\right| > \eta) \longrightarrow 0 \text{ when } \epsilon \downarrow 0. \tag{3.3.14}$$

Indeed, let $v_s = \widetilde{\dot{P}}_s v_i$. Then v_s is also in the extended domain of $\widetilde{\dot{\xi}}$ (see Lemma 1.3.7 of chapter 1) and we have:

$$\begin{aligned} P_w(\sup_{0 \le t \le \epsilon^{-2}}\left|\epsilon v_i(\widetilde{\dot{X}}_t)\right| > \eta) &\le P_w(\sup_{0 \le t \le \epsilon^{-2}}\left|\epsilon (v_i - v_s)(\widetilde{\dot{X}}_t)\right| > \frac{\eta}{2}) \\ &\quad + P_w(\sup_{0 \le t \le \epsilon^{-2}}\left|\epsilon v_s(\widetilde{\dot{X}}_t)\right| > \frac{\eta}{2}). \end{aligned} \tag{3.3.15}$$

Note that $v_s(\dot{x}) = \int_{I_0} v_i(\dot{y}) \widetilde{\dot{p}}_s(\dot{x}, \dot{y}) w(\dot{y}) d\dot{y} \le c(s) \left\|v_i\right\|_{L^2(I_0; w(d\dot{x}))}$ $a.e\ \dot{x} \in I_0$ where $c(s) = \sup_{\dot{x}, \dot{y} \in I_0} \widetilde{\dot{p}}_s(\dot{x}, \dot{y})$. Therefore the second term in (3.3.15) vanishes when ϵ is small enough. By Lemma 1.6.5 applied to the function $v_i - v_s$,

$$\limsup_{\epsilon \to 0} P_w(\sup_{0 \le t \le \epsilon^{-2}}\left|\epsilon (v_i - v_s)(\widetilde{\dot{X}}_t)\right| > \frac{\eta}{2}) \le 2\frac{e^1}{\eta}\sqrt{\widetilde{\dot{\xi}}(v_i - v_s; v_i - v_s)}.$$

This last bound holds for any $s > 0$ and

$$\lim_{s \to 0} \widetilde{\dot{\xi}}(v_i - v_s; v_i - v_s) = 0$$

as follows from Lemma 1.3.7 of chapter 1.

Thus we are done with the proof of (3.3.14) and the proof of the convergence towards zero of (II) and (3.3.12) follows.

□

As in [3], we now deduce the invariance principle for $X^{(\epsilon)}$ from the invariance principle for $\widetilde{X}^{(\epsilon)}$.

3.3.4 Invariance principle for X

In this part of the work, we deduce from the invariance principle for $\widetilde{X}$ that the rescaled process $\big(X^\epsilon(t) = \epsilon X(\frac{t}{\epsilon^2})\big)$ converges in distribution to a Brownian motion. We recall that a family of continuous processes (Y^ϵ) is tight under P if and only if it satisfies the following compactness criterion:

$$\lim_{\gamma\downarrow 0}\limsup_{\epsilon\downarrow 0} P\left(\sup_{\substack{|t-s|\le\gamma\\ 0<s,t<T}} |Y^\epsilon_t - Y^\epsilon_s| > R\right) = 0 \tag{3.3.16}$$

for all $T > 0$ and $R > 0$ (see [16], Theorem 7.5).

Consider the two sequences of processes $(X^{(\epsilon)})$ and $(\widetilde{X}^{(\epsilon)})$. We recall that by definition of $\widetilde{X}$:

$$\widetilde{X}^{(\epsilon)}(t) = X^{(\epsilon)}\left(\epsilon^2 (A)^{-1}(t\epsilon^{-2})\right)$$

Define $A^\epsilon_t := \epsilon^2 A_{t/\epsilon^2}$.

The large time asymptotic of the time changed A is easily deduced from the ergodic theorem as stated in the following Lemma:

Lemma 3.3.2. *There exists a constant k such that, under P_x for almost any x, the sequence of processes A^ϵ almost surely converges to the process $(kt; t \ge 0)$ uniformly on any compact i.e., for all T,*

$$\sup_{t\in[0,T]} |A^\epsilon(t) - kt| \longrightarrow_{\epsilon\to 0} 0 \tag{3.3.17}$$

P_x a.e. for almost all x.

