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Reduction of gauge symmetries: a new geometrical approach

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This vain presumption of understanding everything can have no other basis than never understanding anything. For anyone who had experienced just once the perfect understanding of one single thing, and had truly tasted how knowledge is accomplished, would recognize that of the infinity of other truths he understands nothing. [...]

[...] And when I run over the many and marvelous inventions men have discovered in the arts as in letters, and then reflect upon my own knowledge, I count myself little better than miserable. I am so far from being able to promise myself, not indeed the finding out of anything new, but even the learning of what has already been discovered, that I feel stupid and confused, and am goaded by despair.

Galileo,
Dialogue Concerning the Two Chief World Systems, 1632.

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Contents

Introduction

Geometry and Physics

It is almost a commonplace to state that Geometry and Physics have always shared an intimate link. Historically indeed one can well appreciate that physical problems inspired new geometrical investigations and conversely new geometrical ideas provided tools for solving physical problems. But when we mention this link we actually think of something deeper than this mere exchange of courtesies. Something like a deep connection. A feeling that, when it comes to the foundations, Geometry is the real language of Physics. A feeling turned into a conviction manifested in the program of 'geometrization' of Physics initiated by Einstein and promoted by him and some others.

However the persistence of this link through time should a priori sound quite surprising. Isn't it strange that while Physics progresses, and the concepts we use to understand Nature are deepening, Geometry remains one of the most relevant tools? Not if we appreciate the fact that what today we call geometry would not have been recognized as such by mathematicians of the early XIX$^{\text{th}}$, except perhaps for the *mathematicorum principi* Gauss. And what Gauss considered as geometry would have seemed nonsensical to Galileo and Newton. At their time Geometry was only Euclidean geometry, that is the synthetic method described in Euclid's *Elements of mathematics.* And yet, already the deep relation between Geometry and Physics was praised, as testified by the famous excerpt from Galileo's *Il saggiatore* published in 1623:

> "Philosophy is written in that great book which ever lies before our eyes – I mean the universe – but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written. This book is written in the mathematical language, and *the symbols are triangles, circles and other geometric figures*, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth."

And contrary to what one may think, this did not change with Newton. Indeed, in his *Géométrie* of 1637, Descartes had introduced the method of coordinates, founding analytic geometry. Here is Newton's appreciation,

> " To be sure, their [the Ancients'] method is more elegant by far than the Cartesian one. For he [Descartes] achieved the result by an algebraic calculus which, when transposed into words (following the practice of the Ancients in their writings), would prove to be so tedious and entangled as to provoke nausea, nor might it be understood. But they accomplished it by certain simple propositions, judging that nothing written in a different style was worthy to be read, and in consequence concealing the analysis by which they found their constructions."

Clearly committed to a strict separation of arithmetic and geometry he argues in the appendix of *Arithmetica Universalis* published in 1707 and dedicated to the theory of algebraic equations,

> "Equations are expressions of Arithmetical computation, and properly have no place in Geometry, except as far as quantities truly Geometrical (that is lines, surfaces, solids, and proportions) may be said to be some equal to others. Multiplications, Divisions, and such sort of computations, are newly received into Geometry, and that unwarily, and contrary to the first design of this Science. For whosoever considers the construction of Problems by a right line and a circle, found out by the first Geometricians, will easily perceive that Geometry was invented that we might expeditiously avoid, by drawing lines, the tediousness of computation. Therefore these two Sciences ought not to be confounded. The ancients did so industriously distinguish them from one another, that they never introduced Arithmetical terms into Geometry. And the moderns, by confounding both, have lost the simplicity in which all the elegance of Geometry consists."

This testifies of the high esteem, almost devotional, Newton had (as many of his contemporaries) for the Greek geometers. And as a matter of fact the *Principia* of 1687 are entirely written in the language of Euclidean geometry, and make no explicit use of the Cartesian method or of Newton's calculus of fluxion and fluent (the basis of differential calculus), even if he used it to discover his results in the first place.[1] This of course also responds to the practical constrain of communicating his results to his peers.

So, contrary to Newton's taste, and in a very Kuhnian paradigmatic shift, our notion of geometry had broadened over time. The deep bond between Physics and Geometry is thus not to be seen as a static state of affairs, but as a dynamical, often synergistic relation.

With the advent of quantum physics, Geometry may seem to have stepped back. And indeed Einstein himself, though a passionate defender of the geometrization of Physics (such he intended a unified field theory of gravitation and electromagnetism), admits in a letter to Langevin in 1935,

> "In any case, one does not have the right today to maintain that the foundation must consist in a field theory in the sense of Maxwell. The other possibility, however, leads in my opinion to a renunciation of the space-time continuum and to a purely algebraic physics."

Nevertheless the idea still remains potent that geometrical considerations may unravel the primeval mystery of quantization. Indeed the program of geometrical quantization initiated in the early 60's by Kirilov, Kostant and Souriau, is still an active area of research. In another and less programmatic fashion punctual works, like (Ashtekar and Schilling, 1999), show that geometry is still believed to be a relevant way to shed some lights on the origin of quantization. These are attempts to understand quantum physics by the resources of what is now admitted to be Geometry. But there are also explorations from the mathematical side, that propose to enlarge still further the very notion of Geometry. We can think of the example of Connes non-commutative geometry developed in the 70's up to now, which has been proposed as the most natural framework for the Standard Model. So with new mathematical breakthrough yet to come or germinating right now, we may assist to a broadening of the notion of Geometry that will at last provide tools to push the limits of our current understanding of Nature. Such a deepening of our notion of Geometry, freed of prejudices, is exactly what Riemann argued for in the conclusion of his habilitation dissertation of 1854, *On the hypothesis that lie at the foundation of Geometry*:

> "Now it seems that the empirical notions on which the metric determinations of space are based, the concept of a solid body and that of a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of space in the infinitely small do not conform to the hypothesis of geometry; and in fact one ought to assume this as soon as it permits a simpler way of explaining phenomena.
>
> [...] An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts that cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to insure that this work is not hindered by unduly restricted concepts and that progress in comprehending the connection of things is not obstructed by traditional prejudices.
>
> This leads us away into the domain of another science, the realm of Physics, in which the nature of the present occasion does not allow us to enter."

The example of gauge theories

The history of gauge theories is a fascinating one for it provides a marvelous example of the intertwining of Physics and Geometry: how each developed along different paths, yet starting from a common impulse,

[1] To appreciate the geometric elegance alluded to by Newton, one can take a look at the transcription of the lesson Feynman gave in 1963 at Caltech for first year students on the motion of planets around the Sun, (Feynman et al., 1996). Feynman gives elementary Euclidean geometrical demonstrations of Kepler's laws and shows, following essentially Newton's approach, how the elliptical orbits imply an inverse square law for Gravitation.

Einstein's General Relativity, and meet again. The so-called *gauge principle*, or principle of local symmetry, is at the core of our understanding of the four fundamental interactions. The Standard Model of Particle Physics, as well as the Standard Model of Cosmology, our best theories about the known particles and their interactions and about the evolution the Universe, are based on gauge theories.

In the 20's it was noticed that within the framework of the newborn Quantum Mechanics, demanding the space-time dependence of an initially rigid symmetry described by the abelian $U(1)$ Lie group (one speaks of the 'localization' of the group), seemed to imply, under minimal assumptions, the introduction of the electromagnetic potential, its transformation law as well as its minimal coupling with charged particles.[2] This was a strong result, whose origin is found in the works of London, Fock and Weyl between 1927 and 1929. In 1954 Yang and Mills, and independently Shaw in 1955, generalized the gauge principle to the non-abelian $SU(2)$ group to model the strong interaction. In 1956 Utiyama, still independently, gave the systematic means to construct a gauge theory for any Lie group, showing in particular that General Relativity could be seen as a gauge theory for the Lorentz group $SO(1,3)$. For a nice historical review one can consult (O'Raifeartaigh, 1997) which contains translations of the most important original papers that paved the way toward modern gauge theories.

It happens that gauge theories are perfectly formalized, at the classical level, in term of the geometry of fiber bundles and of principal connections, also called Ehresmann connections, developed in the late 40's and early 50's. It is not excessive to say that one feels a deeper sense of understanding once a clear mathematical, geometrical picture is available. This is so with gauge theories. They provide a unifying framework to work out model theories of the fields in Physics, not only interaction fields but also matter fields of any kind. The first section of the Chapter 1 illustrates this claim and describes the basic notions of bundles geometry.

Einstein's General Relativity has been historically the root of the inspiration of Weyl's work on gauge invariance, which was revived in the context of Quantum Mechanics. But the question of the gauge structure of gravitation had not been genuinely addressed before the work of Utiyama. Arguably though, his results were already encompassed within the common generalization of Klein and Riemann geometries that Cartan developed as early as 1922. It is the aim of the second section of Chapter 1 to show that actually Cartan geometry and Cartan connections (which can be seen as ancestors of principal bundles geometry and Ehresmann connections) are the natural framework that properly acknowledges the singularity of gravitation among the other interactions.

The success of gauge theories is to provide a deeper understanding of the origin of the fundamental interactions as emerging from symmetry principles. Nevertheless it happens that their bare structure presented some difficulties in being reconciled with phenomenology. At bottom the problems came from the very thing that was also the prime appeal of gauge theories: the gauge symmetry. The third section of Chapter 1 discusses briefly these issues as well as the standard solutions devised to overcome it. A broad classification in three items is suggested.

The proposition of this thesis

This thesis proposes to add a new item to this classification, a new geometrical tool to handle gauge symmetry in gauge theories. This method we call the *dressing field method*. It is the aim of Chapter 2 to present the basic definitions and results. In the easiest applications, the latter allows to construct *gauge invariant composite fields* and Lagrangians. We say that in this case the gauge theory has been *geometrized*. This dispenses to fix a gauge, and may add something to the discussion about quantization. As a matter of ancestry it can be related to the so-called *Dirac variables*, (Dirac, 1955), (Dirac, 1958). It turns out that if one follows Dirac's physical interpretation the name "dressing field" may be more relevant than initially expected.

In the less straightforward cases, it is possible to reduce only partially the gauge symmetry so that the composite fields display a *residual gauge freedom*. Several examples of application are proposed; ranging from simple toy models to non-trivial examples like the electroweak sector of the Standard Model (following the pioneering work of (Masson and Wallet, 2011)) and General Relativiy. In the case of the electroweak

[2]See (Martin, 2002) for a philosophical critic of the gauge principle.

model, the method entails an interpretive shift with respect to the standard interpretation of the Englert-Brout-Higgs mechanism, to the point were we are lead to challenge the very terminology of ‘spontaneous symmetry breaking’.

The second part of Chapter 2 proposes already a generalization of the method and an application to the 2^{nd}-order conformal structure, also known as Möbius geometry. See (Sharpe, 1996), (Kobayashi, 1972) and (Ogiue, 1967). The latter can be seen as the geometry underlying a gauge theory of conformal gravity. This example illustrates the possibility, if the dressing fields satisfy some compatibility conditions, to compose the dressing operation several times thus reducing the gauge symmetry by steps. Standard results of conformal geometry are easily recovered. All calculations, from the composite fields to their residual gauge freedom, are performed through an operative matrix formalism.

Since its inception in the 70’s, the celebrated BRS framework has become a standard tool in gauge theories. The BRS algebra of a gauge theory reflects the infinitesimal gauge symmetry. It is expected that the dressing field method should interact with the BRS formalism. Indeed this question is investigated and solved in Chapter 3. The central notion here is that of *composite ghost* which encodes the residual gauge freedom. The corresponding modified BRS algebra provides very easily the infinitesimal transformations under this residual gauge freedom. Again the conformal structure illustrates the scheme. The inclusion of infinitesimal diffeomorphisms of the base manifold in our modified BRS framework is discussed.

Chapter 4 offers some preliminary considerations related to anomalies in Quantum Field Theory. The main, and somewhat obvious, result to be mentioned is the following: if a gauge theory is susceptible to be geometrized, it is *anomaly free*. The relevance of this result is highlighted when one remembers that the electroweak sector of the Standard Model can be treated through the dressing field method. This guarantees the model to be free of $SU(2)$-anomalies.

Along the study, some contacts with the literature (recent or older) are proposed to make clear how the approach advocated here may underlie various constructions in a broad spectrum of topics within modern gauge field theories. Some of them are analyzed in details, or simply commented, in Appendix A.

Chapter 1
Geometry of fundamental interactions

1.1 The geometry of gauge fields

The notion of fibered spaces, or fiber bundles, was developed by (Ehresmann, 1947). He also gave the modern definition of the notion of connection in (Ehresmann, 1950). Fiber bundles and Ehresmann, or principal, connections are the natural language for Yang-Mills fields theory. It appears that the Yang-Mills gauge potential can be interpreted as the local expression of the Ehresmann connection while the field strength is the local expression of the curvature of the connection. Any matter field can be seen as the section of an associated bundle, and the minimal coupling between the gauge field and the matter field is nothing but the covariant derivative of the sections with respect to the connection.

1.1.1 The basics of bundle geometry

We would like to speak of the notion of "fields" in a mathematically precise sense. Consider an example borrowed from (Sharpe, 1996). The velocity field of a point describing a trajectory c in a n-dimensional space is a smooth map $v : c \to \mathbb{R}^n$ which assigns to each $x \in c$ a vector $v(x) \in \mathbb{R}^n$. It is thus a vector-valued function. We can see $v(x)$ as belonging to the 1-dimensional subspace V_x of $\mathbb{R}^n$ tangent to c in the point x. That is, $\left\{V_x | x \in c\right\}$ is a family of subspace parametrized by c. Moreover the velocity field is merely an assignation of a particular element in each subspace of this family for each value x of the "parameter space" c, what is called a *section* of the parametrized family.

Why such a roundabout description? Because the framework of fibered manifolds generalizes this picture: the "parameter space" is the *base manifold* $\mathcal{M}$, the "parametrized family of *something*" is the *total manifold* E and the "something" at each point $x \in \mathcal{M}$ is the *fiber* F_x at x. Moreover the *sections* of a fibered space generalize the notion of "something"-valued functions, and it is the adapted notion to describe fields in Physics. We now proceed to make this more precise and refer to (Sharpe, 1996), (Nakahara, 2003), (Azcarraga and Izquierdo, 1995) or (Göckeler and Schücker, 1987) for more complete treatment and proofs.

Fiber bundle Let E and $\mathcal{M}$ be smooth manifolds, the *total manifold* and *base manifold* respectively. The *canonical projection* is a smooth surjective map $\pi : E \to \mathcal{M}$, that is $\pi(p) = x$ for any $p \in E$ and $x \in \mathcal{M}$. The *typical fiber*, F, is a topological space. The *fiber*, $F_x := \pi^{-1}(x)$, in each $x \in \mathcal{M}$ is diffeomorphic to F. The quadruple $(E, \mathcal{M}, \pi, F)$ is a *locally trivial fiber bundle* with abstract fiber F if, for any open subset in a covering $\{U_i\}$ of $\mathcal{M}$, there is a diffeomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times F$. This diffeomorphism is such that $\pi \circ \phi_i^{-1} = \mathrm{proj}_{U_i}$, i.e $\pi \circ \phi_i^{-1}(x, f) = x$, for any $x \in U_i$ and $f \in F$. The pair (U_i, ϕ_i) is a *local bundle coordinate system*, or *bundle chart*, and the collection $\{(U_i, \phi_i)\}$ is an *atlas* for E.[1]
Despite its local triviality, the bundle E cannot in general be written as the global product $\mathcal{M} \times F$. The latter is the trivial bundle, with projection $\pi = \mathrm{proj}_{\mathcal{M}}$. Actually fiber bundles are introduced to generalize this trivial situation. A necessary and sufficient condition for E being trivial is the existence of a smooth map $t : E \to F$, called a *trivialization*, such that $t : \pi^{-1}(x) = F_x \to F$ is a diffeomorphism for any $x \in \mathcal{M}$.

***G*-bundle** A refinement appears when one introduces a Lie group G, *the structure group*, which acts smoothly and effectively on F (on the left) as a group of diffeomorphisms. That is, there is a group homeomorphism $G \to \mathrm{Diff}(F)$ with trivial kernel. In this case the bundle coordinate changes are controlled by maps with values in G. Indeed, let (U_i, ϕ_i) and (U_j, ϕ_j) be two charts. If $U_i \cap U_j \neq \emptyset$, then we have the change of coordinates

[1] For short, the fiber bundle is noted like its total space, E, or sometimes $E \xrightarrow{\pi} \mathcal{M}$.

$\Phi = \phi_i \circ \phi_j^{-1} : U_i \cap U_j \times F \to U_i \cap U_j \times F$. It explicitly reads $\Phi(x,f) = (x, t_{ij} f)$, for $x \in U_i \cap U_j$ and $f \in F$. The maps $t_{ij} : U_i \cap U_j \to G$ are the *transition functions*. Obviously, the transition functions satisfy, $t_{ii} = e$ the identity of G, $t_{ij} = t_{ji}^{-1}$ and due to $\phi_i \phi_j^{-1} \phi_j \phi_k^{-1} = \phi_i \phi_k^{-1}$, we have the consistency condition $t_{ij} t_{jk} = t_{ik}$.
If we were given only the covering $\{U_i\}$ of $\mathcal{M}$, the typical fiber F, the structure group G and the set of transition functions $\{t_{ij}\}$, we could reconstruct the G-bundle by forming the disjoint union $\bigcup_i U_i \times F$ and dividing out by the equivalence relation $(x,f) \sim (x, t_{ij} f)$. This construction depends on the given covering $\{U_i\}$ and the bundle thus obtained is called a *coordinate bundle*. But the abstract/global G-bundle, with data $(E, \mathcal{M}, \pi, F, G)$, is defined independently of a particular covering.

Sections A *local section* of a fiber bundle is a smooth map, $\sigma : U \to \pi^{-1}(U)$, with $U \subset \mathcal{M}$, satisfying $\pi \circ \sigma = \mathrm{id}_U$. Explicitly, $\sigma(x) = p \in \pi^{-1}(x) = F_x$. The set of local sections of E is noted $\Gamma(E)$. A section is a generalization of a "something"-valued function. Indeed, locally $\phi \circ \sigma(x) = (x, f(x))$, with $x \in U$ and $f(x) \in F$, is the *graph* of a F-valued function. In the case of the trivial bundle, the trivialization t allows to identify the global section $\sigma : \mathcal{M} \to E$ with the corresponding F-valued function, $t \circ \sigma : \mathcal{M} \to F$.

Principal bundles Here is the most important kind (for us at least) of fiber bundles. A principal bundle is a G-bundle whose fibers are homeomorphic or even diffeomorphic to the structure group G. Given a point $f_0 \in F$ the diffeomorphism is the map $G \to F$ sending $f_0 \mapsto g f_0$. Obviously the identification is not canonical, but once an identification is chosen we have the fiber above $x \in U \subset \mathcal{M}$, $\pi^{-1}(x) = G_x$. Of course G_x does not have the group structure of G. A principal bundle is noted $\mathcal{P}(\mathcal{M}, G)$, or $\mathcal{P}$ for short. It supports a fiber preserving free right action $\mathcal{P} \times G \to \mathcal{P}$ given by $(p,g) \mapsto \mathcal{R}_g p := pg$, with $p, pg \in G_{\pi(p)}$. The right action is compatible with the bundle chart: $\mathcal{R}_{g'} \phi^{-1}(x,g) = \phi^{-1}(x,g) g' = \phi^{-1}(x, gg')$.
A local section $\sigma : U \to \mathcal{P}$ determines a canonical bundle chart, $\sigma(x) = \phi^{-1}(x,e)$, so that any $p \in \pi^{-1}(U)$ is associated to its coordinates, $(x,g) \in U \times G$, through, $p = \sigma(x) g$. This is why a local section if often referred to as a *local trivializing section*. Obviously if we have a global section $\sigma : \mathcal{M} \to \mathcal{P}$, the previous construction provides an isomorphism $\mathcal{P} \to \mathcal{M} \times G$, so $\mathcal{P}$ is a trivial principal bundle. Conversely, if $\mathcal{P} = \mathcal{M} \times G$, then a global bundle chart, $\phi^{-1} : \mathcal{M} \times G \to \mathcal{P}$, allows to define the global section $\sigma_g := \phi^{-1}(x,g)$. Hence the proposition: a principal bundle is trivial iff it has a global section.

Vector bundles A vector bundles E is a G-bundle whose typical fiber is a vector space $F = V$ which is a space of representation for the structure group G. The latter then acts effectively on the left on each fiber, thus on E, through a representation ρ by, $(g,v) \mapsto \rho(g) v$, for any $g \in G$ and $v \in V_x = \pi^{-1}(x)$.
A local section $\sigma : U \to \pi^{-1}(U)$ is locally the graph of a V-valued function, $v(x)$. So a local section $\sigma \in \Gamma(E)$ is a *vector field*. Since the null vector $0 \in V_x$ is left invariant by the action of G, a vector bundle admits a global null section, $\sigma_0 : \mathcal{M} \to E$, such that $\sigma_0(x) = \phi^{-1}(x,0)$ in any bundle chart. Note that this in no way implies the triviality of E, since obiously the demonstration achieved for a principal bundle cannot be carried for a vector bundle.
There are operations defined on vector bundles. Given two vector bundles E and E' with typical fibers V and V' respectively, one can form the Whitney sum vector bundle $E \oplus E'$ whose typical fiber is the sum $V \oplus V'$. But perhaps more interestingly one can form the tensor product bundle $E \otimes E'$ whose fiber is $V \otimes V'$. As a notable example, if $T\mathcal{M}$ and $T^*\mathcal{M}$ are bundles with fibers $T_x\mathcal{M}$ and $T_x^*\mathcal{M}$ respectively, then $\mathcal{T}^r{}_s = \bigotimes_{r,s} T\mathcal{M} \otimes \cdots \otimes T\mathcal{M} \otimes T^*\mathcal{M} \otimes \cdots \otimes T^*\mathcal{M}$ is a (r,s)-tensor bundle and $\sigma \in \Gamma(\mathcal{T}^r{}_s)$ is a (r,s)-tensor field.

Associated bundle Given a principal bundle $\mathcal{P}(\mathcal{M}, G)$ and a representation (V, ρ) for the structure group G, one can define a vector bundle associated to $\mathcal{P}$ as follows. First of all, define a right action $(\mathcal{P} \times V) \times G \to (\mathcal{P} \times V)$ by $\big((p,v), g\big) \mapsto \big(pg, \rho(g^{-1}) v\big)$.[2] This action defines an equivalence relation, $(p,v) \sim_G \big(pg, \rho(g^{-1}) v\big)$, and the equivalence class is noted $[p,v]$. The associated vector bundle is then, $E = P \times V / \sim_G := P \times_G V$. A point of E is $[p,v]$, and the projection, $\pi_E\big([p,v]\big) = \pi(p)$, is clearly well defined. An associated vector bundle E is trivial

[2]The inverse $\rho(g^{-1})$ in the second factor is here to secure the fact that we have indeed a right action by successive composition.

if its underlying principal bundle $\mathcal{P}$ is.

From the above data we are able to draw an important conclusion: there is a bijective correspondence between (local) sections of an associated vector bundle and (local) $\rho(G)$-equivariant maps on $\mathcal{P}$:

$$\iota : \Gamma(E) \to \Lambda^0(\mathcal{P}, \rho), \tag{1.1}$$

where by definition $\Lambda^0(\mathcal{P}, \rho) = \left\{\psi : \pi^{-1}(U) \subset \mathcal{P} \to V \mid \psi(pg) = \rho(g^{-1})\psi(p)\right\}$. Indeed given ψ, the local section $\widetilde{\psi} := \iota^{-1}(\psi) : U \subset \mathcal{M} \to E$ is induced by the map, $\mathcal{P} \to \mathcal{P} \times V$ sending $p \mapsto \big(p, \psi(p)\big)$.

Bundle morphisms and gauge group Given two fibered spaces E and E' over base manifolds $\mathcal{M}$ and $\mathcal{M}'$ respectively, a *bundle morphism* is a a pair of smooth maps $(\bar{\varphi}, \varphi)$ where $\bar{\varphi} : E \to E'$ and $\varphi : \mathcal{M} \to \mathcal{M}'$ satisfy $\pi' \circ \bar{\varphi} = \varphi \circ \pi$. The latter condition implies that the map $\bar{\varphi}$ is fiber preserving. This is easily remembered by demanding the commutativity of the following diagram:

$$\begin{tikzcd} E \arrow[r, "\bar{\varphi}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ \mathcal{M} \arrow[r, "\varphi"] & \mathcal{M}' \end{tikzcd}$$

The map $\bar{\varphi}$ is called the covering, or *lift*, of φ. Conversely, φ is called the projection of $\bar{\varphi}$. Two bundles E and E' over the same base manifold $\mathcal{M}$ are equivalent if there is a bundle morphism $(\bar{\varphi}, \mathrm{id}_{\mathcal{M}})$. In the latter case, if $E' = \mathcal{M} \times F$, the bundle morphism shows that E is trivial.
A *bundle automorphism* is a bundle morphism $(\bar{\varphi}, \varphi)$ of a bundle onto itself: $\bar{\varphi}$ is a fiber preserving diffeomorphism of E and $\varphi \in \mathrm{Diff}(\mathcal{M})$. We will note $\mathrm{Aut}(E)$ the group of fiber preserving diffeomorphisms of E which project onto diffeomorphisms of $\mathcal{M}$. The subgroup $\mathrm{Aut}_v(E)$ of *vertical automorphisms* of E, are those that project to $\mathrm{id}_{\mathcal{M}}$; they don't move the base point x and are just diffeomorphisms of the fiber F_x.
If E is a G-bundle and E' a G'-bundle, one needs to add to the above maps a group homomorphism $\widetilde{\varphi} : G \to G'$ compatible with the respective group actions. Simply said, one requires the commutativity of the following diagram:

$$\begin{tikzcd} E \times G \arrow[r, "\bar{\varphi}\times\widetilde{\varphi}"] \arrow[d, "\mathrm{Act}_G"'] & E' \times G' \arrow[d, "\mathrm{Act}_{G'}"] \\ E \arrow[r, "\bar{\varphi}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ \mathcal{M} \arrow[r, "\varphi"] & \mathcal{M}' \end{tikzcd}$$

In the same way as above we have G/G'-bundle equivalence and triviality. It could simplify if $G = G'$, i.e $\widetilde{\varphi} = \mathrm{id}$, so that both bundles are G-bundles. This is of course the case for G-bundle automorphisms and vertical automorphisms.

In the case of a principal bundle $\mathcal{P}(\mathcal{M}, G)$, a vertical automorphism Ψ is a map commuting with $\mathcal{R}_g$ and such that $\pi \circ \Psi = \pi$. We write, $\mathrm{Aut}_v(\mathcal{P}) = \left\{\Psi : \mathcal{P} \to \mathcal{P} \mid \Psi(pg) = \Psi(p)g,\ \pi \circ \Psi(p) = \pi(p)\right\}$.
These Ψ are induced by maps $\gamma : \pi^{-1}(U) \to G$, through $\Psi(p) = p\gamma(p)$. The γ's transform as $\gamma(pg) = g^{-1}\gamma(p)g$, in order to ensure the equivariance of Ψ. This suggests that the γ's can be seen as local sections of the bundle $\mathcal{P} \times_{Ad_G} G$, through the identification, ι, of sections and equivariant maps.
Now, given $\Psi(p) = p\gamma(p)$ and $\Phi(p) = p\alpha(p)$, the composition is,

$$\Phi^*\Psi(p) = \Psi \circ \Phi(p) = \Psi\big(p\alpha(p)\big) = \begin{cases} \Psi(p)\alpha(p) = p\gamma(p)\alpha(p), \\ p\alpha(p)\gamma\big(p\alpha(p)\big) = p\alpha(p)\alpha^*\gamma(p). \end{cases}$$

This shows something worth mentionning: while the Ψ's form a group under composition, despite the apparences displayed by the first line above, the γ's do not form a group under pointwise (G-) mutiplication. Indeed, we have the transformation law,

$$\alpha^*\gamma(p) := \gamma^\alpha(p) = \alpha^{-1}(p)\gamma(p)\alpha(p), \tag{1.2}$$

which expresses the action of α on γ. The maps $\gamma : \pi^{-1}(U) \to G$ are called *gauge transformations*, and the group,

$$\mathcal{G} = \left\{\gamma : \pi^{-1}(U) \to G \mid \gamma(pg) = g^{-1}\gamma(p)g\right\} \simeq \Gamma(\mathcal{P} \times_{Ad_G} G), \tag{1.3}$$

is called the *gauge group* of the principal bundle $\mathcal{P}(\mathcal{M}, G)$. This terminology is mostly used by physicists. Due to the group homomorphism $\mathcal{G} \to \mathrm{Aut}_v(\mathcal{P})$ explicitly given by $\Psi(p) = p\gamma(p)$, it is frequent to identify the two infinite dimensional groups.

Bundle reduction theorem It can happen that a bundle $\mathcal{P}(\mathcal{M}, G)$ is reducible to a subbundle $\mathcal{P}'(\mathcal{M}, H)$, with $H \subset G$. This gives rise to the *bundle reduction theorem*. On this known result we refer to (Sharpe, 1996) for a rigorous mathematical treatment, and to (Trautman, 1979), (Westenholz, 1980) and (Sternberg, 1994) for neat exposures oriented toward the physics of spontaneous symmetry breaking. Actually, in the following we adopt the viewpoint of (Sternberg, 1994) which is slightly sharper and, as it turns out, is better suited to a comparison with the approach presented in Chapter 2.

Given a principal bundle $\mathcal{P}(\mathcal{M}, G)$ and a space V supporting a $\rho(G)$-action (often (V, ρ) is simply a representation of G), we have the associated bundle $E = \mathcal{P} \times_{\rho(G)} V$. A section in of E can be viewed, through (1.1), as an equivariant map,

$$f : \mathcal{P} \to V, \qquad f(pg) = \rho(g^{-1})f(p).$$

Let us admit that the G-action on V is nor transitive nor effective, and that it a has cross section Γ whose isotropy group is $H \subset G$. That is, we have[3] $V \simeq \Gamma \times G/H$. Accordingly, the map f splits as,

$$f \simeq (r, \bar{u}), \tag{1.4}$$

both maps satisfying the equivariance property.

For the map r this reads, $r(pg) = r(p)$, since by definition the G-action is trivial on Γ (each point of which being a G-class). This means that $r : \mathcal{M} \to \Gamma$, i.e it is a G-invariant/*gauge invariant* field on the base manifold.

The map $\bar{u}$ is the one which allows to perform the bundle reduction. To see this easily, we just need to motivate the equality $f^{-1}\big(\rho(g^{-1})v\big) = f^{-1}(v)g$, with $v \in V$. The latter, like the equivariance condition, stems from the commutative diagram

$$\begin{tikzcd}
\mathcal{P} \times G \arrow[r, "f \times \rho"] \arrow[d, "\mathcal{R}_G"'] & V \times \rho(G) \arrow[d, "L_{\rho^{-1}(G)}"] \\
\mathcal{P} \arrow[r, "f"] & V
\end{tikzcd}$$

being followed from the upper-right to bottom-left, and from upper-left to bottom-right respectively.
Now, defining

$$\mathcal{P}' = \left\{p \in \mathcal{P} \mid \bar{u}(p) = eH\right\} = \bar{u}^{-1}(eH), \tag{1.5}$$

we observe that if $h \in H$, then $eH = \rho(h^{-1})eH$, so $\bar{u}^{-1}(eH) = \bar{u}^{-1}\big(\rho(h^{-1})eH\big) = \bar{u}^{-1}(eH)h$. This means, $\mathcal{P}' = \mathcal{P}'h$, thus the set $\mathcal{P}'$ is stable by right H-action. Moreover, given p' and p in the same fiber of $\mathcal{P}'$, there

[3] Here we neglect the possibility for a small G-invariant singular set S, like the 0 element if V is a vector space, since this caveat does not add much to the discussion. In this case we would have, $V - S \simeq \Gamma \times G/H$.

is a $g \in G$ such that $p' = pg$. Applying the map $\bar{u}$ we get, $eH = \bar{u}(p') = \bar{u}(pg) = \rho(g^{-1})\bar{u}(p) = \rho(g^{-1})eH$, so $g \in H$ and the H-action is transitive on fibers of $\mathcal{P}'$. Thus we conclude that $\mathcal{P}'$ is an H-reduction of $\mathcal{P}$. As such it carries its own group of automorphisms $\mathrm{Aut}(\mathcal{P}')$ which is a subgroup of $\mathrm{Aut}(\mathcal{P})$, and in particular the gauge group $\mathcal{G}$ of $\mathcal{P}$ is reduced to the gauge group $\mathcal{H}$ of $\mathcal{P}'$.

Had we chosen another point $gH \in G/H$, we would have defined another subbundle of the bundle $\mathcal{P}$ by $\mathcal{P}'' = \bar{u}^{-1}(gH) = \bar{u}^{-1}\big(\rho(g^{-1})eH\big) = \bar{u}^{-1}(eH)g = \mathcal{P}'g$. Clearly $\mathcal{P}''$ is also an H-reduction. From this, we see that the structure group G can act on the space of H-subbundles, sending one into another. Both $\mathcal{P}''$ and $\mathcal{P}'$ define the same abstract H-bundle, they are simply different realisations linked by an element of G.

Example 1 As mentionned above, the bundle reduction theorem is used to give a geometrical interpretation of the physics of the spontaneous symmetry breaking in the electroweak sector of the standard model. There the data are: the bundle $\mathcal{P}\big(\mathcal{M}, G = U(1) \times SU(2)\big)$, $V = \mathbb{C}^2$, $\Gamma = \left\{\binom{0}{\eta} \mid \eta \in \mathbb{R}^+\right\}$ and H as a subgroup of $SU(2)$ is isomorphic to $U(1)$. We have the decomposition $V \simeq \binom{0}{\eta} \times SU(2)$, and accordingly a map $\varphi : \mathcal{P} \to V$ splits as $\varphi \simeq (\eta, \bar{u})$. The map $\eta : \mathcal{M} \to \mathbb{R}^+$ is a gauge invariant scalar field, the true *observable Higgs field*, and $\bar{u} : \mathcal{P} \to SU(2)$ realizes the subbundle $\mathcal{P}'\big(\mathcal{M}, H = U(1)\big)$.

Example 2: Another noteworthy instance of bundle reduction is provided by the reduction of the *frame bundle*, $\mathcal{P} = (\mathcal{M}, GL) := L\mathcal{M}$, via a (pseudo) riemannian metric. Indeed, a metric is a map $\bar{g} : L\mathcal{M} \to T^*_x\mathcal{M} \otimes T^*_x\mathcal{M}$, with the equivariance property $\bar{g}(pg) = \rho(g^{-1})\bar{g}(p)$. The $\rho(GL)$-action is explicitly given in coordinates as, $\bar{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial x^\alpha}\frac{\partial x^\nu}{\partial x^\beta}\bar{g}_{\mu\nu}$, with $\frac{\partial x^\alpha}{\partial x^\mu} = g \in GL$.[4]

One formulation of the equivalence principle says that locally (in a point of space-time) a free falling observer in a gravitational field cannot notice its presence. That is, locally there is a coordinate change that allows to diagonalise an arbitrary metric tensor $\bar{g}_{\mu\nu}$ into the Minkoswki/flat metric $\bar{\eta}_{ab}$. In coordinates this is the well known identity $\bar{g}_{\mu\nu} = e^a{}_\mu e^b{}_\nu \bar{\eta}_{ab}$. This means that there is a $\rho(GL)$-space decomposition, $T^*_x\mathcal{M} \otimes T^*_x\mathcal{M} \simeq \bar{\eta} \times GL/SO$, where $\bar{\eta}$ is defined as the $\rho(GL)$-class and SO (the Lorentz/rotations group) is its isotropy group. Accordingly we have the splitting $\bar{g} \simeq (\eta, \bar{u})$. The map $\eta : \mathcal{M} \to \bar{\eta}$ can be viewed as the field that gives the Minkoswki/flat metric in each point x. The map $\bar{u} : \mathcal{P} \to GL/SO$ realizes the reduction by defining: $\mathcal{P}' = \{p \in L\mathcal{M} \mid \bar{u}(p) = eSO\} = \bar{u}^{-1}(eSO)$. The subbundle $\mathcal{P}'$ is indeed an SO-reduction of $L\mathcal{M}$.
Remark: When pulled-back on $\mathcal{M}$, the equivariant map $\bar{u}$ is, in coordinates, the *vielbein*: $u = e^a{}_\mu(x)$.[5] Also called the *tetrad field* or *moving frame*, this is a well known object in the Palatini formulation of General Relativity.

The latter example is the reason for some authors to call a riemannian metric a "Higgs field for gravitation". See, e.g (Trautman, 1979) and (Sardanashvily, 2011). Indeed (Sternberg, 1994), (Trautman, 1979), (Westenholz, 1980), all name the equivariant map f a "Higgs field". If so, then indeed the metric $\bar{g}$ is one. But as briefly stressed above, it is actually the G-invariant map t (η in *Example 1*) which is the real observable Higgs field, so deserves the name. We put forward this remark since, starting from another viewpoint, we'll argue for precisely the same conclusion in section 2.3.2 ahead.

1.1.2 Connection & Curvature

In this section we still focus on principal bundles and their associated vector bundles. For the matter covered, we again refer to (Sharpe, 1996), (Nakahara, 2003), (Bertlmann, 1996), (Azcarraga and Izquierdo, 1995) and (Göckeler and Schücker, 1987) for extensive treatment.

As a smooth manifold, a principal bundle $\mathcal{P}$ has a tangent space $T_p\mathcal{P}$ in each point p, spanned by the vectors $\bar{X}_p$ tangent to $\mathcal{P}$ at p. Due to the right G-action there is a canonical subspace, the *vertical tangent space* $V_p\mathcal{P} \subset T_p\mathcal{P}$, whose elements are thus called *fundamental vectors* and are constructed as follows. Given

[4] Here we've overlooked the fact that a local section $\sigma : U \subset \mathcal{M} \to \pi(U) \subset L\mathcal{M}$ is needed in order to pullback the metric $\bar{g}$ on $\mathcal{M}$, giving $\sigma^*\bar{g}$ as (part of) the associated section of $\mathcal{T}^r{}_s$. Only then do we have the asserted coordinate expression.

[5] Actually a SO-class of them.

an element $A \in \mathfrak{g}$, the Lie algebra of G, we have a curve $g_t = \exp(tA)$ in G.[6] Then $\mathcal{R}_{g_t}p = p\exp(tA)$ is a curve in the fiber $G_{\pi(p)} \subset \mathcal{P}$. The vector field defined by,

$$\bar{X}^v f(p) = \frac{d}{dt}\bigg|_{t=0} f\left(p\exp(tA)\right),$$

with $f : \mathcal{P} \to \mathbb{R}$ a smooth map, is tangent to the fiber $G_{\pi(p)}$ at p. Performed at each $p \in \mathcal{P}$, this construction defines the *fundamental vertical vector field* $\bar{X}^v$ generated by $A \in \mathfrak{g}$. The map $A \mapsto \bar{X}^v$ is a Lie algebra isomorphism. Obviously $\mathcal{R}_{g*}\bar{X}^v_p = \bar{X}^v_{pg}$ and $\pi_*\bar{X}^v = 0$. In other words the vertical vector fields are right-invariant, thus *projectable*, with trivial projection.

However there is, in general, no canonical complement of $V_p\mathcal{P}$ in $T_p\mathcal{P}$. Had we such a space we could canonically define a *parallel transport*, that is a way to compare points in different fibers, like vectors in different tangent spaces on a manifold $\mathcal{M}$, in the case of the tangent bundle $T\mathcal{M}$. Unfortunately we do not. So we need to make a *choice* of a complement, called the *horizontal subspace* $H_p\mathcal{P} \subset T_p\mathcal{P}$, so as to be able to compare points in distinct fibers, or in other words, to *connect* them.[7] Hence the notion of *connection on principal bundles* articulated first in (Ehresmann, 1950).

Connection An *Ehresmann connection* is the choice of a smooth distribution $\{H_p\mathcal{P}\}$ such that at any $p \in \mathcal{P}$:

i $H_p\mathcal{P}$ is a complement of $V_p\mathcal{P}$, so $T_p\mathcal{P} = H_p\mathcal{P} \oplus V_p\mathcal{P}$, and $\bar{X}_p = \bar{X}^h_p + \bar{X}^v_p$,

ii $\mathcal{R}_{g*}H_p\mathcal{P} = H_{pg}\mathcal{P}$.

Since $V_p\mathcal{P} = \ker\pi_*$, the restriction $\pi_* : H_p\mathcal{P} \to T_{\pi(p)}\mathcal{M}$ is an isomorphism. Then any $X \in \Gamma(T\mathcal{M})$ with integral curve c can be lifted to a unique $\bar{X}^h \in \Gamma(H\mathcal{P})$ with integral curve $\bar{c}$, such that $\pi_*\bar{X}^h = X$ and $\pi(\bar{c}) = c$. We say that $\bar{X}^h/\bar{c}$ is the *horizontal lift* of X/c. Conversely however, not all horizontal vector fields on $\mathcal{P}$ project as well defined vector fields on $\mathcal{M}$. Only projectable, i.e right-invariant, horizontal vector fields do so.

If $V_p\mathcal{P}$ is closed under the bracket operation, this is not so for $H_p\mathcal{P}$. Indeed given $\bar{X}^h, \bar{Y}^h \in H_p\mathcal{P}$, $[\bar{X}^h, \bar{Y}^h]^v \neq 0$ in general. By definition, only if the connection is *integrable* (or *involutive*) does $H_p\mathcal{P}$ close under the bracket.

As claimed, given a connection we can define parallel transport. Indeed, let $c(t) \in \mathcal{M}$ be a curve with endpoints $\{x_0 = c(0), x_1 = c(1)\}$, and fix a point $p_0 \in \pi^{-1}(x_0) \subset \mathcal{P}$. There is a unique horizontal lift $\bar{c}(t)$ such that $\bar{c}(0) = p_0$ and a unique point $p_1 \in \pi^{-1}(x_1) \subset \mathcal{P}$ such that $p_1 = \bar{c}(1)$. The point p_1 is called the parallel transport of p_0 along $c(t)$. By varying the point p_0 we obtain an isomorphism from $\pi^{-1}(x_0)$ to $\pi^{-1}(x_1)$ called the parallel transport.

Let $\varphi : \mathcal{P} \to V$ be an equivariant V-valued map. We define the *covariant derivative* of φ along the vector field X as, $D_X\varphi := \bar{X}(\varphi)$. It is also a V-valued equivariant function on $\mathcal{P}$. We say that φ is covariantly constant or *parallel* along X if $D_X\varphi = 0$. This notion of covariant derivative of equivariant maps φ translates, through ι, as covariant differentiation of sections $\widetilde{\varphi}$ of associated bundles $E = \mathcal{P} \times_\rho V$.

This is a fine geometric construction, but as it stands it is poorly suited for field theory in Physics. Since a smooth distribution is as well described as the kernel of a smooth vector space-valued 1-form, an equivalent but more tractable dual definition of a connection goes as follows.

An *Ehresmann connection* 1*-form* is a smooth $\mathfrak{g}$-valued 1-form on $\mathcal{P}$, $\omega \in \Lambda^1(\mathcal{P}, \mathfrak{g})$, satisfying the conditions:

i $\omega_p(\bar{X}^v_p) = A \in \mathfrak{g}$, the Lie algebra element A generating $\bar{X}^v$,

ii Ad_G-equivariance: $\mathcal{R}^*_g\omega_{pg} = \mathrm{Ad}_{g^{-1}}\omega_p$.

The horizontal subspace at p is then defined as, $H_p\mathcal{P} = \ker\omega_p$. The first condition then implies that ω realizes a projection of $T_p\mathcal{P}$ onto $V_p\mathcal{P} \simeq \mathfrak{g}$, so that it is equivalent to the geometric condition i above. Moreover

[6] We suppose G connected.

[7] Applied to $T\mathcal{M}$, this is the root of the theory of connections of Schouten and Levi-Civita (1917).

the second condition implies that, $\mathcal{R}_g^*\omega_{pg}(\bar{X}_p) = \omega(\mathcal{R}_{g*}\bar{X}_g) = g^{-1}\omega_g(\bar{X}_g)g = 0$. Said otherwise, if $\bar{X}_p \in \ker\omega$ then $\mathcal{R}_g^*\bar{X}_p \in \ker\omega$. This is equivalent to the geometric condition ii.

Many choices of connection 1-forms are available. Denote the set of connections on $\mathcal{P}$ by $\mathcal{A}_\mathcal{P}$. It is an affine space, not a vector space. Indeed, given two connections ω_0 and ω_1, the sum $\omega_0 + \omega_1$ fails to be a connection since it does not satisfy condition i above. The only possibility for a linear combination $\lambda_0\omega_0+\lambda_1\omega_1$ to be a connection is if $\lambda_1 = 1-\lambda_0$. Such "homotopic connections" are met in the study of characteristic classes of fiber bundles. See e.g section 4.1.3 ahead.

Remarkable p-forms on $\mathcal{P}$ As a preparation for what comes next, here are some useful definitions.

A V-valued r-form on $\mathcal{P}$, $\beta \in \Lambda^r(\mathcal{P}, V)$, is said *horizontal* or *semibasic* when $\beta(\bar{X}_1, \ldots, \bar{X}_r) = 0$ if any $\bar{X}_i \in V_p\mathcal{P}$.

The p-form β is said *equivariant*, or *pseudo-tensorial* , of type (V,ρ) if it satisfies, $\mathcal{R}_g^*\beta = \rho(g^{-1})\beta$. If β is both horizontal and equivariant of type (V,ρ), it is said *tensorial* of type (V,ρ). We denote by $\mathcal{F}_\mathcal{P}$ the set of tensorial forms on $\mathcal{P}$ of all types. Notice that if β is a $(\mathfrak{g}, \mathrm{Ad})$-tensorial 1-form and ω is a connection 1-form, then $\omega + \beta$ is still a connection.

If a r-form β is tensorial of type (V, id), i.e $\mathcal{R}_g^*\beta = \beta$, it is said *projectable*. Projectable forms are a subset of tensorial forms. They are also called *basic* forms since they lie in the image of π^*.

The *graded commutator* of two V-valued r/s-forms α/β is defined as, $[\alpha,\beta]_{\mathrm{grad}} = \alpha\wedge\beta - (-)^{rs}\beta\wedge\alpha$. For r odd we have, $[\alpha,\beta]_{\mathrm{grad}} = 2\alpha\wedge\alpha$.

Covariant derivative Define the projection map $h : \bar{X} \mapsto \bar{X}^h$. The *exterior covariant derivative* of an equivariant r-form β of type (V,ρ) is defined as, $D\beta := d\beta \circ h$. Obviously $D\beta$ is by definition a tensorial $r+1$-form. It satisfies the graded Leibniz rule, $D(\alpha\wedge\beta) = D\alpha\wedge\beta + (-)^{|\alpha|}\alpha\wedge D\beta$, where $|\alpha|$ is the degree of α. On tensorial forms the exterior covariant derivative reads

$$D\beta = d\beta + \rho_*(\omega)\wedge\beta. \tag{1.6}$$

This defines, modulo isomorphism (1.1), the covariant derivative of sections of an associated bundle $E = \mathcal{P}\times_\rho V$ since equivariant functions on $\mathcal{P}$ are tensorial 0-forms.

Curvature The connection ω is an equivariant 1-form of type $(\mathfrak{g}, \mathrm{Ad})$. Through D we can associate to it a tensorial 2-form, the *curvature*, defined as $\Omega := D\omega$. Explicitly evaluated on two vectors, $\Omega_p(\bar{X}_p, \bar{Y}_p) = D\omega_p(\bar{X}_p, \bar{Y}_p) = d\omega_p(\bar{X}_p^h, \bar{Y}_p^h) = \bar{X}_p^h\omega_p(\bar{Y}_p^h) - \bar{Y}_p^h\omega_p(\bar{X}_p^h) - \omega_p([\bar{X}_p^h, \bar{Y}_p^h]) = -\omega_p([\bar{X}_p^h, \bar{Y}_p^h])$. So the curvature Ω is a measure of the non-integrability of the distribution $\{H_p\mathcal{P}\}$.[8] If the latter is integrable, $\Omega = 0$ and the connection ω is said *flat*.

Knowing that $[\bar{X}_p^h, \bar{Y}_p^v] \in H_p\mathcal{P}$, one can show by evaluation on the three possible pairs, $(\bar{X}^h, \bar{Y}^h)$, $(\bar{X}^h, \bar{Y}^v)$ and $(\bar{X}^v, \bar{Y}^v)$, that the curvature satisfies *Cartan's structure equation*:

$$\Omega = d\omega + \frac{1}{2}[\omega,\omega]_{\mathrm{grad}} = d\omega + \omega\wedge\omega, \tag{1.7}$$

where the last equality hold for matrix-valued forms. From this it is easy to prove the *Bianchi identity*: $D\Omega = 0$. The curvature is related to the square of the covariant derivative. Indeed, β being tensorial of type (V,ρ), we have $DD\beta = \rho_*(\Omega)\beta$, which in nonzero unless Ω is.

Action of the gauge group $\mathcal{G}$ Every object on $\mathcal{P}$ is acted upon by its group of vertical automorphisms $\mathrm{Aut}_v(\mathcal{P}) \simeq \mathcal{G}$. A vertical automorphism $\Psi(p) = p\gamma(p)$ acts by pullback on forms.

On tensorial r-forms β of type (V,ρ) we have, $\Psi^*\beta := \beta^\gamma = \rho(\gamma^{-1})\beta$. This tells us that the gauge transformations of a section of E or of a (V,ρ)-tensorial 0-form, of its (V,ρ)-tensorial covariant derivative 1-form and of the $(\mathfrak{g}, \mathrm{Ad})$-tensorial curvature 2-form are respectively,

$$\psi^\gamma = \rho(\gamma^{-1})\psi, \quad (D\psi)^\gamma = \rho(\gamma^{-1})D\psi \quad \text{and} \quad \Omega^\gamma = \gamma^{-1}\Omega\gamma. \tag{1.8}$$

[8] An equivalent formulation is to say that Ω measures the default for $H_p\mathcal{P}$ to be a Lie algebra.

We have tensorial transformations here.

On the connection 1-form ω, which is not tensorial, the action is more involved and the the gauge transformation is non-tensorial,

$$\Psi^*\omega = \omega^\gamma = \gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma. \tag{1.9}$$

It is easily verified that this expression combined with Cartan's structure equation (1.7) gives the right transformation for Ω, and that combined with (1.6) it indeed reproduces the right transformation for the covariant derivative.

Being the products of an active transformation by a vertical automorphism Ψ, the results (1.8) and (1.9) are referred to as *active gauge transformations* to be distinguished from *passive* ones described below.

Localization The connection, the curvature and the equivariant maps are all globally defined objects on $\mathcal{P}$. We would like to describe them as seen from the base manifold $\mathcal{M}$, if only because Physics is done there.

To achieve this goal we only need a local section of the bundle $\mathcal{P}$, $\sigma_i : U_i \subset \mathcal{M} \to \pi^{-1}(U_i) \subset \mathcal{P}$. Then we can define the *local connection* 1*-form* on U_i as the pullback, $\sigma_i^*\omega = A_i = A_{i,\mu}dx^\mu \in \Lambda^1(U_i,\mathfrak{g})$. Similarly the corresponding *local curvature* 2*-form* on U_i is, $\sigma_i^*\Omega = F_i = \frac{1}{2}F_{i,\mu\nu}dx^\mu \wedge dx^\nu \in \Lambda^2(U_i,\mathfrak{g})$. The local version of Cartan's structure equation holds,

$$F_i = dA_i + \frac{1}{2}[A_i, A_i]_{\text{grad}} = dA_i + A_i \wedge A_i, \tag{1.10}$$

which in coordinates reads, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.[9] We recognize the familiar expression of the *field strength* as a function of the *gauge potential*, encountered in non-abelian gauge field theory. The even more familiar case of the Maxwell-Faraday tensor of electromagnetism is obtained when the bracket is zero, that is when the underlying symmetry/structure group G is abelian.

The local version of an equivariant map is, $\sigma_i^*\psi = \varphi_i$. The local version of the covariant derivative is then given by,

$$\sigma_i^* D\psi = d\sigma^*\psi + \rho_*(\sigma^*\omega)\sigma^*\psi, \quad \text{that is,} \quad D_i\varphi_i = d\varphi_i + \rho_*(A_i)\varphi_i. \tag{1.11}$$

In coordinates this reads, $D_\mu\varphi = \partial_\mu\varphi + \rho_*(A_\mu)\varphi$. We recognize the expression of the *minimal coupling* between a *matter field* φ and the *gauge potential* A_μ.

Given another trivialization (U_j, σ_j) such that on $U_i \cap U_j$, $\sigma_j = \sigma_i g_{ij}$ with $g_{ij} : U_i \cap U_j \to G$, we have the gluing properties:

$$A_j = g_{ij}^{-1}A_i g_{ij} + g_{ij}^{-1}dg_{ij}, \quad F_j = g_{ij}^{-1}F_i g_{ij}, \quad \varphi_j = \rho(g_{ij}^{-1})\varphi_i, \quad \text{and} \quad D_j\varphi_j = \rho(g_{ij}^{-1})D_i\varphi_i. \tag{1.12}$$

These are known as *passive gauge transformation* since they arise as two local descriptions of one and the same global object. One object seen in two coordinate systems, sort of.

The local version an active gauge transformation is an entirely different matter. It relates, in a unique local description (a single coordinate system), two different global objects actively transformed into one another. The active transformations, the vertical automorphisms, seen from $U_i \subset \mathcal{M}$ is $\sigma_i^*\Psi := \Psi_i : U_i \to \mathcal{P}$. These are generated by maps $\sigma_i^*\gamma := \gamma_i : U_i \to \mathcal{G}$ such that, $\Psi_i(x) = \sigma_i(x)\gamma_i(x)$. The γ_i's constitute the *local gauge group* $\mathcal{G}_{loc}$. The local version of (1.8) and (1.9) are then,

$$A_i^{\gamma_i} = \gamma_i^{-1}A_i\gamma_i + \gamma_i^{-1}d\gamma_i, \quad F_i^{\gamma_i} = \gamma_i^{-1}F_i\gamma_i, \quad \varphi_i^{\gamma_i} = \rho(\gamma_i^{-1})\varphi_i, \quad \text{and} \quad (D_i\varphi_i)^{\gamma_i} = \rho(\gamma_i^{-1})D_i\varphi_i. \tag{1.13}$$

These are the *local active gauge transformations*, not to be confused with the *passive* ones despite the formal similarity.

[9] From now on we will drop the subscript for the graded commutator. Its identification and distinction from the usual bracket, as here, should be understood from the context.

Connections & tensorial forms on reduced bundles As we've seen at the end of section 1.1.1, given a principal bundle $\mathcal{P}(\mathcal{M},G)$ and an equivariant function $f \simeq (r,u) : \mathcal{P} \to V \simeq \Gamma \times G/H$, the map u realises a subbundle $\mathcal{P}'(\mathcal{M},H)$. The question arises as to what happens for connections and tensorial forms on $\mathcal{P}$ once restricted to $\mathcal{P}'$.

If (V,ρ) is a representation for G, it becomes a representation for its subgroup H by restriction. Moreover the vertical vector fields on $\mathcal{P}'$ are a subset of those on $\mathcal{P}$. So a tensorial form on $\mathcal{P}$ restricts to a tensorial form on $\mathcal{P}'$. Denote $\imath : \mathcal{P}' \to \mathcal{P}$ the inclusion map, we have $\imath^*\mathcal{F}_\mathcal{P} \to \mathcal{F}'_\mathcal{P}$.

For a connection the situation is a bit more subtle. Suppose we can find an Ad_H-invariant complement vector space $\mathfrak{p}$ to $\mathfrak{h}$ in $\mathfrak{g}$.[10] Accordingly the connection form splits as, $\omega_\mathfrak{g} = \omega_\mathfrak{h} + \omega_\mathfrak{p}$. Under restriction to $\mathcal{P}'$ we have $\imath^*\omega_\mathfrak{g} = \widetilde{\omega}_\mathfrak{g} \in \Lambda^1(\mathcal{P}',\mathfrak{g})$. Clearly enough $\widetilde{\omega}_\mathfrak{h} = \imath^*\omega_\mathfrak{h}$ satisfies,

$$\mathcal{R}_h^*\widetilde{\omega}_\mathfrak{h} = \text{Ad}_h\widetilde{\omega}_\mathfrak{h} \quad \text{and} \quad \widetilde{\omega}_\mathfrak{h}(\bar{Y}^v) = y, \tag{1.14}$$

where $y \in \mathfrak{h}$ induce $\bar{Y}^v$ on $\mathcal{P}'$. So $\widetilde{\omega}_\mathfrak{h}$ is a connection 1-form on $\mathcal{P}'$. Besides, $\widetilde{\omega}_\mathfrak{p} = \imath^*\omega_\mathfrak{p}$ satifies,

$$\mathcal{R}_h^*\widetilde{\omega}_\mathfrak{p} = \text{Ad}_h\widetilde{\omega}_\mathfrak{p} \quad \text{and} \quad \widetilde{\omega}_\mathfrak{h}(\bar{Y}^v) = 0. \tag{1.15}$$

That is, $\widetilde{\omega}_\mathfrak{p}$ is a tensorial 1-form of type $(\mathfrak{p},\text{Ad})$. Thus we find that $\imath^* : \mathcal{A}_\mathcal{P} \to \mathcal{A}_{\mathcal{P}'} \times \mathcal{F}_{\mathcal{P}'}$.

The curvature splits as $\Omega_\mathfrak{g} = \Omega_\mathfrak{h} + \Omega_\mathfrak{p}$. And under restriction,

$$\imath^*\Omega_\mathfrak{g} = \widetilde{\Omega}_\mathfrak{g} = \widetilde{\Omega}_\mathfrak{h} + \widetilde{D}\widetilde{\omega}_\mathfrak{p} + \widetilde{\omega}_\mathfrak{p} \wedge \widetilde{\omega}_\mathfrak{p}. \tag{1.16}$$

The first term, $\widetilde{\Omega}_\mathfrak{h}$, is the curvature 2-form associated to the connection $\widetilde{\omega}_\mathfrak{h}$ on $\mathcal{P}'$. The second term $\widetilde{D}\widetilde{\omega}_\mathfrak{p} = d\widetilde{\omega}_\mathfrak{p} + [\widetilde{\omega}_\mathfrak{h},\widetilde{\omega}_\mathfrak{p}]$, is the covariant derivative of the $(\mathfrak{p},\text{Ad})$-tensorial 1-form $\widetilde{\omega}_\mathfrak{p}$ on $\mathcal{P}'$. All terms indeed belong to $\mathcal{F}_{\mathcal{P}'}$, as expected.

The gauge group $\mathcal{H}$ of $\mathcal{P}'$ acts on $\widetilde{\omega}_\mathfrak{h}$, $\widetilde{\Omega}_\mathfrak{h}$, $\widetilde{\omega}_\mathfrak{p}$ and $\widetilde{D}\widetilde{\omega}_\mathfrak{p}$ as described by (1.9), (1.8) (with $\rho = \text{Ad}$). The local version is exactly as described in the previous paragraph.

As for the function f itself, its restriction is $\widetilde{f} = \imath^* f = (\imath^* r, \imath^* u) = (\widetilde{r}, eH)$, where the second component is constant on $\mathcal{P}'$ by definition. Then, on $\mathcal{P}'$, $\widetilde{f}$ can be identified with the map $\widetilde{r} : \mathcal{M} \to \Gamma$. The group H being the isotropy group of each point of Γ, the action $\rho_*(\mathfrak{h})$ on the latter is trivial. Therefore $\widetilde{D}\widetilde{f} \simeq d\widetilde{r} + \rho_*(\widetilde{\omega}_\mathfrak{p})\widetilde{r}$.

Let us briefly illustrate all this by a follow-up of the two examples given at the end of section 1.1.

Example 1 Given the map $\varphi \simeq (\eta,u) : \mathcal{P} \to \mathbb{C}^2 \simeq \Gamma \times SU(2)$, with $\Gamma = \left\{\binom{0}{\eta} \mid \eta \in \mathbb{R}^+\right\}$, we reduce the bundle $\mathcal{P}\big(\mathcal{M}, G = SU(2) \times U(1)\big)$ to the bundle $\mathcal{P}'\big(\mathcal{M}, H = U(1)\big)$.

The group G is indeed compact. Being a direct product, the two factors don't see each other. Thus its Lie algebra splits as $\mathfrak{g} = \mathfrak{u}(1) + \mathfrak{su}(2)$, with no action of $\mathfrak{u}(1)$ on $\mathfrak{su}(2)$. Accordingly the connection on $\mathcal{P}$ splits as $\omega_\mathfrak{g} = a_{\mathfrak{u}(1)} + b_{\mathfrak{su}(2)}$. Under restriction to $\mathcal{P}'$, $\widetilde{\omega}_\mathfrak{g} = \widetilde{a}_{\mathfrak{u}(1)} + \widetilde{b}_{\mathfrak{su}(2)}$, where $\widetilde{a}_{\mathfrak{u}(1)}$ is a $U(1)$-connection and $\widetilde{b}_{\mathfrak{su}(2)}$ is an $\big(\mathfrak{su}(2), L_{U(1)}\big)$-tensorial 1-form.[11] Both are $SU(2)$-invariant and the only gauge freedom left is $U(1)$. Under localization, $\widetilde{a}_{\mathfrak{u}(1)}$ and $\widetilde{b}_{\mathfrak{su}(2)}$ describe the photon A_μ and the weak bosons $W^\pm, Z^0$ respectively. The map $\widetilde{\varphi} \simeq \eta$ and its covariant derivative is $\widetilde{D}\widetilde{\varphi} \simeq d\eta + \widetilde{b}\eta$.

Example 2 The group GL is non-compact, so we couldn't find a complement space of $\mathfrak{so}$ in $\mathfrak{gl}$. However the GL-connection Γ on $\mathcal{P} = L\mathcal{M}$, whose components are the *Christoffel symbols*, still restricts to an SO-connection ω on $\mathcal{P}'$. The latter is known as the *Lorentz connection* or *spin connection* in Palatini formulation of General Relativity (see 2.1.2 ahead).

1.1.3 Lagrangian

Thus far we've presented a geometrical framework where various objects seem to describe adequately the fields of a gauge theory . But the theory itself is only specified once a *Lagrangian* is chosen. A Lagrangian is

[10] Such a complement can always be found if G is compact and semi-simple. Nevertheless this vector space $\mathfrak{p} \simeq T_e(G/H)$ might not be a Lie subalgebra of $\mathfrak{g}$.

[11] Here $L_{U(1)}$ means left multiplication by a 2×2 matrix, $U(1)$ being represented as a subgroup of $SU(2)$.

a scalar function on the space of fields,[12] in this case,

$$\mathcal{L} : \mathcal{A} \times \mathcal{F} \to \mathbb{R},$$

where $\mathcal{A}$ and $\mathcal{F}$ denote the pullback on $\mathcal{M}$ of $\mathcal{A}_{\mathcal{P}}$ and $\mathcal{F}_{\mathcal{P}}$. To construct such a Lagrangian we need quadratic forms on the Lie algebra $\mathfrak{g}$ and on the various representation spaces (V_i, ρ_i) involved in the construction of associated bundles and their sections. For the former the Killing form, multiple of the Trace operator, will do.

We then define the *action functional* $\mathcal{S} = \int_{\mathcal{M}} \mathcal{L}\ d^m x$. By use of the variational principle on $\mathcal{S}$ we obtain the equations of motion for the fields.

A general-relativistic requirement is that $\mathcal{L}$ must be invariant under Diff$(\mathcal{M})$. Which it is since written in term of differential forms. Another requirement is that the Lagrangian must be invariant under the action of the gauge group $\mathcal{G}$ so that $\Psi^* \mathcal{L} := \mathcal{L}^\gamma = \mathcal{L}$. The argument being that, much in the spirit of the general covariance principle, no observable physical quantities could depend on the specific "gauge coordinate system" chosen for the description.[13] Then the Lagrangian is actually a scalar function on the *moduli space of fields*, $(\mathcal{A} \times \mathcal{F})/\mathcal{G}$.

An alternative formulation would be to speak of the *Lagrangian scalar m-form* $\bar{L}$ on $\mathcal{P}$, with $m = \dim\mathcal{M}$. If the form $\bar{L}$ is tensorial of type $(\mathbb{R}, \text{id})$, it projects to a globally defined Lagrangian m-form on $\mathcal{M}$, $L = \mathcal{L}\ d^m x$. Actually the horizontality of $\bar{L}$ is not mandatory, and its $(\mathbb{R}, \text{id})$-type equivariance is enough to provide a well defined L such that the lagrangian $\mathcal{L}$ is clearly invariant under both Diff$(\mathcal{M})$ and $\mathcal{G}$.

The action is just $\mathcal{S} = \int_{\mathcal{M}} L$, thus the Hodge star operator might enter into the definition of L, in which case a metric on $\mathcal{M}$ is needed. If the Hodge operator is not necessary, a metric structure on $\mathcal{M}$ is unnecessary and one says that the theory described by such a Lagrangian is *topological*.

It is worthwhile to note that if additional criterions are met, the strict invariance of $\mathcal{L}$ is not mandatory. Indeed, in the context of Quantum Field Theory e.g, the *path integral* generating the propagators, or *quantum action*, is $Z = \int_{\mathcal{A}\times\mathcal{F}} dA_\mu d\varphi\ e^{\mathcal{S}(\mathcal{A}_\mu, \varphi)}$. So $\mathcal{L}$ could be invariant up to a strictly gauge dependent term that could be rescaled so as to give a pure number which just affects the (anyway arbitrary) normalization of Z. The Chern-Simons Lagrangian is a notorious example.

A more common situation is to ask for $\mathcal{M}$ to be boundaryless. In such a case it is enough to demand the *quasi-invariance* of the Lagrangian, that is, invariance up to a total derivative, $\mathcal{L}^\gamma = \mathcal{L} + \partial_\mu \alpha$. This amounts to require the quasi-invariance of the lagrangian form, that is, invariance up to an exact m-form: $\bar{L}^\gamma = \bar{L} + d\bar{\alpha} \to L^\gamma = L + d\alpha$. Due to Stokes' theorem $\mathcal{S}$ remains invariant, $\mathcal{S}^\gamma = \mathcal{S}$, and the equations of motion are unaffected so that (non-quantum) Physics does not see the gauge symmetry as required.

Quasi-invariance is famously met in Newtonian mechanics, e.g in the case of the free particle Lagrangian under a Galilean transformation. See (Azcarraga and Izquierdo, 1995). It is more scarce in gauge relativistic field theory, and to all practical purpose we will consider strict invariance under $\mathcal{G}$.

1.2 Cartan geometry and Cartan connections: a language for Gravitation

1.2.1 A language for classical gravitation

Geometry of fibered spaces, principal bundles and Ehresmann connection have been presented above. Their generality stems from the fact that the relation between space-time and the fiber is loose: indeed in each point any fiber can be attached. Their efficiency in describing the (classical) physics of Yang-Mills and gravitationnal fields fulfills Einstein's conviction: "One is driven to the belief that both sorts of field must correspond to a unified structure of space"[14].

[12] Actually since derivatives of any order could enter the Lagrangian, the latter is sometimes defined as a scalar map on the *r-th jet bundle* of the space of fields: $\mathcal{L} : J^r(\mathcal{A} \times \mathcal{F}) \to \mathbb{R}$. But this nice subtlety adds little to our present discussion. We admit that the derivatives of fields enter the definition of the Lagrangian function.

[13] And we've seen that active and passive gauge transformations are formally identical. We thus speak of the *gauge invariance* of the Lagrangian, meaning both passive and active.

[14] He refers here to gravitation and electromagnetism. This is the year 1934, twenty years before the paper of Yang and Mills. See (Pesic, 2007), p 193.

There is a precursor to Ehresmann/principal connections associated to princpal bundles, the so-called Cartan connections associated to Cartan geometry. They are somehow less general than their heir, but of immense interest to the physics of gravitation. Indeed, since Einstein we know that gravitation is the geometry of space-time, the base manifold of bundles. And it turns out that Cartan geometry has a close relation to the base manifold. The depth of Cartan geometry is that it unifies the two great advances in geometry of the nineteenth century.

The first, which we could call the *algebraic* viewpoint, is synthetised in Klein's *Erlangen program* of 1872. The idea of Klein is that geometry is the study of the invariants of a space which is homogeneous. Since Cartan such spaces are called symmetric spaces. The invariants are defined with respect to the continuous group of transformations G of the symmetric space and its subgroups. If one defines the isotropy group H of a symmetric space, this space might be identified with the quotient G/H. The pairs (G,H) and the quotient space they define are called *Klein geometries.*

The second, which is the *differential* viewpoint, is essentially Riemann's insight delivered in his habilitation dissertation of 1854, "*On the hypothesis which lie at the foundations of geometry*". Riemann's view is that of a space, a *manifold*, which is neither homogeneous nor isotropic. The geometric properties of such a manifold are encoded in its line element, or *metric*, from which one can derive the Riemann tensor which describes how the curvature of the manifold changes (smoothly) from point to point and thus departs from the flatness of the Euclidean space. A significant contribution to Riemannian geometry was made by Schouten and Levi-Civita when they independently introduced the notions of parallel transport and connection.

What Riemann did with Euclidean space, Cartan did for any Klein geometry, thereby providing the common generalization of both Riemann and Klein geometries. In the following we are going to explain the basics of Cartan geometry, showing that it contains all the essential ingredients for a gauge theory, and illustrate it. In so doing we will prove the following two connected facts; first that Cartan geometry has indeed an organic link to its base manifold, and second, that it is a generalization of Riemannian geometry. This will support the idea that Cartan geometry is the proper framework for gravitational theories.

1.2.2 Global definition of a Cartan geometry

This section is devoted to a synthetic overview of the basics concepts of Cartan geometry. Our presentation is essentially based on the beautiful book by Sharpe (Sharpe, 1996) to which we refer the reader for demonstrations and additionnal material. An alternative presentation, using the framework of higher-order frame bundles, can be found in (Kobayashi, 1972), (Ogiue, 1967) and (Ochiai, 1970).

Cartan connection Given a Klein geometry (G,H), $\mathfrak{g}$ is the Lie algebra of G. Let $\mathcal{P}(\mathcal{M},H)$ be a fiber bundle over the manifold $\mathcal{M}$ with structure Lie group H. We have the following

Definition: a *Cartan connection* is a 1-form on $\mathcal{P}$, $\varpi \in \Lambda^1(\mathcal{P},\mathfrak{g})$, which satisfies:

i $\mathcal{R}_h^*\varpi = Ad_{h^{-1}}\varpi$,

ii $\varpi(\bar{X}^v_\lambda) = \lambda$, where $\bar{X}^v_\lambda$ is a fundamental/vertical vector field and $\lambda \in \mathfrak{h}$ its associated element in $\mathfrak{h}$,

iii for any point $p \in \mathcal{P}$, $\varpi : T_p\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism.

A *Cartan geometry* is the pair $(\mathcal{P},\varpi)$. The first two properties above are those which characterize an Ehresmann connection, as previously seen. The third one is the key property of a Cartan connection, and the very reason why Cartan geometry is much more intimately connected to the base manifold $\mathcal{M}$ than the principal bundle geometry, as we're about to see shortly.

Curvature The *curvature*, $\Omega \in \Lambda^2(\mathcal{P},\mathfrak{g})$, of the Cartan connection is given by Cartan's structure equation,

$$\Omega := d\varpi + \tfrac{1}{2}[\varpi,\varpi].$$

The curvature is a tensorial 2-form and satisfies the *Bianchi identity*, $d\Omega = [\Omega,\varpi]$. Given the projection map $\tau : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, we define the **torsion** as: $\tau(\Omega)$.

The 2-form Ω define the *curvature function* $K : \mathcal{P} \to \mathrm{Hom}\left(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g}\right)$ through,

$$K(p)(\lambda_1, \lambda_2) = \Omega_p \left(\varpi_p^{-1}(\lambda_1),\ \varpi_p^{-1}(\lambda_2)\right),$$

for $\lambda_1, \lambda_2 \in \mathfrak{g}/\mathfrak{h}$ (indeed, for $\lambda \in \mathfrak{h}$, $\bar{Y} = \varpi^{-1}(\lambda)$ is vertical so $\Omega(Y, \_) = 0$).

Klein model A Cartan geometry $(\mathcal{P}, \varpi)$ is said to be based on the *model Klein geometry* (G,H) where G is called the *principal group* of the Klein geometry. Moreover, notice that the Maurer-Cartan form ω_G of G satisfies all the requirements of a Cartan connection. It furthermore satisfies the Maurer-Cartan structure equation,

$$d\omega_G + \tfrac{1}{2}[\omega_G, \omega_G] = 0.$$

From this we see that a Klein geometry is nothing but a Cartan geometry whose principal bundle is the principal group, $\mathcal{P} = G$[15]. It is a bundle over $M \simeq G/H$ with structure group H. Its Cartan connection is the Maurer-Cartan form of G, $\varpi = \varpi_G$, and it is a *flat* connection since $\Omega = \Omega_G = 0$. Thus is justified the assertion that Cartan geometry generalizes Klein geometry.

From this we see that flatness in the sense of Cartan means that the base manifold can be any symmetric space ($\sim G/H$). The flatness in the sense of Riemann means that the manifold can only be the symmetric space $\mathbb{R}^n$. This is already a hint on how Cartan geometry generalizes Riemannian geometry.

Special geometries In general the values of Ω will span the whole of $\mathfrak{g}$. But there might be special cases where the span is a subspace of $\mathfrak{g}$. The most immediate example is the torsion-free case, $\tau(\Omega) = 0$, where the span is $\mathfrak{h}$ and the curvature function has values in $\mathrm{Hom}\left(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}\right)$.

We can distinguish several special geometries as the values of the curvature span submodules of $\mathfrak{g}$ or $\mathrm{Hom}\left(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}\right)$. But among the most interesting are the *normal* geometries and their associated *normal Cartan connection.* To define it we need the following

Definition: the *Ricci homomorphism*, Ricci: $\mathrm{Hom}\left(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}\right) \to (\mathfrak{g}/\mathfrak{h})^* \otimes (\mathfrak{g}/\mathfrak{h})^*$, is the composite map:

$$\begin{aligned}\mathrm{Hom}\left(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}\right) &\simeq \Lambda^2(\mathfrak{g}/\mathfrak{h}) \otimes \mathfrak{h} \xrightarrow{\mathrm{id}\otimes\mathrm{ad}} \Lambda^2(\mathfrak{g}/\mathfrak{h}) \otimes \mathrm{End}(\mathfrak{g}/\mathfrak{h}) \\ &\simeq \Lambda^2(\mathfrak{g}/\mathfrak{h}) \otimes (\mathfrak{g}/\mathfrak{h})^* \otimes (\mathfrak{g}/\mathfrak{h}) \xrightarrow{\text{contraction}} (\mathfrak{g}/\mathfrak{h})^* \otimes (\mathfrak{g}/\mathfrak{h})^*.\end{aligned}$$

A **normal Cartan connection** is the unique ϖ whose curvature Ω satisfies:

- $\tau(\Omega) = 0$, thus ϖ is torsion-free and K takes values in $\mathrm{Hom}\left(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}\right)$,
- $\mathrm{Ricci}(K) = 0$, i.e the curvature function is in the ker of the Ricci homomorphism.

A wide and rich class of special geometries are the *reductive* ones. They are of first importance for physics and will occupy the rest of this section. Before we turn to these, we define the gauge group, say a word about the local version of Cartan geometry and introduce the notions of tensors and covariant derivative.

Gauge group A vertical automorphism of the bundle $\mathcal{P}$ is a map $\psi : \mathcal{P} \to \mathcal{P}$ which maps fibers to fibers. It is induced by a map $\gamma : \mathcal{P} \to H$, such that $\psi(p) = p\gamma(p)$ and satisfying $\gamma(ph) = h^{-1}\gamma(p)h$. The action on ϖ and Ω:

- $\Psi^*\varpi = \varpi^\gamma = \gamma^{-1}\varpi\gamma + \gamma^{-1}d\gamma$,
- $\Psi^*\Omega = \Omega^\gamma = \gamma^{-1}\Omega\gamma$.

These are the active gauge transformations of the Cartan geometry and the infinite dimensional group $\mathcal{H} = \left\{\gamma : \mathcal{P} \to H \mid \gamma(ph) = h^{-1}\gamma(p)\gamma\right\}$ is the *gauge group.*

[15]Appreciate the nice coincidence of both terminologies.

Localization Given an open set $U \subset \mathcal{M}$ and a section $\sigma : U \to \mathcal{P}$, we can pull back the Cartan connection and its curvature,

- $A = \sigma^* \varpi$,
- $F = \sigma^* \Omega = dA + \frac{1}{2}[A, A]$,

The pair (U, A) is called a *Cartan gauge*. The local form A is such that, $\tau(A) : T_x(U) \to \mathfrak{g}/\mathfrak{h}$ is a linear isomorphism for any $x \in U$.

If we have another section $\sigma' : U' \to P$, and a map $h : U \cap U' \to H$ such that $\sigma' = \sigma h$, then we have a new Cartan gauge where,

- $A' = Ad_{h^{-1}} A + h^{-1} dh$,
- $F' = Ad_{h^{-1}} F$.

These are passive gauge transformations related to local descriptions in distinct open sets on $\mathcal{M}$. They are formally identical to the local representation of the active gauge transformations above.

Tensors Given a vector space V and a representation $\rho : H \to GL(V)$ for the structure group, we have the associated vector bundle $E = \mathcal{P} \times_{\rho(H)} V$ and its sections $\Gamma(E)$. Through the isomorphism, $\iota^{-1} : \Lambda^0(P, \rho) \to \Gamma(E)$, between equivariant functions on P and sections of E, we define a *tensor of type* (V, ρ) as a function $\varphi : P \to V$ which transforms under the action of H as $R_h^* \varphi = \rho(h^{-1})\varphi$. The corresponding section of E is $\widetilde{\varphi} = \iota^{-1}(\varphi) = (p, \varphi(p))$.

The action of the gauge group on tensors is given by, $\psi^* \varphi(p) = \varphi(\psi(p)) = \varphi(p\gamma(p)) = \rho(\gamma^{-1}(p))\varphi(p)$. Or for short, $\varphi^\gamma = \rho(\gamma^{-1})\varphi$, which is the active gauge transformation of a tensor field of type (V, ρ).

The representation of a tensor in a Cartan gauge (U, A) is, $\phi = \sigma^* \varphi = \varphi(\sigma)$. In another gauge (U', A') it is, $\phi' = \sigma'^* \varphi = (\sigma h)^* \varphi = \varphi(\sigma h) = \rho(h)^{-1} \varphi(\sigma) = \rho(h)^{-1} \varphi$. This is the passive gauge transformation of a tensor field, formally identical to the local representation of the active gauge transformations.

Universal covariant derivative A Cartan connection allows the definition of a notion of covariant differentiation. Given $\lambda \in \mathfrak{g}$, we have,

Definition: the *universal covariant derivative* is a linear operator, $\widetilde{D}_\lambda : \Lambda^0(P, \rho) \to \Lambda^0(P, \rho)$, which is defined by $\widetilde{D}_\lambda \varphi = \varpi^{-1}(\lambda)\varphi$.

Actually it would be more precise to consider the operator $\widetilde{D} : \Lambda^0(P, \rho) \to \Lambda^0(P, \rho \otimes Ad)$. Indeed we have, $R_h^*(\widetilde{D}_\lambda \varphi)_p = (\widetilde{D}_\lambda \varphi)_{ph} = \varpi_{ph}^{-1}(\lambda)\varphi$. But $R_h^* \varpi_{ph} = \varpi_{ph} R_{h*} = \mathrm{Ad}_{h^{-1}} \varpi_p$. And $\varpi_{ph}^{-1} = R_{h*} \varpi_p^{-1} Ad_h$. Thus $R_h^*(\widetilde{D}_\lambda \varphi)_p = R_{h*} \varpi_p^{-1}(Ad_h \lambda)\varphi$. Besides, for any vector field X of $\mathcal{P}$ we have, $(R_{h*} Y)\varphi = \varphi_* R_{h*} Y = (\varphi R_h)_* Y = \rho(h^{-1})\varpi_* Y = \rho(h^{-1}) Y \varphi$. Now $\varpi_p^{-1}(\mathrm{Ad}_h \lambda)$ is a vector field on $\mathcal{P}$, then $R_h^*(\widetilde{D}_\lambda \varphi)_p = \rho(h^{-1})\varpi_p^{-1}(\mathrm{Ad}_h \lambda)\varphi = \rho(h^{-1})\widetilde{D}_{\mathrm{Ad}_h \lambda}\varphi$.

Remark: For $\lambda \in \mathfrak{h}$, $\varpi^{-1}(\lambda)$ is a vertical vector field. So $\widetilde{D}_\lambda \varphi$ tells us how tensors vary under the action of H, which we already know. Precisely,

$$\begin{aligned}
\widetilde{D}_\lambda \varphi &= \varpi_p^{-1}(\lambda)\varphi = \varphi_*(\varpi_p^{-1}(\lambda)\varphi) \\
&= \left.\frac{d}{dt}\right|_{t=0} \varphi(pe^{t\lambda}) = \left.\frac{d}{dt}\right|_{t=0} \rho(e^{-t\lambda})\varphi(p) = -\rho_*(\lambda)\varphi(p).
\end{aligned}$$

This is nothing but the infinitesimal version of gauge transformation on tensors.

Through the isomorphism ι^{-1}, the universal covariant derivative might be seen as a linear first order operator on the section of the associated bundle E: $\widetilde{D}_x \circ \iota^{-1} : \Gamma(E) \to \Gamma(E)$. We develop this last remark within the reductive geometry, which we are now ready to consider.

Reductive Cartan geometry

A Cartan geometry is said *reductive* if there is a H-module decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Any $\mathfrak{g}$-valued form will split accordingly. This is true in particular of the Cartan connection, which splits as:

$$\varpi = \varpi_{\mathfrak{h}} + \varpi_{\mathfrak{p}}.$$

In this decomposition we see that $\varpi_{\mathfrak{h}}$ is precisely an Ehresmann/principal connection, and $\varpi_{\mathfrak{p}}$ is a tensorial form called the **soldering form**. The local version on $U \subset \mathcal{M}$ splits as $A = A_{\mathfrak{h}} + A_{\mathfrak{p}}$. Since $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{p}$, we see that the form $A_{\mathfrak{p}} = \tau(A)$ is a linear isomorphism between $T_x(U)$ and $\mathfrak{p}$. This makes clear the name "soldering form" usually used. In the context of physics, especially General Relativity, $A_{\mathfrak{p}}$ is called a tetrad or moving frame[16].

If we have a non-degenerate $\mathrm{Ad}(H)$-invariant quadratic form η on $\mathfrak{p}$, the Cartan connection induces a well defined metric $g : T(U) \times T(U) \to \mathbb{R}$ on U (extendible to $\mathcal{M}$) by:

$$g(X_1, X_2) = \eta(A_{\mathfrak{p}}(X_1), A_{\mathfrak{p}}(X_2)).$$

Writting $A_{\mathfrak{p}} = e^a{}_\mu dx^\mu$ (where a is an index in $\mathfrak{p}$) and $X_{1,2} = X^\mu_{1,2}\partial_\mu$, the above expression reads in components: $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$. The latter formula is well known in the tetrad formulation of GR.

The curvature splits as $\Omega = \Omega_{\mathfrak{h}} + \Omega_{\mathfrak{p}}$, where $\Omega_{\mathfrak{p}} = \tau(\Omega)$ is the torsion 2-form. Remembering that Ω is tensorial, we can write,

$$\Omega = K_{cd}\, \varpi^c_{\mathfrak{p}} \wedge \varpi^d_{\mathfrak{p}} = (K^a{}_{b,cd} + K^a{}_{cd})\, \varpi^c_{\mathfrak{p}} \wedge \varpi^d_{\mathfrak{p}},$$

where K is the curvature function. With this at hand we can define a *reductive normal Cartan geometry* by essentially re-expressing the conditions satisfied by a normal Cartan connection, which are:

- $\tau(\Omega) = \Omega_{\mathfrak{p}} = 0$, so $K = K^a{}_{b,cd} \in \mathrm{Hom}\left(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{h}\right)$[17],
- $\mathrm{Ricci}(K) = K^a{}_{b,ad} = 0$.

These two conditions implies $K^a{}_{a,cd} = 0$.

The universal covariant derivative also split as $\widetilde{D}_\lambda = \widetilde{D}_{\lambda^{\mathfrak{h}}} + \widetilde{D}_{\lambda^{\mathfrak{p}}}$. We allready know that $\widetilde{D}_{\lambda^{\mathfrak{h}}}\varphi = -\rho_*(\lambda^{\mathfrak{h}})\varphi$ and correspond to the action of the structure group H. But the other part is very interesting. Indeed $\widetilde{D}_{\lambda^{\mathfrak{p}}}\varphi = \varpi_{\mathfrak{p}}^{-1}(\lambda^{\mathfrak{p}})\varphi$. But $\varpi_{\mathfrak{p}}^{-1}(\lambda^{\mathfrak{p}})$ is an horizontal vector field. So we have the following,

Definition: In a reductive Cartan geometry, the linear first order differential operator $\widetilde{D}_{\lambda^{\mathfrak{p}}}$ is the usual *covariant derivative.*

1.2.3 Reductive geometries and gravity

In this section we consider three examples of reductive Cartan geometries of particular interest for physics. The first is based on the (pseudo-) Euclidean Klein model, the isometry group of the (pseudo-) Euclidean space. This Cartan-Euclid geometry proves to be already a minimal generalization of Riemannian geometry. The second is based on the Möbius model. Its associated normal Cartan-Möbius geometry is just the conformal geometry of the base manifold. The third example is based on the deSitter model. This Cartan-deSitter geometry is arguably the most natural one to do GR with a positive cosmological constant (the physically favored case).

[16]The latter terminology being the historical one, borrowed from Darboux and Cartan.

[17]By the Bianchi identity , $d\Omega = [\Omega, \varpi]$, this implies that K satisfies the *algebraic Bianchi identity*; $K^a{}_{b,cd} + K^a{}_{c,db} + K^a{}_{d,bc} = 0$.

Cartan-Minkowski geometry

The principal group of the model is $G = O(r,s) \ltimes \mathbb{R}^{(r,s)}$ and $H = O(r,s)$. The associated symmetric space is thus $G/H \simeq \mathbb{R}^{(r,s)}$, the Minkowski space. We write in a matrix form,

$$G = \left\{ \begin{pmatrix} S & t \\ 0 & 1 \end{pmatrix} \middle| \, S \in O(r,s), t \in \mathbb{R}^{(r,s)} \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} \middle| \, R \in O(r,s) \right\}.$$

The corresponding Lie algebras are, $\mathfrak{g} = \mathfrak{o}(r,s) + \mathbb{R}^{(r,s)}$ and $\mathfrak{h} = \mathfrak{o}(r,s)$. The quotient space is $\mathfrak{p} = \mathbb{R}^{(r,s)}$. We write in matrix form,

$$\mathfrak{g} = \left\{ \begin{pmatrix} s & \tau \\ 0 & 0 \end{pmatrix} \middle| \, s \in \mathfrak{o}(r,s), \tau \in \mathbb{R}^{(r,s)} \right\} \quad \text{and} \quad \mathfrak{h} = \left\{ \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} \middle| \, s \in \mathfrak{o}(r,s) \right\}.$$

The associated Cartan geometry is thus $\big(\mathcal{P}(M, O(r,s)), \varpi\big)$ with the (local) Cartan connection,

$$\varpi = \begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix}.$$

Here A is a $O(r,s)$-principal connection known as the *Lorentz connection* in GR, and $\theta = \tau(\varpi)$ is the soldering form, or the tetrad in the parlance of GR. Moreover since $\mathfrak{p} = \mathbb{R}^{(r,s)}$ is endowed with an $O(r,s)$-invariant non-degenerate quadratic form which is nothing but the pseudo-euclidean metric η, we have a pseudo-riemannian metric g on M which is, $g(X_1, X_2) = \eta(\theta(X_1), \theta(X_2))$. Or, writing in components as above, $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$.

The curvature reads,

$$\Omega = \begin{pmatrix} R & \Theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} dA + A \wedge A & d\theta + A \wedge \theta \\ 0 & 0 \end{pmatrix}.$$

We recognize the Riemann curvature tensor $R = R^a{}_b$. The torsion $\tau(\Omega) = \Theta$ is just the covariant derivative of the soldering form. If $\Omega = 0$ the manifold is the symmetric space $\mathbb{R}^n$. This is flatness in the sense of Riemann. Nevertheless this Cartan geometry is richer. Indeed the concept of torsion that Cartan introduced does not exist in Riemann's conception of space[18]. Actually Riemannian geometry correspond to the case where ϖ is torsion free, $\Theta = 0$.
We can then list three sub-classes of geometries, each being the framework for a different theory:

- $R \neq 0$, $\Theta = 0$ (Lorentzian geometry): General Relativity (with a null cosmological constant),
- $R = 0$, $\Theta \neq 0$ (teleparallel space): the geometry of the 1929 "unified field theory" of Einstein, which is the object of the Einstein-Cartan correspondance from 1929 to 1932[19],
- $R \neq 0$, $\Theta \neq 0$: the geometry of the so-called Einstein-Cartan theory, rediscovered from a physical perspective by Sciama and Kibble in the 60's.

Cartan-Möbius geometry

The principal group of the model is the Möbius group $G = SO(m,2)/\pm I$, and H its maximal normal subgroup. The associated symmetric space is the n-deSitter space dS_n. In matrix form,

$$G = \left\{ \begin{pmatrix} z & i & 0 \\ t & S & i^t \\ 0 & t^t & z^{-1} \end{pmatrix} \middle| \, zS \in CO(r,s), t \in \mathbb{R}^{(r,s)}, i \in \mathbb{R}^{(r,s)*} \right\} \text{ and } H = \left\{ \begin{pmatrix} z & i & 0 \\ 0 & S & i^t \\ 0 & 0 & z^{-1} \end{pmatrix} \middle| \dots \right\},$$

where the operation of transposition involves the metric of $\mathbb{R}^{(r,s)}$, η. In the case of a vector: $v^t = (\eta v)^T = v^T \eta$.

[18]This is why Cartan coined the term "*espaces généralisés*" for his new geometry.

[19]The correspondance starts with a letter of Cartan at the beginning of which he reminds Einstein that he had the full Cartan-Minkowski geometry (Cartan speaks of "Euclidean connection") as soon as 1922! see (Cartan and Einstein, 1979).

The corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are graded Lie algebras. They have the following decomposition, $\mathfrak{g} = \mathbb{R}^{(r,s)} + \mathfrak{co}(r,s) + \mathbb{R}^{(r,s)*}$, and $\mathfrak{h} = \mathfrak{co}(r,s) + \mathbb{R}^{(r,s)*}$. The quotient space is $\mathfrak{p} = \mathbb{R}^{(r,s)}$. In matrix form we get,

$$\mathfrak{g} = \left\{ \begin{pmatrix} \epsilon & \iota & 0 \\ \tau & s & \iota^t \\ 0 & \tau^t & -\epsilon \end{pmatrix} \middle| \, (s - \epsilon \mathbb{1}) \in \mathfrak{co}(r,s), \tau \in \mathbb{R}^{(r,s)}, \iota \in \mathbb{R}^{(r,s)*} \right\} \quad \text{and} \quad H = \left\{ \begin{pmatrix} \epsilon & \iota & 0 \\ 0 & s & \tau^t \\ 0 & 0 & -\epsilon \end{pmatrix} \middle| \dots \right\}$$

The associated Cartan-Möbius geometry is then $\big(\mathcal{P}(\mathcal{M}, H), \varpi\big)$ with the conformal Cartan connection,

$$\varpi = \begin{pmatrix} a & \alpha & 0 \\ \theta & A & \eta\alpha^T \\ 0 & \theta^T\eta & -a \end{pmatrix}.$$

Here the upper right triangular matrix is a H-principal connection on $\mathcal{P}(\mathcal{M}, H)$, and $\theta = \tau(\varpi)$ is the soldering form. A metric on $\mathcal{M}$ is defined, as above, with the non-degenerate $O(r,s)$-invariant quadratic form η on $\mathfrak{p} = \mathbb{R}^{(r,s)}$: $g = \eta(\theta, \theta)$. Nevertheless, the fact that η is not H-invariant has interesting consequences. Indeed, let us define an active gauge transformation $\gamma : \mathcal{M} \to H$, written in matrix form

$$\gamma = \begin{pmatrix} z & i & 0 \\ 0 & S & i^t \\ 0 & 0 & -z \end{pmatrix}. \qquad \text{Then we have:} \quad \varpi^\gamma = \begin{pmatrix} * & * & 0 \\ zS^{-1}\theta & * & * \\ 0 & z\theta^t S & * \end{pmatrix}.$$

So the metric associated with this new Cartan gauge is, $g^\gamma = \eta(\theta^\gamma, \theta^\gamma) = \eta(zS^{-1}\theta, zS^{-1}\theta) = z^2\eta(\theta, \theta) = z^2 g$, which is nothing but a **Weyl rescaling of the metric**. Thus a Cartan-Möbius geometry induces a *conformal (class of) metric(s)* on $\mathcal{M}$.