Proof. The ergodic theorem implies that

$$\begin{aligned}\frac{A(t)}{t} =& \frac{\int_0^t w(\dot{X}_s)e^{V(\dot{X}_s)}ds}{t}\\ \longrightarrow_{t\to\infty}& \Big(\int_{I_0} e^{-V(\dot{x})}d\dot{x}\Big)^{-1}\int_{I_0} e^{V(\dot{x})}w(\dot{x})e^{-V(\dot{x})}d\dot{x}\\ =& \Big(\int_{I_0} e^{-V(\dot{x})}d\dot{x}\Big)^{-1}\int_{I_0} w(\dot{x})d\dot{x} := k\end{aligned}$$

P_x a.e. for almost all x.

Observe that the map $t \longmapsto kt$ is continuous in $[0, T]$ and, for all ϵ, the map $t \longmapsto A^\epsilon_t$ is non-decreasing.

Thus Dini's theorem applies and we deduce the uniform convergence from the pointwise convergence.

□

Lemma 3.3.3. *For all $T > 0$ and all $R > 0$, we have*

$$\lim_{\epsilon \to 0} P_x(\sup_{t \in [0,T]} \left| X_t^{(\epsilon)} - \widetilde{X}_{kt}^{(\epsilon)} \right| > R) = 0,$$

for almost all x.

Proof. Since $A(t)$ is bijective (continuous and strictly monotone), we have:

$$\widetilde{X}_t^{(\epsilon)} = X^{(\epsilon)} \left(\epsilon^2 (A)^{-1} (t\epsilon^{-2}) \right) \Leftrightarrow X_t^{(\epsilon)} = \widetilde{X}^{(\epsilon)} \left(\epsilon^2 A(t\epsilon^{-2}) \right).$$

Choose $\theta > 0$. If $\sup_{t\in[0,T]} \left| X_t^{(\epsilon)} - \widetilde{X}_{kt}^{(\epsilon)} \right| > R$, then either $\sup_{t\in[0,T]} \left| A_t^{(\epsilon)} - kt \right| > \theta$ or $\sup_{t\in[0,kT];|t-s|\le\theta} \left| \widetilde{X}_t^{(\epsilon)} - \widetilde{X}_s^{(\epsilon)} \right| > R$.

Lemma 3.3.2 implies that the probability of the first event tends to 0 as ϵ goes to 0. The tightness of the sequence $(\widetilde{X}^{(\epsilon)})$, see (3.3.16), ensures that the probability of the second event can be made as small as wanted by taking θ close to 0.

□

The invariance principle for the sequence $(X^{(\epsilon)})$, i.e. Theorem 1, now clearly follows from Lemma 3.3.3 and Proposition 3.3.4.

3.4 Conclusion:

We have proved a quenched invariance principle for diffusions evolving in a periodic potential, without smoothness assumptions and without uniform boundedness assumptions on the potential.

Remark 5. *We note that if we consider the more general case:*

$$\mathcal{L} = div(A\nabla)$$

where $A(x)$ is a $d * d$-symmetric matrix satisfying the following hypothesis: A is periodic and $A \in L^1(I_0)$; there exists V, measurable periodic such that: $e^V \in L^1(I_0)$ and $A \ge e^{-V} Id$, then the result of this paper holds for the diffusions associated with $\mathcal{L}$.

Indeed, recall that the main result of this work is Theorem 1 and the main tool of the proof is Theorem 2 proved in chapter 2. Furthermore, we can deduce easily that the Dirichlet form associated with $\dot{\mathcal{L}}$ on $L^2(I_0; A(\dot{x})d\dot{x})$, is

$$\text{(3.4.1)} \qquad \begin{cases} \dot{\mathcal{J}}(f,f) = \int_{I_0} \langle A(\dot{x})\nabla f(\dot{x}), \nabla f(\dot{x}) \rangle \, d\dot{x}, \\ \mathcal{D}(\mathcal{J}) = \overline{C^\infty(I_0)}^{\dot{\mathcal{J}}_1} \end{cases}$$

$\dot{\mathcal{I}}_1(f,f) = \dot{\mathcal{I}}(f,f) + \|f\|_{L^2(I_0;A(\dot{x})d\dot{x})}$. The existence of the Hunt diffusion process on I_0 is easy and the proof is the same as in Proposition 1.3.3. This following Sobolev inequality is got from Theorem 2 and the two hypotheses: $e^V \in L^1(I_0)$ and $A \geq e^{-V} Id$. It holds:

$$\left(\int_{I_0} |f(\dot{x})|^{r^*} w(\dot{x}) d\dot{x} \right)^{\frac{2}{r^*}} \leq c \int_{I_0} \langle A(\dot{x}) \nabla f(\dot{x}), \nabla f(\dot{x}) \rangle \, d\dot{x} \tag{3.4.2}$$

for all function f defined on I_0, centered and C^1. The function w is exactly the one given in Theorem 2.
We have proved the existence of Hunt process on $\mathbb{R}^d$ associated with $\mathcal{L}$ by lifting the trajectory of Hunt process on the torus I_0 associated with $\dot{\mathcal{L}}$ as done in the begining. We consider, the Sobolev inequality (3.4.2). We copy line by line by starting at Section 3.3.1 to Conclusion, to get the invariance principle for diffusion assoiated with $\mathcal{L}$.