The curvature reads,

$$\Omega = \begin{pmatrix} f & \Pi & 0 \\ \Theta & F & \Pi^t \\ 0 & \Theta^t & -f \end{pmatrix} = \begin{pmatrix} da + \alpha \wedge \theta & d\alpha + \alpha \wedge (A - a\mathbb{1}) & 0 \\ d\theta + (A - a\mathbb{1}) \wedge \theta & dA + A \wedge A + \theta \wedge \alpha + \alpha^t \wedge \theta^t & * \\ 0 & * & * \end{pmatrix}.$$

We recognize the Riemann tensor in $F = R + \theta \wedge \alpha + \alpha^t \wedge \theta^t$. Now suppose we are in a gauge where $a = 0$ and consider the normal geometry. Several facts can be deduced:

- Θ is the torsion for the $O(r,s)$-connection A, and $\Theta = 0 \Rightarrow A$ is the Levi-Civita connection.
- Ricci$(K) = 0$ means $f = 0$, and α is a symmetric tensor: $\alpha_{b,c} = \alpha_{c,b}$. We can furthermore show that, $\alpha_{b,c} = \frac{1}{(n-2)}\left(R^a{}_{b,ac} - \frac{1}{2(n-1)}\eta_{bc}R\right)$, where R is the Ricci scalar. This is the *Schouten tensor*.
- From this immediately follows that Π is the *Cotton tensor*, and that F is the *Weyl tensor*.

These are remarkable tensors of the conformal geometry of M. The normal Cartan-Möbius geometry is thus the natural framework for conformal gravity, whose Yang-Mills like Lagrangian form involves the Weyl tensor, $L \propto \mathrm{Tr}\big(F \wedge *F\big)$.

Cartan-deSitter geometry

The principal group of the model is $G = O(1,n)$ the isometry group of the deSitter space dS_n. Given the isotropy group $H = O(1, n-1)$, the latter is realized as the quotient $O(1,n)/O(1,n-1)$. The corresponding Lie algebras are, $\mathfrak{g} = \mathfrak{o}(1,n) = \mathfrak{o}(1,n-1) + \mathbb{R}^{1,n-1}$ and, $\mathfrak{h} = \mathfrak{o}(1,n-1)$. In matrix form,

$$\mathfrak{g} = \left\{ \begin{pmatrix} s & \tau \\ \tau^t & 0 \end{pmatrix} \middle| s \in \mathfrak{o}(1,n-1), \tau \in \mathbb{R}^{1,n-1} \right\} \quad \text{and} \quad \mathfrak{h} = \left\{ \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} \middle| s \in \mathfrak{o}(1,n-1) \right\}.$$

The associated Cartan-deSitter geometry is, $\big(\mathcal{P}(\mathcal{M}, O(1,n-1)), \varpi\big)$ with,

$$\varpi = \begin{pmatrix} A & \theta \\ \theta^t & 0 \end{pmatrix} \quad \text{and} \quad \Omega = \begin{pmatrix} dA + A \wedge A + \theta \wedge \theta^t & d\theta + A \wedge \theta \\ * & 0 \end{pmatrix}.$$

A metric on $\mathcal{M}$ is induced by $\theta^a = e^a{}_\mu dx^\mu$ as above. We see that $F = R + \theta \wedge \theta^t$ so that if $F = 0$ we would have straightforwardly the Riemann tensor of dS_n, $R = \theta \wedge \theta^t$. Speaking about physics, for dimensional reason we should have $R = \frac{1}{l^2}\theta \wedge \theta^t$ where l would be the "radius" of dS_n (tacitly put to unity above). If we furthermore require that the latter be solution of Einstein's equations with *positive* cosmological constant, then $l^2 = \frac{(n-1)(n-2)}{2\Lambda}$. This would lead us to add a scale factor to the soldering form: $\sqrt{\frac{2\Lambda}{(n-1)(n-2)}}\theta$. Locally the Riemann tensor of dS_n would assume the more familiar form,

$$R^a{}_{b,\mu\nu} = \frac{2\Lambda}{(n-1)(n-2)} \left(e^a{}_\mu e^b{}_\nu - e^a{}_\nu e^b{}_\mu \right).$$

Cartan-deSitter geometry thus seems the most natural one to do gravitation and cosmology with $\Lambda > 0$. Moreover it is also the proper ground to understand MacDowell-Mansouri formulation of gravity (McDowell and Mansouri, 1977). Indeed, for n=4, their Lagrangian reads:

$$L = \frac{-3}{2G\Lambda}\mathrm{Tr}\Big(F \wedge *F\Big) = \frac{-3}{2G\Lambda}\mathrm{Tr}\Big(\big(R - \tfrac{\Lambda}{3}\theta \wedge \theta^t\big) \wedge *\big(R - \tfrac{\Lambda}{3}\theta \wedge \theta^t\big)\Big).$$

Besides, their initial move which consists in "adding" the Lorentz connection and the tetrad to form a single object and treat it as an extended connection is clearly understood within Cartan geometry. All this (and more) is detailled in the nice paper by Wise (Wise, 2010) where the case $\Lambda < 0$ and $\Lambda = 0$ are also treated.

Conclusion

Einstein's General Relativity has been historically the conceptual precursor of gauge theories. Yet, gravitation has been itself considered as a gauge theory, on equal footing with the other interactions, for the first time by Utiyama only in 1956 (see (O'Raifeartaigh, 1997)). Not long after Yang and Mills's work, but 27 years after Weyl's seminal paper on local gauge invariance in Quantum Mechanics. The question wether gravitation is to be considered a true gauge theory has been the object of much discussions, and maybe not definitely settled.

From a formal viewpoint at least, we can agree on one thing: gravitation has a much richer structure than the others gauge fields. Indeed it exhibits features absents from Yang-Mills theories: a metric structure, a notion of torsion beside that of curvature, the possibility to construct more invariants for a Lagrangian etc... All this can be traced back to the existence of a soldering form[20]. The latter ensures the close relation between the base manifold and the bundle above it. We've argued here that in this respect Cartan geometry and Cartan connections clearly do justice to the distinct features of gravitation. Since it furthermore provides all the standard tools for a gauge theory; a gauge group, associated bundles, a covariant derivative etc... Cartan geometry could claim its legitimacy as the adequate framework for gauge theories of gravity and its relevance for current issues in the physics of gravitation[21].

1.3 Downsides of gauge symmetry

1.3.1 Problem with gauge theories?

There are philosophical/epistemological questions raised by local gauge symmetry that are of great interest. See for example (Brading and Castellani, 2003). Nevertheless while entertained by the philosophically inclined part of a physicist's mind, these questions could be ignored by the pragmatic part. However, what the latter cannot overlook is the apparent *technical* shortcomings of gauge field theories. What are they? The main two

[20]See the very clear paper by A. Trautman (Trautman, 1979) for a developed argument on this question.
[21]For a defense of this view, see again the paper by D. Wise (Wise, 2010) and references therein.

that deserve special attention are those concerning quantization in the one hand, and the mass of the gauge fields in the other hand.

With regard to the first, we've quickly mentioned in section 1.1.3 that in QFT we have a quantum action given by Feynman's path integral $Z = \int_{\mathcal{A}\times\mathcal{F}} dA_\mu d\varphi\ e^{\mathcal{S}(A_\mu,\varphi)}$. The integral goes over all the fields A_μ and φ in $\mathcal{A} \times \mathcal{F}$, in particular those connected by *active gauge transformations*.[22] Unfortunately the volume of the gauge group $\mathcal{G}$ is infinite. The path integral thus diverges, providing singular propagators for the fields.

As for the second issue, it originates from the requirement, advocated in 1.1.3, of gauge invariance for a Lagrangian form L. This demand cannot be dispensed with. This implies that a mass term $m^2\mathrm{Tr}(A \wedge *A)$ for the gauge potential in L, which is clearly non-invariant under $\mathcal{G}$, is forbidden. As it stands, the gauge field A must be massless, thus the mediator of a long range interaction. A formal constrain at odds with the empirical fact that both weak and strong interactions are short range, which naturally suggests massive mediator fields. Only Electromagnetism theory is right away compatible with this constrain, since indeed it is a long range interaction carried by a massless field whose quantum is the photon. No wonder EM was the first gauge theory (recognized as such).

1.3.2 What standard solutions?

At bottom, both problems clearly arise from the gauge symmetry of a theory. A general strategy to solve them should be to *reduce* this gauge symmetry. There are however different approaches to do so, adapted to each issue.

For Quantization the strategy is quite obvious, one selects a single representative in each field's gauge orbit by *gauge fixing*. One does so by adding "by hand" a constraint equation on the fields either in the functional measure of the path integral or directly in the Lagrangian. The only consistency condition one has to check is that the physical outcomes of the gauge fixed theory should not depend on the specific gauge chosen.

More formally, a gauge fixing is a choice of a slice in $\mathcal{A}$ (or $\mathcal{F}$) that is transverse to all $\mathcal{G}$-orbits. In other words it is a choice of section of the infinite dimensional bundle $\mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{G}$ over the moduli space. It is known that a global section may not exist due to a non-trivial topology of this bundle, see (Singer, 1978). It is the case in non-abelian gauge theories. A phenomenon known as the *Gribov ambiguity*, see (Gribov, 1978).

The riddle of the short range of the nuclear interactions received each a different answers. The short range of the strong interaction, was explained by the notions of *confinement* and *asymptotic freedom*. There the gauge symmetry is not reduced nor broken, so this case is derogatory to our general strategy.[23]

The short range of the weak interaction was finally explained within the Glashow-Weinberg-Salam model of the electroweak interaction by the means of the Englert-Brout-Higgs-Kibble-Guralnik-Haggen *spontaneous symmetry breaking mechanism*, (Brout and Englert, 1964), (Higgs, 1964), (Guralnik et al., 1964). In this model the underlying gauge symmetry involves the group $G = U(1) \times SU(2)$, but the theory involves a scalar field imbedded in a potential whose minima do not respect the $SU(2)$ symmetry (the famous "Mexican hat" potential). Usually interpreted, the high energy phase of the theory is symmetric under the full group G and the gauge fields are all massless, but as the energy decreases it undergoes a transition to a phase were $SU(2)$ is broken and the residual gauge symmetry is $U(1)$ so that three gauge fields gain masses ($W^\pm$, Z^0) and only one of them remains massless (the photon A). This is clearly a dynamical process, the phase transition is indexed by an external parameter (here the energy or temperature). The SSBM has been devised in a purely field theoretic context.

At the end of section 1.1.1 we've seen that there exists a theorem on bundle reduction, thus on reduction of gauge symmetry. There we've alluded to the fact that there have been attempts to give a geometrical account of the SSBM through this theorem. We should be careful however, for the bundle reduction theorem does not

[22] The *passive* gauge transformation being just the same fields described differently. Remember section 1.1.2.

[23] Yet, we will see that through the work of (Lavelle and McMullan, 1997) that there might be a strong link between confinement, and QCD in general, and the approach advocated below. See section 2.3.2 ahead and, for more precisions, appendix A.1.

depend on the existence of an asymmetric vaccum, nor is it dynamically indexed by an external parameter. Then, if formally alike, the two formulations of the SSBM seem to entail different interpretations as to what physically happens. The bundle reduction theorem seems to support a non-dynamical viewpoint. In section 2.3.2 we will argue for the same position from a different perspective. A perspective that we now introduce.

1.3.3 A new approach

To sum up, if one is willing to acknowledge the differences just mentioned, there are three general tools to reduce the symmetry of a gauge field theory,

- gauge fixing
- spontaneous symmetry breaking
- bundle reduction theorem

In the following chapters we propose a study of an alternative fourth tool to achieve a symmetry reduction. It relies on the existence of what we call a *dressing field*.

In the easiest applications, the latter allows to construct *gauge invariant* composite fields and Lagrangians. It can be related to the so-called *Dirac variables*. See (Dirac, 1955), (Dirac, 1958).

In the less straightforward cases, it is possible to reduce only partially the gauge symmetry so that the composite fields display a *residual gauge freedom*. Non-trivial examples can (and will) be given, like the electroweak sector of the Standard Model and others from the framework of Cartan connections on first or second order frame bundles.

It is expected that the dressing field should alter the BRS algebra of a gauge theory, (Bertlmann, 1996). Indeed this issue is investigated and solved. The infinitesimal counterparts of the above mentionned examples are worked out, and the modified BRS algebra handles the residual gauge freedom. Finally we will consider how this new approach may interact with the question of anomalies in QFT.

Chapter 2
The dressing field

In this chapter we expose the dressing field method of gauge symmetry neutralization. A short first section relies on the notions seen in 1.1 to give a working definition of a gauge theory, together with some examples of successful such theories.

The second section describes the formalism of the dressing field and the associated two main lemmas. The differences with the three standard approaches mentioned in 1.3.3 are highlighted, and the link with the bundle reduction theorem is drawn.

In the third section the method is applied to several examples ranging from simple toy models to the mentioned successful theories, and passing by others found in the literature and analyzed in appendix A. In each case it is shown how the method suggests an interpretive shift.

The fourth and last section of the chapter describes a generalization of the method to higher-order G-structures. The best way to appreciate the scheme is to go through the worked out example of Cartan-Möbius geometry. In this example we recover some known results of conformal geometry usually obtained through the jet formalism, see (Kobayashi, 1972), (Ogiue, 1967), but with a more systematic and handy matrix formalism.

2.1 Gauge theories: a recipe

2.1.1 What is a gauge theory?

It seems that there are several possible answers to this question, each relying on a different formalism.[1] One may consider e.g the constrained Hamiltonian formalism which is sometimes a preferred one to handle quantization.[2] However as argued in 1.1 the geometrical formalism of fiber bundles seems the proper ground for gauge field theories, at least at the classical level. So our answer to the question holds in the following basic ingredients for a sound gauge theory:

- A principal fiber bundle $\mathcal{P}(\mathcal{M},H)$ over a base manifold $\mathcal{M}$ (space-time).
- An *Ehresmann connection* 1*-form* ω and its *curvature* 2-form Ω on $\mathcal{P}$. Through a local trivializing section $\sigma : U \subset \mathcal{M} \to \mathcal{P}$ these two are pulled-back on $\mathcal{M}$ to give A and F, which describe the *gauge potential* and its *field strength* respectively.
- Representations (V_i, ρ_i) for H in order to built associated vector bundles, $E_i = \mathcal{P} \times_{\rho_i} V_i$, whose *sections* $\varphi_i \in \Gamma(E_i) \simeq \Lambda^0(\mathcal{P}, \rho_i)$ represent various *matter fields.*
- A *covariant derivative* on sections arises naturally, which represents the *minimal coupling* between the matter fields and the gauge potential.
- The *gauge group* $\mathrm{Aut}_v(\mathcal{P}) \simeq \mathcal{H}$, is the space of *local symmetry* required by the so-called *gauge principle.* Its action on A, F and φ_i implements the (active) *gauge transformations.*

Once given this geometrical setup, Physics is not yet described. A gauge theory is specified when a *Lagrangian m-form*, $L = \mathcal{L}\ d^m x$, is chosen. We ask for the form L to be strictly *invariant* under the action of $\mathcal{H}$, so that $L^\gamma = L$. This is a consistency requirement for a choice of gauge being truly an “abstract reference

[1] For example (Creutz, 1985) distinguishes four formal definitions.

[2] See the reference textbook (Henneaux and Teitelboim, 1994). As a matter of fact, constrained Hamiltonian formalism is largely used in Loop Quantum Gravity to tackle the long standing issue of the quantization of gravitation.

frame", gauge invariance, in much the same spirit as general covariance, states that nothing physical could depend on the choice of a specific gauge coordinate system.

A gauge theory could thus be interpreted as a theory where, in addition to the purely external or "basic" (that is, spatio-temporal) degrees of freedom on $\mathcal{M}$, there are additional "inner" degrees of freedom associated to an abstract *internal space*, the fibers, in each point of $\mathcal{M}$. We thus distinguish the *natural geometry* associated with the base manifold $\mathcal{M}$ from the *gauge geometry* associated with the whole of $\mathcal{P}$.

This receives some support from (Kolar et al., 1993) who distinguish the category of *natural bundles* and the category of *gauge natural bundles*. The category of natural bundles itself covers closely the notion of G-structures and higher order G-structures. A G-structure being a G-reduction of the frame bundle $L\mathcal{M}$ and an r^{th}-G-structure being a reduction of the r^{th}-frame bundle of $\mathcal{M}$. See (Kobayashi, 1972), (Ogiue, 1967).

However this very precise nomenclature makes hard to talk about gravitation in gauge terms, as is our intention in this essay. Indeed gravitation (as we know it classically) is all about the geometry of the space-time base manifold $\mathcal{M}$. Clearly the second example below would belong to the category of natural bundles and not to the category of gauge natural bundles. Moreover Cartan geometry, argued to be the natural framework for gravitation, belongs to higher G-structures and natural bundles. Nevertheless on a strictly formal ground, section 1.2 showed that Cartan geometry contains all the ingredients listed above. So to all practical ends, gravitation fits our definition of a gauge theory and will be treated as such.

2.1.2 Examples

The Electroweak sector of the Standard Model Here we discard the spinors of the theory and consider only the Lagrangian describing the gauge potentials and the scalar field.

We've already encountered the principal bundle of the model $\mathcal{P}\big(\mathcal{M}, H = U(1)\times SU(2)\big)$. The connection on $\mathcal{P}$ is $\omega_{\mathfrak{g}} = \bar{a}_{\mathfrak{u}(1)} + \bar{b}_{\mathfrak{su}(2)}$, its pullback on $U \subset \mathcal{M}$ is $A = a+b$.[3] The associated curvature is simply $F = f_a + g_b$. The fundamental representation is $(\mathbb{C}^2, L_H)$, with L_H the left matrix multiplication. The associated vector bundle is then $E = \mathcal{P} \times_{L_H} \mathbb{C}^2$, and a section is $\varphi : U \to \mathbb{C}^2$. The covariant derivative is thus $D\varphi = d\varphi + (g'a + gb)\varphi$, with g', g the coupling constant of $U(1)$ and $SU(2)$ respectively. The action of (the pullback of) the gauge group $\mathcal{H}_{loc} = \mathcal{U}(1)_{loc} \times \mathcal{SU}(2)_{loc}$,[4] with element $\gamma = (\alpha, \beta)$, is

$$\begin{aligned} a^\alpha &= a + \frac{1}{g'}\alpha^{-1}d\alpha, \qquad b^\alpha = b, \qquad \text{and} \qquad \varphi^\alpha = \alpha^{-1}\varphi, \\ a^\beta &= a, \qquad b^\beta = \beta^{-1}b\beta + \frac{1}{g}\beta^{-1}d\beta, \qquad \text{and} \qquad \varphi^\beta = \beta^{-1}\varphi. \end{aligned} \tag{2.1}$$

The structure of direct product group is clear. Denoting $\langle , \rangle$ the scalar product on $\mathbb{C}^2$ and using the trace operator Tr on $\mathfrak{h}$, the Lagrangian scalar m-form of the theory is,

$$L = \langle D\varphi, *D\varphi\rangle + V(\varphi)\text{vol} + \frac{1}{2}\text{Tr}(F \wedge *F). \tag{2.2}$$

Here vol $= \sqrt{|\det(g_{\mu\nu})|}d^m x$ is the volume form on $\mathcal{M}$. Moreover, due to the direct product structure, the Yang-Mills term splits as, $\text{Tr}(F\wedge *F) = \text{Tr}(f \wedge *f) + \text{Tr}(g \wedge *g)$, the Yang-Mills terms associated to the gauge potentials a and b respectively. Giving the quartic potential $V(\varphi) = -\mu^2\langle\varphi,\varphi\rangle - \lambda\langle\varphi,\varphi\rangle^2$, we obtain the well known Lagrangian of the electroweak sector of the Standard Model,

$$L = \mathcal{L}\, d^m x = \left(\langle D_\mu\varphi, D^\mu\varphi\rangle - \mu^2\langle\varphi,\varphi\rangle - \lambda\langle\varphi,\varphi\rangle^2 - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4}\text{Tr}(g_{\mu\nu}g^{\mu\nu})\right)\sqrt{|\det(g_{\mu\nu})|}d^m x. \tag{2.3}$$

This gauge theory describes the interaction of a doublet scalar field φ with two gauge potentials a and b. The field φ should be named "Englert-Brout-Higgs-Guralnik-Hagen-Kibble field", or EBHGHK-field for short, from the authors who, between June and October 1964, have independently discovered its importance for particle Physics. See, (Brout and Englert, 1964), (Higgs, 1964) and (Guralnik et al., 1964).

[3] If $\mathcal{M} = \mathbb{R}^{(1,m-1)}$ the Minkowski space, $\mathcal{P}$ is trivial (all principal bundles over contractible spaces are trivial) then a and b are globally defined on $\mathcal{M}$.

[4] Again globally defined if $\mathcal{M} = \mathbb{R}^{(1,m-1)}$.

The tetrad formulation of General Relativity The tetrad formulation of General Relativity is necessary to describe the interaction of gravity with spinor fields. A classic argument is that GL as no spinorial representation contrary to SO, hence the necessity to use a formalism with *local Lorentz invariance.* This is precisely what is achieved by reducing the frame bundle $L\mathcal{M}$ to an SO-subbundle. Something we have encountered already.

The underlying bundle is $\mathcal{P}(\mathcal{M}, SO)$. The connection ω on $\mathcal{P}$ pulls-back on $U \subset \mathcal{M}$ as A, where it is known as the *Lorentz connection* (or *spin connection*). The curvature Ω pulls-back as R, the Riemann tensor. Given a spinorial representation (V, ρ), one constructs an associated bundle $E = \mathcal{P} \times_\rho V$ and $\psi \in \Gamma(E)$ is a spinor field on $\mathcal{M}$. The covariant derivative is $D\psi = d\psi + \rho(\omega)\psi$. The gauge group $\mathcal{SO}_{loc}$ acts on A, ψ and $D\psi$ as prescribed by (1.13). Nevertheless here again we discard spinors and consider only the free gravitational field. The latter is quite unique among gauge fields since, in the most general case, it is described jointly by the connection A and by the tetrad field θ.[5] As a matter of fact the Lagrangian form is not of Yang-Mills type,

$$L = \frac{1}{32\pi G}\mathrm{Tr}\Big(R \wedge *(\theta \wedge \theta^t)\Big) = \frac{1}{32\pi G}\mathrm{Tr}\Big(R \wedge *(\theta \wedge \theta^T \eta)\Big) = \frac{1}{32\pi G} R^a{}_b \wedge *(\theta^b \wedge \theta^c \eta_{ac}). \tag{2.4}$$

with η the metric of $\mathbb{R}^{1,m-1}$ and G the gravitational constant. This is known, with slight abuse of language, as the *Palatini Lagrangian*, $L = L_{\text{Pal}}$. If we add an arbitrary Lagrangian form for matter L_{Matter} so that the action is $\mathcal{S} = \int L_{\text{Pal}} + L_{\text{Matter}}$, variation of $\mathcal{S}$ with respect to θ gives Einstein's equations relating the curvature R of $\mathcal{M}$ to the energy-momentum density and variation with respect to ω gives an equation relating the *torsion* $\Theta = D\theta$ to the spin density of the matter. If $\Theta = 0$ we have standard General Relativity. If $\Theta \neq 0$ we have the so-called *Einstein-Cartan theory*. See (Göckeler and Schücker, 1987) and (Trautman, 1979) for details, also our discussion of Cartan geometry in section 1.2.

2.2 Dressing field

2.2.1 Easy propositions

Let us begin with an easy proposition before considering refinements. Consider a gauge theory (as above) with underlying bundle $\mathcal{P}(\mathcal{M}, H)$, connection ω, curvature Ω and sections ψ_i. Then we have the following

Lemma 1 (Main Lemma). *Given a Lie group $G \supseteq H$ sharing the same representations (V_i, ρ_i) as H, suppose that there exists a map $\bar{u} : \mathcal{P} \to G$ with $\mathcal{H}$-gauge transformation $\bar{u}^\gamma = \gamma^{-1}\bar{u}$. Then the following global objects,*

$$\widehat{\omega} := \bar{u}^{-1}\omega\bar{u} + \bar{u}^{-1}d\bar{u}, \quad \widehat{\Omega} = \bar{u}^{-1}\Omega\bar{u}, \quad \widehat{\psi_i} := \rho_i(\bar{u}^{-1})\psi_i \quad \text{and} \quad \widehat{D\psi_i} = \rho_i(\bar{u}^{-1})D\psi_i, \tag{2.5}$$

are projectable, that is horizontal and $\mathcal{H}$-gauge invariant.

Furthermore, the definition $\widehat{\Omega} := d\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}]$ implies the second equality above. Similarly, the definition $\widehat{D\psi_i} := \widehat{D}\widehat{\psi_i}$, with the invariant derivative $\widehat{D} = \rho_i(\bar{u}^{-1})D\rho_i(\bar{u}) = d + \rho_{i*}(\widehat{\omega})$, implies the fourth equality.

Proof. Given the set of gauge transformations (1.8) and (1.9) the proof of $\mathcal{H}$-invariance is straightforward. Compute explicitly the first transformation,

$$\begin{aligned}\widehat{\omega}^\gamma &:= (\bar{u}^\gamma)^{-1}\omega^\gamma\bar{u}^\gamma + (\bar{u}^\gamma)^{-1}d\bar{u}^\gamma = (\bar{u}^{-1}\gamma)(\gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma)(\gamma^{-1}\bar{u}) + \bar{u}^{-1}\gamma d(\gamma^{-1}\bar{u}),\\ &= \bar{u}^{-1}\omega\bar{u} + \bar{u}^{-1}d\bar{u} = \widehat{\omega}.\end{aligned}$$

In the same way one proves, $\widehat{\Omega}^\gamma = \widehat{\Omega}$ and $\widehat{\psi}_i^\gamma = \widehat{\psi}_i$. Thus, since from above $\widehat{D}^\gamma = \widehat{D}$, one has $\widehat{D\psi_i}^\gamma = \widehat{D\psi_i}$.

Now the horizontality of $\widehat{\Omega}$ and $\widehat{\Psi}$ is clear. Only for $\widehat{\omega}$ and $\widehat{D}\widehat{\psi_i}$ is it less immediate. Given $\bar{X}^v \in V_p\mathcal{P}$ the vertical vector field generated by $X \in \mathfrak{h}$ one has,

$$\begin{aligned}\widehat{\omega}(\bar{X}^v) &= \bar{u}^{-1}\omega(\bar{X}^v)\bar{u} + \bar{u}^{-1}d\bar{u}(\bar{X}^v) = \bar{u}^{-1}X\bar{u} + \bar{u}^{-1}\bar{X}^v(\bar{u}),\\ &= \bar{u}^{-1}X\bar{u} + \bar{u}^{-1}\frac{d}{dt}\bigg|_{t=0}\bar{u}\Big(p\exp(tX)\Big) = \bar{u}^{-1}X\bar{u} + \bar{u}^{-1}\frac{d}{dt}\bigg|_{t=0}\exp(-tX)\bar{u}(p),\\ &= \bar{u}^{-1}X\bar{u} + \bar{u}^{-1}(-X)\bar{u} = 0.\end{aligned}$$

[5] See our opening discussion on the example of General Relativity in section 2.3.2 ahead.

This helps to prove the horizontality for $\widehat{D}\widehat{\psi}_i$, indeed,

$$\begin{aligned}\widehat{D}\widehat{\psi}_i(\bar{X}^v) &= d\widehat{\psi}_i(\bar{X}^v) + \rho_{i*}\big(\widehat{\omega}(\bar{X}^v)\big)\widehat{\psi}_i = \bar{X}^v(\widehat{\psi}_i) = \bar{X}^v\big(\rho_i(u^{-1})\psi_i\big),\\ &= \bar{X}^v\big(\rho_i(u^{-1})\big)\psi_i + \rho_i(u^{-1})\bar{X}^v\psi_i = \rho_i(u^{-1}X)\psi_i + \rho_i(u^{-1})\big(-\rho_{i*}(X)\psi_i\big) = 0.\end{aligned}$$

Here we used the fact that the action of $\bar{X}^v$ on $\bar{u}$ and ψ is the infinitesimal version of their equivariance property.

The second part of the lemma is also straightforward for the calculations are strictly analogous to gauge transformations. Let us prove the fourth equality,

$$\begin{aligned}\widehat{D}\widehat{\psi}_i &= d(\rho_i(u^{-1})\psi_i) + \rho_{i*}(u^{-1}\omega u + u^{-1}du)\rho_i(u^{-1})\psi_i,\\ &= d\rho_i(u^{-1})\psi_i + \rho_i(u^{-1})d\psi_i + \rho_i(u^{-1})\rho_{i*}(\omega)\psi_i + \rho_i(u^{-1})d\rho_i(u)\rho_i(u^{-1})\psi_i,\\ &= \rho_i(u^{-1})\big(d\psi_i + \rho_{i*}(\omega)\psi_i\big) = \rho_i(u^{-1})D\psi_i.\end{aligned}$$

The second equality goes similarly. □

The objects (2.5) on $\mathcal{P}$ being projectable, they induce globally defined fields on the base manifold $\mathcal{M}$. Given a local trivializing section σ, the pullback $u = \sigma^*\bar{u}$ we call the **dressing field**[6] for we have the

Corollary 1. *Denote by $A = \sigma^*\omega$ the gauge potential, $F = \sigma^*\Omega$ the field strength and $\varphi_i = \sigma^*\psi_i$ the matter fields. The following **composite fields**,*

$$\widehat{A} := u^{-1}Au + u^{-1}du, \quad \widehat{F} = u^{-1}Fu, \quad \widehat{\varphi}_i := \rho_i(u^{-1})\varphi_i \quad \text{and} \quad \widehat{D\varphi_i} = \rho_{i*}(u^{-1})D\varphi_i, \tag{2.6}$$

are $\mathcal{H}_{loc}$-gauge invariant and globally defined on $\mathcal{M}$.

Exactly as in Lemma 1, one has $\widehat{F} := d\widehat{A} + \frac{1}{2}[\widehat{A},\widehat{A}]$ implies the second equality above. And $\widehat{D\varphi_i} := \widehat{D}\widehat{\varphi}_i$, with the invariant derivative $\widehat{D} = \rho_i(u^{-1})D\rho_i(u) = d + \rho_{i*}(\widehat{A})$, implies the fourth equality.

Proof. The $\mathcal{H}_{\text{loc}}$-invariance is inherited from the global objects and is straightforwardly obtained from (1.13). The globality is obvious since the gluing properties, or passive gauge transformations, (1.12), are formally like active gauge transformations (1.13). Then the composite fields being invariant under active gauge transformations should have trivial gluing properties. Let us demonstrate this for the "hard" case of the field $\widehat{A}$.

Let $\sigma_{i/j} : U_{i/j} \to \mathcal{P}$ be two local trivializing sections such that $\sigma_j = \sigma_i g_{ij}$ on $U_i \cap U_j$. We have $\widehat{A}_{i/j} = \sigma^*_{i/j}\widehat{\omega}$ and $u_{i/j} = \sigma^*_{i/j}\bar{u}$. Also $A_j = g_{ij}^{-1}A_i g_{ij} + g_{ij}^{-1}dg_{ij}$ and $u_j = \sigma_j^*\bar{u} = (\sigma_i g_{ij})^*\bar{u} = \bar{u}(\sigma_i g_{ij}) = g_{ij}^{-1}\bar{u}(\sigma_i) = g_{ij}^{-1}\sigma_i^*\bar{u} = g_{ij}^{-1}u_i$. So,

$$\begin{aligned}\widehat{A}_j &= \sigma_j^*\omega = (\sigma_j^*\bar{u})^{-1}(\sigma_j^*\omega)(\sigma_j^*\bar{u}) + (\sigma_j^*\bar{u})^{-1}d(\sigma_j^*\bar{u}) = (g_{ij}^{-1}u_i)^{-1}A_j(g_{ij}^{-1}u_i) + \big((g_{ij}^{-1}u_i)^{-1}\big)d(g_{ij}^{-1}u_i),\\ &= u_i^{-1}g_{ij}(g_{ij}^{-1}A_i g_{ij} + g_{ij}^{-1}dg_{ij})g_{ij}^{-1}u_i + u_i^{-1}g_{ij}dg_{ij}^{-1}u_i + u_i^{-1}du_i,\\ &= u_i^{-1}A_i u_i + u_i^{-1}du_i = \widehat{A}_i.\end{aligned}$$

Similar but easier computations prove that $\widehat{F}_j = \widehat{F}_i$, $\widehat{\varphi}_{k,j} = \widehat{\varphi}_{k,i}$ and $\widehat{D}\widehat{\varphi}_{k,j} = \widehat{D}\widehat{\varphi}_{k,i}$.

The second part of the corollary goes as the second part of Lemma 1. □

[6] By abuse we may often call $\bar{u}$ a dressing field.

What this construction is... The corollary shows that the composite fields are globally defined as far as the gauge structure is concerned. That is, they do not undergo any transformation from one trivializing open set to another as long as these are encompassed in a single coordinate chart of the base manifold. If the trivializing open sets $\{U_i\}$ are also coordinate charts of $\mathcal{M}$, then by passing from one to another, the composite fields remain unchanged for they are differential forms. At worst they undergo coordinates change if the dressing u carries base manifold indices. That's all. This implies that the composite fields (2.6) belong to the **natural geometry** of the base manifold $\mathcal{M}$ and are blind to any gauge structure on top of it. Thus the process of **dressing** the gauge fields A, F, and φ_i with the field u can be seen as a **neutralization** of the gauge symmetry of the original theory that ends up in a **basic/natural geometrization** of these gauge fields.

...and is not Due to the formal likeness between (1.8)-(1.9) and (2.5) on the one hand, and between (1.13) or (1.12) and (2.6) on the other hand, it may be easy in each case to mistake the latter for the former. But let us stress this as clearly as possible: (2.5) *are not* active gauge transformations and (2.6) *are not* local active gauge transformations nor passive gauge transformations. This was already quite clear from the above discussion. But let us add two decisive arguments.

First, the dressing field $\bar{u}$ does not belong to the gauge group $\mathcal{H}$ since it takes values in a group that may be larger than H. And even if $\bar{u}$ had values in H, it still has the wrong equivariance. By its very definition, the dressing field transforms as $\bar{u}^\gamma = \gamma^{-1}\bar{u}$, whereas an element $\alpha \in \mathcal{H}$ transforms as $\alpha^\gamma = \gamma^{-1}\alpha\gamma$, see (1.2). In any case $\bar{u} \notin \mathcal{H}$, and (2.5) are not gauge transformations.

Secondly, Lemma 1 showed that $\widehat{\omega}$ is horizontal, so that $\widehat{\omega} \notin \mathcal{A}_\mathcal{P}$. A gauge transformation cannot get us out of the space of connections of $\mathcal{P}$ since $\mathcal{H}$ is a group of transformation of $\mathcal{A}_\mathcal{P}$. Again, equations (2.5) are not gauge transformations.

Now it is obvious that the process of forming (2.5) or (2.6) has nothing to do with a gauge fixing. Indeed a gauge fixing is a choice of a section in $\mathcal{A}_\mathcal{P} \xrightarrow{\pi} \mathcal{A}_\mathcal{P}/\mathcal{H}$. Yet $\widehat{\omega} \notin \mathcal{A}_\mathcal{P}$ is not a representative of a gauge orbit, even if by definition there is a *one-to-one correspondence* between the $\widehat{\omega}$'s and gauge orbits of the ω's, so that in this respect the dressing method is a perfect substitute for gauge fixing.

With the above caveats we have positioned the dressing field method clearly apart from two of the three tools of symmetry reduction mentioned in 1.3. What about the third? It is natural to ask what are the links, if any, of the dressing field to the Bundle Reduction Theorem. To answer this question we first need to introduce some refinement.

2.2.2 Dressing field and residual gauge freedom

Here we relax the restriction on the target group of the dressing field and we'll see that its (global) existence constrains the topology of the bundle $\mathcal{P}$. We also restrict the equivariance property of $\bar{u}$, that is its gauge transformation. Intuitively one expects a **residual gauge symmetry**. It is indeed the case.

Lemma 2. • *Let K be a Lie subgroup of H. There exists a map $\bar{u} : \mathcal{P} \to K$ with K-equivariance $\mathcal{R}_k^*\bar{u} = k^{-1}\bar{u}$, if and only if there is a right-K-space isomorphism, $\mathcal{P} \simeq \mathcal{P}/K \times K$.*

• *Let ω be a connection on $\mathcal{P}$. Define the map $f_{\bar{u}} : \mathcal{P} \to \mathcal{P}$ by $p \mapsto p\bar{u}(p)$. The 1-form $\widehat{\omega} = f_{\bar{u}}^*\omega$ is K-invariant and K-horizontal, so projects to a well defined 1-form on $\mathcal{P}/K$.*

Proof. • $\Leftarrow$ If there is a K-space isomorphism $\mathcal{P} \to \mathcal{P}/K \times K$ given by $p \mapsto \big([p]_K,\ k\big)$, the map $\bar{u} : \mathcal{P} \to K$ defined by $\bar{u}(p) = k^{-1}$ has the asserted equivariance. Indeed, for $k' \in K$, $pk' \mapsto \big([pk']_K,\ kk'\big) = \big([p]_K,\ kk'\big)$, so that $\bar{u}(pk') = (kk')^{-1} = k'^{-1}k^{-1} = k'^{-1}\bar{u}(p)$.

$\Rightarrow$ Suppose $\bar{u}$ exists. For any $p \in \mathcal{P}$ we see that $\bar{u}\big(p\bar{u}(p)k^{-1}\big) = k\bar{u}(p)^{-1}\bar{u}(p) = k$. So $\bar{u}$ is surjective. Then we can define the non-empty set $Q = \bar{u}^{-1}(e)$, with e the identity in K. We clearly have the isomorphism $Q \to \mathcal{P}/K$, given by $p\bar{u}(p) \mapsto [p]_K$.

The map $\mathcal{P} \to Q\times$ K, given by $p \mapsto \big(p\bar{u}(p),\ \bar{u}(p)^{-1}\big)$ is a right-K-space isomorphism. Indeed by the right action of K we have, $pk \mapsto \big(pk\bar{u}(pk),\ \bar{u}(pk)^{-1}\big) = \big(p\bar{u}(p), \bar{u}(p)^{-1}k\big) = \big(p\bar{u}(p), \bar{u}(p)^{-1}\big)k$. The inverse map $Q \times K \to \mathcal{P}$ is given by, $(q,k) \mapsto qk$.

• Given the definition of $f_{\bar{u}}$, the calculation of $f_{\bar{u}}^*\omega$ is exactly analogous to that of an active gauge transformation and gives, $\widehat{\omega} = f_{\bar{u}}^*\omega = \bar{u}^{-1}\omega\bar{u} + \bar{u}^{-1}d\bar{u}$. This is the first 1-form in (2.5). The proof of K-invariance and K-horizontality goes as in Lemma 1. □

Had we defined $\mathcal{Q} = \bar{u}^{-1}(k_0)$ for any $k_0 \in K$, *mutatis mutandis* the construction would still hold.

The first part of the lemma states that the existence of the K-valued map $\bar{u}$ implies the triviality of $\mathcal{P}$ along the direction of the K subgroup. So if $\bar{u}$ takes values in H itself , we have $\mathcal{P} \simeq \mathcal{P}/H \times H \simeq \mathcal{M} \times H$. This was expected from the fact that $\bar{u}$ may be seen as a section of the associated bundle $E = \mathcal{P} \times_H K$. If $K = H$, E is the bundle $\mathcal{P}$ itself, and a global section of the principal bundle $\mathcal{P} = E$ means its triviality. Moreover, in this case the second part of the lemma entails that $\widehat{\omega}$ is the projectable 1-form of Lemma 1. Then, the global existence of the map $\bar{u}$ and global existence of (2.5) are settled only if $\mathcal{P}$ is trivial.

In the general case, $\widehat{\omega}$ does not project on $\mathcal{M}$ but on $\mathcal{P}/H$. It then ought to display a $\mathcal{H}/\mathcal{K}$-**residual gauge freedom** inherited from the non-neutralized gauge transformation of ω *and* of the gauge transformation/equivariance of $\bar{u}$ under H/K. The latter is here left unspecified, but we will see examples before long.

Link with the Bundle Reduction Theorem We summarize the essential features of the Bundle Reduction Theorem (BRT) and of Lemma 2. We take J and K as subgroups of H.
In the BRT, the map $\bar{u} : \mathcal{P} \to H/J$ with H-equivariance $\mathcal{R}_h^*\bar{u} = h^{-1}\bar{u}$, realises J-reductions $\mathcal{P}' = \bar{u}^{-1}(eJ)$ parametrized by elements of H/J (for we can define other reductions $\mathcal{P}'' = \bar{u}^{-1}(hJ)$ as subbundles of $\mathcal{P}$). In Lemma 2, the map $\bar{u} : \mathcal{P} \to K$ with K-equivariance $\mathcal{R}_k^*\bar{u} = k^{-1}\bar{u}$, realises H/K-bundles $\mathcal{Q} = \bar{u}^{-1}(e)$ parametrized by elements of K (for we can define other quotient bundles $\mathcal{Q}' = \bar{u}^{-1}(k)$ as subbundles of $\mathcal{P}$).

From this it is obvious that both approaches are the same when we have the *product structure* $H = K \times J$, with $h \times h' = (k,j) \times (k',j') = (kk',jj')$. If this is so, $H/J = K$ and the H-equivariance of $\bar{u}$ reduces to a K-equivariance. Therefore, $\mathcal{Q} \simeq \mathcal{P}/K$ is a J-bundle, so that $\bar{u}$ realises a J-reduction of $\mathcal{P}$.

Furthermore the connection on $\mathcal{P}$ splits as $\omega_{\mathfrak{h}} = \omega_{\mathfrak{k}} + \omega_{\mathfrak{j}}$. Then, still on $\mathcal{P}$, $\widehat{\omega}_{\mathfrak{h}} = \widehat{\omega}_{\mathfrak{k}} + \omega_{\mathfrak{j}}$ since $\bar{u}$ is K-valued. Clearly enough, $\omega_{\mathfrak{j}}$ is a $\mathcal{J}$-connection but is $\mathcal{K}$-invariant. On the other hand, whereas $\widehat{\omega}_{\mathfrak{k}}$ is $\mathcal{K}$-invariant as expected, since $\omega_{\mathfrak{k}}$ is $\mathcal{J}$-invariant the gauge transformation of $\widehat{\omega}_{\mathfrak{k}}$ under $\mathcal{J}$ depends entirely on the $\mathcal{J}$-equivariance of the map $\bar{u}$.

The Electroweak sector of the Standard Model of Particle Physics, with its group structure $SU(2) \times U(1)$, is an example of such a case, where the BRT and the Lemma 2 coincide, at least for the construction of a $U(1)$-reduction. But this remark anticipates on the next section.

2.3 Applications to Physics

We test the relevance of the above described framework for Physics. It turns out that the dressing field method provides and unifying scheme for several constructions found scattered in the literature on gauge field theories and may clarify the interpretive baggage accompanying some of them. In order to have a construction as natural as possible, we would like to find the dressing field already somewhere in the Lagrangian of the theory under study. In that way, the process of forming the composite fields by dressing the various fields of the theory might be seen a mere **change of field variables**. Toy models as well as more physically relevant examples will illustrate this.

2.3.1 Abelian theories and Dirac variables

Applied to Electromagnetism, that is by working with the bundle $\mathcal{P}\big(\mathcal{M}, H = U(1)\big)$, the *composite fields* (2.6) are known as **Dirac variables**. Well aware of the first problem alluded to in 1.3 above, Dirac advocated in (Dirac, 1955), and subsequently in the final chapter of his book (Dirac, 1958), the idea that reconstructing Electrodynamic Theory with gauge invariant variables would be better suited for quantization. In (Dirac, 1955) he defined the field,

$$\psi^*(x) := \psi(x)e^{iC} = \psi(x)\exp\left(i\int c(x,x')A(x')d^3x'\right) \qquad \text{with} \qquad c(x,x') = \frac{e}{4\pi\hbar}\frac{x'-x}{|x'-x|^3}$$

(equations [16] and [19] of Dirac's paper) as well as the derivative,

$$\Big(\partial_r\psi - i\frac{e}{\hbar}A^r\psi\Big)e^{iC}, \qquad \text{with } r \text{ a spatial index,}$$

(equation [21]). These are special cases of $\widehat{\varphi}_i$ and $\widehat{D\varphi_i}$ in (2.6) respectively. Furthermore, in the line proving the gauge invariance of equation [21] appears the quantity $A^r + \hbar/e\ \partial_r C$ which is nothing but the abelian case of the composite field $\widehat{A}$. We see there that the phase factor, the *abelian dressing field* $u = e^{iC}$, is nonlocal so that gauge invariance is obtained at the expense of the locality of the fields.[7]

Then, studying quantization, he argued that: "$\psi^*(x)$ is the operator of creation of an electron *together* with its Coulomb field, or possibly [...] of absorption of a positron *together* with its Coulomb field". According to Dirac's interpretation: "A theory that works entirely with gauge-invariant operators has its electrons and positrons always accompanied by Coulomb fields around them [...]".[8] A statement reaffirmed almost verbatim at bottom of p.303 in (Dirac, 1958). In view of this interpretation, the terminology "dressing field", devised before any knowledge of Dirac's work on the subject, could assume an unexpected physical significance.

Toy model 1 The Stueckelberg formalism can, modulo a remark below, also be seen as a special case of our approach. See (Ruegg and Ruiz-Altaba, 2004) for a review. Let us give an easy example. Consider the following prototype Stueckelberg Lagrangian form for the abelian gauge potential A and the Stueckelberg scalar field B,

$$L(A,B) = \frac{1}{2}F \wedge *F + \frac{m^2}{2}\Big(A - \frac{1}{m}dB\Big) \wedge *\Big(A - \frac{1}{m}dB\Big),$$

where F is the field strength of A. This Lagrangian is invariant under the infinitesimal gauge transformations $\delta A = -d\alpha$ and $\delta B = -m\alpha$, with $\alpha \in C^\infty(\mathcal{M})$. Now consider the $U(1)$-valued dressing field given by $u = e^{\frac{i}{m}B}$. It transforms under $\gamma = e^{i\alpha} \in \mathcal{U}(1)$ as, $u^\gamma = \gamma^{-1}u = e^{\frac{i}{m}(B-m\alpha)}$. The composite field, or Dirac variable, associated to the gauge potentiel A reads, $\widehat{A} = A + iu^{-1}du = A - \frac{1}{m}dB$. So that the Lagrangian form can be rewritten,

$$L(\widehat{A}) = \frac{1}{2}\widehat{F} \wedge *\widehat{F} + \frac{m^2}{2}\widehat{A} \wedge *\widehat{A}.$$

This is a Proca Lagrangian form for a massive vector field $\widehat{A}$.

An important remark: the Stueckelberg trick usually consists in implementing a $U(1)$-gauge symmetry on a Proca Lagrangian at the expense of introducing a new scalar field whose degree of freedom compensates the introduced gauge freedom. Using the dressing method we've done precisely the opposite move, that is we've merely made a *change of variables* and shifted from a $U(1)$-gauge theory of the fields A and B where the constant m has no clear meaning, to a theory of the field $\widehat{A}$ with mass m where the $U(1)$-gauge symmetry has been factorized out.

This toy model illustrates Lemmas 1/2 in the case $u : \mathcal{M} \to H$, as well as the requirement of finding the dressing field directly in the Lagrangian. There the dressing is the Stueckelberg field, given right away. Things might not always be so easy. The dressing field could be in the Lagrangian indeed, but hidden in some **auxiliary field**. This can be shown through another, less trivial, toy model.

Toy model 2 Consider the abelian Higgs model of a $U(1)$-gauge potential A interacting with a $\mathbb{C}$-scalar field φ,

$$L(A,\varphi) = \Big[D\varphi^\dagger D\varphi + V(\varphi)\Big]\text{vol} + \frac{1}{2}F \wedge *F, \quad \text{with} \quad D\varphi = d\varphi + A\varphi \quad \text{and} \quad V(\varphi) = -\mu^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2.$$

[7] The trade between gauge invariance and non-locality is not new, Wilson loops are textbook cases. It is also part of the discussion around the Arahonov-Bohm effect. See the contribution of A. M. Nounou in (Brading and Castellani, 2003).

[8] (Dirac, 1955), p.657. The sentence ends by "[...] which is very reasonable from the physical point of view." Who would object?

The Lagrangian form is invariant under the finite gauge transformations $A^\gamma = A + \gamma^{-1}d\gamma$ and $\varphi^\gamma = \gamma^{-1}\varphi$, for any $\gamma = e^{i\alpha} \in \mathcal{U}(1)$.

The dressing field is found from the auxiliary field $\varphi : \mathcal{M} \to \mathbb{C}$ by polar decomposition, $\varphi = \eta u$, where $\eta = \varphi^\dagger\varphi$. The field, $u : \mathcal{M} \to U(1)$ transforms as $u^\gamma = \gamma^{-1}u$ on account of the transformation law of φ. Thus u is our dressing field. The field $\eta : \mathcal{M} \to \mathbb{R}^+$ is clearly $\mathcal{U}(1)$-invariant. Since it can be written as $\eta = \widehat{\varphi} = u^{-1}\varphi$ it is a composite field/Dirac variable associated to φ. The composite field/Dirac variable associated to the gauge potential A is thus, $\widehat{A} = A + u^{-1}du$. The corresponding abelian field strength is $\widehat{F} = F$. The invariant derivative is $\widehat{D}\widehat{\varphi} = \widehat{D}\eta = u^{-1}D\varphi$. Finally the Lagrangian form can be rewritten as,

$$L(\widehat{A}, \eta) = \left[\widehat{D}\eta^\dagger \widehat{D}\eta + V(\eta)\right]\mathrm{vol} + \frac{1}{2}\widehat{F} \wedge *\widehat{F}.$$

It describes a massless vector field $\widehat{A}$ coupled to a $\mathbb{R}^+$-scalar field η embedded in a potential V. It is gauge-invariant since it contains only gauge-invariant fields. The dressing method is again shown to be a mere **change of variables** $(A, \varphi) \to (\widehat{A}, \eta)$ which conveniently redistributes the degrees of freedom of the theory.

Notice the **residual field** η, absent in the previous example, left after the extraction of the dressing field u from the auxiliary field φ. This residual field η is the true **observable** Higgs field, since gauge-invariance is mandatory for observability. We'll return to this point latter. The residual field may not always be identified this easily. In general the extraction of the dressing from an auxiliary field might involve an arbitrary choice (in a sense to be made precise shortly) so that this identification is blurred. It seems that it is the Lagrangian of the theory which allows to firmly ascertain the residual field. See the example of the electroweak model below.

2.3.2 Non-abelian theories and generalized Dirac variables

Within the literature of Quantum ChromoDynamics, hadronic Physics and more generally in the context of non-abelian gauge theories, there have been attempts to generalize the construction of Dirac. Not surprisingly, we find instances of (2.6) which are then naturally called **generalized Dirac variables**.

In (Lavelle and McMullan, 1997) e.g , which touches upon the problem of defining gauge invariant coloured states for the quarks in QCD and combining them in hadronic bound states, the term *generalized Dirac variable* is not used but the will to extend his idea is well assumed. Moreover by an interesting terminological coincidence, if ψ is a quark, the gauge-invariant quantity defined by $\psi_{\text{phys}} = h^{-1}\psi$ (equation [5.2] of their paper) is called a "dressed quark". And the field h transforming according to $h^U = U^{-1}h$, for $U \in \mathcal{SU}(3)$ (equation [5.1]), is called it a "dressing". The composite field, or generalized Dirac variable, associated to the $SU(3)$-gauge field A is defined as, $(A_{\text{phys}})_i = h^{-1}A_i h + h^{-1}\partial_i h$ (equation [5.5]), with i a spatial index.

Notice how gauge-invariance and observability are tied (if not identified) by the subscript "phys" used to denote the composite fields. In accordance with Dirac's remark cited above, and by virtue of its explicit construction as a (still non-local) function of the gauge potential, the dressing field h is interpreted as a gluon sea surrounding the bare quark ψ *and* the bare gluon A. This is announced from the introduction of the paper: "One views dressings as surrounding the charged particles with a cloud of gauge field". The aim of the authors is then stressed: "[...] such dressed quarks may be combined to form colourless hadrons in the way commonly done in the constituent quark model. [...] gluons may also be dressed".

A most salient point of this paper is the will to draw a link between "dressings" and gauge fixing. Nevertheless, on account of our closing discussion in 2.2.1, we are bound to dispute this will. However the related nice suggestion of a connection between Gribov ambiguity and quark confinement might be left untouched. A discussion of this interpretive issue is proposed in appendix A.1.

The paper (Lorcé, 2013b), which reviews and addresses the problem of the proton spin decomposition in terms of gauge-invariant contributions with clear partonic interpretation, is an example of an approach whose link with generalized Dirac variables went at first unnoticed. Though, a careful analysis shows that the first part of the paper can be entirely founded on the dressing field method. Nevertheless the construction presented there is interpreted as a specific gauge tranformation, a viewpoint that we must again dispute on account of 2.2.1. I propose the full analysis in appendix A.2.

The authors of this paper and of (Fournel et al., 2014) came to a short correspondence (iniated by the former) on the contact points of both works. As a mark of fruitful exchanges, the dressing field and the composite fields/Dirac variables are more clearly identified in the subsequent paper (Lorcé, 2013a). In there, the transformation law of the matrix valued field U_{pure}, $\widetilde{U}_{\text{pure}}(x) = U^{-1}(x)U_{\text{pure}}(x)$ (slight correction of equation [22] of Lorcé's paper), identifies it with a dressing field. Moreover the fields $\widehat{\phi}(x) = U^{-1}_{\text{pure}}(x)\phi(x)$ (equation [44]) and $\widehat{A}_\mu(x) = U^{-1}_{\text{pure}}(x)\left[A_\mu(x) + \frac{i}{g}\partial_\mu\right]U_{\text{pure}}(x)$ (equation [45]) are instances of (2.6).

The remarks made by the author are relevant : "[...] despite appearances, eqs [44] and [45] *are not gauge transformations*. In practive, the matrices $U_{\text{pure}}(x)$ can be expressed in terms of the gauge fields $A_\mu(x)$ (Lorcé, 2013c), and can be thougth of as *dressing fields*". In view of this clear statement the next sentence is quite surprising: "From a geometrical point of view $U_{\text{pure}}(x)$ simply determines a reference configuration in the internal space. The gauge-invariant field $\widehat{\phi}(x)$ then represents "physical" deviations from this reference configuration". We have to object to this. If equations [44] and [45] are not gauge transformations, as we plainly agree on, then the dressing U_{pure} does not belong to the gauge group and does not determine a point ("reference configuration", or "abstract reference frame" as we termed it) in the fiber ("internal space") of the underlying bundle.

The transformation of the Lagrangian is considered in the last part of the paper and we find the key words: "[...] one can switch between gauge-covariant and invariant canonical formalism by a *mere change of variables*". All this is repeated and concisely synthesized, with the same misinterpretation though, on p52-53 of the extensive review (Leader and Lorcé, 2014) on the problem of the proton spin decomposition.

We should notice two important facts. First, while in (Dirac, 1955), (Lavelle and McMullan, 1997) and (Lorcé, 2013b) the dressing fields are constructed as non-local functions of the gauge potential A, this was not the case in the two simple toy models presented above where the dressings were local functions of an auxiliary field. But this is not an artefact of over-simplistic models. From now on we will see physically substantial examples (to say the least) where the dressing is not a function of the gauge potential or, if it is, it is still local.

Secondly, in the aforementioned works the question of the loss of the manifest Lorentz covariance of the composite fields is of constant worry and much energy is deployed in order to settle this question in each specific construction. Notice that due to our differential geometric, thus intrinsic, formulation of the dressing method, the question of the Lorentz covariance and even of the general covariance of our composite fields never arises. Once again, from now on the examples we consider are free of such concerns.

One last remark. All constructions above were instances of a dressing with value in H, the full structure group, therefore instances of complete neutralization of gauge symmetry and full geometrization. The composite fields/generalized Dirac variables had no residual gauge freedom and belonged to the natural geometry of the space-time base manifold $\mathcal{M}$. An interesting case of partial neutralization, thus of residual gauge freedom, is presented below. A more complicated illustration is to be worked out in the next section. For the moment let us go to our two most relevant examples, the Electroweak sector of the Standard Model and General Relativity.

The ElecroWeak sector of the Standard Model

The geometric setup of the model has been detailled in 2.1.2, we simply remind the notations. The principal bundle of the model is $\mathcal{P}\big(\mathcal{M}, H = U(1)\times SU(2)\big)$. The connection on $\mathcal{P}$ splits as $\omega_{\mathfrak{g}} = \bar{a}_{\mathfrak{u}(1)} + \bar{b}_{\mathfrak{su}(2)}$, its pullback on $U \subset \mathcal{M}$ is $A = a + b$. The associated curvature is simply $F = f_a + g_b$. The scalar field is $\varphi : U \to \mathbb{C}^2$, and its covariant derivative $D\varphi = d\varphi + (g'a + gb)\varphi$. Since we are interested in Physics, which takes place on the base manifold (space-time), we will apply Lemmas 1 and 2 on the localized fields, that is write the composite fields (2.6). Giving the quartic potential $V(\varphi) = -\mu^2\langle\varphi,\varphi\rangle - \lambda\langle\varphi,\varphi\rangle^2$, the initial Lagrangian of the theory is,

$$L(\varphi, a, b) = \langle D\varphi, *D\varphi\rangle + V(\varphi)\text{vol} + \frac{1}{2}\text{Tr}(f_a \wedge *f_a) + \frac{1}{2}\text{Tr}(g_b \wedge *g_b). \tag{2.7}$$

As it stands nor a nor b can be massive, and indeed L contains no mass term for them. It is not a problem for a since we expect to have at least one massless field to carry the electromagnetic interaction. But the weak interaction is short range, so its associated field must be massive. Hence the necessity to reduce the $SU(2)$ gauge symmetry in the theory in order to allow a mass term for the weak field. Of course we know that one can achieve this by the celebrated Spontaneous Symmetry Breaking Mechanism. Actually the SSBM is used in conjunction with a gauge fixing, the so-called *unitary gauge*,[9] see e.g (Becchi and Ridolfi, 2006). We even know that some authors gave a more geometrical account of the model based on the Bundle Reductions Theorem, see the end of section 1.1.1 and refer to (Trautman, 1979), (Westenholz, 1980), (Sternberg, 1994) for literature.

As a fourth way to the symmetry reduction, can we treat the model with the dressing field method? Since we've seen that in the case of a principal bundle whose structure group is a simple product of groups, the Bundle Reduction Theorem and our Lemma 2 coincide, it is natural to expect an affirmative answer. How do we proceed? The answer was first given in (Masson and Wallet, 2011), where the spinor fields are also included. We give here an account adapted from (Fournel et al., 2014).

First, sticking to our requirement of naturalness mentioned earlier, we ought to find a candidate dressing field as a part of some *auxiliary field* already present in the theory. The most obvious place to search for it, given the gauge transformations (2.1), is on the side of the doublet scalar field φ. To mimic the simple abelian case (toy model 2) we can use a generalization of the polar decomposition in $\mathbb{C}^2$. But unlike the abelian case, this time the decomposition involves an arbitrary choice of **reference point**, in this case a vector $v \in \mathbb{C}^2$. Let us choose $v = \binom{0}{1}$. With respect to this vector, one can decompose φ as,

$$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \eta \cdot u[\varphi] \cdot v = \eta \cdot \frac{1}{\eta}\begin{pmatrix}\bar\varphi_2 & \varphi_1\\ -\bar\varphi_1 & \varphi_2\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} \qquad \text{with,}\quad \eta = |\varphi| = \left(|\varphi_1|^2 + |\varphi_2|^2\right)^{\frac{1}{2}}. \tag{2.8}$$

Now we should want to know how this decomposition behaves under gauge transformation $\alpha \in \mathcal{U}(1)$ and $\beta \in \mathcal{SU}(2)$. First, realizing $U(1)$ as a subgroup of 2×2 matrices, we have:

$$\varphi^\alpha = \alpha^{-1}\varphi = \begin{pmatrix}\alpha^{-1} & 0\\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \eta\cdot\frac{1}{\eta}\begin{pmatrix}\alpha\bar\varphi_2 & \alpha^{-1}\varphi_1\\ -\alpha\bar\varphi_1 & \alpha^{-1}\varphi_2\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \eta\cdot\frac{1}{\eta}\begin{pmatrix}\bar\varphi_2 & \varphi_1\\ -\bar\varphi_1 & \varphi_2\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} := \eta\cdot u\widehat{\alpha}\cdot v.$$

Thus the $SU(2)$-valued field u has a $\mathcal{U}(1)$-gauge transformation $u^\alpha = u\widehat{\alpha}$. This is not a dressing transformation so u cannot be used to neutralize the $U(1)$ symmetry. This is no trouble to us, as already explained, to the contrary this transformation under $U(1)$ will be important thereafter. Let us see the action of $\mathcal{SU}(2)$:

$$\begin{aligned}\varphi^\beta = \beta^{-1}\varphi = \begin{pmatrix}x & -y\\ \bar y & \bar x\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}x\varphi_1 - y\varphi_2\\ \bar y\varphi_1 + \bar x\varphi_2\end{pmatrix} &= \eta\cdot\frac{1}{\eta}\begin{pmatrix}y\bar\varphi_1 + x\bar\varphi_2 & x\varphi_1 - y\varphi_2\\ -\bar x\bar\varphi_1 + \bar y\bar\varphi_2 & \bar y\varphi_1 + \bar x\varphi_2\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix},\\ &= \eta\cdot\frac{1}{\eta}\begin{pmatrix}x & -y\\ \bar y & \bar x\end{pmatrix}\begin{pmatrix}\bar\varphi_2 & \varphi_1\\ -\bar\varphi_1 & \varphi_2\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix} := \eta\cdot\beta^{-1}u\cdot v.\end{aligned}$$

So finally we find for u the $\mathcal{SU}(2)$-gauge transformation $u^\beta = \beta^{-1}u$. Then u is our $SU(2)$-valued dressing field. The latter taking values in a subgroup of the structure group of the bundle, we are in the situation of Lemma 2. We thus expect to be able to neutralise $SU(2)$ and to have a $U(1)$-residual gauge freedom whose exact nature will depend on the transformation law $u^\alpha = u\widehat{\alpha}$. Remember the discussion following Lemma 2 in 2.2.2. Now we can dress the fields of the theory and form the composite fields according to (2.6),

$$\begin{aligned}&\widehat{A} = u^{-1}Au + \frac{1}{g}u^{-1}du = u^{-1}(a+b)u + \frac{1}{g}u^{-1}du = a + (u^{-1}bu + \frac{1}{g}u^{-1}du) := a + B, \qquad \text{since } u^{-1}au = a,\\ &\widehat{F} = u^{-1}(f_a + g_b)u = f_a + u^{-1}g_b u := f_a + G, \qquad \text{with } G = dB + gB^2,\\ &\widehat{\varphi} = u^{-1}\varphi = \begin{pmatrix}0\\ \eta\end{pmatrix} := \eta,\\ &\widehat{D}\widehat{\varphi} = u^{-1}D\varphi = \widehat{D}\eta = d\eta + (g'a + gB)\eta, \qquad \text{where } \eta \text{ means the above vector.}\end{aligned}$$

[9]The unitary gauge is used at the tree level. For loop perturbative calculations the Feynman–'t Hooft gauge, or the more general R_ξ gauge, are often used as more convenient.

Notice that these are easy matrix calculations. The composite field B and its field strength G, as well as η and its covariant derivative $\widehat{D}\eta$, are SU(2)-gauge invariant by construction. How do these fields transform under the residual $U(1)$-gauge symmetry? Well, it depends of course on the $U(1)$-transformation of the original fields but also, as already stressed, on the $U(1)$-transformation of the dressing field. For B, we have by definition,

$$B^\alpha = (u^\alpha)^{-1} b^\alpha (u^\alpha) + \frac{1}{g}(u^\alpha)^{-1} d(u^\alpha) = \widehat{\alpha}^{-1} u^{-1} \cdot b \cdot u\widehat{\alpha} + \frac{1}{g}\widehat{\alpha}^{-1}(u^{-1}du)\widehat{\alpha} + \frac{1}{g}\widehat{\alpha}^{-1}d\widehat{\alpha},$$
$$B^\alpha = \widehat{\alpha}^{-1} B\widehat{\alpha} + \frac{1}{g}\widehat{\alpha}^{-1}d\widehat{\alpha}.$$

Explicitly, we use the decomposition $B = B_a\sigma^a$ where σ^a are the hermitian Pauli matrices so that for B to be truly in $\mathfrak{su}(2)$ we have $B_a \in i\mathbb{R}$, so that $\bar{B}_a = -B_a$. Then,

$$B = B_a\sigma^a = \begin{pmatrix} B_3 & B_1 - iB_2 \\ B_1 + iB_2 & -B_3 \end{pmatrix} := \begin{pmatrix} B_3 & W^- \\ W^+ & -B_3 \end{pmatrix}, \qquad \text{and} \qquad B^\alpha = \begin{pmatrix} B_3 + \frac{1}{g}\alpha^{-1}d\alpha & \alpha^{-2}W^- \\ \alpha^2 W^+ & -B_3 - \frac{1}{g}\alpha^{-1}d\alpha \end{pmatrix}.$$

It is a good news to have $W^\pm$ transforming tensorially, like a matter field, because this, added to their $SU(2)$-invariance, means that it is possible to have mass terms for these two fields. Precisely what we wanted from the beginning. Nevertheless we see that B_3 transforms as a $U(1)$-connection[10], no mass term allowed at first sight. Actually the situation is more favorable than it appears. One can indeed make a further change of variables in the space of fields. Given the so called *Weinberg angle* by $\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$ and $\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$, one can define the two 1-forms,

$$\begin{pmatrix} A \\ Z^0 \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} a \\ B_3 \end{pmatrix} = \begin{pmatrix} \cos\theta_W a + \sin\theta_W B_3 \\ \cos\theta_W B_3 - \sin\theta_W a \end{pmatrix}.$$

Since $a^\beta = a$ and $B_3^\beta = B_3$, then $(Z^0)^\beta = Z^0$. Moreover we have,

$$\left(Z^0\right)^\alpha = \frac{g}{\sqrt{g^2+g'^2}}\left(B_3 + \frac{1}{g}\alpha^{-1}d\alpha\right) - \frac{g'}{\sqrt{g^2+g'^2}}\left(a + \frac{1}{g'}\alpha^{-1}d\alpha\right) = \frac{g}{\sqrt{g^2+g'^2}}B_3 - \frac{g'}{\sqrt{g^2+g'^2}}a = Z^0.$$

So this field is $\big(SU(2)\times U(1)\big)$-invariant. It can thus be massive and obervable. We'll see that it appears naturally in the norm of the $SU(2)$-*invariant* derivative $\widehat{D}\eta$. Clearly we have $A^\beta = A$ and,

$$A^\alpha = \frac{g'}{\sqrt{g^2+g'^2}}\left(B_3 + \frac{1}{g}\alpha^{-1}d\alpha\right) + \frac{g}{\sqrt{g^2+g'^2}}\left(a + \frac{1}{g'}\alpha^{-1}d\alpha\right) = A + \frac{1}{e}\alpha^{-1}d\alpha,$$

where the coupling constant is $e = \frac{gg'}{\sqrt{g^2+g'^2}} = g'\cos\theta_W = g\sin\theta_W$. So A still transforms as a $\mathcal{U}(1)$-connection, it is thus the massless carrier of the electromagnetic interaction and e is the elementary electric charge.

We should now see what are the $U(1)$-transformations of G, η, $\widehat{D}\eta$ and see what happens to the Lagrangian form. The $U(1)$-transformation of the $SU(2)$-invariant field strength is,

$$G = u^{-1}g_b u, \qquad \Rightarrow \qquad G^\alpha := (u^\alpha)^{-1}g_b^\alpha(u^\alpha) = (u\widehat{\alpha})^{-1}g_b(u\widehat{\alpha}) = \widehat{\alpha}^{-1}G\widehat{\alpha}.$$

This tensorial transformation is again a good thing for it will allow us to write a genuine Yang-Mills term for B, both $SU(2)$ and $U(1)$-invariant. Indeed we have,

$$\mathrm{Tr}(F\wedge *F) = \mathrm{Tr}(f_a\wedge *f_a) + \mathrm{Tr}(g_b\wedge *g_b) = \mathrm{Tr}(\widehat{F}\wedge *\widehat{F}) = \mathrm{Tr}(f_a\wedge *f_a) + Tr(G\wedge *G).$$

[10]With a different coupling constant through.

It is quite easy to show that the $f_a \wedge * f_a$ and $G \wedge * G$ are actually diagonal so that nothing is lost by tracing. The above term gives all possible interactions between the four electroweak fields. For the sake of completeness we give the explicit expression,

$$
\begin{aligned}
\frac{1}{2}\mathrm{Tr}(\widehat{F} \wedge *\widehat{F}) = dZ^0 \wedge *dZ^0 \quad &+ \quad dA \wedge *dA \quad + \quad dW^- \wedge *dW^+ \\
&+2g\Big\{ \sin\theta_W \Big(dA \wedge *(W^-W^+) + dW^- \wedge *(W^+A) + dW^+ \wedge *(AW^-) \Big) \\
&\quad + \cos\theta_W \Big(dZ^0 \wedge *(W^-W^+) + dW^- \wedge *(W^+Z^0) + dW^+ \wedge *(Z^0W^-) \Big) \Big\} \\
&+4g^2\Big\{ \sin^2\theta_W \; AW^- \wedge *(W^+A) + \sin\theta_W \cos\theta_W \; AW^- \wedge *(W^+Z^0) \\
&\quad + \cos^2\theta_W \; Z^0W^- \wedge *(W^+Z^0) + \sin\theta_W \cos\theta_W \; Z^0W^- \wedge *(W^+A) \\
&\quad + \frac{1}{4}\; W^-W^+ \wedge *(W^-W^+) \Big\}.
\end{aligned}
$$

The $U(1)$-invariance is less easily seen than in the compact expression, but it is definitely there. Notice that there is no direct coupling between the field A and Z^0 as we would expect on physical ground.

The case of the scalar field $\eta : U \to \mathbb{R}^+$ is easy, it is both $SU(2)$ and $U(1)$-invariant by definition, $\eta = |\varphi|$. So the potential term in (2.7) is,

$$
V(\varphi) = -\mu^2\bar{\varphi}\varphi - \lambda(\bar{\varphi}\varphi)^2, \qquad \Rightarrow \qquad V(\eta) = -\mu^2\eta^2 - \lambda\eta^4.
$$

It is trivially $\big(SU(2) \times U(1)\big)$-invariant, thus an observable field. Notice that η is the **residual field** left after extraction of the dressing field u from the **auxiliary field** φ. It is also the analogue of the map r in the split $f = (r, \bar{u})$, equation (1.4), in our discussion of the Bundle Reduction Theorem, section 1.1.1.

Now we can consider the $SU(2)$-*invariant* covariant derivative which is,

$$
\widehat{D}\eta = d\begin{pmatrix} 0 \\ \eta \end{pmatrix} + g\begin{pmatrix} B_3 & W^- \\ W^+ & -B_3 \end{pmatrix}\begin{pmatrix} 0 \\ \eta \end{pmatrix} + g'\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 \\ \eta \end{pmatrix} = \begin{pmatrix} gW^-\eta \\ d\eta - gB_3\eta + g'a\eta \end{pmatrix}, \quad \text{so that} \quad (\widehat{D}\eta)^\alpha = \begin{pmatrix} g\alpha^{-2}W^-\eta \\ d\eta - gB_3\eta + g'a\eta \end{pmatrix}.
$$

The last equality is easily seen. It is easy to calculate the norm of $\widehat{D}\eta$ which is,

$$
\begin{aligned}
\langle D\varphi, *D\varphi \rangle : &= D\varphi^\dagger \wedge *D\varphi = \widehat{D}\eta^\dagger \wedge *\widehat{D}\eta = \Big(-gW^+\eta,\; d\eta + gB_3\eta - g'a\eta \Big) u \wedge *u^{-1} \begin{pmatrix} gW^-\eta \\ d\eta - gB_3\eta + g'a\eta \end{pmatrix}, \\
&= d\eta \wedge *d\eta - g^2\eta^2\; W^+ \wedge *W^- - (g^2 + g'^2)\eta^2\; Z^0 \wedge *Z^0.
\end{aligned}
$$

We see mass-like terms appearing for both $W^\pm$ and Z^0 fields.

Let us sum-up what has been done so far. After identifying a $SU(2)$-dressing field u out of the auxiliary field φ already provided by the theory, we've dressed the fields of the aforementioned theory in order to produce $SU(2)$-invariant composite fields. This construction is the local counterpart of the global reduction of the the $SU(2) \times U(1)$ bundle $\mathcal{P}$ to a $U(1)$-subbundle as proved in Lemma 2. We've performed a *change of variables* in the space of fields on $\mathcal{M}$, that is $L(\varphi, a, b) \to L(\eta, a, B_3, W^\pm)$. Furthermore, knowing the transformation of the dressing field under the $U(1)$-*residual gauge freedom* we were able to identify an adequate second change of variables $L(\eta, a, B_3, W^\pm) \to L(\eta, A, Z^0, W^\pm)$ to handle the residual $U(1)$-symmetry. This change is anyway suggested by the calculation of the norm of $\widehat{D}\eta$, that is by the Lagrangian form which finally reads,

$$
\begin{aligned}
L(\eta, A, Z^0, W^\pm) &= \langle \widehat{D}\eta, *\widehat{D}\eta \rangle + V(\eta)\mathrm{vol} + \frac{1}{2}\mathrm{Tr}(\widehat{F} \wedge *\widehat{F}), \qquad (2.9) \\
&= d\eta \wedge *d\eta \quad - g^2\eta^2\; W^+ \wedge *W^- - (g^2 + g'^2)\eta^2\; Z^0 \wedge *Z^0 \quad - \mu^2\eta^2 - \lambda\eta^4 \quad + \frac{1}{2}\mathrm{Tr}(f_a \wedge *f_a) + \frac{1}{2}\mathrm{Tr}(G \wedge *G).
\end{aligned}
$$

Each field in it is $SU(2)$-invariant. Being also $U(1)$-invariant, η and Z^0 are also **observable**. A residual $U(1)$-gauge symmetry remains on A and $W^\pm$ so that to qualify them as true observables one should find a $U(1)$-dressing à la Dirac.

True mass terms for the fields Z^0 and $W^\pm$ are obtained when the $\mathbb{R}^+$-valued scalar field η is expanded around its **unique** configuration minimizing the potential $V(\eta)$, the so-called Vacuum Expectation Value. Nevertheless the VEV depends on the sign of μ^2. The VEV of η being zero in the phase of the theory where $\mu^2 > 0$, the fields $Z^0/W^\pm$ are massless there. But in the phase where $\mu^2 < 0$, there is a non-vanishing VEV which is $\eta_0 = \sqrt{\frac{-\mu^2}{2\lambda}}$. If one writes $\eta = \eta_0 + H$ in (2.9) one obtains the electroweak Lagrangian form of the Standard Model in the so-called **unitary gauge** where the masses of the fields Z^0 and $W^\pm$ are,

$$m_{Z_0} = \eta_0 \sqrt{(g^2 + g'^2)} \quad \text{and} \quad m_{W^\pm} = \eta_0 g, \quad \text{with ratio} \quad \frac{m_{W^\pm}}{m_{Z^0}} = \frac{g}{\sqrt{g^2 + g'^2}} = \cos\theta_W. \tag{2.10}$$

Remember that as a result of the dressing field method, **no gauge fixing** is involved to obtain (2.9) and (2.10).

A difference with the usual viewpoint is worth stressing. The SSBM as usually construed goes as follows. At high energy (i.e in the phase $\mu^2 > 0$) the symmetric VEV $\varphi_0 = \binom{0}{0}$ of $\varphi \in \mathbb{C}^2$ respect the full $SU(2) \times U(1)$ gauge symmetry group so that no gauge potential in the theory can be massive. At low energy (i.e in the phase $\mu^2 < 0$) the field φ must fall somewhere in the space of configurations that minimize the potential $V(\varphi)$. A space which is a circle in $\mathbb{C}^2$ defined by $M_0 = \left\{\varphi \in \mathbb{C}^2 \mid \bar\varphi_1\varphi_1 + \bar\varphi_2\varphi_2 = -\mu^2/\lambda\right\}$, and is not invariant under $SU(2)$. Then, once an arbitrary minimum $\varphi_0 \in M_0$ is randomly selected, the gauge group is broken down to $U(1)$ and mass terms for $SU(2)$-gauge potentials are generated. See e.g (Zinn-Justin, 2011).

Notice that our construction allows to clearly distinguish the neutralization of $SU(2)$ and the generation of the masses as two distinct operations. The first being a prerequisite for the second for sure, but not the direct *cause* as usually assumed. The reduction of the $SU(2)$ symmetry by the dressing field method shows that the display of $SU(2)$-invariant fields in the theory is a matter of change of variables. Much in the same way as the simple example of the Stueckelberg Lagrangian, we showed that, so to speak, the $SU(2)$-symmetry was an artefact of a poor choice of field variables. With the right choice, the $SU(2)$-invariance is obvious and the residual $U(1)$ symmetry appears as the only non-trivial symmetry of the theory. And this is so in *both* phases of the theory, $\mu^2 > 0$ and $\mu < 0^2$. The transition between these phases, massless to massive, is still a dynamical process parametrized by the sign of μ^2. But notice that in our scheme there is no arbitrariness in the choice of the VEV of η in the massive phase. The VEV is *unique* since this field is $\mathbb{R}^+$-valued.

According to (Westenholz, 1980) the very statement of *spontaneous* symmetry breaking is embodied by the fact that M_0 is not reduced to a point. If we take this remark seriously, the present construction leads us to deny the soundness of this terminology in the electroweak case. First because the symmetry reduction is not related to the choice of a VEV in M_0, then because the latter being reduced to a point, the choice of VEV is always unique. If, on the basis of these remarks, and to be constructive, I would dare to propose another terminology it would be something like 'mass generation through (vacuum) phase transition'.

Discussion: It remains to discuss the, a priori, genuine points left to arbitrariness in the above application of the dressing field method. Essentially there are two such points.

First, when we sought the dressing field we chose it so as to be $SU(2)$-valued, that is minimal for the task of reducing the $SU(2)$ gauge symmetry. But to do so, Lemma 1 insists that a dressing field with values in a larger group would be as effective. Then we could have chosen the dressing defined by $\widetilde{u} = \eta u$, and such that $\widetilde{u}^\dagger\widetilde{u} = \eta^2\mathrm{id}_2$, which can be seen as an element of the larger group $G = SU(2) \times \mathbb{R}^*_+$. It is not hard to see that A dresses as $\widehat{\widetilde{A}} = a + \widetilde{B}$, where $\widetilde{B} = B + \frac{1}{g}\eta^{-1}d\eta$ is a $SU(2)$-invariant $\mathfrak{g}$-valued 1-form. Notice that since $\eta^{-1}d\eta$ is a pure-gauge-like scalar 1-form, it does not affect the field strength associated to B, that is $\widetilde{G} = G$, and the Yang-Mills term in the Lagrangian form remains unchanged. The scalar field φ dresses as $\widehat{\widetilde{\varphi}} = \widetilde{u}^{-1}\varphi = v$, which is just the constant *reference point* used to find the dressing in the first place.

The covariant derivative $D\varphi$ dresses as

$$
\begin{aligned}
\widehat{\widetilde{D}}v = \widehat{\widetilde{D}}\widehat{\widetilde{\varphi}} = \widetilde{u}^{-1}D\varphi = \widetilde{u}^{-1}\Big(d(\widetilde{u}v) + gb\widetilde{u}v + g'a\widetilde{u}v\Big) &= \widetilde{u}^{-1}d\widetilde{u}\cdot v + dv + g\widetilde{u}^{-1}b\widetilde{u}\cdot v + g'av, \\
&= g\widetilde{B}v + g'av = gBv + \eta^{-1}d\eta v + g'av, \\
&= g\begin{pmatrix} B_3 & W^- \\ W^+ & B_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \eta^{-1}d\eta\begin{pmatrix} 0 \\ 1 \end{pmatrix} + g'a\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} gW^- \\ -gB_3 + \eta^{-1}d\eta + g'a \end{pmatrix} = \eta^{-1}\begin{pmatrix} gW^-\eta \\ -gB_3\eta + d\eta + g'a\eta \end{pmatrix}.
\end{aligned}
$$

That is $\widehat{\widetilde{D}}v = \eta^{-1}\widehat{D}\eta$, so that we have,

$$
< D\varphi, *D\varphi >:= D\varphi^\dagger \wedge *D\varphi = \widehat{\widetilde{D}}v^{\dagger}\,\widetilde{u}^\dagger \wedge *\widetilde{u}\widehat{\widetilde{D}}v = \widehat{\widetilde{D}}v^{\dagger}\cdot \eta^2\mathrm{id}_2 \wedge *\widehat{\widetilde{D}}v = \widehat{D}\eta^{\dagger}\eta^{-1}\cdot\eta^2\mathrm{id}_2 \wedge *\eta^{-1}\widehat{D}\eta = \widehat{D}\eta^{\dagger} \wedge *\widehat{D}\eta.
$$

The exact same term where η appears as a *residual gauge invariant observable field.*

The conclusion we draw from this digression is that the final field content of the theory after change of variables seems to not depend on the choice of the dressing, minimal or not. In particular the status of η as a residual field owes nothing, in this case, to its identification as a bit of an auxiliary field left after extraction of the dressing field.

With the minimal choice u, the field η is seen from the beginning as a residual field. The change of variables is $(\varphi, b, a) \xrightarrow{u} (\eta, B, a) = (\eta, B_3, W^\pm, a)$. What's more, by appealing to the Lagrangian form, the further change $(\eta, B_3, W^\pm, a) \xrightarrow{L} (\eta, Z_0, W^\pm, A)$ is suggested, providing identification for more observable fields.

With the non-minimal choice $\widetilde{u}$ we have a priori no idea that η would emerge as a residual observable field. The change of variables is $(\varphi, b, a) \xrightarrow{\widetilde{u}} (\widetilde{B}, a) = (\widetilde{B}_3, W^\pm, a)$. Then by appealing to the Lagrangian form we end up with the change of variables $(\widetilde{B}_3, W^\pm, a) \xrightarrow{L} (\eta, Z^0, W^\pm, A)$.

This seems to point toward a certain robustness of the dressing field method applied to Physics when it comes to identify the true physical degrees of freedom of a theory. That is, its observable fields. In this respect, if the symbolical equation "$L \circ \widetilde{u} = L \circ u$" holds generally, this would cancel the a priori arbitrariness in the choice of the dressing field.

The second focus of arbitrariness is in the choice of *reference point* v used to decompose the auxiliary field φ and extract the dressing field u. Indeed if we were to choose another reference point in $\mathbb{C}^2$, a vector v' related to the initial one v by a constant rotation $r \in SU(2)$ so that $v' = rv$, then the new dressing field would be $u' = ur^{-1}$. It would still be a dressing under $SU(2)$-transformations, $u'^\beta = \beta^{-1}u'$. And the $U(1)$-transformation would be, $u'^\alpha = u\widehat{\alpha}r^{-1}$. In any case the dressed b field would be $B' = rBr^{-1}$ so that $G' = rGr^{-1}$, the dressed φ would be $r\widehat{\varphi} = r\eta$ and their respective gauge transformations would be well behaved. Actually the inclusion of the fermion fields in the model would provide the argument that for the left-components of the spinors to have definite $U(1)$-charges the rotation matrix should be $r = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$ or $r = \begin{pmatrix} 0 & -s^{-1} \\ s & 0 \end{pmatrix}$, with $c, s \in U(1)$, reducing the arbitrariness to two constants. See (Masson and Wallet, 2011) for this point.

In any case the new choice of reference point just corresponds to a *rigid* $SU(2)$ transformation. The structural $SU(2)$-invariance of the Lagrangian form removes this dependence, so that the final theory is independent of r. We then end up with the theory already described. Again, the formal equation "$L \circ u' = L \circ u$" holds, which means that the arbitrariness in the choice of the dressing is in this case canceled by the $SU(2)$-invariance of the Lagrangian. This is another clear sign of robustness of the dressing method applied to Physics.

Conclusion: If viewed locally, this application of the dressing field method allows to recover the standard Lagrangian of the electroweak model in the unitary gauge but *without* any gauge fixing and provides an interpretative shift that could lead to a calling into question of the 'spontaneous', that is dynamical and random, character of $SU(2)$-symmetry breaking. If viewed globally, it entails (Lemma 2) that the underlying bundle is *trivial* in the $SU(2)$ direction so that the only non-trivial geometry is the one of a $U(1)$-bundle. This supports the idea that the $SU(2)$-symmetry was an artefact of a poor choice of field variables.

We close this section about physical applications of the dressing field method by a last neat example, formally easier to treat than the above one but more subtle in its interpretation: the case of Gravitation as described by General Relativity.

General Relativity

We should stress again that the status of GR as a gauge theory is still a controversial topic. The invariance group of GR is Diff($\mathcal{M}$) which, in its active representation, is global and not a local group like a gauge group would be. But Diff($\mathcal{M}$) has a passive representation, the change of coordinates, so that locally there is a group morphism with the local gauge group $\mathcal{GL} = \Gamma(L\mathcal{M} \times_{\text{Ad}} GL)$.[11] It becomes then possible to treat gravitation as a gauge theory on the frame bundle $L\mathcal{M}$ with the linear connection Γ. See (Göckeler and Schücker, 1987) p.76 for caveats on the matter. Granted this, one may face a mere terminological contention. If by 'gauge theory' one means 'Yang-Mills' theory, then of course there's a problem. Indeed many deep differences between GR and Yang-Mills theories are to be acknowledged.[12] Or perhaps one main difference which can be stated in various ways and has mutiple important consequences.

GR teaches us that gravitation is the dynamics of space-time, the base manifold, itself. It deals with spatio-temporal, 'external', degrees of freedom, not inner ones like Yang-Mills-type gauge theories. In the most general case there exists a notion of *torsion*, a concept absent in Yang-Mills theories. On a formal viewpoint, there are more possible invariants one can use in a Lagrangian due to index contractions impossible in Yang-Mills theories. As a matter of fact, the actual Lagrangian form for GR is not of Yang-Mills type. All this issues from the existence in gravitational theories of the *soldering form*, also known as (co-)tetrad field, which realises an isomorphism between the tangent space at each point of space-time and the Minkowski space. See, (Trautman, 1979) for a defense of this viewpoint. The soldering form can be seen as the formal implementation of the 1907 "happiest thought" of Einstein's life, the *Equivalence Principle*, which sent him on the right road toward GR. It states[13] that *locally*, infinitesimally indeed, a gravitational field can be erased by free fall, that is a geodesic motion. In other words, at a point of an arbitrarily curved space-time it is always possible to find a set of local Minkowkian coordinates.[14] Exactly what does the soldering form/tetrad field. It is interesting, though not so surprising, that the key specific physical feature of the gravitational interaction with respect to the three others, the Equivalence Principle, has a mathematical counterpart that is the root of all additional richness of the theory of gravitation compared to Yang-Mills theories.

Hopefully our definition of gauge theories, stressing their key features, given at the beginning of this chapter allows us to speak meaningfully of a gauge formulation of gravitation. But on account of the above caveats, while the Yang-Mills fields are described by an Ehresmann connection ω on a principal bundle only, the gravitationnal field is described by both an Ehresmann connection, the Lorentz/spin-connection, *and* a soldering form, (ω, e). This is known as the 'first order formalism' of GR, or tetrad/Palatini formulation of GR. The concatenation of the (local versions of the) connection and of the soldering form is treated as a single gauge potential in (McDowell and Mansouri, 1977). The mathematical foundation of this move is to be found in the realm of Cartan geometry. And indeed section 1.2 presented a defense of the idea that Cartan geometry is the proper framework for gravitation for it deeply does justice to the above mentionned specificity of this interaction. See also (Wise, 2009) and (Wise, 2010) for this viewpoint. Moreover in 1.2 we've seen that Cartan geometry displays all the necessary ingredients that enter our definition of a gauge theory. So Cartan geometry is the right framework for *gauge theories of gravitation.*

Therefore we now recast the first-order/tetrad/Palatini formulation of GR in terms of the adequate Cartan geometry, and then see how to apply the dressing field method. We stress that we will systematically use a matrix formalism. By the way, from now on, Cartan geometry will be our main landscape.

The adequate Cartan geometry is based on the Klein pair (G, H) given by $G = SO \ltimes \mathbb{R}^{1,n-1}$, the Poincaré

[11] Actually in the most general case coordinate changes do not form a group for there might be singular ones.

[12] Here we consider electromagnetism as a Yang-Mills fields too, even if sometimes the term is reserved to non-abelian gauge fields. The distinction abelian/non-abelian gauge fields is not as deep, physically *and* mathematically, as the distinction Yang-Mills/Gravitational interactions.

[13] A formulation of it. They are many.

[14] This allows hopefully to treat the interaction of spinor fields with gravitation, as mentionned in 2.1.2.

group, and $H = SO$, the Lorentz group, so that the associated homogeneous space is $G/H = \mathbb{R}^{1,n-1}$. In other words this Cartan geometry is just the usual Lorentz geometry or $(1, n-1)$-pseudo-Riemannian geometry (with torsion), whose flat limit is the Minkowski plane. The infinitesimal Klein pair is $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g} = \mathfrak{so} + \mathbb{R}^{1,n-1}$ and $\mathfrak{h} = \mathfrak{so}$. The principal bundle of this Cartan geometry is $\mathcal{P}(\mathcal{M}, SO)$. The Cartan connection and its curvature are the 1-forms $\bar{\varpi} \in \Lambda^1(\mathcal{P}, \mathfrak{g})$ and $\bar{\Omega} \in \Lambda^2(\mathcal{P}, \mathfrak{g})$ respectively, which can be written in matrix form,

$$\bar{\varpi} = \omega + \bar{\theta} = \begin{pmatrix} \omega & \bar{\theta} \\ 0 & 0 \end{pmatrix}, \qquad \bar{\Omega} = \Omega + \bar{\Theta} = \begin{pmatrix} \Omega & \bar{\Theta} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d\omega + \omega \wedge \omega & d\bar{\theta} + \omega \wedge \bar{\theta} \\ 0 & 0 \end{pmatrix},$$

where $\omega \in \Lambda^1(\mathcal{P}, \mathfrak{so})$ is an Ehresmann connection on $\mathcal{P}$, $\bar{\theta} \in \Lambda^1(\mathcal{P}, \mathbb{R}^{1,n-1})$ is the soldering form, $\Omega \in \Lambda^2(\mathcal{P}, \mathfrak{so})$ is the Riemann curvature 2-form and $\bar{\Theta} \in \Lambda^2(\mathcal{P}, \mathbb{R}^{1,n-1})$ is the torsion 2-form.