Remark 3.4.1. *One may compare our approach with the one used in [1].*

We recall that [1] proves a quenched invariance principle for random walks with random conductances under L^p integrability conditions on the conductances and their inverses where p is much larger than 1.

The proof of [1] also relies on Sobolev inequalities. Since the environment may not be periodic, there is no finite scale that controls everything. Therefore, rather than one single Sobolev inequality, one needs a sequence of Sobolev inequalities on a growing family of balls centered at the origin. In [1], these are obtained from the classical (discrete) Sobolev embedding as in Remark 2.5.1. This explains why the integrability condition in [1] is not optimal. On the other hand, combining our technics with those of [1] in the random environment setting would require some information on the constant appearing in our Theorem 2.1.1 of chapter 2.

Chapter 4

Appendix

4.1 Introduction

This chapter is devoted to give some supplements on the third chapters below.

4.2 (B) on chapter 2

4.2.1 The maximal function is not necessary integrable

In this part, we give a simple example which prove that the maximal function Mf is not necessary integrable. Indeed, let $d = 1$, $f(y) = \chi_{(0,1/2)} \frac{d}{dy}(-log(y))$. Then $f(y) \geq 0$ and $f \in L(T; dx)$ where $T = [-\pi; \pi]$. Now for $x \in (-1/2, 0)$ we

$$Mf(x) \geq \frac{1}{2\eta} \int_{(x-\eta,x+\eta)} f(y)dy, \qquad \eta > 0.$$

In particular setting $\eta = 2\,|x|$, we get

$$Mf(x) \geq \frac{1}{4\,|x|} \int_{(0,|x|)} f(y)dy = \frac{1}{4\,|x|} \frac{1}{-log(|x|)},$$

and this function is not integrable in a neighborhood of 0.

4.2.2 A_1 weight

As a first step, we consider powers of $|x|$, i.e $|x|^\eta$. When $d = 1$ and $\eta > 0$, by letting $I = (o, b)$ we note that $(\frac{1}{b} \int_I |x|^\eta\, dx = \frac{b^\eta}{\eta+1} \to 0$ when $b \to \infty$ whereas $\inf_I |x|^\eta = 0$. The positive power of $|x|^\eta$ are ruled out, but how about negative powers? We must have $-d < \eta \leq 0$ for otherwise $|x|^\eta$ is not locally integrable. But this is the only restriction. Indeed, we have

Proposition 4.2.1. *Suppose that $-d < \eta \leq 0$. Then $|x|^\eta \in A_1(I_0)$, more precisely there exists a constant c independent of I such that*

$$\frac{1}{|I|}\int_I |x|^\eta \leq c \inf_I |x|^\eta . \tag{4.2.1}$$

Proof. Fix I and denote by I_0 the translate of I centred at 0; we consider two mutually exclusive cases, to wit (i) $2I_0 \cap I \neq \varnothing$ and $2I_0 \cap I = \varnothing$. In case (i) we have that $6I_0 \supseteq I$ and $(\frac{1}{|I|})\int_I |x|^\eta \, dx \leq (\frac{1}{|I|})\int_{6I_0} |x|^\eta \, dx \leq c\,|x|^{\frac{1}{\eta}}$, where c is a dimensional constant, independent of I; clary (4.2.1) holds in this case. Case (ii) is easier, for then $|x|$ is equivalent to $|y|$, for $x, y \in I$; indeed we have $|x| \leq |x-y| + |y| \leq c\,|y|$, and the opposite inequality follows by exchanging x and $y$above. Thus $|y|^\eta \leq c \inf_I |x|^\eta$, all $y \in I$, and averaging over $y \in I$, (4.2.1) holds for (ii) as well. □

4.3 (C) On chapter 3

We have asked in Remark 3 of chapter 1 the question: is $H^1(I_0; e^{-V}) = \mathcal{H}^1(I_0; e^{-V})$? where

$$H^1(I_0; e^{-V}) := \overline{C^\infty(I_0)}^{\dot{\xi}_1}$$

and

$$\mathcal{H}^1(I_0; e^{-V}) := \left\{ f \in L^2(I_0; e^{-V(\dot{x})} d\dot{x}) : \forall i, \text{ la derivée faible } \partial_i f \text{ est dans } L^2(I_0; e^{-V(\dot{x})} d\dot{x}) \right\} .$$

The answer is no here, see counterexample below, and in general the two spaces don't correspond (see [19]).