Let us do Physics and pull-back all the objects on a trivializing open set $U \subset \mathcal{M}$ via a local trivializing section $\sigma : U \to \mathcal{P}$. We get,

$$\varpi = A + \theta = \begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix}, \qquad \Omega = R + \Theta = \begin{pmatrix} R & \Theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} dA + A \wedge A & d\theta + A \wedge \theta \\ 0 & 0 \end{pmatrix},$$

where $A = \sigma^* \omega \in \Lambda^1(U, \mathfrak{so})$ is the *Lorentz/spin*-connection, $\theta = \sigma^* \bar{\theta} \in \Lambda^1(U, \mathbb{R}^{1,n-1})$ is the *(co-)tetrad* 1*-form* also known as the *vielbein* 1*-form*. We can thus consider the (local) Cartan connection ϖ as the *gravitational gauge potential*. The (local) gauge group of the theory is $\mathcal{SO}_{\text{loc}} = \Gamma_{\text{loc}}(\mathcal{P} \times_{\text{Ad}} SO)$ and its action by an element $\gamma : U \to SO$, assuming the matrix form $\gamma = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}$, is given by the matrix calculation,

$$\varpi^\gamma = \gamma^{-1} \varpi \gamma + \gamma^{-1} d\gamma = \begin{pmatrix} S^{-1} \omega S + S^{-1} dS & S^{-1} \theta \\ 0 & 0 \end{pmatrix}, \qquad \Omega^\gamma = \gamma^{-1} \Omega \gamma = \begin{pmatrix} S^{-1} R S & S^{-1} \Theta \\ 0 & 0 \end{pmatrix}.$$

Given these geometrical data, the associated Lagrangian form of GR is given by,

$$L_{\text{Pal}}(A, \theta) = \frac{-1}{32\pi G} \text{Tr}\Big(R \wedge *(\theta \wedge \theta^t)\Big) = \frac{-1}{32\pi G} \text{Tr}\Big(R \wedge *(\theta \wedge \theta^T \eta)\Big), \tag{2.11}$$

with η the metric of $\mathbb{R}^{1,m-1}$ and G the gravitational constant. Given $\mathcal{S} = \int L$, variation under θ gives Einstein's equation in vacuum and variation under ω gives an equation for the torsion which in the vacuum is null (even in the presence of matter, the torsion does not propagate). Let us show explicitly the SO-gauge invariance of the Lagrangian form,

$$\begin{aligned} L^\gamma_{\text{Pal}} &\propto \text{Tr}\left(R^S \wedge *\left(\theta^S \wedge (\theta^S)^t\right)\right) = \text{Tr}\left(S^{-1} R S \wedge *\left(S^{-1}\theta \wedge (S^{-1}\theta)^t\right)\right) = \text{Tr}\left(S^{-1} R \wedge *\left(\theta \wedge (S^{-1}\theta)^T \eta\right)\right) \\ &= \text{Tr}\left(S^{-1} R \wedge *\left(\theta \wedge \theta^T \eta S\right)\right) = \text{Tr}\left(R \wedge *\left(\theta \wedge \theta^T \eta\right)\right) \\ &= \text{Tr}\Big(R \wedge *(\theta \wedge \theta^t)\Big) \propto L_{\text{Pal}}. \end{aligned}$$

There we used the cyclicity of the Trace and $(S^{-1}\theta)^T \eta = \theta^T (S^{-1})^T \eta = \theta^T \eta S$ due to $S^T \eta S = \eta$ since $S \in SO$.

We now want to find a dressing field liable to neutralize the SO-gauge symmetry of our theory. We still stick to the requirement of naturalness and hope to find it in the theory. Moreover it may not be given right away, but could be hidden in an **auxiliary field**, as was the case in the previous example. Given the gauge transformation of the Cartan connection ϖ, we see that the soldering form transforms as $\theta^S = S^{-1}\theta$ which makes it a natural candidate. Being a form on T^*U, not a map on U, and being $\mathbb{R}^{1,n-1}$-valued, which is not a group related in any way to SO, θ is at best our auxiliary field. Given the coordinates $\{x_\mu\}_{\mu=1\ldots m}$ on $U \subset \mathcal{M}$ we can develop θ on the natural basis $\{dx^\mu\}$ of T^*U,

$$\theta^a = e^a{}_\mu dx^\mu \quad \text{with,} \quad e^a{}_\mu : U \to GL. \qquad \text{In index free notation we shall write,} \quad \theta = e \cdot dx. \tag{2.12}$$

The GL-valued map e is known as the *tetrad-field* or the *vielbein*. The basis $\{dx^\mu\}$ associated to the coordinate system $\{x_\mu\}_{\mu=1\dots m}$ is the analogue of the **reference point** $v \in \mathbb{C}^2$ used to decompose the auxiliary field φ in equation (2.8). Since by definition the gauge group does not affect the basis $\{dx^\mu\}$ we have,

$$(\theta^a)^S = (e^a{}_\mu)^S dx^\mu = (S^{-1})^a{}_b e^b{}_\mu dx^\mu. \qquad \text{Or, in index free notation,} \quad \theta^S = e^S \cdot dx = S^{-1}e \cdot dx.$$

Thus we've found in the vielbein a GL-valued dressing field, u. We are thus in the case specified by Lemma 1, $GL \supset SO$, and we expect complete neutralization of the SO-gauge symmetry and complete **geometrization** of the gauge fields of the theory.

We can now use the dressing to form the gauge-invariant composite fields (2.6). Write our dressing field in matrix form $u = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$, so that the *dressed Cartan connection* is,

$$\widehat{\varpi} = u^{-1}\varpi u + u^{-1}du = \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} d \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-1}Ae + e^{-1}de & e^{-1}\theta \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} \Gamma & dx \\ 0 & 0 \end{pmatrix}.$$

We see that the dressed soldering form gives just the natural basis $\{dx^\mu\}$ of T^*U. More interestingly, the dressed Lorentz connection A is now the linear connection 1-form on $U \subset \mathcal{M}$, whose components are the Christoffel symbols. Explicitly $\Gamma = \Gamma^\mu{}_\nu = \Gamma^\mu{}_{\nu,\rho}dx^\rho = e^\mu{}_a A^a{}_b e^b{}_\nu + e^\mu{}_a de^a{}_\nu = e^\mu{}_a A^a{}_{b,\rho} e^b{}_\nu dx^\rho + e^\mu{}_a \partial_\rho e^a{}_\nu dx^\rho$. In the same way we have the dressed cuvature,

$$\widehat{\Omega} = u^{-1}\Omega u = \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R & \Theta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-1}Re & e^{-1}\Theta \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} \widehat{R} & T \\ 0 & 0 \end{pmatrix},$$

where $\widehat{R}$ and T are the Riemann curvature and torsion 2-forms respectively, written in the coordinate system $\{x^\mu\}$ on $U \subset \mathcal{M}$. We have their explicit expressions as functions of the components of the dressed Cartan connection $\widehat{\varpi}$ on account of,

$$\widehat{\Omega} = \widehat{D}\widehat{\varpi} = d\widehat{\varpi} + \widehat{\varpi} \wedge \widehat{\varpi} = \begin{pmatrix} d\Gamma & d^2x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Gamma \wedge \Gamma & \Gamma \wedge dx \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d\Gamma + \Gamma \wedge \Gamma & \Gamma \wedge dx \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{R} & T \\ 0 & 0 \end{pmatrix}.$$

We obtain the Riemann tensor and the torsion in terms of the Christoffel symbols. We see clearly that if Γ is symmetric on its lower indices, the torsion vanishes. Indeed, $T^\mu{}_{\rho\sigma} = \Gamma^\mu{}_{\sigma,\rho} dx^\rho \wedge dx^\sigma$. These are well known results, here easily obtained.

A Cartan connection always induces a metric on the base manifold $U \subset \mathcal{M}$ by $g(X,Y) = \eta\big(\theta(X), \theta(Y)\big)$, with $X, Y \in T_xU$. In component this reads $g_{\mu\nu} = e_\mu{}^a \eta_{ab} e^b{}_\nu$, or in index free notation $g = e^T\eta e$. Notice that by definition g is SO-gauge-invariant. It is easy to show that in this formalism, the metricity condition is necessarily satisfied,

$$\begin{aligned} \widehat{D}g := \nabla g = dg - \Gamma^T g - g\Gamma &= d(e^T\eta e) - \left(e^T A^T (e^{-1})^T + de^T (e^{-1})^T\right) e^T \eta e - e^T \eta e \left(e^{-1}Ae + e^{-1}de\right) \\ &= de^T \cdot \eta e + e^T \eta de - e^T A^T \eta e - de^T \cdot \eta e - e^T \eta A e - e^T \eta de \\ &= -e^T \left(A^T \eta + \eta A\right) e = 0, \end{aligned} \tag{2.13}$$

where we use the fact that the 1-form A is $\mathfrak{so}$-valued. Therefore, if $T = 0$, Γ is the Levi-Civita connection and can be written in term of the metric g.

The SO-invariant fields g, $\widehat{\varpi} \simeq (\Gamma, dx)$ and $\widehat{\Omega} = (\widehat{R}, T)$ belong to the *natural geometry* of the base manifold $\mathcal{M}$.[15] From the point of view of gauge theory, they are closer to observability. Closer but not observable yet. Indeed there remains the subtle question of the coordinate invariance, or of the Diff($\mathcal{M}$)-invariance on $\mathcal{M}$, that needs to be addressed before asserting the physical observability of a field. A well known concern in GR. We shall briefly touch the question, and stress the difference with the electroweak sector of the Standard Model, when we will discuss the change of reference point in the decomposition (2.12).

[15] See our discussion on natural geometry at the end of section 2.1.1.

It is now time to show how the classic calculation that allows to pass from the Palatini/gauge Lagrangian form (2.11) to Einstein's original formulation, is a special case of the change of variables within the dressing field method.

$$
\begin{aligned}
L_{\text{Pal}}(A,\theta) = \frac{-1}{32\pi G}\text{Tr}\Big(R\wedge *(\theta\wedge\theta^t)\Big) = \frac{-1}{32\pi G}\text{Tr}\Big(R\wedge *(\theta\wedge\theta^T\eta)\Big) = \frac{-1}{32\pi G}\text{Tr}\Big(e\widehat{R}e^{-1}\wedge *(e\cdot dx\wedge dx^T\cdot e^T\eta)\Big)\\
= \frac{-1}{32\pi G}\text{Tr}\Big(e\widehat{R}e^{-1}\,e\,e^T\eta\Big)\wedge *(dx\wedge dx^T) = \frac{-1}{32\pi G}\text{Tr}\Big(\widehat{R}\,\delta\cdot g\Big)\wedge *(dx\wedge dx^T)\\
= \frac{-1}{32\pi G}\text{Tr}\Big(\widehat{R}g\Big)\wedge *(dx\wedge dx^T) =: L_{\text{EH}}(\Gamma,g),
\end{aligned}
$$

where $\delta = e^{-1}e \rightarrow \delta^\mu{}_\nu = e^\mu{}_a e^a{}_\nu$, and we used $g = e^T\eta e$. The last equation defines the Einstein-Hilbert Lagrangian form, depending on Γ and g. It is easy to show that it gives indeed the usual Einstein-Hilbert Lagrangian *function*,

$$
\begin{aligned}
L_{\text{EH}}(\Gamma,g) &= \frac{-1}{32\pi G}\text{Tr}\Big(\widehat{R}g\Big)\wedge *(dx\wedge dx^T) = \frac{-1}{32\pi G}\,R^\lambda{}_{\alpha,\rho\sigma}g_{\lambda\beta}\,dx^\rho\wedge dx^\sigma\wedge *(dx^\alpha\wedge dx^\beta),\\
&= \frac{-1}{32\pi G}\,R^\lambda{}_{\alpha,\rho\sigma}g_{\lambda\beta}\,\frac{\sqrt{|g|}}{(m-2)!}\epsilon^{\alpha\beta}{}_{\nu_1\cdots\nu_{m-2}}\epsilon^{\rho\sigma\nu_1\cdots\nu_{m-2}}dx^1dx^2\cdots dx^m\\
&= \frac{-1}{32\pi G}\,R^{\beta\alpha}{}_{\rho\sigma}\,\frac{\sqrt{|g|}}{(m-2)!}\,\frac{(m-2)!}{(m-m)!}\,\delta^{\rho\sigma}_{\alpha\beta}d^mx = \frac{-1}{32\pi G}\,\sqrt{|g|}d^mx\,R^{\beta\alpha}{}_{\rho\sigma}\,\left(\delta^\rho_\alpha\delta^\sigma_\beta-\delta^\rho_\beta\delta^\sigma_\alpha\right)\\
&= \frac{-1}{32\pi G}\,\sqrt{|g|}d^mx\,\left(R^{\sigma\rho}{}_{\rho\sigma}-R^{\rho\sigma}{}_{\rho\sigma}\right) = \frac{1}{16\pi G}\,\sqrt{|g|}d^mx\,R_{\text{icc}} = \mathcal{L}_{EH}\,d^mx,
\end{aligned}
$$

where R_{icc} is the Ricci scalar curvature. This is standard calculation, but several points are worth discussing.

Discussion: First of all, the vielbein as a dressing field is not in the gauge group $\mathcal{SO}_{\text{loc}}$ of the theory. So, this is not new to us, the invariant composite field $\widehat{\varpi}$ is not a gauge transformation of the Cartan connection ϖ. In particular this means that, contrary to the usual claim, found e.g in (Bertlmann, 1996) p.490, Γ is not a gauge transformed of the Lorentz connection A. We should say that it is not a SO-gauge transformation to be precise. Indeed Γ is an SO-invariant $\mathfrak{gl}$-valued 1-form on $\mathcal{M}$, clearly it does not belong to the space of connections $\mathcal{A}$ of the theory. If one considers the gauge symmetry of GR as the coordinate changes, one may consider that Γ and A as gauge related. This would be accurate if one was willing to study GR as a gauge theory on $L\mathcal{M}$ with gauge group $\mathcal{GL}$. But once $L\mathcal{M}$ is reduced to a SO-subbundle, through the Bundle Reduction Theorem (see section 1.1.1), there's no way to recover Γ from A by a gauge transformation. To do so, rigorously, one needs the dressing field method. Notice that, while in the case of the electroweak sector of the Standard Model the Bundle Reduction Theorem and the dressing field method coincide, in the case of GR they are reciprocal constructions.

From this viewpoint therefore, we go from L_{Pal} displaying SO-gauge symmetry to L_{EH} where the SO-gauge symmetry is neutralized, by a mere change of field variables from the gauge potential ϖ, i.e (A,θ), to the invariant field $\widehat{\varpi}$, i.e (Γ, dx). And not, rigorously speaking, by gauge transformation.

Second, notice some resemblances between the metric field g and the Higgs field η of the Standard Model as previously treated. The metric could be seen as a 'dressing' of the flat metric η_{ab} constructed out of θ, our auxiliary field. Moreover its a priori unexpected appearance[16] as a final invariant field of the theory is dictated by the Lagrangian form, much in the same way as η formerly was. So that g is truly an invariant **residual field** here. These facts show that, in our scheme, g is the gravitational formal analogue of the field η in the Standard Model. We could thus say that the metric g is the **Higgs field of gravitation** in this precise sense. This conclusion was already advocated e.g in (Trautman, 1979) and (Sardanashvily, 2011) on the basis of the Bundle Reduction Theorem. See our discussion on this matter at the end of section 1.1.1. But the same conclusion is reached by different formal means which carry different interpretations.

[16]That is, unexpected if we had known at first only the gauge formulation of GR given by the Lagrangian form (2.11).

From the viewpoint of the Bundle Reduction Theorem the $\mathbb{C}^2$-scalar field φ of the SM is a map that splits as (η, u) where u performs the bundle reduction and the $\mathbb{R}^+$-scalar field η is the invariant map liable to be interpreted as an observable Higgs field. From the viewpoint of the Bundle Reduction Theorem still, the metric g splits as $(\eta_{ab}, [e])$ where $[e]$, a SO-class of vielbeins, performs the bundle reduction and η_{ab} is an invariant flat metric at each point. So with the Bundle Reduction Theorem, if φ is called the Higgs field then indeed g achieves a similar role. Both are in the business of providing the mean to reduce a bundle, that is a gauge-symmetry. If on the other hand, as is physically more sensible, η is called the Higgs field, then g cannot be given the same name except for a loose use of the terminology.

From our viewpoint, φ is an auxiliary field from which is extracted a dressing field, u or $\widetilde{u}$, and η appears in the Lagragian as a invariant residual field. In the same way, θ is an auxiliary field from which is extracted the dressing field, e, and g appears in the Lagrangian as an invariant residual field. In our scheme both η and g have the very same formal status, with the similar sensible physical property of being gauge-invariant so liable of observability.[17] So in our scheme it is sound to say that the metric g is a Higgs field for gravitation.

Remember moreover that our denomination for Higgs fields does not carry the meaning of 'symmetry breaker'. In our treatment of the electroweak sector of the SM, η is just the 'mass giver' for the gauge-invariant fields. If g is also a Higgs field, can we expect it to give a mass to the SO-invariant field $\widehat{\varpi} \sim \Gamma$? The answer is, of course, no. And the reason is connected to the question of the arbitrariness in the choice of the dressing field.

In the decomposition (2.12) we used the natural basis $\{dx^\mu\}$ induced by the coordinate system $\{x^\mu\}_{\mu=1\cdots m}$ as a reference point to extract the dressing $e = e^a{}_\mu$. What if we had chosen another reference point, that is, what if we had used another coordinate system? An equivalent way to formulate the question is to ask what would happen at the overlap of two patches of coordinates, $\{x^\mu\}_{\mu=1\cdots m}$ and $\{y^\mu\}_{\mu=1\cdots m}$, coexisting in our trivializing open set $U \subset \mathcal{M}$. Or yet, if the trivializing open sets $\{U_i\}$ are also coordinate charts of $\mathcal{M}$, what happens on $U_i \cap U_j$?

We've seen in 2.2.1 that the composite fields (2.6) do not see any gauge structure but also that if the dressing u carries base manifold indices, they would undergo transformations by passing from a coordinate chart to another. The present situation is an obvious example of this. Indeed suppose we choose the natural basis $\{dy^\mu\}$ associated to $\{y^\mu\}_{\mu=1\cdots m}$ as another reference point for the decomposition of θ, we have

$$\theta^a = e^a{}_\mu dx^\mu = e'^a{}_\nu dy^\nu, \quad \text{where,} \quad e'^a{}_\nu = e^a{}_\mu \frac{\partial x^\mu}{\partial y^\nu} = e^a{}_\mu G^\mu{}_\nu,$$

$$\text{or in coordinate free notations,} \quad \theta = e\, dx = e'\, dy \quad \rightarrow \quad e' = eG,$$

$$\text{and still,} \quad \theta^S = S^{-1}\theta \quad \rightarrow \quad e'^S = S^{-1}e'.$$

So e' is another dressing field, writting it in matrix form as $u' = \begin{pmatrix} e' & 0 \\ 0 & 1 \end{pmatrix} = uG = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} G & 0 \\ 0 & 1 \end{pmatrix}$ we have the dressed Cartan connection,

$$\begin{aligned}
\widehat{\varpi}' &= u'^{-1}\varpi u' + u'^{-1}du' = \begin{pmatrix} e'^{-1}Ae + e'^{-1}de & e'^{-1}\theta \\ 0 & 0 \end{pmatrix}, \\
&= G^{-1}u^{-1} \cdot \varpi \cdot uG + G^{-1}u^{-1}d(uG), \\
&= G^{-1}u^{-1} \cdot \varpi \cdot uG + G^{-1}(u^{-1}du)G + G^{-1}dG, \\
&= G^{-1}\widehat{\varpi}G + G^{-1}dG = \begin{pmatrix} G^{-1}\Gamma G + G^{-1}dG & G^{-1}dx \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} \Gamma' & dy \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Similarly we have $\widehat{\Omega}' = u'^{-1}\Omega u = G^{-1}\widehat{\Omega}G \rightarrow \begin{pmatrix} \widehat{R}' & T' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} G^{-1}\widehat{R}G & G^{-1}T \\ 0 & 0 \end{pmatrix}$, and $g' = e'^T\eta e' = G^T g G$. These are the well known transformations of the metric, of the Christoffel symbols, of the Riemann curvature tensor and of the torsion tensor under general changes of coordinates. Of course the Einstein-Hilbert Lagrangian form, L_{EH}, is invariant under coordinate changes. A requirement that warrants the covariance of Einstein's

[17] Granted for η, and with the mentionned caveat about coordinate transformations for g. See below for further remarks.

equations, thus implements the 'generalized relativity principle'. By virtue of its transformation law, it is clear that a mass term for the SO-invariant field Γ in the Lagrangian form L_{EH}, which would read $m^2\text{Tr}(\Gamma \wedge *\Gamma)$, is still forbidden. The Higgs metric field g can do nothing about it, hence the masslessness of the gravitational field even after symmetry reduction.

Notice that with the GL-valued dressing e, the dressed auxiliary field is $e^{-1}\theta = dx$, which is nothing but the reference point chosen to extract the dressing in the first place. This situation is in every respect similar to what we've done in the SM with the $SU(2) \times \mathbb{R}^+$-valued dressing $\widetilde{u}$ used to form the dressed auxiliary field $\widehat{\varphi} = \widetilde{u}^{-1}\varphi = v$, where $v = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \in \mathbb{C}^2$ was the reference point used to extract the dressing $\widetilde{u}$ in the first place. But in the SM we had first found a *minimal* $SU(2)$-valued dressing field that made apparent the invariant residual field from the start, even before finding it in the Lagrangian. This raises the question of the possibility to find a *minimal* SO-valued dressing in the present case. The answer is yes, but the final construction is badly behaved under coordinate changes. This is an interesting exercice to undertake and the reader can refer to the short appendix B for a sketch of it, but in the end it is more tractable and elegant to work with the usual non-minimal dressing e.

Conclusion: We've seen that the classic calculation that allows to pass from the gauge formulation of GR to its 'metric' formulation can be seen as an instance of the dressing field method. Comparison with the electroweak sector of the Standard Model allows to draw the conclusion that the metric g, as a residual field, is the gravitational analogue of the Higgs fields. Notice however that while in the Standard Model the dressing field was found from an auxiliary field in the matter sector of the theory, in GR the dressing is found as a part of the gauge potential itself, namely the Cartan connection ϖ. This will be a constant feature of the subsequent examples of applications of the method to Cartan geometry.

Find in the tables below a summary of the main treatment of the previous two examples and providing a count of the physical degrees of freedom in each case.

Initial gauge theory	EW sector of the SM	General Relativity
Structure group	$U(1) \times SU(2)$	$SO(1, m-1)$
dimension (1)	$1+3$	$\frac{m(m-1)}{2}$
Connection (principal)	$a+b$	A
dimension (2)	$m+3m$	$\frac{m^2(m-1)}{2}$
Auxiliary field	φ	θ
dimension (3)	4	m^2
Reference point	$v := \binom{0}{1}$	$dx := \{dx^\mu\}$
Dressing field	$u \in SU(2)$ such that, $\varphi = \eta u v$	$e \in GL$ such that, $\theta = e\, dx$
Degrees of freedom of the theory, (2) + (3) − (1)	$4m$	$\frac{m(m^2+1)}{2}$

Table 2.1: Initial data of the two main examples of gauge theories treated, before applying the dressing field method.

Final theory	EW sector of the SM	General Relativity
Residual group	$U(1)$	$\{\mathrm{id}\}$
dimension (1)	1	0
Gauge invariant fields/ generalized Dirac variables	$a + B \to (A, Z^0, W^\pm)$	Γ such that, $\nabla g = 0$
dimension (2)	$m+3m$	$\frac{m^2(m-1)}{2}$
Residual field	η	g
dimension (3)	1	$\frac{m(m+1)}{2}$
Degrees of freedom of the theory, (2) + (3) − (1)	$4m$	$\frac{m(m^2+1)}{2}$

Table 2.2: Final data of the two main examples of gauge theories treated, after applying the dressing field method.

2.4 Application to geometry

In the first part of this section the dressing field is generalized to structures of higher order. The general conditions for performing successively several dressings are derived. The second part of the section deals with the worked out example of the 2nd-order conformal structure.

2.4.1 The dressing and higher-order G-structures

Multiple dressing fields

Suppose the structure group of our typical pincipal bundle $\mathcal{P}(\mathcal{M}, H)$ can be written as a product of subgroups $H = K_0 K_1 \cdots K_n = \prod_{i=0}^{n} K_i$. An element of the gauge group $\mathcal{H} = \prod_{i=0}^{n} \mathcal{K}_i$ reads $\gamma = \gamma_0 \gamma_1 \cdots \gamma_n = \prod_{i=0}^{n} \gamma_i$, and the connection form $\omega \in \Lambda^1(\mathcal{P}, \mathfrak{h})$ satisfies

$$
\begin{aligned}
\omega^{\gamma_i} &= \gamma_i^{-1} \omega \gamma_i + \gamma_i^{-1} d\gamma_i, \quad \text{for } \gamma_i \in \mathcal{K}_i, \text{ and } i \in [0, \cdots n], \\
\omega^{\gamma} &= \gamma^{-1} \omega \gamma + \gamma^{-1} d\gamma, \quad \text{for } \gamma \in \mathcal{H}.
\end{aligned}
\tag{2.14}
$$

Then $\mathcal{P}$ is a 'multi-story' bundle, so to speak. It has $n+1$ stories, and it is a 'multiply fibered manifold'. Indeed, beside being a bundle over $\mathcal{M}$ with structure group $H = K_0 \cdots K_n =: \bar{K}_n$, so that we note $\mathcal{P} = \mathcal{P}_n$, it is also a bundle over $\mathcal{P}_{n-1}$ with structure group K_n. More generally,

- $\mathcal{P}_n$ is a bundle over $\mathcal{P}_{n-i}$ with group $K_{n-i+1} K_{n-i+2} \cdots K_n = \prod_{j=1}^{i} K_{n-i+j}$,
- every $\mathcal{P}_{n-i}$ is a bundle over $\mathcal{P}_{n-i-1}$ with group K_{n-i},
- every $\mathcal{P}_{n-i}$ is a bundle over $\mathcal{M}$ with structure group $\bar{K}_{n-i} := K_0 \cdots K_{n-i} = \prod_{j=0}^{n-i} K_j$.

Moreover ω is a $\bar{K}_{n-i}$-connection on any $\mathcal{P}_{n-i}$ by restriction.

The question now arises: is it possible to find a first dressing to reduce $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$, a second one to reduce $\mathcal{P}_{n-1}$ to $\mathcal{P}_{n-2}$... and so on down to $\mathcal{M}$? The answer is yes, but the various dressing fields have to satisfy quite restrictive conditions. Before giving this set of conditions in full generality, let us appreciate how they arise by working out the first three steps.

Beginning of the recursive scheme Suppose we have a dressing field $u_n : \mathcal{P}_n \to K_n$ which, by definition, satisfies $u_n^{\gamma_n} = \gamma_n^{-1} u_n$.[18] By Lemma 2 we know that it realises $\mathcal{P}_n / K_n \sim \mathcal{P}_{n-1}$ as a subbundle of $\mathcal{P}_n$, so that $\mathcal{P}_n \simeq \mathcal{P}_{n-1} \times K_n$. The dressed connection $\omega^{u_n} =: \omega_n$ is K_n-invariant and K_n-horizontal so that it passes down to a well-defined 1-form on $\mathcal{P}_{n-1}$.
Now, given (2.14), if one wants ω_n to transform as a K_{n-1}-connection so that a new dressing operation would make sense, one finds that the first dressing must satisfy

$$
u_n^{\gamma_{n-1}} = \gamma_{n-1}^{-1} u_n \gamma_{n-1}.
$$

That is, the first dressing has gauge-like transformations under K_{n-1}. It is easily checked that one has,

$$
(\omega_n)^{\gamma_{n-1}} = \gamma_{n-1}^{-1} \omega_n \gamma_{n-1} + \gamma_{n-1} d\gamma_{n-1}^{-1}.
\tag{2.15}
$$

Suppose we have a second dressing field $u_{n-1} : \mathcal{P}_{n-1} \to K_{n-1}$ which by definition satisfies $u_{n-1}^{\gamma_{n-1}} = \gamma_{n-1}^{-1} u_{n-1}$. By Lemma 2 we know it realizes $\mathcal{P}_{n-1}/K_{n-1} \sim \mathcal{P}_{n-2}$ as a subbundle of $\mathcal{P}_{n-1}$, so that $\mathcal{P}_{n-1} \simeq \mathcal{P}_{n-2} \times K_{n-1}$ and $\mathcal{P}_n \simeq \mathcal{P}_{n-2} \times K_{n-1} K_n$. By virtue of (2.15) we can dress ω_n and form $\omega_n^{u_{n-1}} = (\omega^{u_n})^{u_{n-1}} = \omega^{u_n u_{n-1}} =: \omega_{n-1}$ which is K_{n-1}-invariant and K_{n-1}-horizontal. But in order to pass it down to a well-defined

[18] This map should be noted $\bar{u}_n$ and the notation u_n should be reserved for its pull-back on $U \subset \mathcal{M}$. By convenience I keep u_n for the global map(s) in all this section. By convenience still, I call it 'dressing field(s)' even if this terminology is strictly speaking reserved for the pull-back.

1-form on $\mathcal{P}_{n-2}$, it must also be K_n-invariant. Since ω_n is already so, we only need the second dressing to satisfy

$$u_{n-1}^{\gamma_n} = u_{n-1}.$$

So the second dressing should be invariant under the above-order gauge transformations. This condtition also insures that $\omega_{n-1}(X_n^v) = 0$ so that ω_{n-1} is K_n-horizontal and definitely passes down to $\mathcal{P}_{n-2}$. Now given (2.14), if one wants ω_{n-1} to transform as a K_{n-2}-connection so that a new dressing operation would make sense, one finds that the double dressing $u_n u_{n-1}$ must satisfy

$$(u_n u_{n-1})^{\gamma_{n-2}} = \gamma_{n-2}^{-1}(u_n u_{n-1})\gamma_{n-2},$$

or equivalently each dressing must satisfy,

$$u_n^{\gamma_{n-2}} = \gamma_{n-2}^{-1} u_n \gamma_{n-2} \quad \text{and} \quad u_{n-1}^{\gamma_{n-2}} = \gamma_{n-2}^{-1} u_{n-1} \gamma_{n-2}.$$

That is, each dressing has gauge-like transformations under K_{n-2}. It is easily checked that,

$$(\omega_{n-1})^{\gamma_{n-2}} = \gamma_{n-2}^{-1}\omega_{n-1}\gamma_{n-2} + \gamma_{n-2} d\gamma_{n-2}^{-1}. \tag{2.16}$$

Suppose we have a third dressing field $u_{n-2} : \mathcal{P}_{n-2} \to K_{n-2}$ which by definition satisfies $u_{n-2}^{\gamma_{n-2}} = \gamma_{n-2}^{-1} u_{n-2}$. By Lemma 2 we know it realizes $\mathcal{P}_{n-2}/K_{n-2} \sim \mathcal{P}_{n-3}$ as a subbundle of $\mathcal{P}_{n-2}$, so that $\mathcal{P}_{n-2} \simeq \mathcal{P}_{n-3} \times K_{n-2}$ and $\mathcal{P}_n \simeq \mathcal{P}_{n-3} \times K_{n-2}K_{n-1}K_n$. By virtue of (2.16) we can dress ω_{n-1} and form $\omega_{n-1}^{u_{n-2}} = (\omega^{u_n u_{n-1}})^{u_{n-2}} = \omega^{u_n u_{n-1} u_{n-2}} =: \omega_{n-2}$ which is K_{n-2}-invariant and K_{n-2}-horizontal. But in order to pass it down to a well-defined 1-form on $\mathcal{P}_{n-3}$, it must also be K_n and K_{n-1}-invariant. Since ω_{n-1} is already so, we only need the third dressing to satisfy

$$u_{n-2}^{\gamma_n} = u_{n-2}, \quad \text{and} \quad u_{n-2}^{\gamma_{n-1}} = u_{n-2}.$$

These conditions secure the K_n- and K_{n-1}-horizontality of ω_{n-2}. Then again, one could require that ω_{n-2} transforms as a K_{n-3} connection and one would find, on account of (2.14), gauge-like transformations under K_{n-3} for u_n, u_{n-1} and u_{n-2}. This would suggest a new dressing, and so on. The scheme could be recursively repeated all the way down to $\mathcal{M}$ with $n+1$ dressings.

The compatibility conditions The above steps are enough for us to draw the compatibility conditions on the dressings in their full generality. We collect the various transformation laws we've found,

$$u_n^{\gamma_n} = \gamma_n^{-1} u_n \tag{2.17}$$
$$u_n^{\gamma_{n-1}} = \gamma_{n-1}^{-1} u_n \gamma_{n-1} \tag{2.18}$$
$$u_n^{\gamma_{n-2}} = \gamma_{n-2}^{-1} u_n \gamma_{n-2} \tag{2.19}$$
$$u_{n-1}^{\gamma_n} = u_{n-1} \tag{2.20}$$
$$u_{n-1}^{\gamma_{n-1}} = \gamma_{n-1}^{-1} u_{n-1} \tag{2.21}$$
$$u_{n-1}^{\gamma_{n-2}} = \gamma_{n-2}^{-1} u_{n-1} \gamma_{n-2} \tag{2.22}$$
$$u_{n-2}^{\gamma_n} = u_{n-2} \tag{2.23}$$
$$u_{n-2}^{\gamma_{n-1}} = u_{n-2} \tag{2.24}$$
$$u_{n-2}^{\gamma_{n-2}} = \gamma_{n-2}^{-1} u_{n-2} \tag{2.25}$$

We clearly see the pattern. Equations (2.17), (2.21) and (2.25) express the defining transformation properties of the dressings at each order,

$$u_{n-i}^{\gamma_{n-i}} = \gamma_{n-i}^{-1} u_{n-i}, \tag{2.26}$$

This must be so in order to be able to reduce $\mathcal{P}_{n-i}$ to $\mathcal{P}_{n-i-1}$. Equations (2.18), (2.19) and (2.22) express the fact that each dressing should have gauge-like transformations under the lower order groups,

$$u_{n-i}^{\gamma_{n-j}} = \gamma_{n-j}^{-1} u_{n-i} \gamma_{n-j}, \quad \text{for } j > i. \tag{2.27}$$

This must be so in order for the K_{n-i}-dressed form ω_{n-i} to behave as true (lower order) K_{n-j}-gauge fields, so that a new dressing operation makes sense. Equations (2.20), (2.23) and (2.24) express the fact that each dressing should be invariant under the higher order groups,

$$u_{n-i}^{\gamma_{n-j}} = u_{n-i}, \quad \text{for } j < i. \tag{2.28}$$

This must be so for each K_{n-i}-dressed field to be invariant under (higher order) K_{n-j}-gauge transformations and to be K_{n-j}-horizontal so as to pass down to well defined forms on $\mathcal{P}_{n-i-1}$.

These constrains conspire to make a succession of dressing fields of any length a dressing field by itself under the corresponding succession of groups and, of course, under any of these groups alone. We show this for our three first steps before giving the general demonstration. Consider the succession $u_n u_{n-1}$. We have,

$$\begin{aligned}(u_n u_{n-1})^{\gamma_n} &= u_n^{\gamma_n} \cdot u_{n-1}^{\gamma_n} = \gamma_n^{-1} u_n \cdot u_{n-1} = \gamma_n^{-1}(u_n u_{n-1}), \qquad \text{by (2.17) and (2.20).}\\ (u_n u_{n-1})^{\gamma_{n-1}} &= u_n^{\gamma_{n-1}} u_{n-1}^{\gamma_{n-1}} = \gamma_{n-1}^{-1} u_n \gamma_{n-1} \cdot \gamma_{n-1}^{-1} u_{n-1} = \gamma_{n-1}^{-1}(u_n u_{n-1}), \qquad \text{by (2.18) and (2.21).}\end{aligned}$$

And finally,

$$\begin{aligned}(u_n u_{n-1})^{\gamma_{n-1}\gamma_n} &= \left((u_n u_{n-1})^{\gamma_{n-1}}\right)^{\gamma_n} = \left(\gamma_{n-1}^{-1}(u_n u_{n-1})\right)^{\gamma_n} = (\gamma_{n-1}^{\gamma_n})^{-1}(u_n u_{n-1})^{\gamma_n} = \gamma_n^{-1}\gamma_{n-1}^{-1}\gamma_n \cdot \gamma_n^{-1}(u_n u_{n-1})\\ &= \gamma_n^{-1}\gamma_{n-1}^{-1}(u_n u_{n-1}).\end{aligned}$$

Consider now the succession $u_n u_{n-1} u_{n-2}$. We have,

$$\begin{aligned}(u_n u_{n-1} u_{n-2})^{\gamma_n} &= u_n^{\gamma_n} \cdot u_{n-1}^{\gamma_n} \cdot u_{n-2}^{\gamma_n} = \gamma_n^{-1}(u_n u_{n-1} u_{n-2}), \qquad \text{by (2.17), (2.20) and (2.23).}\\ (u_n u_{n-1} u_{n-2})^{\gamma_{n-1}} &= u_n^{\gamma_{n-1}} \cdot u_{n-1}^{\gamma_{n-1}} \cdot u_{n-2}^{\gamma_{n-1}} = \gamma_{n-1}^{-1} u_n \gamma_{n-1} \cdot \gamma_{n-1}^{-1} u_{n-1} \cdot u_{n-2}\\ &= \gamma_{n-1}^{-1}(u_n u_{n-1} u_{n-2}), \qquad \text{by (2.18), (2.21) and (2.24).}\\ (u_n u_{n-1} u_{n-2})^{\gamma_{n-2}} &= u_n^{\gamma_{n-2}} \cdot u_{n-1}^{\gamma_{n-2}} \cdot u_{n-2}^{\gamma_{n-2}} = \gamma_{n-2}^{-1} u_n \gamma_{n-2} \cdot \gamma_{n-2}^{-1} u_{n-1} \gamma_{n-2} \cdot \gamma_{n-2}^{-1} u_{n-2}\\ &= \gamma_{n-2}^{-1}(u_n u_{n-1} u_{n-2}), \qquad \text{by (2.19), (2.22) and (2.25).}\end{aligned}$$

And finally,

$$\begin{aligned}(u_n u_{n-1} u_{n-2})^{\gamma_{n-2}\gamma_{n-1}\gamma_n} &= \left(\left((u_n u_{n-1} u_{n-2})^{\gamma_{n-2}}\right)^{\gamma_{n-1}}\right)^{\gamma_n} = \left(\left(\gamma_{n-2}^{-1}(u_n u_{n-1} u_{n-2})\right)^{\gamma_{n-1}}\right)^{\gamma_n}\\ &= \left((\gamma_{n-2}^{\gamma_{n-1}})^{-1}(u_n u_{n-1} u_{n-2})^{\gamma_{n-1}}\right)^{\gamma_n} = \left(\gamma_{n-1}^{-1}\gamma_{n-2}^{-1}\gamma_{n-1} \cdot \gamma_{n-1}^{-1}(u_n u_{n-1} u_{n-2})\right)^{\gamma_n}\\ &= (\gamma_{n-1}^{\gamma_n})^{-1}(\gamma_{n-2}^{\gamma_n})^{-1}(u_n u_{n-1} u_{n-2})^{\gamma_n} = \gamma_n^{-1}\gamma_{n-1}^{-1}\gamma_n \cdot \gamma_n^{-1}\gamma_{n-2}^{-1}\gamma_n \cdot \gamma_n^{-1}(u_n u_{n-1} u_{n-2})\\ &= \gamma_n^{-1}\gamma_{n-1}^{-1}\gamma_{n-2}^{-1}(u_n u_{n-1} u_{n-2}).\end{aligned}$$

These relations mean that we could reduce $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$ or to $\mathcal{P}_{n-2}$ in a single step thanks to the dressing fields $u_n u_{n-1}$ and $u_n u_{n-1} u_{n-2}$ respectively.

Now for a string of any length $\prod_{k=0}^{i} u_{n-k}$, we have under transformation γ_{n-j}, $j \in [0 \cdots i]$,

$$\begin{aligned}\left(\prod_{k=0}^{i} u_{n-k}\right)^{\gamma_{n-j}} &= \left(\prod_{k=0}^{j-1} u_{n-k}\right)^{\gamma_{n-j}} \cdot u_{n-j}^{\gamma_{n-j}} \cdot \left(\prod_{k=j+1}^{i} u_{n-k}\right)^{\gamma_{n-j}}\\ &= \prod_{k=0}^{j-1}\left(\gamma_{n-j}^{-1} u_{n-k}\gamma_{n-j}\right) \cdot \gamma_{n-j}^{-1} u_{n-j} \cdot \prod_{k=j+1}^{i} u_{n-k} \qquad \text{by (2.27), (2.26) and (2.28),}\\ &= \gamma_{n-j}^{-1}\left(\prod_{k=0}^{j-1} u_{n-k}\right)\gamma_{n-j} \cdot \gamma_{n-j}^{-1} u_{n-j} \cdot \prod_{k=j+1}^{i} u_{n-k}\\ \left(\prod_{k=0}^{i} u_{n-k}\right)^{\gamma_{n-j}} &= \gamma_{n-j}^{-1}\left(\prod_{k=0}^{i} u_{n-k}\right).\end{aligned} \tag{2.29}$$

Equation (2.29) indicates that the string of dressing fields $\prod_{k=0}^{i} u_{n-k}$ is a dressing field on its own right under any subgroup K_{n-j}, $j \in [0 \cdots i]$. We finally show,

$$
\begin{aligned}
\Big(\prod_{k=0}^{i} u_{n-k}\Big)^{\prod_{j=i}^{0}\gamma_{n-j}} &= \Big(\big(\prod_{k=0}^{i} u_{n-k}\big)^{\gamma_{n-i}}\Big)^{\prod_{j=i-1}^{0}\gamma_{n-j}} = \Big(\gamma_{n-i}^{-1}\big(\prod_{k=0}^{i} u_{n-k}\big)\Big)^{\prod_{j=i-1}^{0}\gamma_{n-j}} \\
&= \Big(\big(\gamma_{n-i}\big)^{\prod_{j=i-1}^{0}\gamma_{n-j}}\Big)^{-1}\cdot\Big(\prod_{k=0}^{i} u_{n-k}\Big)^{\prod_{j=i-1}^{0}\gamma_{n-j}} = \cdots = \\
&= \big(\prod_{j=i-1}^{0}\gamma_{n-j}\big)^{-1}\gamma_{n-i}^{-1}\,\big(\prod_{j=i-1}^{0}\gamma_{n-j}\big)\cdot\big(\prod_{j=i-1}^{0}\gamma_{n-j}\big)^{-1}\Big(\prod_{k=0}^{i} u_{n-k}\Big)
\end{aligned}
$$

$$
\Big(\prod_{k=0}^{i} u_{n-k}\Big)^{\prod_{j=i}^{0}\gamma_{n-j}} = \big(\prod_{j=i}^{0}\gamma_{n-j}\big)^{-1}\Big(\prod_{k=0}^{i} u_{n-k}\Big) = \prod_{j=0}^{i}\gamma_{n-j}^{-1}\,\Big(\prod_{k=0}^{i} u_{n-k}\Big). \tag{2.30}
$$

Equation (2.30) indicates that the string of dressing fields $\prod_{k=0}^{i} u_{n-k}$ is a dressing field on its own right under the string of subgroups $\prod_{j=i}^{0} K_{n-j}$, so that it can be used to reduce in a single step $\mathcal{P}_n$ to $\mathcal{P}_{n-i}$.

If the compatibility conditions (2.26), (2.27) and (2.28) hold to any order then one can form the dressing field $\prod_{k=0}^{n} u_{n-k}$ and reduce the multistage bundle $\mathcal{P}_n$ down to $\mathcal{M}$, thereby geometrizing all the gauge fields to fields of the natural geometry of $\mathcal{M}$ as in Lemma 1.

If, for any reason, the compatibility conditions cease to hold at a given order i, then the dressing $\prod_{k=0}^{i} u_{n-k}$ realises $\mathcal{P}_n/\prod_{j=i}^{0} K_{n-j} \sim \mathcal{P}_{n-i-1}$ as a subbundle of $\mathcal{P}_n$ so that $\mathcal{P}_n \simeq \mathcal{P}_{n-i-1} \times \prod_{j=i}^{0} K_{n-j}$. As in Lemma 2, a residual gauge freedom under $\bar{K}_{n-i-1} = \prod_{j=0}^{n-i-1} K_j$ is expected for dressed fields on $\mathcal{P}_{n-i-1}$ depending on the usual gauge transformations of the gauge fields and of the gauge transformations of $\prod_{k=0}^{i} u_{n-k}$ under $\bar{K}_{n-i-1}$.

Higher order G-structures

It may seem that the above conditions are too restrictive to be realized in any interesting context. Yet, all this is not an empty generalization. Indeed the realm of the jet-bundle formalism, in particular of the higher-order G-structures, provides an example where this construction may be realized. We give a brief overview of the jet-formalism in order to be able to define the r^{th}-frame bundle of a manifold $\mathcal{M}$ and finally r^{th}-order G-structures. The next paragraphs are based mainly on the neat exposure by (Kobayashi, 1961), but also on (Kobayashi, 1972) and (Ogiue, 1967).

Jet formalism Kobayashi attributes the creation of the formalism to Charles Ehresmann (who was a student of Élie Cartan, the same who generalized his master's notion of connection), and speaks in (Kobayashi, 1961) of the 'theory of Ehresmann'. As for Ogiue, he speaks in (Ogiue, 1967) of the 'theory of Ehresmann-Kobayashi'.

Consider two manifolds N and M of dimension n and m respectively. Let U and V be two neighborhoods of a point $p \in N$, along with two maps $f : U \to M$ and $g : V \to M$. The maps f and g are said to define the same r-**jet** at p if $\left.\frac{\partial^i f}{\partial x^i}\right|_p = \left.\frac{\partial^i g}{\partial x^i}\right|_p$ for $i \in [1 \cdots r]$. This definition is independent of the coordinate system. A r-jet thus defines an equivalence class and is noted $j^r_p(f)$.

The set of all r-jets $j^r_p(f)$ at p is noted $J^r_p(N,M)$. And $J^r(N,M) = \bigcup_p J^r_p(N,M)$ has a manifold structure naturally inherited from N and M. Define $s : J^r(N,M) \to N$ and $t : J^r(N,M) \to M$, the *source* and *target* maps respectively. We have $s\big(j^r_p(f)\big) = p$ and $t\big(j^r_p(f)\big) = f(p)$ the source and target of the r-jet $j^r_p(f)$ respectively. $J^r(N,M)$ is the total space of a fiber bundle over N with projection s, or over M with projection t.[19]

Given $j^r_p(f) \in J^r(N,M)$ and $j^r_q(g) \in J(M,L)$ with $q = f(p)$ (the source of the second jet is the target of the first), one defines the **jet product** as the jet of the composed function: $j^r_q(g) \cdot j^r_p(f) = j^r_p(g \circ f)$.
An r-jet $j^r_p(f) \in J^r(N,M)$ is invertible if there exists a r-jet $j^r_{f(p)}(g) \in J^r(M,N)$ such that,

$$j^r_{f(p)}(g) \cdot j^r_p(f) = j^r_p(g \circ f) = j^r_p(\mathrm{id}_N), \qquad \text{and} \qquad j^r_p(f) \cdot j^r_{f(p)}(g) = j^r_{f(p)}(f \circ g) = j^r_{f(p)}(\mathrm{id}_M).$$

If $r > s$, there is a natural projection ${\mathrm{pr}^s}_r : J^r(N,M) \to J^s(N,M)$, simply defined by ${\mathrm{pr}^s}_r\big(j^r_p(f)\big) = j^s_p(f)$. For $r > s > t$ we have, ${\mathrm{pr}^t}_s \circ {\mathrm{pr}^s}_r = {\mathrm{pr}^t}_r$.

The r^{th}-order frame bundle and r^{th}-order G-structures Let us specialize the above construction. Take $N = \mathbb{R}^m$ and $M = \mathcal{M}$ with $m = \dim\mathcal{M}$. Choose $p = 0$ and $f(0) = x \in \mathcal{M}$. An invertible r-jet $j^r_0(f) \in J^r_0(\mathbb{R}^m, \mathcal{M})$ is a **r-frame** of $\mathcal{M}$ at x. The set $J^r_0(\mathbb{R}^m, \mathcal{M})$ of r-frames of M is noted $L^r\mathcal{M}$. It is a fiber bundle over $\mathcal{M}$ with projection $\pi = t : L^r\mathcal{M} = J^r_0(\mathbb{R}^m, \mathcal{M}) \to \mathcal{M}$ given by, $\pi\big(j^r_0(f)\big) = t\big(j^r_0(f)\big) = f(0) = x$.

Take now $N = \mathbb{R}^m = M$ and choose $p = 0$, $f(0) = 0$. The set $J^r_0(\mathbb{R}^m, \mathbb{R}^m)$ of invertible r-jets $j^r_0(f)$ with source and target at 0 is noted G^r. It is a group under the jet multiplication, and is called the **r^{th}-differential group**. We have furthermore the natural sequence,

$$G^r \xrightarrow{{\mathrm{pr}^{r-1}}_r} G^{r-1} \xrightarrow{{\mathrm{pr}^{r-2}}_{r-1}} G^{r-2} \cdots \longrightarrow G^1.$$

The kernel of ${\mathrm{pr}^s}_r$, for $r > s$, is noted ${K^r}_s$. It is a nilpotent subgroup. The subgroup ${K^{i+1}}_i$ is abelian. We have naturally, $G^r/{K^r}_s \simeq G^s$. Hence the decomposition of the r^{th}-differential group,

$$G^r \simeq \prod_{i=0}^{r-1} {K^{i+1}}_i. \tag{2.31}$$

Remark that G^r could play the role of $H = \bar{K}_n$ and that each ${K^{i+1}}_i$ could play the role of a K_i, in the notation of the previous subsection.

Actually $L^r\mathcal{M}$ is a principal bundle over $\mathcal{M}$ with structure group G^r. Given a point $u = j^r_0(f) \in L^r\mathcal{M}$ and $g = j^r_0(g) \in G^r$, the right action of G^r on $L^r\mathcal{M}$ is $ug = j^r_0(f) \cdot j^r_0(g) = j^r_0(f \circ g) \in L^r\mathcal{M}$. This action is simply transitive on each fiber. $L^r\mathcal{M}$ is known as the **r^{th}-order frame bundle**. Clearly enough $G^1 \simeq GL$ and $L^1\mathcal{M}$ is the bundle of linear frames of $\mathcal{M}$. For $r > s$ there is again a natural projection ${\mathrm{pr}^s}_r : L^r\mathcal{M} \to L^s\mathcal{M}$ so that,

$$L^r\mathcal{M}/{K^r}_s \simeq L^s\mathcal{M}, \qquad \text{or} \qquad L^r\mathcal{M} \simeq L^s\mathcal{M} \times {K^r}_s. \tag{2.32}$$

Given a subgroup $G \subset GL$, a G-structure is a G-reduction of the frame bundle $L\mathcal{M}$. These are now to be understood as *first order* G-structure. A **r^{th}-order G-structure** will be a reduction of the r^{th}-order frame bundle $L^r\mathcal{M}$. Any higher order G-structure is thus expected to display features like (2.31) and (2.32). So they are a natural realm for the application of the construction exposed at the beginning of this section.

In the following I will work out the example of the 2^{nd}-order conformal structure which is a reduction of the 2^{nd}-order frame bundle $L^2\mathcal{M}$. But first, as closing words for this section and introducing words for the next, remark the following. As already noted, modulo some caveats, one could be willing to consider GR as a gauge theory on $L\mathcal{M}$. A genuine gauge formulation of GR on the first order SO-structure is possible, as we know. One could then try to formulate generalized gravitational gauge theories on $L^r\mathcal{M}$ or on one of its reductions. Actually this is naturally the case when one is interested in conformal gravity, since conformal geometry is canonically a 2^{nd}-order structure.

[19] It is also a fiber bundle over $N \times M$ with projection $s \times t$.

2.4.2 Dressing fields in the Cartan-Möbius geometry

As far as Physics is concerned, one ought to be interested in conformal geometry since conformal transformations of a Lorentz manifold preserve the class of null geodesics. In other words the causal structure of spacetime is preserved by conformal transformations. As a consequence, conformal symmetry is a fundamental symmetry of any relativistic theory of massless fields: Maxwell's theory, General Relativity in vacuum and both of these interactions coupled to each other and/or to massless scalar or fermion fields give conformally invariant theories. In high energy particle physics, that is in accelerators, in violent astrophysical phenomena or in early cosmology, all situations where the masses of the particles are negligible compared to their total energy, the conformal symmetry may be considered as an approximate symmetry.

From the viewpoint of Mathematics, conformal geometry is now classically approached through the jet formalism. It also has a natural Cartan geometric structure. Indeed the bundle of any Cartan geometry is a reduction of an adequate higher-order structure, and is studied from both perspectives in (Ogiue, 1967) and (Kobayashi, 1972).[20] Nevertheless, the jet formalism is intricate to use. The mere jet multiplication, which is the partial derivative of the composition of functions, is already complex at second order and becomes formidable at higher order. Moreover the jets usually do not allow torsion. So, following (Sharpe, 1996) we will use a matrix formalism which allows to focus on the Cartan geometric aspect, allows torsion, is more handy and better suited to apply the dressing field method. We will occasionally stress the equivalence between the two formalisms at chosen moments, by reference to the mentioned classic papers on the jet approach.

Cartan-Möbius geometry: a reminder

Conformal geometry seen as Cartan geometry has been discussed in 1.2, the exposure followed (Sharpe, 1996). We briefly recall the setup.

The Klein model geometry is the pair (G,H) where the *principal group* is the Möbius group defined by $G = SO(m,2)/\pm I$. It is the isometry group of of the deSitter space dS^m, and H is the isotropy group of the points of dS^m. The latter can be given the matrix representation,

$$H = K_0K_1 = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & r & \frac{1}{2}rr^t \\ 0 & \mathbb{1} & r^t \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; z \in \mathbb{R}^*,\ S \in SO(r,s),\ zS \in CO(r,s),\ r \in \mathbb{R}^{(r,s)*} \right\}.$$

So the group H as a structure of type (2.31). There is a natural projection ${\rm pr^0}_1 : H = K_0K_1 \to K_0$ whose kernel is K_1. The latter is abelian, since it is a matrix representation of $\mathbb{R}^{(r,s)*}$, as it should be from discussion on the $r^{\rm th}$-differential group.[21]

The infinitesimal Klein pair is $(\mathfrak{g},\mathfrak{h})$, where both are graded Lie algebras. They can be decomposed as $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 = \mathbb{R}^{(r,s)} + \mathfrak{co}(r,s) + \mathbb{R}^{(r,s)*}$, and $\mathfrak{h} = \mathfrak{g}_0 + \mathfrak{g}_1 = \mathfrak{co}(r,s) + \mathbb{R}^{(r,s)*}$. There, $\mathfrak{g}_1$ is the *first prolongation* of $\mathfrak{g}_0$. See (Kobayashi, 1972). The quotient space is just $\mathfrak{g}/\mathfrak{h} := \mathfrak{p} = \mathfrak{g}_{-1} = \mathbb{R}^{(r,s)}$. In matrix form we get,

$$\mathfrak{g} = \left\{ \begin{pmatrix} \epsilon & \iota & 0 \\ \tau & s & \iota^t \\ 0 & \tau^t & -\epsilon \end{pmatrix} \;\middle|\; (s - \epsilon\mathbb{1}) \in \mathfrak{co}(r,s),\ \tau \in \mathbb{R}^{(r,s)},\ \iota \in \mathbb{R}^{(r,s)*} \right\} \text{ and } \mathfrak{h} = \left\{ \begin{pmatrix} \epsilon & \iota & 0 \\ 0 & s & \iota^t \\ 0 & 0 & -\epsilon \end{pmatrix} \;\middle|\; \dots \right\}.$$

The graded structure of the Lie algebras, $[\mathfrak{g}_i,\mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, is automatically handled by the matrix product. Actually the Lie-graded commutator will be the mere matrix commutator.

[20]The latter consider the projective geometry as well, first treated through jets in (Kobayashi and Nagano, 1964). Projective and conformal geometry were the first important examples studied by Cartan himself as examples of his theory of 'generalized spaces'. See (Cartan, 1923), (Cartan, 1924b), (Cartan, 1924a).

[21] Note that since K_1 is an upper triangular $n \times n$ matrix group with 1's on the diagonal, one would expect that it is nilpotent of class $n-1$. But an easy calculation shows that the commutator of two elements is $[g,g'] = g^{-1}g'^{-1}gg' = e$, then the the lower central series $\left\{C^{n+1}(G) = [G,C^n(G)] \mid C^1(G) = G\right\}$ ends for $n = 1$, as it should be for a non-trivial abelian group.

The Cartan-Möbius geometry modeled on this Klein-Möbius model is a principal bundle, $\mathcal{P}(\mathcal{M},H)$, with group H, together with a (local) Cartan connection $\varpi \in \Lambda^1(U,\mathfrak{g})$ with curvature $\Omega \in \Lambda^2(U,\mathfrak{g})$. The latter have the matrix representation,

$$\varpi = \begin{pmatrix} a & \alpha & 0 \\ \theta & A & \alpha^t \\ 0 & \theta^t & -a \end{pmatrix}, \quad \text{and} \quad \Omega = \begin{pmatrix} f & \Pi & 0 \\ \Theta & F & \Pi^t \\ 0 & \Theta^t & -f \end{pmatrix} = \begin{pmatrix} da + \alpha\theta & d\alpha + \alpha(A - a\mathbb{1}) & 0 \\ d\theta + (A - a\mathbb{1})\theta & dA + A^2 + \theta\alpha + \alpha^t\theta^t & d\alpha^t + (A + a\mathbb{1})\alpha^t \\ 0 & d\theta^t + \theta^t(A + a\mathbb{1}) & -da + \theta^t\alpha^t \end{pmatrix}.$$

Here the wedge product is tacitly understood. The **Normal Cartan connection** is the unique ϖ whose curvature is such that $\Theta = 0$ (torsion-free geometry), $f = \frac{1}{m}\mathrm{Tr}(\Omega^0) = 0$ and $\mathrm{Ric}(\Omega^0) = (\Omega^0)^a{}_{bad} = 0$. In the last two conditions $\Omega^0 \in \mathfrak{g}_0$. The Lie algebra homomorphism $\mathfrak{k}_0 \to \mathfrak{g}_0 = \mathfrak{co}(r,s)$ is given by $(s,\epsilon) \to s - \epsilon\mathbb{1}$, so that $\Omega^0 = F - f\mathbb{1}$.
An element γ of the gauge group $\mathcal{H}_{\text{loc}}$ is,

$$\gamma = \gamma_0\gamma_1 : U \to H = K_0K_1, \quad \text{with} \begin{cases} \gamma_0 & : U \to K_0 \\ \gamma_1 & : U \to K_1 \end{cases}$$

By keeping the same matrix notation for the groups $K_{0,1}$ and the maps $\gamma_{0,1}$, the gauge transformations of the Cartan connection are, with respect to $\mathcal{K}_0$

$$\varpi^{\gamma_0} = \gamma_0^{-1}\varpi\gamma_0 + \gamma_0^{-1}d\gamma_0 = \begin{pmatrix} a + z^{-1}dz & z^{-1}\alpha S & 0 \\ S^{-1}\theta z & S^{-1}AS + S^{-1}dS & S^{-1}\alpha^t z^{-1} \\ 0 & z\theta^t S & -a + zdz^{-1} \end{pmatrix}, \tag{2.33}$$

and with respect to $\mathcal{K}_1$

$$\varpi^{\gamma_1} = \gamma_1^{-1}\varpi\gamma_1 + \gamma_1^{-1}d\gamma_1 = \begin{pmatrix} a - r\theta & ar - r\theta r + \alpha - rA + \frac{1}{2}rr^t\theta^t + dr & 0 \\ \theta & \theta r + A - r^t\theta^t & \theta\frac{1}{2}rr^t + Ar^t - r^t\theta^t r^t + \alpha^t + r^t a + dr^t \\ 0 & \theta^t & \theta^t r^t - a \end{pmatrix}. \tag{2.34}$$

These are (local) instances of equation (2.14). The gauge transformation under $\gamma = \gamma_0\gamma_1$ is obvious.

Clearly enough $\mathcal{P}(\mathcal{M},H)$ is a 2^{th}-order G-structure, a reduction of $L^2\mathcal{M}$, a 'two stage bundle'. We have $\mathcal{P} = \mathcal{P}_1$ a principal bundle over $\mathcal{M}$ with group H, but it is also a bundle over $\mathcal{P}_0$ with structure group K_1, the latter being itself a bundle over $\mathcal{M}$ with structure group K_0.

The whole structure group $H = K_0K_1 = CO(r,s) \ltimes \mathbb{R}^{(r,s)*}$ has dimension $1 + \frac{m(m-1)}{2} + m$. We will show how, thanks to the dressing field method, we can reduce this group down to the 1-dimensional group of Weyl rescaling parameterized by z in K_0. We will also see how the Cartan connection ϖ and its curvature Ω are dressed so as to become composite fields (2.6) containing well known tensors of the conformal geometry. Clearly the 1-dimensionnal Weyl group will be the residual gauge freedom of these composite fields which will give the conformal transformations of the mentioned tensors.

First dressing: neutralizing K_1

Finding the first dressing In order to neutralize K_1 and to 'reduce' H down to K_0 we need a dressing field, that is

$$\text{a map } u_1 : U \to K_1 \text{ such that } u_1^{\gamma_1} = \gamma_1^{-1}u_1. \quad \text{In matrix form, } u_1 = \begin{pmatrix} 1 & q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix}.$$

How are we to find it? It turns out that the solution lies in a 'gauge-fixing-like' constraint imposed on the Cartan connection, $\chi(\varpi^{u_1}) = 0$. This strategy should remind the reader of the analysis of the paper of (Lavelle and McMullan, 1997) in appendix A.1. This constraint reads explicitly $\chi(\varpi^{u_1}) = a^{u_1} = a - q\theta = 0$. Its effect is to set the component a^{u_1} of ϖ^{u_1} to zero. Sharpe says to do so by gauge fixing and works with the resulting form of a ϖ with $a = 0$. This plays an important role in proving that for the normal Cartan connection, the $\mathfrak{g}_1$ entry α is actually the so-called Schouten tensor. See (Sharpe, 1996) p.287-288 or (Kobayashi, 1972) p.136-137. Neither of them explicitly identifies the Schouten tensor as such,[22] but its expression cannot be mistaken. We will reproduce here this result following the line of (Kobayashi, 1972).

Solving the constrain for q we get,

$$a - q\theta = a - q_a\theta^a = a_\mu dx^\mu - q_a e^a{}_\mu dx^\mu = 0 \quad \longrightarrow \quad q_a = a_\mu (e^{-1})^\mu{}_a, \quad \text{or in index free notation} \quad q = a \cdot e^{-1}.$$

Be careful, in this index free notation we should remember that a is the *scalar coefficient* of the 1-form a. The distinction should be clear from the context. Also notice that in this index free notation the point "$\cdot$" often means (greek) index summation. Now the K_1-gauge transformation of the Cartan connection (2.34) gives us,

$$\begin{aligned} a^{\gamma_1} &= a - r\theta \quad \longrightarrow a^{\gamma_1}_\mu = a_\mu - r_a e^a{}_\mu, \quad \text{or in index free notation} \quad a^{\gamma_1} = a - re, \\ \theta^{\gamma_1} &= \theta \quad \longrightarrow \quad e^{a\gamma_1}_{\mu} = e^a{}_\mu \quad \text{in index free notation} \quad e^{\gamma_1} = e. \end{aligned}$$

This implies $q^{\gamma_1} = a^{\gamma_1} \cdot e^{-1\gamma_1} = (a - re)\cdot e^{-1} = a\cdot e^{-1} - r = q - r$. The two other entries of $u_1^{\gamma_1}$ are $(q^t)^{\gamma_1} = q^t - r^t$ and $\frac{1}{2}(qq^t)^{\gamma_1} = \frac{1}{2}(qq^t + rr^t) - rq^t$. These are precisely the right K_1-gauge transformations for our dressing field u_1. Indeed in matrix form we want,

$$u_1^{\gamma_1} = \gamma_1^{-1} u_1 = \begin{pmatrix} 1 & -r & \frac{1}{2}rr^t \\ 0 & \mathbb{1} & -r^t \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & q - r & \frac{1}{2}(qq^t + rr^t) - rq^t \\ 0 & \mathbb{1} & q^t - r^t \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.35}$$

Notice once again that on account of this transformation law, $u_1 \notin \mathcal{K}_1$. Indeed for $\gamma_1, \alpha_1 \in \mathcal{K}_1$, a simple matrix calculation shows that $\alpha_1^{\gamma_1} := \gamma_1^{-1}\alpha_1\gamma_1 = \alpha_1$, due to the abelian nature of K_1.

Remark: In my analysis of (Lavelle and McMullan, 1997) in appendix A.1, I stressed the fact that it was the *requirement* of the invariance of the gauge-like constraint, $\chi\Big((A^\gamma)^{u^\gamma}\Big) = \chi(A^u)$, that implied the transformation $u^\gamma = \gamma^{-1}u$ for the field u. An instance of this fact is found here. Indeed we have $\chi(\varpi^{u_1}) = a - q\theta$, and

$$\chi\left((\varpi^{\gamma_1})^{u_1^{\gamma_1}}\right) = \left(a^{\gamma_1}\right)^{u_1^{\gamma_1}} = \left(a - r\theta\right)^{\gamma_1^{-1}u_1} = (a - r\theta) - q^{\gamma_1}\theta = (a - r\theta) - (q - r)\theta = a - q\theta = \chi(\varpi^{u_1}).$$

The equality is satisfied thanks to the (abelian) dressing transformation law of $u_1 \sim q$.

Now that we've found our K_1-dressing field u_1, we can now proceed to dress the Cartan connection,

$$\begin{aligned} \varpi_1 &:= \varpi^{u_1} = u_1^{-1}\varpi u_1 + u_1^{-1}du_1 \\ &= \begin{pmatrix} a - q\theta & aq - q\theta q + \alpha - qA + \frac{1}{2}qq^t\theta^t + dq & 0 \\ \theta & \theta q + A - q^t\theta^t & \theta\frac{1}{2}qq^t + Aq^t - q^t\theta^t q^t + \alpha^t + q^t a + dq^t \\ 0 & \theta^t & \theta^t q^t - a \end{pmatrix} =: \begin{pmatrix} 0 & \alpha_1 & 0 \\ \theta & A_1 & \alpha_1^t \\ 0 & \theta^t & 0 \end{pmatrix}. \end{aligned}$$

Similarly for the dressed curvature, it gives,

$$\Omega_1 := \Omega^{u_1} = u_1^{-1}\Omega u_1 = \begin{pmatrix} f - q\Theta & \Pi - qF_1 + fq - \frac{1}{2}qq^t\Theta^t & 0 \\ \Theta & \Theta q + F - q^t\Theta^t & \Pi^t - F_1^t q^t + q^t f - \frac{1}{2}\Theta qq^t \\ 0 & \Theta^t & \Theta^t q^t - f \end{pmatrix} =: \begin{pmatrix} f_1 & \Pi_1 & 0 \\ \Theta & F_1 & \Pi_1^t \\ 0 & \Theta^t & -f_1 \end{pmatrix}.$$

By definition ϖ_1 and Ω_1 are K_1-invariant composite fields. The skeptic reader can check this entry by entry. This means in particular that the *form* of ϖ_1 is invariant, that is $a_1 = 0 = a_1^{\gamma_1}$. Moreover if ϖ is the Normal Cartan connection, then ϖ_1 also satisfies the axioms of normality. Indeed the torsion-free condition $\Theta = 0$ is unchanged. The trace-free condition $f = 0$ for ϖ implies then the trace-free condition for ϖ_1, $f_1 = f - q\Theta = 0$. Finally the Ricci-null condition is also satisfied, $\text{Ric}\Big((\Omega_1)^0\Big) = \text{Ric}(F_1 - f_1\mathbb{1}) = \text{Ric}(\Theta q - F + q^t\Theta^t) = \text{Ric}(F) = 0$.

[22] The calculation of Kobayashi is intended to prove the uniqueness of the normal Cartan connection, not to find the tensor for itself. Conversely, Sharpe aims at showing that the normality of the Cartan connection allows to write its $\mathfrak{g}_1$ component as a function of the other.

The action of $SO \subset K_0$ We could now ask how these composite fields behave under the action of K_0. This depends of course on how ϖ and u_1 transform under K_0. The answers to both of these questions can be found from (2.33). But it turns out that it is easier and conceptually much clearer to decompose the action of K_0 as the action of the Lorentz subgroup $S = SO(r,s)$ on the one hand, and the action of the Weyl subgroup $W = \mathbb{R}^*$ on the other hand. We have,

$$\gamma_0 = W \cdot S = \begin{pmatrix} z & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider first the action of the Lorentz subgroup,

$$\varpi^S = S^{-1}\varpi S + S^{-1}dS = \begin{pmatrix} a & \alpha S & 0 \\ S^{-1}\theta & S^{-1}AS + S^{-1}dS & S^{-1}\alpha^t \\ 0 & \theta^t S & -a \end{pmatrix}, \quad \text{and} \quad \Omega^S = S^{-1}\Omega S = \begin{pmatrix} f & \Pi S & 0 \\ S^{-1}\Theta & S^{-1}FS & S^{-1}\Pi^t \\ 0 & \Theta^t S & -f \end{pmatrix}.$$

Since $\theta^S = S^{-1}\theta$ means $e^S \cdot dx = S^{-1}e \cdot dx$, we have $q^S = a^S(e^{-1})^S = a(e^{-1})S = qS$, $q^t = S^{-1}q^t$ and $\frac{1}{2}(qq^t)^S = \frac{1}{2}qq^t$. This implies that under the Lorentz subgroup, we have the matrix transformation law for u_1,

$$u_1^S = S^{-1} \cdot u_1 \cdot S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & qS & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & S^{-1}q^t \\ 0 & 0 & 1 \end{pmatrix} \tag{2.36}$$

This is nothing but an instance of the compatibility condition (2.27), the first dressing field has gauge-like transformation under the action of the lower-order subgroup $SO \subset K_0$. For now the first important conclusion that we draw is that the K_1-composite field ϖ_1 behaves as a true SO-gauge field under the action of the Lorentz group. Indeed,

$$\begin{aligned}
\varpi_1^S &= (u_1^S)^{-1} \cdot \varpi^S \cdot u_1^S + (u_1^S)^{-1}du_1^S = S^{-1}u_1^{-1}S \cdot \left(S^{-1}\varpi S + S^{-1}dS\right) \cdot S^{-1}u_1 S + S^{-1}u_1^{-1}Sd(S^{-1}u_1 S),\\
&= S^{-1}u_1^{-1} \cdot \varpi \cdot u_1 S + S^{-1}u_1^{-1} \cdot dSS^{-1} \cdot u_1 S + S^{-1}u_1^{-1} \cdot SdS^{-1} \cdot u_1 S + S^{-1}u_1^{-1}du_1 S + S^{-1}dS,\\
\varpi_1^S &= S^{-1} \cdot \varpi_1 \cdot S + S^{-1}dS.
\end{aligned}$$

In the same way we have, $\Omega_1^S = S^{-1}\Omega_1 S$. We can conclude from this that the entries of ϖ_1 and Ω_1 are the *true* SO-gauge fields of the theory. Hereby I mean that A_1 is the real Lorentz/spin connection already encountered in our treatment of General Relativity at the end of section 2.3.2, whereas $A \in \varpi$ is not. As a consequence $dA_1 + A_1^2 =: R_1$ is the real Riemann curvature 2-form while $R = dA + A^2$ is not, and $F_1 \in \Omega_1$ is the real Weyl curvature 2-form while $F \in \Omega$ is not.[23] Precisely R_1 and F_1 are 'gauge versions' of the Riemann and Weyl curvatures.[24] This is plainly in accordance with the section '*The Möbius Geometry Associated to a Riemann Geometry*' p.290-291 of (Sharpe, 1996) where he shows how the **normal** conformal Cartan connection with entry $a = 0$ is related to the Cartan connection we introduced for GR. The reader can indeed compare the latter to the upper-left sub-matrix of ϖ_1,

$$\varpi_{GR} = \begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix}, \quad \text{and } \varpi_1\Big|_{\text{up-left}} = \begin{pmatrix} 0 & \alpha_1 \\ \theta & A_1 \end{pmatrix}.$$

The soldering form θ is the same in both cases. The 1-forms A and A_1 have the same gauge transformations under the Lorentz group, they denote the same object: the Lorentz/spin connection. The only difference is the $\mathfrak{g}_1$-valued 1-form α_1 in ϖ_1, absent in the geometry of GR for $\mathfrak{so}$ has vanishing first prolongation contrary to $\mathfrak{g}_0 = \mathfrak{co}$. See (Kobayashi, 1972) p.8-9, or (Ogiue, 1967) p.194-195. If ϖ is the normal Cartan connection of the Cartan-Möbius geometry, ϖ_1 is also normal and the 1-form α_1 is actually the gauge version of the Schouten tensor, as we now prove.

[23]Except in the normal case where $F_1 = F$. This statement is equivalent to the Proposition 19 p.214 in (Ogiue, 1967).

[24] Said otherwise they are written in a pseudo-orthogonal basis, denoted by latin indices.

Normality and the Schouten tensor To prove this result we need the expression of Ω_1 in function of ϖ_1, which is,

$$\Omega_1 = d\varpi_1 + \varpi_1 \wedge \varpi_1 = \begin{pmatrix} \alpha_1\theta & d\alpha_1 + \alpha_1 A_1 & 0 \\ d\theta + A_1\theta & dA_1 + A_1^2 + \theta\alpha_1 + \alpha_1^t\theta^t & d\alpha_1^t + A_1\alpha_1^t \\ 0 & d\theta^t + \theta^t A_1 & \theta^t\alpha_1^t \end{pmatrix} = \begin{pmatrix} f_1 & \Pi_1 & 0 \\ \Theta & F_1 & \Pi_1^t \\ 0 & \Theta^t & -f_1 \end{pmatrix}, \tag{2.37}$$

where the wedge product is tacitly understood in the second equality. Notice in passing that if the torsion seems to have changed here, it is not the case. On account of $\Omega = d\varpi + \varpi^2$ the torsion reads $\Theta = d\theta + (A - a\mathbb{1})\theta$. Here it is $\Theta = d\theta + A_1\theta = d\theta + (\theta q + A - q^t\theta^t)\theta$. But $\theta^t\theta = 0$ and $\theta q\theta = \theta a e^{-1}\theta = \theta a e^{-1} e \cdot dx = \theta a \cdot dx = -a\theta$.

Since $\{\theta^a\}$ is the basis for the horizontal forms we can decompose ϖ_1 and Ω_1 on it. In particular if ϖ_1 is normal we have $f_1 = \alpha_1\theta = (\alpha_1)_{ab}\theta^b \wedge \theta^a = 0$. Then the components of α_1 form a symmetric tensor, $\alpha_{ab} = \alpha_{ba}$. This futher implies $(\Omega_1)^0 = F_1$, so

$$\begin{aligned} (\Omega_1)^0 &= F_1 = R_1 + \theta\alpha_1 + \alpha_1^t\theta^t \\ &= \frac{1}{2}\Big((F_1)^a{}_{b,cd}\Big)\theta^c \wedge \theta^d = \frac{1}{2}\Big((R_1)^a{}_{b,cd}\Big)\theta^c \wedge \theta^d + \delta^a_c\theta^c \wedge (\alpha_1)_{bd}\theta^d + \eta^{ai}(\alpha_1)_{id}\theta^d \wedge \delta^j_c\theta^c\eta_{jb}, \end{aligned}$$

In components only,

$$(F_1)^a{}_{b,cd} = (R_1)^a{}_{b,cd} + \delta^a_c(\alpha_1)_{bd} - \delta^a_d(\alpha_1)_{bc} - \eta^{ai}(\alpha_1)_{id}\delta^j_c\eta_{jb} + \eta^{ai}(\alpha_1)_{ic}\delta^j_d\eta_{jb}.$$

If we now apply the Ricci homomorphism we get (remember ϖ_1 is assumed to be normal),

$$\begin{aligned} 0 &= (R_1)^a{}_{b,ad} + m(\alpha_1)_{bd} - (\alpha_1)_{bd} - \delta^i_b(\alpha_1)_{id} + \eta^{ai}(\alpha_1)_{ia}\eta_{db}, \\ &= (R_1)_{bd} + (m-2)(\alpha_1)_{bd} + (\alpha_1)^a{}_a\eta_{bd}, \quad \text{a further contraction gives,}\ \ 0 = R + 2(m-1)(\alpha_1)^a{}_a, \end{aligned}$$

where R_{bd} is the (gauge version of) the Ricci tensor, and R is the Ricci scalar. With these last two equalities we finally obtain,

$$(\alpha_1)_{bd} = \frac{-1}{(m-2)}\left((R_1)_{bd} - \frac{R}{2(m-1)}\eta_{bd}\right).$$

This is the expression for (the gauge version of) the Schouten tensor.

To sum up, ϖ_1 and Ω_1 are K_1-invariant composite fields but SO-gauge fields still. Moreover if ϖ is **normal**, ϖ_1 is normal too and we have,

$$\varpi_1 = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \theta & A_1 & \alpha_1^t \\ 0 & \theta^t & 0 \end{pmatrix}, \quad \text{and} \quad \Omega_1 = \begin{pmatrix} 0 & \Pi_1 & 0 \\ 0 & F_1 & \Pi_1^t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d\alpha_1 + \alpha_1 A_1 & 0 \\ 0 & R_1 + \theta\alpha_1 + \alpha_1^t\theta^t & d\alpha_1^t + A_1\alpha_1^t \\ 0 & 0 & 0 \end{pmatrix},$$

where θ is the soldering form, A_1 is the **Lorentz/spin-connection**, α_1 is the **Schouten** $\mathfrak{g}_1$-valued 1-form, R_1 is the **Riemann curvature** 2-form, F_1 is the **Weyl curvature** 2-form and finally Π_1 is the **Cotton** 2-form, that is the covariant derivative of the Schouten 1-form with respect to the spin-connection A_1.

The SO-gauge fields ϖ_1 and Ω_1 (normal or not) live on the *first-order G-structure* $\mathcal{P}_0 := \mathcal{P}(\mathcal{M}, K_0)$, since $SO \in K_0$, which is realized as a subbundle of our initial 2^{nd}*-order G-structure* $\mathcal{P}(\mathcal{M}, H) =: \mathcal{P}_1 \simeq \mathcal{P}_0 \times K_1$, as Lemma 2 and the section 2.4.1 teach us. It then makes sense to ask if we can neutralize the Lorentz gauge symmetry as we did for the case of GR, by finding an adequate *second* dressing field. This is indeed possible as is shown in the next section.

Second dressing: neutralizing the Lorentz subgroup of K_0

Our starting point is thus the first-order structure $\mathcal{P}_0 := \mathcal{P}(\mathcal{M}, K_0)$ on which live the SO-gauge fields ϖ_1 and Ω_1. We want to neutralize the Lorentz subgroup of K_0, leaving only the abelian Weyl subgroup as the final residual gauge symmetry. This is then a situation that falls under the purview of the original Lemma 2.

The suitable dressing field is not hard to find. From our experience of GR we suspect the soldering form to be the right guess. Indeed from ϖ^S, and ϖ_1^S, we have $\theta^S = S^{-1}\theta$ which provides the transformation law for the vielbein $e \in GL(r,s)$, $e^S = S^{-1}e$. Hence the definition of

$$\text{a map } u_0 : U \to GL \supset K_0, \text{ with matrix form, } u_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{such that} \quad u_0^S = S^{-1}u_0 \quad \to \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S^{-1}e & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.38}$$

Moreover, hopefully, from (2.34) or from the fact that ϖ_1 is K_1-invariant, we have that

$$\theta^{\gamma_1} = \theta \quad \to \quad e^{\gamma_1} = e, \quad \text{which implies} \quad u_0^{\gamma_1} = u_0. \tag{2.39}$$

The latter equation is an instance of the compatibility condition (2.28) of section 2.4.1.

Equation (2.38) and (2.39) tell us that we can dress ϖ_1 to form the K_1- and SO-invariant $\mathfrak{gl}$-valued 1-form[25]

$$\varpi_0 := \varpi_1^{u_0} = u_0^{-1}\varpi_1 u_0 + u_0^{-1}du_0 = \begin{pmatrix} 0 & \alpha_1 e & 0 \\ e^{-1}\theta & e^{-1}A_1 e + e^{-1}de & e^{-1}\alpha_1^t \\ 0 & \theta^t e & 0 \end{pmatrix} = \begin{pmatrix} 0 & P & 0 \\ dx & \Gamma & g^{-1}P^T \\ 0 & dx^T \cdot g & 0 \end{pmatrix}, \tag{2.40}$$

where g is the metric on $\mathcal{M}$ induced by the Cartan connection through $e^T\eta e = g$. Actually in the last matrix equality, $\Gamma := e^{-1}A_1 e + e^{-1}de$ and $P := \alpha_1 e$ are *definitions*, but the other entries result from calculations. We detail it,

$$\begin{aligned} e^{-1}\theta &= e^{-1}e \cdot dx = dx, \\ \theta^t e &= \theta^T\eta e = dx^T \cdot e^T\ \eta e = dx^T \cdot e^T\eta e = dx^T \cdot g, \\ e^{-1}\alpha_1^t &= e^{-1}\eta^{-1}\alpha_1^T = g^{-1}e^T\alpha_1^T = g^{-1}(\alpha_1 e)^T = g^{-1}P^T. \end{aligned}$$

If we express ϖ_0 in components we get,

$$\varpi_0 = \begin{pmatrix} 0 & P_{\mu\nu} & 0 \\ \delta_\mu^\rho & \Gamma^\rho{}_{\mu\nu} & g^{\rho\lambda}P_{\lambda\mu} \\ 0 & g_{\mu\nu} & 0 \end{pmatrix} dx^\mu, \tag{2.41}$$

It is nice to have the metric tensor directly as part of the composite field. The 1-form $g \cdot dx$ can be seen as the form lowering the index of the components of a vector field of $\mathcal{M}$. Actually the transformation under coordinate change of ϖ_0, in part due to u_0 as in GR, allows to identify Γ as the linear connection. These calculations can be found in Appendix C.2.1.

The curvature associated to ϖ_0 is the K_1- and SO-invariant 2-form[26]

$$\Omega_0 := \Omega_1^{u_0} = u_0^{-1}\Omega_1 u_0 = \begin{pmatrix} f_1 & \Pi_1 e & 0 \\ e^{-1}\Theta & e^{-1}F_1 e & e^{-1}\Pi^t \\ 0 & \Theta^t e & -f_1 \end{pmatrix} =: \begin{pmatrix} f_1 & C & 0 \\ T & W & C^t \\ 0 & T^t & -f_1 \end{pmatrix}.$$

[25] Strictly speaking it is not a differential form anymore since it is not invariant under coordinate changes. See Appendix C.2.1.
[26] See note 25.

Using (2.37), the above matrix explicitly reads

$$\Omega_0 = \begin{pmatrix} f_1 & C & 0 \\ T & W & C^t \\ 0 & T^t & -f_1 \end{pmatrix} = \begin{pmatrix} P \wedge dx & dP + P \wedge \Gamma & 0 \\ \Gamma \cdot dx \wedge dx & R + dx \wedge P + g^{-1}P^T \wedge dx^T \cdot g & \nabla g^{-1} \wedge P^T + g^{-1}C^T \\ 0 & -dx^T \wedge \left(\nabla g + \Gamma^T g\right) & dx^T \wedge P^T \end{pmatrix}, \tag{2.42}$$

where $R = d\Gamma + \Gamma^2$ is the Riemann curvature of the base manifold $\mathcal{M}$, $\nabla g = dg - \Gamma^T g - g\Gamma$ is the covariant derivative of the metric with respect to the linear connection, see (Göckeler and Schücker, 1987) p.64, from which is deduced the reciprocal relation $\nabla g^{-1} = dg^{-1} + g^{-1}\Gamma^T + \Gamma g^{-1}$. I refer the reader to Appendix C.1 for the detailed calculations. In components we have,

$$\begin{aligned} \Omega_0 &= \frac{1}{2}\begin{pmatrix} (f_1)_{\mu\sigma} & C_{\nu,\mu\sigma} & 0 \\ T^\rho{}_{\mu\sigma} & W^\rho{}_{\nu,\mu\sigma} & C^\rho{}_{\mu\sigma} \\ 0 & T_{\nu,\mu\sigma} & -(f_1)_{\mu\sigma} \end{pmatrix} dx^\mu \wedge dx^\sigma, \\ &= \frac{1}{2}\begin{pmatrix} P_{[\mu\sigma]} & \partial_{[\mu}P_{\sigma]\nu} + P_{[\mu\lambda}\Gamma^\lambda{}_{\sigma]\nu} & 0 \\ \Gamma^\rho{}_{[\mu\sigma]} & R^\rho{}_{\nu,\mu\sigma} + \delta^\rho_{[\mu}P_{\sigma]\nu} + g^{\rho\lambda}P_{\lambda[\mu}g_{\sigma]\nu} & \nabla_{[\mu}g^{\rho\lambda}P_{\lambda\sigma]} + g^{\rho\lambda}C_{\lambda,\mu\sigma} \\ 0 & \nabla_{[\mu}g_{\sigma]\nu} + \Gamma_{[\mu\sigma]}{}^\lambda g_{\lambda\nu} & -P_{[\mu\sigma]} \end{pmatrix} dx^\mu \wedge dx^\sigma. \end{aligned} \tag{2.43}$$

Actually we know that in this framework the metricity condition $\nabla g = 0$ is automatically satisfied.[27] If we are furthermore in the normal case, then $T = 0$ implies the symmetry of Γ in its lower indices, and it can be shown in the usual way that Γ can be expressed as a function of g. Then Γ is the **Levi-Civita connexion** on $\mathcal{M}$. In the normal case still, $f_1 = 0$ and $P_{\mu\sigma}$ is the symmetric **Schouten tensor**, so that $C_{\nu,\mu\sigma} = \nabla_\mu P_{\sigma\nu}$ is the **Cotton tensor** and $W^\rho{}_{\nu,\mu\sigma}$ is the **Weyl tensor**. To sum-up, in the **normal** case we have,

$$\varpi_0 = \begin{pmatrix} 0 & P_{\mu\nu} & 0 \\ \delta^\rho_\mu & \Gamma^\rho{}_{\mu\nu} & g^{\rho\lambda}P_{\lambda\mu} \\ 0 & g_{\mu\nu} & 0 \end{pmatrix} dx^\mu, \quad \text{with} \quad P_{\mu\nu} = \frac{-1}{(m-2)}\left(R_{\mu\nu} - \frac{R}{2(m-1)}g_{\mu\nu}\right), \tag{2.44}$$

$$\Omega_0 = \frac{1}{2}\begin{pmatrix} 0 & C_{\nu,\mu\sigma} & 0 \\ 0 & W^\rho{}_{\nu,\mu\sigma} & g^{\rho\lambda}C_{\lambda,\mu\sigma} \\ 0 & 0 & 0 \end{pmatrix} dx^\mu \wedge dx^\sigma. \tag{2.45}$$

This is fully equivalent to Proposition 15 p.210,[28] Propositions 26-27 and equation (24) p.221-223 in (Ogiue, 1967), and (2.44) is the so-called *Riemannian parameterization of the normal conformal Cartan connection.*

The $\mathfrak{gl}$-valued gauge fields ϖ_0 and Ω_0 (normal or not) live on the *first-order G-structure* $\mathcal{P}_W := \mathcal{P}(\mathcal{M}, W)$, call it the **Weyl bundle**, which is realized as a subbundle of $\mathcal{P}(\mathcal{M}, K_0) =: \mathcal{P}_0 \simeq \mathcal{P}_W \times S$, as Lemma 2 and the section 2.4.1 teach us.

Remark that reducing $\mathcal{P}_0$, with structure group $K_0 = WS \simeq CO(r,s)$, to $\mathcal{P}_W$ with abelian structure group $W \simeq \mathbb{R}^*$, is quite analogous to the case of the electroweak sector of the Standard Model where the initial bundle with structure group $SU(2) \times U(1)$ was reduced to a subbundle with abelian structure group $U(1)$.

Two steps in one

At this point the initial 2^{nd}*-order G-structure* $\mathcal{P}_1 := \mathcal{P}(\mathcal{M}, H = K_0K_1)$, on which live the Cartan connection ϖ and its curvature Ω, has been first reduced, by means of the dressing field $u_1 : U \to K_1$, to the *first-order G-structure* $\mathcal{P}_0(\mathcal{M}, K_0 = WS)$ on which live the K_1-invariant SO-gauge composite 1-forms ϖ_1 and Ω_1. Then the latter has been further reduced, through the dressing field $u_0 : U \to GL$, to the *first-order G-structure* $\mathcal{P}_W(\mathcal{M}, W)$, the *Weyl bundle*, on which live the K_1- and SO-invariant composite fields ϖ_0 and Ω_0. Along the process, the *normality is preserved.* That is, if ϖ is the *normal Cartan connection*, then ϖ_1 and ϖ_0 also satisfy the axioms of normality.

[27] See the equation (2.13) in our treatment of General Relativity.

[28] The dressing field u_1u_0 perform the same job as the so-called *natural cross section*, or *natural frame* described in that Proposition.

Now, as elaborated in previous section 2.4.1 above it would have been possible to reduce $\mathcal{P}_1$ to $\mathcal{P}_W$ in a single step. Indeed the two dressing fields that we used obey the necessary compatibility conditions. We collect them,

$u_1^{\gamma_1} = \gamma_1^{-1}u_1, \quad u_0^S = S^{-1}u_0$ equation (2.35) and (2.38) respectively, which are instances of the general (2.26).

$u_1^S = S^{-1}u_1S, \quad u_0^{\gamma_1} = u_0$ equation (2.36) and (2.39), respective instances of the general (2.27) and (2.28).

These imply,

$$\begin{aligned} \big(u_1u_0\big)^{\gamma_1} &= u_1^{\gamma_1}\cdot u_0^{\gamma_1} = \gamma_1^{-1}u_1\cdot u_0 = \gamma_1^{-1}\big(u_1u_0\big), \\ \big(u_1u_0\big)^{S} &= u_1^{S}\cdot u_0^{S} = S^{-1}u_1S\cdot S^{-1}u_0 = S^{-1}\big(u_1u_0\big). \end{aligned}$$

Here we have instances of the general equation (2.29): the composite dressing u_1u_0 is a dressing for both the groups K_1 and SO. In turn this implies,

$$\begin{aligned} \big(u_1u_0\big)^{S\gamma_1} &= \Big(\big(u_1u_0\big)^S\Big)^{\gamma_1} = \Big(S^{-1}\big(u_1u_0\big)\Big)^{\gamma_1} = (S^{\gamma_1})^{-1}\big(u_1u_0\big)^{\gamma_1} = \gamma_1^{-1}S^{-1}\gamma_1\cdot\gamma_1^{-1}\big(u_1u_0\big), \\ \big(u_1u_0\big)^{S\gamma_1} &= \gamma_1^{-1}S^{-1}\big(u_1u_0\big) = (S\gamma_1)^{-1}\big(u_1u_0\big). \end{aligned}$$

This is an instance of the general equation (2.30): the composite dressing field u_1u_0 is a dressing for the full group SK_1.

So it is possible to use this sole composite dressing field, $u_1u_0 : U \to K_1S$, to reduce in a single step the 2nd-order G-structure $\mathcal{P}_1 = \mathcal{P}(\mathcal{M}, H = WSK_1)$ to the first-order G-structure $\mathcal{P}_W := \mathcal{P}(\mathcal{M}, W)$. Moreover it is possible to dress the Cartan connection on $\mathcal{P}_1$ and its curvature so as to obtain the final composite fields: $\varpi_0 := \varpi^{u_1u_0}$ and $\Omega_0 := \Omega^{u_1u_0}$, with all the properties we've studied already. In particular if ϖ is the normal Cartan connection, then ϖ_0 is normal too. We can summarize all this in the following diagram,

$$\begin{tikzcd}
\begin{array}{c}\text{Group } H = K_0K_1 = WSK_1 \\ 2^{\text{nd}}\text{-order } G\text{-structure } \mathcal{P}_1 := \mathcal{P}(\mathcal{M},H) \\ \text{Cartan connection \& curvature: } \varpi,\ \Omega\end{array} \arrow[dd, "u_1u_0"'] \arrow[dr, "u_1"] & \\
& \begin{array}{c}\text{Group } K_0 = WS \\ 1^{\text{st}}\text{-order } G\text{-structure } \mathcal{P}_0 := \mathcal{P}(\mathcal{M},K_0) \\ K_1\text{-invariant } SO\text{-gauge forms: } \varpi_1,\ \Omega_1\end{array} \arrow[dl, "u_0"] \\
\begin{array}{c}\text{Abelian group } W = \mathbb{R} \\ 1^{\text{st}}\text{-order } G\text{-structure } \mathcal{P}_W := \mathcal{P}(\mathcal{M},W) \\ K_1\text{- and } SO\text{-invariant composite fields: } \varpi_0,\ \Omega_0\end{array} &
\end{tikzcd}$$

Figure 2.1: Diagram of the different steps of the process of reduction.