Counterexample

Let $d = 2$, fix I_0 the unit torus of $\mathbb{R}^2$ and let $\Omega = \{\dot{x} \in I_0 : |\dot{x}| < 1\}$ be the unit disc. We choice the following structure of e^{-V}.

$$e^{-V(\dot{x})} = \begin{cases} a^{-1}(|\dot{x}| & \dot{x}_1\dot{x}_2 > 0 \\ a(|\dot{x}|) & \dot{x}_1\dot{x}_2 < 0 \end{cases} \tag{4.3.1}$$

Where $a(r) = r^\alpha, 0 \leq \alpha < 2$
$a(r), a(r)^{-1} \geq c(\epsilon)$ for $r > \epsilon$.
Obviously,

$$\int_{I_0} e^{-V(\dot{x})} d\dot{x} + \int_{I_0} e^{V(\dot{x})} d\dot{x} < \infty.$$

We remark also that $\int_0^1 \frac{a(r)}{r} dr < \infty$. We require several test functions. In the polar variables $r = |\dot{x}|, \theta = \cos^{-1}(\dot{x}_1/r)$. We set

$$u(\dot{x}) = \begin{cases} 1 & \text{for } \dot{x}_1 > 0, \dot{x}_2 > 0 \\ 0 & \text{for } \dot{x}_1 < 0, \dot{x}_2 < 0 \\ \sin(\theta) & \text{for } \dot{x}_1 < 0, \dot{x}_2 > 0 \\ \cos(\theta) & \text{for } \dot{x}_1 > 0, \dot{x}_2 < 0, \end{cases} \tag{4.3.2}$$

$\widetilde{u}(\dot{x}) = u(-\dot{x}_2, \dot{x}_1)$ (a rotation through an angle of $\pi/2$),

$$g(\dot{x}) = \left\{ \frac{-\partial \widetilde{u}}{\partial \dot{x}_2}, \frac{\partial \widetilde{u}}{\partial \dot{x}_1} \right\}.$$

By construction,

$$|u| \leq 1, \qquad |\nabla u| < c/r, \qquad \int_\Omega |\nabla u(\dot{x})|^2 e^{-V(\dot{x})} d\dot{x} < \infty \Rightarrow u \in \mathcal{H}^1(I_0; e^{-V}).$$

The vector field g satisfies:

$$g.\nabla u = 0, \qquad g.n|_{\partial\Omega} = -\frac{\partial \widetilde{u}}{\partial \dot{x}_2} cos(\theta) + \frac{\partial \widetilde{u}}{\partial \dot{x}_1} sin(\theta) = \frac{\partial \widetilde{u}}{\partial \theta},$$

where n is the outer unit normal vector to the boundary of Ω. Now, let us prove that $u \notin H^1(I_0; e^{-V})$. Assume that $u \in H^1(I_0; e^{-V})$ and let $u_\epsilon \in C^\infty(\overline{\Omega})$ be a approximating sequence of u. Then $\forall \epsilon \geq 0$, by divergence theorem

$$\int_\Omega \nabla u_\epsilon(\dot{x}).g(\dot{x}) d\dot{x} = \int_{\partial\Omega} u_\epsilon(\dot{x}) g(\dot{x}).n d\Gamma = -\int_{\partial\Omega} u_\epsilon \frac{\partial \widetilde{u}}{\partial \theta} d\theta \tag{4.3.3}$$

Since $u_\epsilon|_{\partial\Omega} \to u|_{\partial\Omega}$ in $L^2(\partial\Omega)$, taking the limite in (4.3.3) we get:

$$\int_\Omega \nabla u(\dot{x}).g(\dot{x}) d\dot{x} = -\int_{\partial\Omega} u \frac{\partial \widetilde{u}}{\partial \theta} d\theta = -\int_0^{\pi/2} sin(\theta) d\theta = 1.$$

In other hand $\nabla u_\epsilon \to \nabla u$ in $L^2(\Omega, e^{-V})$ and $g \in \left(L^2(\Omega, e^V)\right)^2$. Thus $\int_\Omega \nabla u_\epsilon(\dot{x}).g(\dot{x}) d\dot{x} \to \int_\Omega \nabla u(\dot{x}).g(\dot{x}) d\dot{x} = 0$ by the identity $\nabla u.g = 0$. This contradiction shows that $H^1(I_0; e^{-V}) \neq \mathcal{H}^1(I_0; e^{-V})$.