Of course since we end-up, not on the base manifold $\mathcal{M}$, but on the **Weyl bundle** $\mathcal{P}_W$, we readily expect ϖ_0 and Ω_0 to have a **residual Weyl gauge symmetry**. This is the object of the following final section.

Final step: the residual Weyl gauge freedom

The final goal of our analysis is to obtain the transformations of the final composite fields ϖ_0 and Ω_0 under the residual Weyl gauge symmetry. Of course this result depends on the transformations under the Weyl group of the Cartan connection and its curvature on the one side, and of the two dressing fields on the other side. Indeed,

$$\varpi_0^W = \left(\varpi^{u_1 u_0}\right)^W = \left(\varpi^W\right)^{u_1^W \cdot u_0^W}, \quad \text{and} \quad \Omega_0^W = \left(\Omega^{u_1 u_0}\right)^W = \left(\Omega^W\right)^{u_1^W \cdot u_0^W}. \tag{2.46}$$

The Weyl transformation of the Cartan connection is,

$$\varpi^W = W^{-1}\varpi W + W^{-1}dW = \begin{pmatrix} a + z^{-1}dz & z^{-1}\alpha & 0 \\ z\theta & A & \alpha^t z^{-1} \\ 0 & z\theta^t & -a + zdz^{-1} \end{pmatrix}. \tag{2.47}$$

From this we see that $\theta^W = z\theta \ \rightarrow \ e^W = ze$, so that we can define,

$$u_0^W = \widehat{W} u_0 \quad \text{in matrix notation} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^W & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & ze & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.48}$$

The element $\widehat{W}$ is thus *another matrix representation* of the Weyl group, different from the initial representation W, adapted to the dressing field u_0.[29] This is so in order to keep the strength of the matrix formalism as far as possible. Nevertheless we encounter a problem with the dressing u_1. Indeed from (2.47) we have that $a^W = a + z^{-1}dz$, where a is a 1-form, so that $a^W = a + z^{-1}\partial z$, where a stands for the scalar coefficients of the 1-form $a := a \cdot dx$.[30] Then, given $u_1 \sim q := a \cdot e^{-1}$, we have

$$\begin{aligned}
q^W &:= a^W \cdot (e^{-1})^W = (a + z^{-1}\partial z) \cdot z^{-1}e^{-1} = z^{-1}\Big(q + z^{-1}\partial z \cdot e\Big), \\
(q^t)^W &:= \eta^{-1}(e^{-1})^{T\,W} \cdot a^W = \eta^{-1}(e^{-1})^T z^{-1} \cdot (a + z^{-1}\partial z) = z^{-1}\Big(q^t + \eta^{-1}(e^{-1})^T \cdot z^{-1}\partial z\Big), \\
\frac{1}{2}(qq^t)^W &= \frac{1}{2}z^{-2}\Big(qq^t + q\eta^{-1}(e^{-1})^T \cdot z^{-1}\partial z + z^{-1}\partial z \cdot e^{-1}q^t + z^{-1}\partial z \cdot g^{-1} \cdot z^{-1}\partial z\Big), \\
&= \frac{1}{2}z^{-2}\Big(qq^t + 2z^{-1}\partial z \cdot e^{-1}q^t + z^{-1}\partial z \cdot g^{-1} \cdot z^{-1}\partial z.\Big)
\end{aligned} \tag{2.49}$$

On account of this, it seems excluded to find a simple matrix writing for u_1^W. Nonetheless we are not stuck yet. Indeed (2.46) can also be written,

$$\varpi_0^W = \left(\varpi_1^{u_0}\right)^W = \left(\varpi_1^W\right)^{u_0^W}, \quad \text{and} \quad \Omega_0^W = \left(\Omega_1^{u_0}\right)^W = \left(\Omega_1^W\right)^{u_0^W}. \tag{2.50}$$

Therefore, a tractable strategy is to calculate the entries of the matrices ϖ_1^W and Ω_1^W and use the matrix calculus in the final step.

Residual Weyl gauge symmetry of $\boldsymbol{\varpi_0}$ For the composite field ϖ_0 we have,

$$\begin{aligned}
\varpi_0^W &= \left(\varpi_1^W\right)^{u_0^W} = \left(\varpi_1^W\right)^{\widehat{W}u_0} = u_0^{-1}\widehat{W}^{-1} \cdot \left(\varpi_1^W\right) \cdot \widehat{W}u_0 + u_0^{-1}\widehat{W}^{-1}d(\widehat{W}u_0), \\
&= u_0^{-1}\widehat{W}^{-1}\left(\varpi_1^W\right)\widehat{W}u_0 + u_0^{-1} \cdot \widehat{W}^{-1}d\widehat{W} \cdot u_0 + u_0^{-1}du_0.
\end{aligned}$$

[29] Since z is a scalar and commutes with the GL-matrix e, $\widehat{W}$ and u_0 commute. Thus we could have written (2.48) as $u_0^W = u_0\widehat{W}$, to stress again the resemblance with our treatment of the electroweak sector of the Standard Model where the $SU(2)$-dressing field u had residual $U(1)$-gauge transformation given by $u^\alpha = u\widehat{\alpha}$.

[30] Again, the difference should be clear from the context. Also remember that the point "$\cdot$" often means index summation in what follows.

Now given,

$$\varpi_1^W = \begin{pmatrix} 0 & \alpha_1^W & 0 \\ \theta^W & A_1^W & \alpha_1^{t\,W} \\ 0 & \theta^{t\,W} & 0 \end{pmatrix},$$

whose entries are explicitly calculated in Appendix C.2.2, we have

$$\varpi_0^W := \begin{pmatrix} 0 & P^W & 0 \\ dx^W & \Gamma^W & (g^{-1}P^T)^W \\ 0 & (dx^T\cdot g)^W & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1^W ze & 0 \\ e^{-1}z^{-1}\theta^W & e^{-1}A_1^W e + e^{-1}e\, z^{-1}dz + e^{-1}de & e^{-1}z^{-1}\alpha_1^{t\,W} \\ 0 & z\theta^{t\,W}e & 0 \end{pmatrix},$$

Entry (2, 1) gives $dx^W = dx$, which is the obvious invariance of the coordinate chart under Weyl rescaling. Entry (3, 2) gives,

$$(dx^T\cdot g)^W = dx^T\cdot(z^2 g), \quad \text{in components} \quad \left(g_{\mu\nu}\right)^W = z^2 g_{\mu\nu}, \tag{2.51}$$

which is the **Weyl rescaling of the metric tensor**. Entry (2, 2) gives,

$$\Gamma^W = \Gamma + \delta\, z^{-1}dz + \delta\cdot dx\, z^{-1}\partial z - g^{-1}\cdot z^{-1}\partial z\, dx^T\cdot g,$$
$$\text{in components} \quad \left(\Gamma^\rho{}_{\mu\nu}\right)^W = \Gamma^\rho{}_{\mu\nu} + \delta^\rho_\nu\, z^{-1}\partial_\mu z + \delta^\rho_\mu\, z^{-1}\partial_\nu z - g^{\rho\lambda}\, z^{-1}\partial_\lambda z\, g_{\mu\nu}.$$

By convenience we define $\gamma_\mu := z^{-1}\partial_\mu z$ so that the above equation reads,

$$\left(\Gamma^\rho{}_{\mu\nu}\right)^W = \Gamma^\rho{}_{\mu\nu} + \delta^\rho_\nu\,\gamma_\mu + \delta^\rho_\mu\,\gamma_\nu - g^{\rho\lambda}\,\gamma_\lambda\, g_{\mu\nu}. \tag{2.52}$$

Entry (1, 2) gives,

$$P^W = P + \left(d(z^{-1}\partial z) - z^{-1}\partial z\cdot\Gamma\right) - z^{-1}dz\, z^{-1}\partial z + \frac{1}{2}(z^{-1}\partial z\cdot g\cdot z^{-1}\partial z)dx^T\cdot g,$$
$$\text{in components} \quad \left(P_{\mu\nu}\right)^W = P_{\mu\nu} + \nabla_\mu(z^{-1}\partial_\nu z) - z^{-1}\partial_\mu z\, z^{-1}\partial_\nu z + \frac{1}{2}(z^{-1}\partial_\lambda z\, g^{\lambda\alpha}\, z^{-1}\partial_\alpha z)\, g_{\mu\nu},$$
$$\text{or} \quad \left(P_{\mu\nu}\right)^W = P_{\mu\nu} + \nabla_\mu\gamma_\nu - \gamma_\mu\,\gamma_\nu + \frac{1}{2}\,\gamma_\lambda\gamma^\lambda\, g_{\mu\nu}. \tag{2.53}$$

with $\gamma^\lambda := g^{\lambda\alpha}\gamma_\alpha$. Entry (2, 3) gives,

$$\left(g^{-1}P^T\right)^W = z^{-2}g^{-1}\left(P^T + \left(d(z^{-1}\partial z) - \Gamma^T\cdot z^{-1}\partial z\right) - z^{-1}\partial z\cdot z^{-1}dz + \frac{1}{2}\, g\cdot dx(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z)\right),$$
$$\text{in components} \quad \left(g^{\rho\lambda}P_{\lambda\mu}\right)^W = z^{-2}g^{\rho\lambda}\left(P_{\lambda\mu} + \nabla_\mu\gamma_\lambda - \gamma_\lambda\gamma_\mu + \frac{1}{2}\, g_{\lambda\mu}\,\gamma_\alpha\gamma^\alpha\right), \tag{2.54}$$

This is then redundant with (2.51) and (2.53).

Remark: Equations (2.52) and (2.53) look like the familiar conformal transformations of the Christoffel symbols and of the Schouten tensor. Notice however that in this framework, the metricity condition $\nabla g = 0$ being automatic, Γ is the Levi-Civita connection and P is the Schouten 1-form *only in the normal case.* But the present calculations hold even without this assumption! We then obtain at once the Weyl variation of the symmetric *and* anti-symmetric parts of Γ and P. Explicitly, if $\Gamma^\rho{}_{\mu\nu} = {}^\Lambda\Gamma^\rho{}_{\mu\nu} + {}^S\Gamma^\rho{}_{\mu\nu}$ and $P_{\mu\nu} = {}^\Lambda P_{\mu\nu} + {}^S P_{\mu\nu}$, then

$$\left({}^\Lambda\Gamma^\rho{}_{\mu\nu}\right)^W = {}^\Lambda\Gamma^\rho{}_{\mu\nu}, \qquad \text{and} \qquad \left({}^S\Gamma^\rho{}_{\mu\nu}\right)^W = {}^S\Gamma^\rho{}_{\mu\nu} + \delta^\rho_\nu\,\gamma_\mu + \delta^\rho_\mu\,\gamma_\nu - g^{\rho\lambda}\,\gamma_\lambda\, g_{\mu\nu} \tag{2.55}$$

$$\left({}^\Lambda P_{\mu\nu}\right)^W = {}^\Lambda P_{\mu\nu} - \gamma_\lambda{}^\Lambda\Gamma^\lambda{}_{\mu\nu} \qquad \text{and} \qquad \left({}^S P_{\mu\nu}\right)^W = {}^S P_{\mu\nu} + \left(\partial_\mu\gamma_\nu - \gamma_\lambda{}^S\Gamma^\lambda{}_{\mu\nu}\right) - \gamma_\mu\,\gamma_\nu + \frac{1}{2}\,\gamma_\lambda\gamma^\lambda\, g_{\mu\nu},$$

where the two equalities on the right are indeed the **transformations of the Christoffel symbols and of the Schouten tensor under Weyl rescaling of the metric**. Thus, with (2.52) and (2.53), not only do we recover classical results in a much more operative way, but we do so on a more general footing: we don't need the assumption that Γ and P are functions of the metric tensor g, as is usually the case when one works with the Levi-Civita connection. We find this transformations 'from above'.

Residual Weyl gauge symmetry of Ω_0 For the composite field Ω_0 we have,

$$\Omega_0^W = \left(\Omega_1^W\right)^{u_0^W} = \left(\Omega_1^W\right)^{\widehat{W}u_0} = u_0^{-1}\widehat{W}^{-1}\cdot\left(\Omega_1^W\right)\cdot\widehat{W}u_0.$$

Now given the transformation of the cuvature of the Cartan connection under the Weyl group,

$$\Omega^W = \begin{pmatrix} f & z^{-1}\Pi & 0 \\ z\Theta & F & z^{-1}\Pi^t \\ 0 & z\Theta^t & 0 \end{pmatrix}, \quad \text{one easily obtains} \quad \Omega_1^W = \begin{pmatrix} f_1^W & \Pi_1^W & 0 \\ \Theta^W & F_1^W & \Pi_1^{t\,W} \\ 0 & \Theta^{t\,W} & -f_1^W \end{pmatrix},$$

whose entries are calculated in Appendix C.2.2 (knowing (2.49) and $\Omega_1 = u_1^{-1}\Omega u_1$ given shortly after (2.35)). With the latter at hand, it is simple to find,

$$\Omega_0^W = \begin{pmatrix} f_1^W & C^W & 0 \\ T^W & W^W & C^{t\,W} \\ 0 & T^{t\,W} & -f_1^W \end{pmatrix} = \begin{pmatrix} f_1^W & z\Pi_1 e & 0 \\ e^{-1}z^{-1}\Theta^W & e^{-1}F_1 e & e^{-1}z^{-1}\Pi^{t\,W} \\ 0 & z\Theta^{t\,W}e & -f_1^W \end{pmatrix},$$

whose entries are, again, calculated in Appendix C.2.2.
Entries (1, 1) and (3, 3) give,

$$f_1^W = f_1 - z^{-1}\partial z\cdot T \quad \to \quad \left(P_{[\mu\sigma]}\right)^W = P_{[\mu\sigma]} - \gamma_\lambda T^\lambda{}_{\mu\sigma}, \tag{2.56}$$

which just reproduces the third equation in (2.55) above, since $P_{[\mu\sigma]} = {}^\Lambda P_{\mu\sigma}$ and $T^\lambda{}_{\mu\sigma} = \Gamma^\rho{}_{[\mu\sigma]} = {}^\Lambda\Gamma^\rho{}_{\mu\sigma}$.
Entry (2, 1) gives,

$$T^W = T \quad \to \quad \left(T^\rho{}_{\mu\sigma}\right)^W = T^\rho{}_{\mu\sigma} \tag{2.57}$$

which is not surprising on account of $T^\rho{}_{\mu\sigma} = \Gamma^\rho{}_{[\mu\sigma]} = {}^\Lambda\Gamma^\rho{}_{\mu\sigma}$, and of the first equation in (2.55).
Entry (3, 2) gives,

$$T^{t\,W} = z^2 T^t \quad \to \quad (T^T g)^W = T^T\, z^2 g \quad \text{which is redundant with (2.57) and (2.51).} \tag{2.58}$$

Entry (2, 2) gives,

$$W^W = W + T\,z^{-1}\partial z - g^{-1}\cdot z^{-1}\partial z\,T^T g,$$

$$\text{in components} \quad \left(W^\rho{}_{\nu,\mu\sigma}\right)^W = W^\rho{}_{\nu,\mu\sigma} + T^\rho{}_{\mu\sigma}\,\gamma_\nu - g^{\rho\lambda}\gamma_\lambda\,T_{\mu\sigma}{}^\alpha g_{\alpha\nu} \tag{2.59}$$

The transformations of the entry (1, 2) is more complicated,

$$C^W = C - z^{-1}\partial z\cdot W + z^{-1}\partial z\cdot\delta f + \frac{1}{2}\left(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\right)T^T g - \left(a + z^{-1}\partial z\right)\cdot T\,z^{-1}\partial z.$$

In components,

$$\left(C_{\nu,\mu\sigma}\right)^W = C_{\nu,\mu\sigma} - \gamma_\lambda W^\lambda{}_{\nu,\mu\sigma} + \gamma_\lambda\,\delta^\lambda_\nu\,f_{\mu\sigma} + \frac{1}{2}\left(\gamma_\lambda g^{\lambda\alpha}\gamma_\alpha\right)T_{\mu\sigma}{}^\beta g_{\beta\nu} - \left(a_\lambda + \gamma_\lambda\right)T^\lambda{}_{\mu\sigma}\,\gamma_\nu. \tag{2.60}$$

At last, entry (2, 3) gives,

$$\begin{aligned} C^{t\,W} &= (g^{-1}C^T)^W, \\ &= z^{-2}g^{-1}\cdot\Big(C^T - W^T\cdot z^{-1}\partial z \,+\, f\delta\cdot z^{-1}\partial z \,+\, \tfrac{1}{2}gT\big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\big) \,-\, z^{-1}\partial z T^T\big(a+z^{-1}\partial z\big)\Big). \end{aligned}$$

In components,

$$\begin{aligned} \left(g^{\rho\lambda}C_{\lambda,\mu\sigma}\right)^W &= \left(g^{\rho\lambda}\right)^W\left(C_{\lambda,\mu\sigma}\right)^W, \\ &= z^{-2}g^{\rho\lambda}\Big(C_{\lambda,\mu\sigma} \,-\, W_\lambda{}^\alpha{}_{\mu\sigma}\gamma_\alpha \,+\, \delta^\alpha_\lambda\, f_{\mu\sigma}\gamma_\alpha \,+\, \tfrac{1}{2}g_{\lambda\alpha}T^\alpha{}_{\mu\sigma}\big(\gamma_\beta g^{\beta\delta}\gamma_\delta\big) \,-\, \gamma_\lambda T_{\mu\sigma}{}^\alpha\big(a_\alpha+\gamma_\alpha\big)\Big). \end{aligned} \tag{2.61}$$

which is redundant with (2.51) and (2.60).

Residual Weyl gauge freedom of ϖ_0 and Ω_0 in the Normal case It is now trivial to specialize the above transformations to the case where the initial Cartan connection ϖ is normal. Indeed the normality of ϖ means,

$$\Theta = 0, \quad f = \frac{1}{m}\mathrm{Tr}(\Omega^0) = 0, \quad \text{and} \quad \mathrm{Ric}\big(\Omega^0\big) := \mathrm{Ric}\big(F - f\mathbb{1}\big) = \mathrm{Ric}\big(F\big) = 0,$$

by definition, as we know. These conditions are invariant under the action of the full group H of the Cartan-Möbius geometry, as it can be seen from Ω^{γ_1}, Ω^S and Ω^W .[31] This implies,

$$\Theta_1 = \Theta = 0, \quad f_1 = f - q\Theta = 0, \quad \text{and} \quad \mathrm{Ric}\big((\Omega_1)^0\big) := \mathrm{Ric}\big(F_1 - f_1\mathbb{1}\big) = \mathrm{Ric}\big(F_1\big) = \mathrm{Ric}\big(F\big) = 0.$$

The last equality follows from $F_1 = F + \Theta q - q^t\Theta^t$ and the above first condition. This means that ϖ_1 is normal too, as we've already seen. Moreover since ϖ_1 and Ω_1 are K_1-invariant these conditions are trivialy K_1-invariant. On account of Ω_1^S (see just above (2.36)), they are also invariant under SO. Finally, the above imply,

$$T = e^{-1}\Theta_1 = 0, \quad f_1 = P_{[\mu\sigma]} = 0, \quad \text{and} \quad \mathrm{Ric}\big((\Omega_0)^0\big) = \mathrm{Ric}\big(W - f_1\mathbb{1}\big) = \mathrm{Ric}\big(W\big) := \mathrm{Ric}\big(e^{-1}F_1e\big) = \mathrm{Ric}\big(e^{-1}Fe\big) = 0.$$

Then ϖ_0 is also normal, as already noted, and in this case remember that we have,

$$\varpi_0 = \begin{pmatrix} 0 & P_{\mu\nu} & 0 \\ \delta^\rho_\mu & \Gamma^\rho{}_{\mu\nu} & g^{\rho\lambda}P_{\lambda\mu} \\ 0 & g_{\mu\nu} & 0 \end{pmatrix} dx^\mu, \quad \text{with} \quad P_{\mu\nu} = \frac{-1}{(m-2)}\left(R_{\mu\nu} - \frac{R}{2(m-1)}g_{\mu\nu}\right), \tag{2.62}$$

$$\Omega_0 = \frac{1}{2}\begin{pmatrix} 0 & C_{\nu,\mu\sigma} & 0 \\ 0 & W^\rho{}_{\nu,\mu\sigma} & g^{\rho\lambda}C_{\lambda,\mu\sigma} \\ 0 & 0 & 0 \end{pmatrix} dx^\mu\wedge dx^\sigma. \tag{2.63}$$

The torsion free condition implies that Γ has only a symmetric part and, since $\nabla g = 0$ is automatic, it is the **Levi-Civita connexion** on $\mathcal{M}$. The second and third conditions imply that $P_{\mu\sigma}$ is the symmetric **Schouten tensor**, so that $C_{\nu,\mu\sigma} = \nabla_\mu P_{\sigma\nu}$ is the **Cotton tensor** and $W^\rho{}_{\nu,\mu\sigma}$ is the **Weyl tensor**. The normality of ϖ_0 is preserved by the action of the Weyl gauge group, as can be seen from Ω_0^W in the general case.

The gauge transformations of the **normal** composite fields ϖ_0 and its curvature Ω_0 living on the Weyl bundle $\mathcal{P}_W := \mathcal{P}(\mathcal{M}, W)$ then provide,

$$\left(dx^\mu\right)^W = dx^\mu, \quad \left(g_{\mu\nu}\right)^W = z^2 g_{\mu\nu}, \quad \text{and} \quad \left(g^{\mu\nu}\right)^W = z^{-2}g^{\mu\nu}. \tag{2.64}$$

[31](Sharpe, 1996) mentions this fact very briefly p.289.

The obvious invariance of the coordinate chart and the **Weyl rescaling of the metric tensor**.

$$\left(\Gamma^\rho{}_{\mu\nu}\right)^W = \Gamma^\rho{}_{\mu\nu} + \delta^\rho_\nu\, \gamma_\mu + \delta^\rho_\mu\, \gamma_\nu - g^{\rho\lambda}\, \gamma_\lambda\, g_{\mu\nu}, \tag{2.65}$$

with $\gamma_\mu := z^{-1}\partial_\mu z$. This is the **transformation of the Levi-Civita connection under Weyl rescaling**.

$$\left(P_{\mu\nu}\right)^W = P_{\mu\nu} + \nabla_\mu\gamma_\nu - \gamma_\mu\, \gamma_\nu + \frac{1}{2}\, \gamma_\lambda\gamma^\lambda\, g_{\mu\nu}, \tag{2.66}$$

which is the **transformation of the Schouten tensor under Weyl rescaling**.

$$\left(W^\rho{}_{\nu,\mu\sigma}\right)^W = W^\rho{}_{\nu,\mu\sigma}. \tag{2.67}$$

This is the **invariance of the Weyl tensor under Weyl rescaling**.

$$\left(C_{\nu,\mu\sigma}\right)^W = C_{\nu,\mu\sigma} - \gamma_\lambda W^\lambda{}_{\nu,\mu\sigma}, \tag{2.68}$$

which is the **transformation of the Cotton tensor under Weyl rescaling**.

Conclusion: Let us sum-up what have been done in this application of the dressing field method to Higher-order G-structure.

We started with the 2^{th}-order conformal structure also know as the Cartan-Möbius geometry, that is a principal bundle $\mathcal{P}_1 := \mathcal{P}(\mathcal{M}, H := WSK_1)$ equipped with a Cartan connection ϖ and its associated curvature Ω. Then, thanks to a gauge-like constraint $\chi\left(\varpi^{u_1}\right)$ (see appendix A.1) we were able to find a K_1-dressing field u_1. Exactly as in Lemma 1 the latter is used to dress ϖ and Ω and to produce the K_1-invariant and K_1-horizontal $\mathfrak{g}$-valued composite fields ϖ_1 and Ω_1. The latter live on a subbundle $\mathcal{P}_0 = \mathcal{P}(\mathcal{M}, K_0 := WS)$ of the initial bundle which can be thus written $\mathcal{P}_1 \simeq \mathcal{P}_0 \times K_1$, and is clearly trivial in the K_1 direction. The group K_1 is thus indeed decoupled from the relevant residual gauge geometry. We showed that the normality was preserved by the dressing operation.

Then, we checked that u_1 has the right gauge-like transformation under the Lorentz subgroup $S = SO(r,s)$ so as to induce connection-like and adjoint SO-gauge transformations for ϖ_1 and Ω_1 respectively. A necessary condition for a new dressing operation to make sense, as the previous section showed. Moreover this showed that ϖ_1 and Ω_1 contain the real objects already found in gauge formulation of Riemaniann geometry: the spin connection and its associated Riemann tensor. It was tempting to try to reduce the Lorentz subgroup S of the conformal group $CO(r,s) \sim WS$, exactly as in Lemma 2 as it was already illustrated in the electroweak sector of the Standard Model. Thanks to the experience taken from our treatment of General Relativity, we easily identified a suited candidate as second dressing field: $u_0 \sim e$, the vielbein. We checked that it was indeed invariant under K_1, so as to not lose the previously gained invariance, and used it to produce the K_1- and S-invariant/horizontal composite fields ϖ_0 and Ω_0. The latter live on the Weyl bundle $\mathcal{P}_W := \mathcal{P}(\mathcal{M}, W)$, a subbundle of $\mathcal{P}_0$, thus of $\mathcal{P}_1$ which can then be written $\mathcal{P}_1 = \mathcal{P}_0 \times K_1 = \mathcal{P}_W \times S \times K_1$, and is clearly trivial in the S and K_1 direction. Once again the normality was preserved by this second dressing.

The fields ϖ_0 and Ω_0 were shown to contain tensors of $\mathcal{M}$.[32] In the normal case those are the metric, the Levi-Civita connection and the Schouten tensor for ϖ_0, and the torsion, the Weyl tensor and the Cotton tensor for Ω_0. The dressing field method then ends-up on what is usually called the *Riemaniann parameterization of the normal conformal Cartan connection.* A classical result. Finally we derived the residual Weyl gauge symmetry of ϖ_0 and Ω_0 and found the well known transformations of the mentionned tensor under Weyl rescaling. But we do so with considerable economy of effort, in a quite systematic way thanks to the matrix formalism and on a more general footing that dispenses with the assumption that all tensors are functions of the sole metric. As far as I know, these transformations are not derived in the jet formalism.

Notice that, as was the case for General Relativity, the two dressing fields are found from the Cartan connection itself. Therefore, by analogy with Physics, the process was a change of variables, a convenient redistribution of the degrees of freedom. The initial and final geometries should then be the same in this

[32] One pseudo-tensor among them.

precise sense of having the same number of degrees of freedom. This is indeed shown by table 2.3 and 2.4.

With this example we close this rather long chapter. We've seen that the dressing field method is the geometric foundation of the notion of Dirac variables, that it already underlies several constructions found in the literature on hadronic physics and occasionally helps to clarify, deepen or correct their interpretive baggage. We've also seen that it can even offer new insight on well established theories. Finally we've considered its generalization on higher-order G-structures and seen its effectiveness in deriving important results in the example of conformal geometry.

The dressing field method is a new way to deal with gauge symmetry in an effective fashion. Until now we've considered *finite* gauge transformations, but in Physics it is often more convenient to work with the linearized version, that is *infinitesimal* gauge transformations. A formalism that handles the latter very effectively and has far-reaching applications in Quantum Fields Theory, like the cohomological treatment of anomalies, is the celebrated BRS formalism. It should not come as to much of a surprise that there is a neat interaction between the BRS formalism and the dressing field method. This is the object of the next chapter.

	Initial geometry	Degrees of freedom	Final geometry	Degrees of freedom
(1) Variables	ϖ	$m\Big(1+\frac{m(m-1)}{2}+2m\Big)$	$g_{\mu\nu}$ $\Gamma^{\rho}{}_{\mu\nu}$ $P_{\mu\nu}$	$\frac{m(m+1)}{2}$ m^3 m^2
(2) Symmetry	$H \sim WSK_1$	$1+\frac{m(m-1)}{2}+m$	Weyl group $W=\mathbb{R}^*$	1
(3) Constrains	$\emptyset$	0	$\nabla g = 0$	$m\frac{m(m+1)}{2}$
Total d.f (1) − ((2) + (3))		$\frac{1}{2}\Big(m^3+2m^2+m-2\Big)$		$\frac{1}{2}\Big(m^3+2m^2+m-2\Big)$

Table 2.3: Count of the degrees of freedom of the **non normal** geometry before and after the dressing operation. Recall that ϖ is $\mathfrak{g}$-valued, and that the metricity is automatic here.

	Initial geometry	Degrees of freedom	Final geometry	Degrees of freedom
(1) Variables	ϖ	$m\Big(1+\frac{m(m-1)}{2}+2m\Big)$	$g_{\mu\nu}$ $\Gamma^{\rho}{}_{\mu\nu}$ $P_{\mu\nu}$	$\frac{m(m+1)}{2}$ m^3 0
(2) Symmetry	$H \sim WSK_1$	$1+\frac{m(m-1)}{2}+m$	Weyl group $W=\mathbb{R}^*$	1
(3) Constrains	$f=0$ $\Theta=0$ Ric(F)=0	$\frac{m(m-1)}{2}$ $m\frac{m(m-1)}{2}$ $\frac{m(m+1)}{2}$	$\nabla g = 0$ $T=0$	$m\frac{m(m+1)}{2}$ $m\frac{m(m-1)}{2}$
Total d.f (1) − ((2) + (3))		$\frac{m(m+1)}{2}-1$		$\frac{m(m+1)}{2}-1$

Table 2.4: Count of the degrees of freedom of the **normal** geometry before and after the dressing operation. It is easy to see how the first and third requirement defining normality entail that $P_{\mu\nu}$ becomes the Schouten tensor, a non-independant variable expressed in terms of $g_{\mu\nu}$. Finally, remark that the normal geometry is the most natural one for its total degrees of freedom are those of a conformal class of metric $[g_{\mu\nu}]$.

Chapter 3
Dressing field and BRS formalism

This chapter is devoted to the analysis of the interaction between the dressing field method and the BRS approach to gauge theories.

The first section motivates the introduction of the BRS symmetry from the viewpoint of Physics and then outlines the mathematical underpinning of the heuristic BRS formalism. This is done essentially to fix the notations and to show that BRS algebra is actually the algebra of infinitesimal gauge transformations.

The second section will show how the dressing field method alters the BRS algebra of a theory, producing what we will call a *composite ghost*. The very first appearances of such an object in particular cases are to be found in (Garajeu et al., 1995) and (Lazzarini and Tidei, 2008). We will present the infinitesimal version of the multiple compositions of dressing fields seen in the previous chapter, that is the tower of successive reduced BRS algebras, and I will give the BRS version of the compatibility conditions that should hold in such a case.

In a third section, the scheme is applied to GR and to the Cartan-Möbius geometry. In the latter case the BRS algebra produces the infinitesimal version of the results derived in 2.4.2.

Finally the fourth section will show how the diffeomorphisms of the base manifold can be incorporated in the formalism and how this spontaneously provides the shifted BRS algebra à la (Langouche et al., 1984). The application to the Cartan-Möbius geometry is given, the modified BRS algebra handles both the Weyl rescaling and the diffeomorphisms.

3.1 The BRS approach, an outline

Since its inception in the mid 70's by Becchi, Rouet and Stora in (Becchi et al., 1975) and (Becchi et al., 1976), the BRS formalism has seeded considerable work and became a standard tool in the analysis of gauge theories and their most subtle properties, especially with respect to their quantization and their non-perturbative features like the anomalies. See (Bertlmann, 1996) and the next chapter for a word about this aspect. As a consequence the physical literature on the subject is vast and it is hard to get a clear and synthetic picture of the state of the art. As to the attempt to give a rigorous mathematical foundation and meaning to the BRS approach, they have been many and if it is now finally recognized that it is related to the cohomology of the Lie algebra of the gauge group of the theory, the exposures often differ so that it is not easy to find a canonical presentation.

Given this state of affairs, in this section I will first briefly remind the motivation for the introduction of the BRS transformations in gauge theories, without entering the details though, and give the minimal BRS algebra that we will use in the chapter. Then I will give my own account of how I understand the link between the BRS formalism and the Lie algebra cohomology of the gauge group.

3.1.1 The symmetry of the gauge fixed effective Lagrangian

As we've seen several times, the gauge symmetry poses a problem for anyone who wants to apply the path integral algorithm for quantization. We've also mentionned the fact that an obvious move is to select a single representative in each gauge orbit by gauge fixing. This was the celebrated contribution of (Faddeev and Popov, 1967) to find a clever general way to do so. There is no need here to be involved in the technical details so I just sketch the idea, following (Bertlmann, 1996).

The path integral $Z = \int_{\mathcal{A}\times\mathcal{F}} dAd\varphi\ e^{iS(A,\varphi)}$, where $S(A,\varphi) = \int_{\mathcal{M}} L_{\text{YM}} + L_{\text{matter}}$ is the action of the classical gauge field theory, is ill-defined. It diverges due to the gauge symmetry. A gauge fixing condition $\chi(A^\gamma) = 0$ is then chosen and is inserted in Z as the identity $\int d\gamma \det\left(\frac{\delta\chi(A^\gamma)}{\delta\gamma}\right)\delta\Big(\chi(A^\gamma)\Big) = 1$, where $\det\left(\frac{\delta\chi(A^\gamma)}{\delta\gamma}\right)$ is the *Faddeev-Popov determinant.* Then the gauge transformation $\gamma' = \gamma^{-1}$ is performed on Z and due to the invariance of the measure and of $S(A,\varphi)$ it gives,[1]

$$Z = \int_{\mathcal{A}\times\mathcal{F}} d\gamma dA d\varphi \;\det\left(\tfrac{\delta\chi(A)}{\delta\gamma}\right)\delta\Big(\chi(A)\Big)\; e^{iS(A,\varphi)}.$$

The integrand does not explicitly depend on the gauge group element γ so integration over it factorizes and produces and infinite normalization that can be conventionally removed. The Faddeev-Popov determinant is shown to be the determinant of a differential operator (depending on the chosen gauge fixing condition χ) and the next step is to express it a Gaussian integral over the $\mathfrak{g}$-valued Grassmann variables v and $\bar{v}$, the so-called **Faddeev-Popov ghost and antighost**, so that finally one obtains,

$$Z = \int_{(\mathcal{A}\times\mathcal{F})/\mathcal{G}} dA d\varphi dv d\bar{v} \;\; e^{iS(A,\varphi,v,\bar{v})}, \quad \text{with} \quad S(A,\varphi,v,\bar{v}) = \int_{\mathcal{M}} L_{\text{eff}}(A,\varphi,v,\bar{v}),$$
$$\text{and} \quad L_{\text{eff}} = L_{\text{YM}} + L_{\text{matter}} + L_{\text{gauge-fix}} + L_{\text{ghost}}. \tag{3.1}$$

The gauge fixing condition is implemented by $L_{\text{gauge-fix}}$, and the dynamics of the anticommuting ghost scalar fields and their interaction with the gauge potential A is given by L_{ghost}.

The initial Lagrangian $L = L_{\text{YM}} + L_{\text{matter}}$ was already invariant under gauge transformations whose infinitesimal (linear) expression reads,

$$\delta A = d\lambda + A\lambda - \lambda A = D\lambda, \qquad \delta F = [F,\lambda], \qquad \text{and} \qquad \delta\varphi = -\rho_*(\lambda)\varphi,$$

where λ is $\mathfrak{g}$-valued and such that $\gamma = e^{\lambda}$, and ρ_* is the representation of $\mathfrak{g}$ on V. The observation of Becchi, Rouet and Stora was that, while the gauge invariance seems lost because of $L_{\text{gauge-fix}}$, the effective Lagrangian (3.1) is invariant under an extended set of transformations, now called **BRS transformations**. The structure is given as follows.

A *ghost degree* is attributed to our fields: 0 for the usual gauge fields A, F and φ, 1 to the ghots field v and -1 to the antighost field $\bar{v}$. The ghost degree added to the form degree constitutes the total degree and we consider the graded algebra, generated by the fields, with respect to this total degree. We define the graded commutator in this algebra in the usual way and we have a graded Jacobi identity,

$$[a,b] = ab - (-)^{|a|\cdot|b|}ba, \qquad (-)^{|a|\cdot|c|}\Big[a,[b,c]\Big] + (-)^{|b|\cdot|a|}\Big[b,[c,a]\Big] + (-)^{|c|\cdot|b|}\Big[c,[a,b]\Big] = 0$$

The exterior derivative increases by one the form degree and acts as a nilpotent antiderivative of the graded algebra. One defines symmetrically the **BRS operator**, s, which increments by one the ghost number and also acts as an antiderivative in the graded algebra. Its action on the fields is given by,

$$sA = -Dv = -dv - [A,v] = -dv - Av - vA, \qquad sF = [F,v] = Fv - vF, \qquad \text{and} \qquad s\varphi = -\rho_*(v)\varphi.$$

We see that the BRS operator acts on the gauge fields so as to reproduce gauge transformations with the anticommuting $\mathfrak{g}$-valued ghost field v as infinitesimal parameter. In this way we thus have,

$$s\left(L_{\text{YM}} + L_{\text{matter}}\right) = 0.$$

The requirement of the nilpotency of the operator $\widetilde{d} = d + s$ implies $\{d,s\} = 0$ and $s^2 = 0$. The nilpotentcy of s can be achieved though an adequate choice of its action on the ghost and antighost. The calculation of the square of s on A gives,

$$s^2A = -sdv - [sA,v] + [A,sv] = d(sv) + [A,sv] + [Dv,v] = D(sv) + \tfrac{1}{2}D[v,v] = D\Big(sv + \tfrac{1}{2}[v,v]\Big).$$

[1]Remark that the gauge fixing condition is not invariant under the gauge transformation, $\chi(A^{\gamma}) \to \chi(A)$. We've moved from a section in $\mathcal{A}\times\mathcal{F}$ to another. This should remind the reader of our discussion in appendix A.1.

In this example, to require $s^2 = 0$ suggests the transformation for the ghost,

$$sv = -\tfrac{1}{2}[v, v], \tag{3.2}$$

This transformation is easily seen to secure the nilpotency of the BRS operator on φ and, thanks to the graded Jacobi identity, on F and v itself. The antighost is a bit isolated and its transformation is chosen as,

$$s\bar{v} = b, \qquad \text{where } b \text{ is } \mathfrak{g}\text{-valued, and} \qquad sb = 0, \tag{3.3}$$

so as to secure once again the nilpotency of s. Now if the auxiliary field b is chosen so as to be proportional to the gauge fixing condition χ, (3.2) and (3.3) allows to rewrite the remaining pieces of the effective Lagrangian as $L_{\text{gauge-fix}} + L_{\text{ghost}} = f(b) + sg$, where f is a polynomial function and g a term whose precise expression depends on χ. In this way, thanks to the nilpotency of s, one obtains the invariance of the gauge-fixed effective Lagrangian under BRS transformations

$$sL_{\text{eff}} = 0.$$

Apart from uncovering a symmetry of the effective Lagrangian, often referred to as a 'enlarged gauge symmetry', the relevance of the BRS formalism lies in two main points. First, given the *quantum action* $W = -i \ln Z$, the Ward identity is concisely written as $sW = 0$. The latter must hold for the quantum theory to be renormalizable. Second, if the Ward identity is violated so that $sW = G$, the renormalizability is lost. The term G is an (integrated) *anomaly* and the BRS cohomology (modulo d) is an effective tool to calculate and classify the anomalies. We will say a bit more on the subject in the last chapter.

3.1.2 Cohomological viewpoint

The minimal BRS algebra that we will use throughout this chapter involves only the standard gauge field together with the ghost,

$$sA = -Dv, \qquad sF = [F, v], \qquad s\varphi = -v\varphi, \qquad \text{and} \qquad sv = -\tfrac{1}{2}[v, v]. \tag{3.4}$$

We've indeed seen that the antighost sector is isolated and this minimal BRS algebra is the relevant one when the problem of anomalies is considered and cohomological tools are used. The resemblance of the last equation with the Cartan-structure equation suggests to interpret the $\mathfrak{g}$-valued anticommuting field v as a Maurer-Cartan form. This is indeed a correct interpretation and it is possible to show that the above BRS algebra is actually a simple instance of the cohomology of the infinite dimensional Lie algebra of the gauge group. The following account is inspired by the neat paper of (Bonora and Cotta-Ramusino, 1983) which adopts a local viewpoint while here the construction is kept global and turns out to be similar to the viewpoint found in (Chevalley and Eilenberg, 1948). Alternative but related presentation can be found in (Azcarraga and Izquierdo, 1995). A summary on the variety of equivalent approaches to the same topic, by Stora, Zumino and others, is given in chapter 8 and 9 of (Bertlmann, 1996) (see references therein).

Recall that the gauge group of a principal bundle $\mathcal{P}(\mathcal{M}, G)$, isomorphic to its group of vertical automorphisms, was defined as $\mathcal{G} = \left\{\gamma : \mathcal{P} \to G \mid \mathcal{R}_g^* \gamma(p) = Ad_{g^{-1}}\gamma(p)\right\}$. We now also define the infinite dimensional **Lie algebra of the gauge group** as,

$$\mathscr{g} = \left\{\lambda : P \to \mathfrak{g} = T_e G \mid \mathcal{R}_g^* \lambda(p) = Ad_{g^{-1}}\lambda(p)\right\} = T_{\text{id}}\mathcal{G}.$$

We assume that $\mathcal{G}$ is connected so that the exponential map generates the gauge group or its universal cover. The Lie algebra $\mathscr{g}$ is isomorphic to the set of right-invariant vector fields on $\mathcal{G}$, noted $\{X_\gamma^R\} \subset T_\gamma\mathcal{G}$, and the isomorphism is provided by the **Maurer-Cartan form** of the gauge group,

$$\omega_{\mathcal{G}} := R_{\gamma^{-1}*} : T_\gamma\mathcal{G} \to T_{\text{id}}\mathcal{G} = \mathscr{g}.$$

The Maurer-Cartan form is right-invariant. Indeed for $X_\gamma \in T_\gamma\mathcal{G}$ and $\alpha \in \mathcal{G}$,

$$R_\alpha^* \omega_{\mathcal{G}}\big(X_\gamma\big) = \omega_{\mathcal{G}}(R_{\alpha*}X_\gamma) = R_{(\gamma\alpha)^{-1}*}R_{\alpha*}X_\gamma = R_{(\alpha^{-1}\gamma^{-1})*}R_{\alpha*}X_\gamma = R_{\gamma^{-1}*}R_{\alpha^{-1}*}R_{\alpha*}X_\gamma = R_{\gamma^{-1}*}X_\gamma = \omega_{\mathcal{G}}(X_\gamma).$$

As such it is the basis for all $\mathcal{V}$-valued right-invariant forms on $\mathcal{G}$, noted $\Omega^*_R(\mathcal{G},\mathcal{V})$, where $\mathcal{V}$ is a representation space for $\mathcal{G}$ and $\mathfrak{g}$. A r-form, $\eta \in \Omega^r_R(\mathcal{G},\mathcal{V})$ decomposes as $\eta = \frac{1}{r!}\eta_i \cdot \Lambda^r \omega^i_{\mathcal{G}}$ with $i \in \{1,\cdots,r\}$ and $\eta_i \in \mathcal{V}$. We have that,

$$\mathcal{V} \otimes \Lambda^r \omega_{\mathcal{G}} : T_\gamma \mathcal{G} \times \cdots \times T_\gamma \mathcal{G} \to \mathcal{V} \otimes \Lambda^r \mathfrak{g}.$$

Denote by c^r the $\mathcal{V}$-valued r-cochain defined as a skew-symmetric r-linear mapping,

$$c^r : \mathfrak{g}^* \times \cdots \times \mathfrak{g}^* \to \mathcal{V},$$

where $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$ defined in an obvious manner. The set of all such r-cochains is noted $C^r(\mathfrak{g}^*,\mathcal{V})$ and the full complex is $C^*(\mathfrak{g}^*,\mathcal{V})$. We see that there is an isomorphism between the algebra of $\mathcal{V}$-valued right-invariant forms on $\mathcal{G}$ and the algebra of $\mathcal{V}$-valued cochain on $\mathfrak{g}^*$,

$$\Omega^*_R(\mathcal{G},\mathcal{V}) \simeq C^*(\mathfrak{g}^*,\mathcal{V}).$$

This is referred to as the process of "localization" in (Chevalley and Eilenberg, 1948). Moreover the **exterior derivative** on $\mathcal{G}$, $\delta : \Omega^r(\mathcal{G},\mathcal{V}) \to \Omega^{r+1}(\mathcal{G},\mathcal{V})$, defined as,

$$\begin{aligned}\delta\alpha(X_1,\cdots,X_{r+1}) = &\sum_{i=1}^{r+1}(-)^{i+1}\zeta(X_i)\cdot\alpha(x_1,\cdots,\widehat{X}_i,\cdots,X_{r+1})\\ &+\sum_{i<j}(-)^{i+j}\alpha([X_i,X_j],\cdots,\widehat{X}_1,\cdots,\widehat{X}_j,\cdots X_{r+1}),\end{aligned}$$

for $\alpha \in \Omega^r(\mathcal{G},\mathcal{V})$ and where ζ is an action of X_i on $\mathcal{V}$, when restricted to $\Omega^*_R(\mathcal{G},V)$ plays precisely the role of the **coboundary operator** for the complex $C^*(\mathfrak{g}^*,V)$. This means that the set of **closed** $\mathcal{V}$-valued right-invariant r-forms is isomorphic to the set of $\mathcal{V}$-valued r-**cocycles**, $Z^*_R(\mathcal{G},H) \simeq Z^*(\mathfrak{g}^*,\mathcal{V})$, and the set of **exact** $\mathcal{V}$-valued right-invariant r-forms is isomorphic to the set of $\mathcal{V}$-valued r-**coboundaries**, $B^*_R(\mathcal{G},\mathcal{V}) \simeq B^*(\mathfrak{g}^*,\mathcal{V})$. This in turns implies the isomorphism of **cohomology** $H^*_R(\mathcal{G},\mathcal{V}) \simeq H^*(\mathfrak{g}^*,\mathcal{V})$.