Invariance principal in one dimensional case.

The result given in this thesis, Theorem 1 and Remark 5 of chapter 3, which is proved essentially by Theorem 2.1.1, main result of chapter 2; holds only for $d \geq 2$: the diffusion process associated with the divergence-form operator defined in (0.0.4), satisfies the invariance principle. We ask the question: what's happened if $d = 1$? The answer is yes, and the proof is simple. Indeed, the hypothesis $d \geq 2$ is used in this work only in the proof of Theorem 2.1.1. We prove without used harmonic analysis, that Theorem 2.1.1 of chapter 2 holds also if $d = 1$ see below. Indeed,

$$\begin{aligned}\left|f(x) - \int_0^1 f(y)dy\right| &\leq \int_0^1 |f(y) - f(x)|\, dy \\ &\leq \int_0^1 dy \int_x^y \left|f^{'}(z)\right| dz \\ &\leq \int_0^1 \left|f^{'}(z)\right| e^{-\frac{1}{2}V(z)} e^{\frac{1}{2}V(z)} dz \\ &\leq \left(\int_0^1 \left|f^{'}(z)\right|^2 e^{-V(z)} dz\right)^{\frac{1}{2}} \left(\int_0^1 e^{V(z)} dz\right)^{\frac{1}{2}} \\ &= c\left(\int_0^1 \left|f^{'}(z)\right|^2 e^{-V(z)} dz\right)^{\frac{1}{2}}\end{aligned}$$

$\Rightarrow \forall p > 2, \left|f(x) - \int_0^1 f(y)dy\right|^p \leq c\left(\int_0^1 \left|f^{'}(z)\right|^2 e^{-V(z)} dz\right)^{\frac{p}{2}}$.
Taking the integral with respect to Lebesgue measure, the desired inequality holds.
Now we are in case where the Sobolev inequality proved in Theorem 2.1.1 in chapter 2 holds with $w = 1$. The Lebesgue measure satisafies all hypotheses of $d\mu$ in Section 1.6: the Lebesque measure is positive finite on I_0, charging no set of zero capacity. Thus the construction of time changed process whose reference measure is Lebesgue measure can be done as the same way as in Section 1.6 and the invariance principle for the process (X, P_x) associated wit L follows by copying line by line starting at Section 3.3.

4.4 Open problem

A problem which could be interesting is to see a random case: when the potential is a realisation of the environment. Here we define a random media as a probability space (Ω, F, Q) in which acts a group of measure preserving transformations $G = \{\tau_x : x \in \mathbb{R}^d\}$ ie
the stationarity property: $\forall x \in \mathbb{R}^d$ and $\forall A \in F, Q(\tau_x.A) = Q(A)$,
the ergodicity property: If $A = \tau_x.A, \forall x \in \mathbb{R}^d$, then $Q(A) \in \{0; 1\}$.
We consider, for a given $\omega \in \Omega$, the divergence form operator defined by:

$$L^\omega = div(A(., \omega)\nabla).$$

One question could be to prove a **quenched invariance principle** for diffusion associated with this operator, with the hypothesis:

$$\begin{cases} A \text{ stationary} \\ A(0, .) \in L^1(\Omega, Q) \end{cases}$$

We assume also that there exists V measurable periodic such that $e^V \in L^1(I_0)$ where I_0 is the unit torus of $\mathbb{R}^d$; and $A(\omega, x) \geq e^{-V(x)} Id$ for almost all $\omega \in \Omega$.
The **reference** of this work is the paper: random conductance model in a degenerate ergodic environment. **S. Andres, J-D Deuschel, and M. Slowik.**
They study a continuous time random walk, $(X_t, t \geq 0)$, on $\mathbb{Z}^d$ in an environment of random conductances taking values in $(0; \infty)$. The law of the conductances is ergodic with respect to space shifts. They prove a quenched invariance principle for X under some moment conditions on the environment:

$$\begin{cases} \mathbb{E}(\omega(e)^p) < \infty \\ \mathbb{E}(\frac{1}{\omega(e)^q}) < \infty \\ \frac{1}{p} + \frac{1}{q} < \frac{2}{d} \qquad d \geq 2. \end{cases}$$

The key result on the sublinearity of the corrector is obtained by Moser's iteration scheme.

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