Consider now the case $\mathcal{V} = \Omega^0_{\text{loc}}(\mathcal{A}_\mathcal{P}\times\mathcal{F}_\mathcal{P})$, that is, we will consider right-invariant forms on $\mathcal{G}$ with values in the functionals on the space of connections and of tensorial forms of the principal bundle $\mathcal{P}(\mathcal{M},G)$. The subscript "loc" means that we only consider the space of *local* functionals on $\mathcal{A}_\mathcal{P}\times\mathcal{F}_\mathcal{P}$ which are polynomials, as is most relevant for Physics. We need to define the action ζ of a vector field $X_\gamma \in T_\gamma\mathcal{G}$ on $\Omega^0_{\text{loc}}(\mathcal{A}_\mathcal{P}\times\mathcal{F}_\mathcal{P})$. We know that $\mathcal{A}_\mathcal{P}\times\mathcal{F}_\mathcal{P}$ can be seen as a principal bundle with structure group $\mathcal{G}$ acting on the right as,

$$\Big((\mathcal{A}_\mathcal{P}\times\mathcal{F}_\mathcal{P})\times\mathcal{G}\Big) \xrightarrow{\mathcal{R}} (\mathcal{A}_\mathcal{P}\times\mathcal{F}_\mathcal{P}), \qquad \Big((\omega,\beta)\times\gamma\Big) \mapsto \mathcal{R}_\gamma\cdot(\omega,\beta) := (\omega^\gamma,\beta^\gamma)$$

Then $\mathfrak{g}$ naturally induces fundamental vector fields on $\mathcal{A}_\mathcal{P}\times\mathcal{F}_\mathcal{P}$ through the action $\mathcal{R}_*$ defined as,

$$X^v_\lambda f\big(\omega,\beta\big) := \mathcal{R}_*(\lambda)\cdot f\big(\omega,\beta\big) = \frac{d}{dt}\bigg|_{t=0} f\big(\omega^{\exp t\lambda},\beta^{\exp t\lambda}\big) = \frac{d}{dt}\bigg|_{t=0} f\big(\omega + tD\lambda, \rho(1-t\lambda)\beta\big),$$

with $f \in \Omega^0(\mathcal{A}_\mathcal{P}\times\mathcal{F}_\mathcal{P})$ and β tensorial of type (V,ρ). It seems therefore natural to define the action ζ as,

$$\zeta(X_\gamma) = \mathcal{R}_*\big(\omega_{\mathcal{G}}(X_\gamma)\big).$$

With this action at hand we can calculate the δ exterior derivative of $\omega \in \mathcal{A}_\mathcal{P}$ and $\Omega,\psi \in \mathcal{F}_\mathcal{P}$ which belong to $\Omega^0_R\Big(\mathcal{G},\Omega^0_{\text{loc}}(\mathcal{A}_\mathcal{P}\times\mathcal{F}_\mathcal{P})\Big)$. For the connection first,

$$\begin{aligned}&\delta\omega(X_\gamma) = \zeta(X_\gamma)\cdot\omega = \mathcal{R}_*\big(\omega_{\mathcal{G}}(X_\gamma)\big)\cdot\omega = \frac{d}{dt}\bigg|_{t=0}\Big(\omega + tD\omega_{\mathcal{G}}(X_\gamma)\Big) = D\omega_{\mathcal{G}}(X_\gamma),\\ &\to \qquad \delta\omega = D\omega_{\mathcal{G}} = d\omega_{\mathcal{G}} + [\omega,\omega_{\mathcal{G}}].\end{aligned}$$

For the curvature next,

$$\delta\Omega(X_Y) = \zeta(X_Y)\cdot\Omega = \mathcal{R}_*\big(\omega_{\mathcal{G}}(X_Y)\big)\cdot\Omega = \frac{d}{dt}\Big|_{t=0}\Big(\mathrm{Ad}_{\left(1-t\omega_{\mathcal{G}}(X_Y)\right)}\Omega\Big) = [\Omega,\omega_{\mathcal{G}}(X_Y)],$$
$$\rightarrow \qquad \delta\Omega = [\Omega,\omega_{\mathcal{G}}].$$

For the equivariant map ψ,

$$\delta\psi = \zeta(X_Y)\cdot\psi = \mathcal{R}_*\big(\omega_{\mathcal{G}}(X_Y)\big)\cdot\psi = \frac{d}{dt}\Big|_{t=0}\Big(\big(1-t\omega_{\mathcal{G}}(X_Y)]\big)\psi\Big) = -\rho_*\big(\omega_{\mathcal{G}}(X_Y)\big)\psi,$$
$$\rightarrow \qquad \delta\psi = -\rho_*\big(\omega_{\mathcal{G}}\big)\psi.$$

Obviously the δ exterior derivative for the Maurer-Cartan form $\omega_{\mathcal{G}} \in \Omega^1_R(\mathcal{G})$, with the action $\zeta(X_Y) = X_Y$, gives

$$\delta\omega_{\mathcal{G}}(X^1_Y,X^2_Y) = -\omega_{\mathcal{G}}[X^1_Y,X^2_Y] = -[\omega_{\mathcal{G}}(X^1_Y),\omega_{\mathcal{G}}(X^2_Y)] = -\tfrac{1}{2}[\omega_{\mathcal{G}},\omega_{\mathcal{G}}](X^1_Y,X^2_Y),$$
$$\rightarrow \qquad \delta\omega_{\mathcal{G}} = -\tfrac{1}{2}[\omega_{\mathcal{G}},\omega_{\mathcal{G}}],$$

which is just the structure equation. The above equations already closely resemble a global version of the BRS algebra. We are dealing here with a bicomplex for $\Omega^*_R\big(\mathcal{G},\Omega^0_{\mathrm{loc}}(\mathcal{A}_{\mathcal{P}}\times\mathcal{F}_{\mathcal{P}})\big) \simeq \Omega^*_R(\mathcal{G})\otimes\Lambda^*(\mathcal{P})$. In this bicomplex d raises by one the usual form degree and δ does so for the group form degree. Nevertheless the two operators commute, $[d,\delta] = 0$. To make contact more closely with the BRS algebra we define the **BRS operator** as

$$s = (-)^p\delta,$$

where p is the usual form degree. In this way we have the anticommutation of the standard exterior derivative and of the BRS operator, $\{d,s\} = 0$, and the above formulae read,

$$s\omega = -D\omega_{\mathcal{G}}, \qquad s\Omega = [\Omega,\omega_{\mathcal{G}}], \qquad s\psi = -\rho_*\big(\omega_{\mathcal{G}}\big)\psi \qquad \text{and} \qquad s\omega_{\mathcal{G}} = -\tfrac{1}{2}[\omega_{\mathcal{G}},\omega_{\mathcal{G}}].$$

Now we have the bicomplex $\Omega^*_R(\mathcal{G})\otimes\Lambda^*(\mathcal{P})$ graded by the total degree: de Rham form degree + group form degree, where d and s act as antiderivatives and such that $\widetilde{d}^{\,2} = (d+s)^2 = 0$. This is the global version of the BRS algebra. The above expresses in a compact form all possible infinitesimal active gauge transformations of the connection, the curvature and of the equivariant maps on $\mathcal{P}$. A specific transformation is picked for a given vector field on $\mathcal{G}$ such that $\omega_{\mathcal{G}}(X_Y) = \lambda \in \mathfrak{g}$.

We recover the local (meaning 'basic', on $\mathcal{M}$) BRS algebra discussed in the previous section considering the bicomplex $\Omega^*_R\big(\mathcal{G},\Omega^0_{\mathrm{loc}}(\mathcal{A}\times\mathcal{F})\big) \simeq \Omega^*_R(\mathcal{G})\otimes\Lambda^*(\mathcal{M})$, where,

$$sA = -Dv_{\mathcal{G}}, \qquad sF = [F,v_{\mathcal{G}}], \qquad s\varphi = -\rho_*\big(v_{\mathcal{G}}\big)\varphi \qquad \text{and} \qquad sv_{\mathcal{G}} = -\tfrac{1}{2}[v_{\mathcal{G}},v_{\mathcal{G}}].$$

Here we defined symbolically $v_{\mathcal{G}} = \sigma^*\omega_{\mathcal{G}}$, with $\sigma : \mathcal{M} \to \mathcal{P}$, so that $v_{\mathcal{G}}(X_Y) = \sigma^*\lambda : \mathcal{M} \to \mathfrak{g}$. We have an associated cohomology of s modulo d, $H^*_{s,d}\big(\mathcal{G},\Omega^0(\mathcal{A}\times\mathcal{F})\big)$. The elements of $H^0_{s,d}\big(\mathcal{G},\Omega^0(\mathcal{A}\times\mathcal{F})\big)$ are gauge quasi-invariant local functional of the gauge fields (Lagrangian). The elements of $H^1_{s,d}\big(\mathcal{G},\Omega^0(\mathcal{A}\times\mathcal{F})\big)$ are closed modulo d local functional of the gauge fields and of a ghost, that is anomalous terms satisfying the so-called *Wess-Zumino* consistency condition.

In the following we will always use a matrix formalism to that the graded bracket will be handled by the matrix bracket. Moreover we will simply write the Maurer-Cartan form/the ghost as a matrix $\mathfrak{g}$-valued field v, so that we will have,

$$s\omega = -Dv = -dv - \omega v - v\omega, \qquad s\Omega = [\Omega,v] = \Omega v - v\Omega, \qquad s\psi = -v\psi \qquad \text{and} \qquad sv = -v^2. \tag{3.5}$$

If we forget a moment about the section ψ, and since the second equation follows from the first, the BRS algebra can be summarized in a single relation. Defining the so-called *algebraic connection* $\widetilde{\omega} = \omega + v$, and $\widetilde{d} = d + s$ on $\mathcal{P} \times \mathcal{G}$, one has the relation,

$$\widetilde{\Omega} = \widetilde{d}\widetilde{\omega} + \widetilde{\omega} \wedge \widetilde{\omega} = \Omega.$$

The latter is known as the 'Russian formula', a name due to Stora. It plays an important role in the derivation of anomalies. We will say more on the subject in the next chapter.

3.2 The dressing field method in the BRS formalism

In this section we show how the dressing field method alters the BRS algebra of a gauge theory. In a manner very similar to what happens for the connection, dressed to become a composite field, the ghost is dressed to become a 'composite ghost'. The corresponding BRS algebra, if not trivial, handles the residual gauge freedom of the composite fields. The extension to higher-order G-structure is worked out and the BRS version of the compatibility conditions seen in 2.4.1 are given.

3.2.1 Modifying the BRS algebra

Easy proposition

Consider a gauge theory given by the bundle $\mathcal{P}(\mathcal{M},H)$ with a connection ω and its curvature Ω, together with a section ψ of any associated bundle. Given the $\mathfrak{h}$-valued ghost v, the BRS algebra is,

$$s\omega = -Dv, \qquad s\Omega = \left[\Omega, v\right], \qquad s\psi = -\rho_*(v)\psi, \qquad \text{and} \qquad sv = -\tfrac{1}{2}[v,v].$$

Then the basic formal result is the following,

Lemma 3 (Modified BRS algebra)**.** *For $\widehat{\omega}$, $\widehat{\Omega}$ and $\widehat{\psi}$, the composite forms of Lemma 1 and 2, there is a BRS algebra given by,*

$$s\widehat{\omega} = -\widehat{D}\widehat{v}, \qquad s\widehat{\Omega} = \left[\widehat{\Omega}, \widehat{v}\right], \qquad s\widehat{\psi} = -\rho_*(\widehat{v})\widehat{\psi}, \qquad \textit{and} \qquad s\widehat{v} = -\tfrac{1}{2}[\widehat{v},\widehat{v}], \tag{3.6}$$

for the ***composite ghost****,*

$$\widehat{v} = \bar{u}^{-1} v \bar{u} + \bar{u}^{-1} s\bar{u}. \tag{3.7}$$

Proof.

- Start with the first relation. Given $s\omega = -dv - \left[\omega, v\right]$, the first member of the equality reads

$$\begin{aligned}
s\omega &= s\left(\bar{u}\widehat{\omega}\bar{u}^{-1} + \bar{u}d\bar{u}^{-1}\right) = s\bar{u}\cdot\widehat{\omega}\bar{u}^{-1} + \bar{u}s\widehat{\omega}\cdot\bar{u}^{-1} - \bar{u}\widehat{\omega}s\bar{u}^{-1} + s\bar{u}d\bar{u}^{-1} - \bar{u}ds\bar{u}^{-1},\\
&= s\bar{u}\bar{u}^{-1}\cdot\bar{u}\widehat{\omega}\bar{u}^{-1} + \bar{u}s\widehat{\omega}\bar{u}^{-1} + \bar{u}\widehat{\omega}\bar{u}^{-1}s\bar{u}\bar{u}^{-1} + s\bar{u}d\bar{u}^{-1} + \bar{u}d\left(\bar{u}^{-1}s\bar{u}\bar{u}^{-1}\right),\\
&= \left[\bar{u}\widehat{\omega}\bar{u}^{-1}, s\bar{u}\bar{u}^{-1}\right] + \bar{u}s\widehat{\omega}\bar{u}^{-1} + s\bar{u}d\bar{u}^{-1} + \bar{u}d\left(\bar{u}^{-1}s\bar{u}\right)\bar{u}^{-1} - s\bar{u}d\bar{u}^{-1},\\
&= \left[\bar{u}\widehat{\omega}\bar{u}^{-1}, s\bar{u}\bar{u}^{-1}\right] + \bar{u}s\widehat{\omega}\bar{u}^{-1} + \bar{u}d\left(\bar{u}^{-1}s\bar{u}\right)\bar{u}^{-1}.
\end{aligned}$$

The second member on the other hand just reads, $-dv - \left[\bar{u}\widehat{\omega}\bar{u}^{-1}, v\right] - \left[\bar{u}d\bar{u}^{-1}, v\right]$. So the equality is,

$$\begin{aligned}
s\omega &= -dv - \left[\omega, v\right],\\
\left[\bar{u}\widehat{\omega}\bar{u}^{-1}, s\bar{u}\bar{u}^{-1}\right] + \bar{u}s\widehat{\omega}\bar{u}^{-1} + \bar{u}d(\bar{u}^{-1}s\bar{u})\bar{u}^{-1} &= -dv - \left[\bar{u}\widehat{\omega}\bar{u}^{-1}, v\right] - \left[\bar{u}d\bar{u}^{-1}, v\right].
\end{aligned}$$

Now we just isolate the second term in the first member,

$$\bar{u}s\widehat{\omega}\bar{u}^{-1} = -dv - \left[\bar{u}d\bar{u}^{-1}, v\right] - \bar{u}d\left(\bar{u}^{-1}s\bar{u}\right)\bar{u}^{-1} - \left[\bar{u}\widehat{\omega}\bar{u}^{-1}, v + s\bar{u}\bar{u}^{-1}\right],$$

which in turn gives,

$$
\begin{aligned}
s\widehat{\omega} &= -\bar{u}^{-1}dv\bar{u} - \left[d\bar{u}^{-1}\bar{u}, \bar{u}^{-1}v\bar{u}\right] - d\left(\bar{u}^{-1}s\bar{u}\right) - \left[\widehat{\omega}, \bar{u}^{-1}v\bar{u} + \bar{u}^{-1}s\bar{u}\right],\\
&= -\bar{u}^{-1}dv\bar{u} - d\bar{u}^{-1}v\bar{u} + \bar{u}^{-1}vd\bar{u} - d\left(\bar{u}^{-1}s\bar{u}\right) - \left[\widehat{\omega}, \bar{u}^{-1}v\bar{u} + \bar{u}^{-1}s\bar{u}\right],\\
&= -d\left(\bar{u}^{-1}v\bar{u} + \bar{u}^{-1}s\bar{u}\right) - \left[\widehat{\omega}, \bar{u}^{-1}v\bar{u} + \bar{u}^{-1}s\bar{u}\right],\\
s\widehat{\omega} &= -d\widehat{v} - \left[\widehat{\omega}, \widehat{v}\right] = -\widehat{D}\widehat{v}
\end{aligned}
$$

• The second relation is easier to treat. Starting with $s\Omega = \left[\Omega, v\right]$ whose first member is,

$$
\begin{aligned}
s\Omega &= s(\bar{u}\widehat{\Omega}\bar{u}^{-1}\big) = s\bar{u}\cdot\widehat{\Omega}\bar{u}^{-1} + \bar{u}s\widehat{\Omega}\cdot\bar{u}^{-1} + \bar{u}\widehat{\Omega}s\bar{u}^{-1},\\
&= s\bar{u}\bar{u}^{-1}\cdot\bar{u}\widehat{\Omega}\bar{u}^{-1} + \bar{u}s\widehat{\Omega}\bar{u}^{-1} - \bar{u}\widehat{\Omega}\bar{u}^{-1}s\bar{u}\bar{u}^{-1} = -\left[\bar{u}\widehat{\Omega}\bar{u}^{-1}, s\bar{u}\bar{u}^{-1}\right] + \bar{u}s\widehat{\Omega}\bar{u}^{-1}.
\end{aligned}
$$

The second member just reads, $\left[\Omega, v\right] = \left[\bar{u}\widehat{\Omega}\bar{u}^{-1}, v\right]$. So it is easily found that,

$$
\bar{u}s\widehat{\Omega}\bar{u}^{-1} = \left[\bar{u}\widehat{\Omega}\bar{u}^{-1}, v - \bar{u}s\bar{u}^{-1}\right],
$$

which in turn gives,

$$
\begin{aligned}
s\widehat{\Omega} &= \left[\widehat{\Omega}, \bar{u}^{-1}(v - \bar{u}s\bar{u}^{-1})\bar{u}^{-1}\right],\\
&= \left[\widehat{\Omega}, \bar{u}^{-1}v\bar{u} + \bar{u}^{-1}s\bar{u}\right],\\
s\widehat{\Omega} &= \left[\widehat{\Omega}, \widehat{v}\right].
\end{aligned}
$$

• The third relation is even easier.

$$
\begin{aligned}
s\psi &= -\rho_*(v)\psi,\\
s\left(\rho(\bar{u})\widehat{\psi}\right) &= -\rho_*(v)\rho(\bar{u})\widehat{\psi},\\
s\rho(\bar{u})\cdot\widehat{\psi} + \rho(\bar{u})s\widehat{\psi} &= -\rho_*(v)\rho(\bar{u})\widehat{\psi}.
\end{aligned}
$$

It is then easily seen that,

$$
\begin{aligned}
s\widehat{\psi} &= -\rho(\bar{u}^{-1})\rho_*(v)\rho(\bar{u})\widehat{\psi} - \rho(\bar{u}^{-1})s\rho(\bar{u})\widehat{\psi},\\
&= -\rho_*\left(\bar{u}^{-1}v\bar{u} + \bar{u}^{-1}s\bar{u}\right)\widehat{\psi},\\
s\widehat{\psi} &= -\rho_*(\widehat{v})\widehat{\psi}.
\end{aligned}
$$

• Finally, the fourth relation is easily checked,

$$
\begin{aligned}
-\tfrac{1}{2}\left[\widehat{v}, \widehat{v}\right] &= -\tfrac{1}{2}\left[\bar{u}^{-1}v\bar{u} + \bar{u}^{-1}s\bar{u}, \bar{u}^{-1}v\bar{u} + \bar{u}^{-1}s\bar{u}\right],\\
&= -\tfrac{1}{2}\left[\bar{u}^{-1}v\bar{u}, \bar{u}^{-1}v\bar{u}\right] - \left[\bar{u}^{-1}v\bar{u}, \bar{u}^{-1}s\bar{u}\right] - -\tfrac{1}{2}\left[\bar{u}^{-1}s\bar{u}, \bar{u}^{-1}s\bar{u}\right],\\
&= \bar{u}^{-1}\Big(-\tfrac{1}{2}[v, v]\Big)\bar{u} - \bar{u}^{-1}vs\bar{u} - \bar{u}^{-1}s\bar{u}\,\bar{u}^{-1}v\bar{u} + s\bar{u}^{-1}s\bar{u},\\
&= u^{-1}sv\bar{u} - \bar{u}^{-1}vs\bar{u} + s\bar{u}^{-1}v\bar{u} + s(\bar{u}^{-1}s\bar{u}) = s(\bar{u}^{-1}v\bar{u}) + s(\bar{u}^{-1}s\bar{u}),\\
-\tfrac{1}{2}\left[\widehat{v}, \widehat{v}\right] &= s\widehat{v}.
\end{aligned}
$$

□

Corollary 2 (Modified Russian formula)**.** *From the above new BRS algebra follows an associated Russian formula,*

$$
(d+s)\left(\widehat{\omega} + \widehat{v}\right) + \tfrac{1}{2}\left[\widehat{\omega} + \widehat{v},\ \widehat{\omega} + \widehat{v}\right] = d\widehat{\omega} + \tfrac{1}{2}\left[\widehat{\omega}, \widehat{\omega}\right] = \widehat{\Omega}. \tag{3.8}
$$

The three most relevant possibilities

The modified BRS algebra of Lemma 3 can take various forms according to the actual expression of the **composite ghost**, that is according to the BRS transformation of the map $\bar{u}$. But only three are of direct importance.

First suppose that the map $\bar{u} : \mathcal{P} \to H$ has **gauge-like** finite transformation under $\mathcal{H}$ so that $\bar{u}^\gamma = \gamma^{-1}\bar{u}\gamma$, for $\gamma \in \mathcal{H}$. Then its BRS transformation, mimicking infinitesimal gauge transformations with the ghost v as $\mathfrak{h}$-valued parameter, is

$$s\bar{u} = \left[\bar{u}, v\right]. \tag{3.9}$$

This implies that the *composite ghost* (3.7) reads,

$$\widehat{v} = \bar{u}^{-1}v\bar{u} + \bar{u}^{-1}s\bar{u} = \bar{u}^{-1}v\bar{u} + \bar{u}^{-1}\left[\bar{u}, v\right] = \bar{u}^{-1}v\bar{u} + \bar{u}^{-1}\bar{u}v - \bar{u}^{-1}v\bar{u} = v.$$

The ghost is thus invariant. This should not surprise us since with gauge-like transformation $\bar{u} = \alpha \in \mathcal{H}$, so that $\widehat{\omega} = \omega^\alpha$, $\widehat{\Omega} = \Omega^\alpha$ and $\widehat{\psi} = \psi^\alpha$ are still H-gauge fields which are actually gauge transformations of ω, Ω and ψ respectively. The BRS algebra of the theory is of course the same, namely (3.6) gives,

$$s\omega^\alpha = -D^\alpha v, \qquad s\Omega^\alpha = \left[\Omega^\alpha, v\right], \qquad s\psi^\alpha = -\rho_*(v)\psi^\alpha \qquad \text{and} \qquad sv = -\tfrac{1}{2}\left[v, v\right].$$

Besides the algebraic connection $\omega + v$ on $\mathcal{P} \times \mathcal{H}$ provides a *composite algebraic connection* $\widehat{\omega} + \widehat{v} = \omega^\alpha + v$ which still lives on $\mathcal{P} \times \mathcal{H}$.

Suppose now that $\bar{u} : \mathcal{P} \to H$ is a **dressing field**[2] **for the full group** H so that $\bar{u}^\gamma = \gamma^{-1}\bar{u}$, for $\gamma \in \mathcal{H}$. Then its BRS transformation is,

$$s\bar{u} = -v\bar{u}. \tag{3.10}$$

This implies that the *composite ghost* reads,

$$\widehat{v} = \bar{u}^{-1}v\bar{u} + \bar{u}^{-1}s\bar{u} = \bar{u}^{-1}v\bar{u} - \bar{u}v\bar{u} = 0.$$

The ghost is thus annihilated by a full dressing field, and the *modified BRS algebra* (3.6) reduces to the **trivial algebra**,

$$s\widehat{\omega} = 0, \qquad s\widehat{\Omega} = 0, \qquad \text{and} \qquad s\widehat{\psi} = 0. \tag{3.11}$$

This is no surprise for we know that $\bar{u}$ being a full dressing, $\widehat{\omega}$, $\widehat{\Omega}$ and $\widehat{\psi}$ are H-**invariant forms** on $\mathcal{P}$. This is precisely what expresses the trivial BRS algebra (3.11), which is the infinitesimal counterpart of Lemma 1. Besides the algebraic connection $\omega + v$ on $\mathcal{P} \times \mathcal{H}$ dresses as $\widehat{\omega} + \widehat{v} = \widehat{\omega}$ which lives on $\mathcal{P}/H \simeq \mathcal{M}$.

Finally the last but not least interesting case is when $\bar{u} : \mathcal{P} \to K$, for $K \subset H$ a subgroup, such that $\bar{u}^{\gamma_1} = \gamma_1^{-1}\bar{u}$ for $\gamma_1 \in \mathcal{K}$. For now we left unspecified the transformation $\bar{u}^{\gamma_0}$ for $\gamma_0 \in \mathcal{H}/\mathcal{K}$. Suppose we can find a complement, $\mathfrak{p}$, to $\mathfrak{k}$ in $\mathfrak{h}$, so that $\mathfrak{h} = \mathfrak{k} + \mathfrak{p}$. The BRS operator would then split as $s = s_\mathfrak{h} = s_\mathfrak{k} + s_\mathfrak{p}$ and the associated ghost as $v = v_\mathfrak{h} = v_\mathfrak{k} + v_\mathfrak{p}$. The BRS transformations of the dressing field are then,

$$s\bar{u} = s_\mathfrak{h}\bar{u} = s_\mathfrak{k}\bar{u} + s_\mathfrak{p}\bar{u} = -v_\mathfrak{k}\bar{u} + s_\mathfrak{p}\bar{u}, \tag{3.12}$$

with the transformation under $s_\mathfrak{p}$ unspecified for now. This implies for the *composite ghost*,

$$\begin{aligned}
\widehat{v} &= \bar{u}^{-1}v_\mathfrak{h}\bar{u} + \bar{u}^{-1}s_\mathfrak{h}\bar{u}, \\
&= \bar{u}^{-1}v_\mathfrak{k}\bar{u} + \bar{u}^{-1}v_\mathfrak{p}\bar{u} - \bar{u}^{-1}v_\mathfrak{k}\bar{u} + \bar{u}^{-1}s_\mathfrak{p}\bar{u}, \\
\widehat{v} &= \bar{u}^{-1}v_\mathfrak{p}\bar{u} + \bar{u}^{-1}s_\mathfrak{p}\bar{u} =: \widehat{v}_\mathfrak{p}.
\end{aligned} \tag{3.13}$$

[2] Again we allow the terminological freedom to call the map $\bar{u}$ on $\mathcal{P}$ 'dressing field' when strictly speaking the name is for the pull-back $u = \sigma^*\bar{u}$ on $U \subset \mathcal{M}$.

The K subgroup is neutralized in a way similar to the previous case, and the composite ghost $\widehat{v}_{\text{p}}$ only depends on the remaining gauge symmetry. So that the *modified BRS algebra* (3.6) is,

$$s\widehat{\omega} = -\widehat{D}\widehat{v}_{\text{p}}, \qquad s\widehat{\Omega} = \left[\widehat{\Omega}, \widehat{v}_{\text{p}}\right], \qquad s\widehat{\psi} = -\rho_*(\widehat{v}_{\text{p}})\widehat{\psi}, \qquad \text{and} \qquad s\widehat{v}_{\text{p}} = -\tfrac{1}{2}\left[\widehat{v}_{\text{p}}, \widehat{v}_{\text{p}}\right]. \tag{3.14}$$

This **reduced BRS algebra** handles the **infinitesimal residual gauge freedom** of the composite forms $\widehat{\omega}$, $\widehat{\Omega}$ and $\widehat{\psi}$. This is the infinitesimal counterpart of Lemma 2. Besides the algebraic connection $\omega + v$ on $\mathcal{P} \times \mathcal{H}$ reduces to the *composite algebraic connection* $\widehat{\omega} + \widehat{v} = \widehat{\omega} + \widehat{v}_{\text{p}}$ on $\mathcal{P}/H \times \mathcal{H}/\mathcal{K}$.

Of course it is easy to pull-back these constructions on $U \subset \mathcal{M}$ and to obtain, with the dressing field $u = \sigma^*\bar{u}$, the very same modified BRS algebras in each case replacing $\widehat{\omega}$ by $\widehat{A}$, $\widehat{\Omega}$ by $\widehat{F}$, $\widehat{\psi}$ by $\widehat{\varphi}$ and noting the pull-back of the ghost by the same letter, $v = \sigma^*v : \mathcal{M} \to \mathfrak{h}$.

These cover the three cases of reduction of gauge symmetry (none, total, partial) and in the last case it is shown how the reduced BRS algebra handles the residual gauge freedom. We are now ready to apply the scheme to higher-order G-structure and give the BRS counterpart of the construction performed in section 2.4.1

3.2.2 Reduced BRS algebra and higher-order G-structures

Let us remind the setup. We suppose our principal bundle $\mathcal{P}(\mathcal{M}, H)$ has a structure group which can be written as a product of subgroups $H = K_0K_1 \cdots K_n = \prod_{i=0}^{n} K_i$. This means that its Lie algebra can be written as $\sum_{i=0}^{n} \mathfrak{k}_i$. The BRS operator and the corresponding ghost split as,

$$s = \sum_{i=0}^{n} s_i =: \bar{s}_n, \qquad \text{and} \qquad v = \sum_{i=0}^{n} v_i =: \bar{v}_n. \tag{3.15}$$

Given a connection 1-form ω on $\mathcal{P} =: \mathcal{P}_n$, which is a connection on any $\mathcal{P}_{n-i}$ by restriction, its BRS transformations are,

$$\begin{aligned} s_i\omega &= -Dv_i = -dv_i - \left[\omega, v_i\right], \quad i \in [0, \dots, n], \\ s\omega &= -Dv = -dv - \left[\omega, v\right]. \end{aligned} \tag{3.16}$$

These are the infinitesimal counterparts of (2.14). The bundle $\mathcal{P} := \mathcal{P}_n$ is a bundle over $\mathcal{M}$ with structure group H, but also a bundle over $\mathcal{P}_{n-1}$ with structure group K_n. I refer to 2.4.1 for the description of the various fibrations.

We've then supposed that we had several dressing fields, $u_i : \mathcal{P}_i \to K_i$,[3] liable to help neutralizing the various subgroups K_i and reduce, step by step, $\mathcal{P}_i$ to $\mathcal{P}_{i-1}$. For this to be possible the dressing fields have to satisfy compatibility conditions whose BRS versions are,

$$s_iu_i = -v_iu_i, \tag{3.17}$$

which is just the dressing field BRS transformation law that warrant the possibility to reduce $\mathcal{P}_i$ to $\mathcal{P}_{i-1}$ for any i. This is the BRS counterpart of (2.26). Then,

$$s_ju_i = \left[u_i, v_j\right] \quad \text{for} \quad j < i, \tag{3.18}$$

which is necessary for the i^{th} dressed fields to behave as genuine gauge fields under the lower order j^{th} gauge subgroups so that a new dressing operation makes sense. This is the BRS conterpart of (2.27). Finally,

$$s_ju_i = 0 \qquad \text{for} \quad j > i, \tag{3.19}$$

[3] Again, for convenience I drop the bar over $\bar{u}_i$ in this section. Keep then in mind that we should not confuse the map with its pull-back. This should be clear from the context.

which must be so in order for the i^{th} dressed fields to remain invariant under higher order j^{th} gauge transformations and to pass down to well defined forms on the reduced bundle $\mathcal{P}_{i-1}$. This is the BRS counterpart of (2.28).

We already know that thanks to this compatibility conditions the successive composite forms, ω_i, Ω_i and ψ_i are well behaved. To understand what happens at the level of the BRS algebra, we then only need to control the behavior of the *composite ghost.* This time again it is easier to appreciate the scheme by working out the first steps of the recursive process.

Reductions step by step Given ω, Ω and ψ on $\mathcal{P}_n := \mathcal{P}(M,\bar{K}_n)$, with $\bar{K}_n := \prod_{i=0}^{n} K_i = H$, and (3.15), we note $\overline{BRS}_n$ the initial total algebra and BRS_i the subalgebras at each stage $i \in [0,\dots n]$, given respectively by,

$$\bar{s}_n\omega = -D\bar{v}_n, \qquad \bar{s}_n\Omega = \big[\Omega,\bar{v}_n\big], \qquad \bar{s}_n\psi = -\rho_*(\bar{v}_n)\psi, \qquad \text{and} \qquad \bar{s}_n\bar{v}_n = -\tfrac{1}{2}\big[\bar{v}_n,\bar{v}_n\big],$$
$$s_i\omega = -Dv_i, \qquad s_i\Omega = \big[\Omega,v_i\big], \qquad s_i\psi = -\rho_*(v_i)\psi, \qquad \text{and} \qquad s_iv_i = -\tfrac{1}{2}\big[v_i,v_i\big].$$

Now with the map $u_n : \mathcal{P}_n \to K_n$ we form the composite forms $\omega_n := \omega^{u_n}$, $\Omega_n := \Omega^{u_n}$ and $\psi_n = \psi^{u_n}$. We already know that they live on $\mathcal{P}_{n-1} := \mathcal{P}(\mathcal{M},\bar{K}_{n-1}) \simeq \mathcal{P}_n/K_n$ where they behave as genuine gauge forms. This should be reflected by the BRS algebras. Let us see what happens to the **composite ghost**.

$$\begin{aligned}
\widehat{v}_n :&= \bar{v}_n^{u_n} = u_n^{-1}\bar{v}_n u_n + u_n^{-1}\bar{s}_n u_n,\\
&= u_n^{-1}\sum_{k=0}^{n} v_k u_n + u_n^{-1}\sum_{k=0}^{n} s_k u_n,\\
&= u_n^{-1}\sum_{k=0}^{n-1} v_k u_n + u_n^{-1}v_n u_n \ + \ u_n^{-1}\sum_{k=0}^{n-1} s_k u_n + u_n^{-1}s_n u_n,\\
&= u_n^{-1}\sum_{k=0}^{n-1} v_k u_n + u_n^{-1}v_n u_n \ + \ u_n^{-1}\sum_{k=0}^{n-1}\big[u_n,v_k\big] + u_n\big(-v_n u_n\big), \qquad \text{where (3.18) and (3.17) are used,}\\
\widehat{v}_n &= \sum_{k=0}^{n-1} v_k =: \bar{v}_{n-1}.
\end{aligned}$$

We thus see that in the composite ghost, the ghost v_n associated to the group K_n is killed by the dressing u_n, and remains only the ghost $\bar{v}_{n-1}$ associated to the group $\bar{K}_{n-1}$ which is the structure group of $\mathcal{P}_{n-1}$. The first subalgebra of the **modified BRS algebra** of Lemma 3 is therefore,

$$s_n\omega_n = 0, \qquad s_n\Omega_n = 0 \qquad \text{and} \qquad s_n\psi_n = 0,.$$

We note $BRS_{n,n} = 0$ to signify its triviality. This reflects the expected K_n-invariance of the composite forms. The non-trivial part of the modified BRS algebra is the residual total algebra $\overline{BRS}_{n-1,n}$ given by,

$$\bar{s}_{n-1}\omega_n = -D_n\bar{v}_{n-1}, \qquad \bar{s}_{n-1}\Omega_n = \big[\Omega_n,\bar{v}_{n-1}\big], \qquad \bar{s}_{n-1}\psi_n = -\rho_*(\bar{v}_{n-1})\psi_n, \qquad \bar{s}_{n-1}\bar{v}_{n-1} = -\tfrac{1}{2}\big[\bar{v}_{n-1},\bar{v}_{n-1}\big],$$

where $D_n := d + \big[\omega_n,\ \big]$. And we have the subalgebras $BRS_{i,n}$ for $i \in [0,\dots n-1]$,

$$s_i\omega_n = -D_nv_i, \qquad s_i\Omega_n = \big[\Omega_n,v_i\big], \qquad s_i\psi_n = -\rho_*(v_i)\psi_n, \qquad \text{and} \qquad s_iv_i = -\tfrac{1}{2}\big[v_i,v_i\big].$$

This modified BRS algebra expresses the fact that the K_n-invariant composite forms are genuine $\bar{K}_{n-1}$-gauge forms on $\mathcal{P}_{n-1}$. Schematically we have the reduction,

$$\mathcal{P}_n \xrightarrow{u_n} \mathcal{P}_{n-1} \qquad\qquad \overline{BRS}_n \xrightarrow{u_n} \overline{BRS}_{n-1,n}$$

We can carry on the process a step further with a map $u_{n-1} : \mathcal{P}_{n-1} \to K_{n-1}$. We have the composite forms $\omega_{n-1} := \omega_n^{u_{n-1}}$, $\Omega_{n-1} := \Omega_n^{u_{n-1}}$ and $\psi_{n-1} = \psi_n^{u_{n-1}}$. Since according to (3.19) $s_n u_{n-1} = 0$, we have,

$$s_n\omega_{n-1} = 0, \qquad s_n\Omega_{n-1} = 0 \qquad \text{and} \qquad s_n\psi_{n-1} = 0.$$

We note $BRS_{n,n-1} = 0$. So the new composite forms live on $\mathcal{P}_{n-2} := \mathcal{P}(\mathcal{M}, \bar{K}_{n-2}) \simeq \mathcal{P}_{n-1}/K_{n-1}$. A calculation exactly similar to the above one proves that the new composite ghost is,

$$\widehat{v}_{n-1} : = \bar{v}_{n-1}^{u_{n-1}} = u_{n-1}^{-1}\bar{v}_{n-1}u_{n-1} + u_{n-1}^{-1}\bar{s}_{n-1}u_{n-1} = \ldots = \sum_{k=0}^{n-2} v_k =: \bar{v}_{n-2}.$$

Then in this new composite ghost, the ghost v_{n-1} associated to the group K_{n-1} is killed by the dressing u_{n-1}, and it remains only the ghost $\bar{v}_{n-2}$ associated to the group $\bar{K}_{n-2}$ which is the structure group of $\mathcal{P}_{n-2}$. The first subalgebra of the new modifield BRS algebra is therefore,

$$s_{n-1}\omega_{n-1} = 0, \qquad s_{n-1}\Omega_{n-1} = 0 \qquad \text{and} \qquad s_{n-1}\psi_{n-1} = 0, .$$

We note $BRS_{n-1,n-1} = 0$. This reflects the expected K_{n-1}-invariance of the composite forms. The non-trivial part of the new modified BRS algebra is the residual total algebra $\overline{BRS}_{n-2,n-1}$ given by,

$$\bar{s}_{n-2}\omega_{n-1} = -D_{n-1}\bar{v}_{n-2}, \qquad \bar{s}_{n-2}\Omega_{n-1} = \left[\Omega_{n-1}, \bar{v}_{n-2}\right], \qquad \bar{s}_{n-2}\psi_{n-1} = -\rho_*(\bar{v}_{n-2})\psi_{n-1},$$
$$\text{and} \qquad \bar{s}_{n-2}\bar{v}_{n-2} = -\tfrac{1}{2}\left[\bar{v}_{n-2}, \bar{v}_{n-2}\right],$$

where $D_{n-1} := d + \left[\omega_{n-1},\ \right]$. And we have the subalgebras $BRS_{i,n-1}$ for $i \in [0, \ldots n-2]$,

$$s_i\omega_{n-1} = -D_{n-1}v_i, \qquad s_i\Omega_{n-1} = \left[\Omega_{n-1}, v_i\right], \qquad s_i\psi_{n-1} = -\rho_*(v_i)\psi_{n-1}, \qquad \text{and} \qquad s_i v_i = -\tfrac{1}{2}\left[v_i, v_i\right].$$

This reflects the fact that the K_{n-1}-invariant composite forms are genuine $\bar{K}_{n-2}$-gauge forms on $\mathcal{P}_{n-2}$. With this second step, schematically we have the reductions,

$$\mathcal{P}_{n-1} \xrightarrow{u_{n-1}} \mathcal{P}_{n-2} \qquad\qquad \overline{BRS}_{n-1,n} \xrightarrow{u_{n-1}} \overline{BRS}_{n-2,n-1}$$

Clearly this is an iterative process, and step by step we have the succession of reductions,

$$\mathcal{P}_n \xrightarrow{u_n} \mathcal{P}_{n-1} \xrightarrow{u_{n-1}} \mathcal{P}_{n-2} \xrightarrow{u_{n-2}} \ldots \xrightarrow{u_i} \mathcal{P}_{i-1} \xrightarrow{u_{i-1}} \ldots$$

$$\overline{BRS}_n \xrightarrow{u_n} \overline{BRS}_{n-1,n} \xrightarrow{u_{n-1}} \overline{BRS}_{n-2,n-1} \xrightarrow{u_{n-2}} \ldots \xrightarrow{u_i} \overline{BRS}_{i-1,i} \xrightarrow{u_{i-1}} \ldots$$

Of course it is possible to skip intermediate steps and to reduce several stages in a single move.

Reduction in a single step We want to reduce $\mathcal{P}_n$ to $\mathcal{P}_{i-1}$ so $\overline{BRS}_n$ to $\overline{BRS}_{i-1,i}$, for any $i < n$ in a single step. In section 2.4.1 of chapter 2 it was shown that the bundle reduction can be achieved by the dressing $\prod\limits_{k=n}^{i} u_k$, with well defined forms ω_i, Ω_i and ψ_i. It then just remains to calculate the composite ghost under such a dressing. To this end we will need two results which we now prove. The first gives the BRS transformation of the dressing under any K_j with $j \geq i$,

$$\begin{aligned}
s_j \prod_{k=n}^{i} u_k &= s_j u_n \cdot \prod_{k=n-1}^{i} u_k + u_n s_j u_{n-1} \cdot \prod_{k=n-2}^{i} u_k + u_n u_{n-1} s_j u_{n-2} \cdot \prod_{k=n-3}^{i} u_k + \ldots, \\
&= s_j u_n \cdot \prod_{k=n-1}^{i} u_k + \sum_{l=n-1}^{i} \left(\prod_{k=n}^{l+1} u_k \cdot s_j u_l \cdot \prod_{k=l-1}^{i} u_k \right), \\
&= s_j u_n \cdot \prod_{k=n-1}^{i} u_k + \sum_{l=n-1}^{j+1} \left(\prod_{k=n}^{l+1} u_k \cdot s_j u_l \cdot \prod_{k=l-1}^{i} u_k \right) + \prod_{k=n}^{j+1} u_k \cdot s_j u_j \cdot \prod_{k=j-1}^{i} u_k + \underbrace{\sum_{l=j-1}^{i} \left(\prod_{k=n}^{l+1} u_k \cdot s_j u_l \cdot \prod_{k=l-1}^{i} u_k \right)}_{=0, \text{ according to } (3.19)},
\end{aligned}$$

Now according to (3.18) and (3.17) we have,

$$
\begin{aligned}
s_j \prod_{k=n}^{i} u_k = \big[u_n, v_j\big] \cdot \prod_{k=n-1}^{i} u_k + \sum_{l=n-1}^{j+1} \Big(\prod_{k=n}^{l+1} u_k \cdot u_l v_j \cdot \prod_{k=l-1}^{i} u_k \Big) - \sum_{l=n-1}^{j+1} \Big(\prod_{k=n}^{l+1} u_k \cdot v_j u_l \cdot \prod_{k=l-1}^{i} u_k \Big) \\
- \prod_{k=n}^{j+1} u_k \cdot v_j u_j \cdot \prod_{k=j-1}^{i} u_k, \\
= \big[u_n, v_j\big] \cdot \prod_{k=n-1}^{i} u_k + \sum_{l=n-1}^{j+1} \Big(\prod_{k=n}^{l+1} u_k \cdot u_l v_j \cdot \prod_{k=l-1}^{i} u_k \Big) - \sum_{l=n-1}^{j} \Big(\prod_{k=n}^{l+1} u_k \cdot v_j u_l \cdot \prod_{k=l-1}^{i} u_k \Big).
\end{aligned}
$$

Developing the first and last term,

$$
\begin{aligned}
s_j \prod_{k=n}^{i} u_k = \boldsymbol{u_n v_j} \cdot \prod_{\boldsymbol{k=n-1}}^{\boldsymbol{i}} \boldsymbol{u_k} - v_j u_n \cdot \prod_{k=n-1}^{i} u_k + \sum_{l=n-1}^{j+1} \Big(\prod_{k=n}^{l+1} u_k \cdot u_l v_j \cdot \prod_{k=l-1}^{i} u_k \Big) - \prod_{\boldsymbol{k=n}}^{\boldsymbol{n}} \boldsymbol{u_k} \cdot \boldsymbol{v_j u_{n-1}} \cdot \prod_{\boldsymbol{k=n-2}}^{\boldsymbol{i}} \boldsymbol{u_k} \\
- \sum_{l=n-2}^{j} \Big(\prod_{k=n}^{l+1} u_k \cdot v_j u_l \cdot \prod_{k=l-1}^{i} u_k \Big), \\
= -v_j \prod_{k=n}^{i} u_k + \sum_{l=n-1}^{j+1} \Big(\prod_{k=n}^{l+1} u_k \cdot u_l v_j \cdot \prod_{k=l-1}^{i} u_k \Big) - \sum_{l=n-2}^{j} \Big(\prod_{k=n}^{l+1} u_k \cdot v_j u_l \cdot \prod_{k=l-1}^{i} u_k \Big).
\end{aligned}
$$

By inspection it is clear that the last two terms are the same so we finally obtain,

$$
s_j \prod_{k=n}^{i} u_k = -v_j \prod_{k=n}^{i} u_k, \qquad \text{for} j \geq i. \tag{3.20}
$$

This is the infinitesimal analogue of (2.29) and indicates that the string of dressing fieds $\prod_{k=n}^{i} u_k$ is a dressing field in its own right under any subgroup K_j with $j \geq i$. The second result we need is a variant; the BRS transformation of the above dressing under any K_j with $j < i$,

$$
\begin{aligned}
s_j \prod_{k=n}^{i} u_k &= s_j u_n \cdot \prod_{k=n-1}^{i} u_k + \sum_{l=n-1}^{i} \Big(\prod_{k=n}^{l+1} u_k \cdot s_j u_l \cdot \prod_{k=l-1}^{i} u_k \Big), \\
&= \big[u_n, v_j\big] \cdot \prod_{k=n-1}^{i} u_k + \sum_{l=n-1}^{i} \Big(\prod_{k=n}^{l+1} u_k \cdot \big[u_l, v_j\big] \cdot \prod_{k=l-1}^{i} u_k \Big), \quad \text{according to (3.18)}, \\
&= \big[u_n, v_j\big] \cdot \prod_{k=n-1}^{i} u_k + \sum_{l=n-1}^{i} \Big(\prod_{k=n}^{l+1} u_k \cdot u_l v_j \cdot \prod_{k=l-1}^{i} u_k \Big) - \sum_{l=n-1}^{i} \Big(\prod_{k=n}^{l+1} u_k \cdot v_j u_l \cdot \prod_{k=l-1}^{i} u_k \Big).
\end{aligned}
$$

Developing the first and last term,

$$
\begin{aligned}
s_j \prod_{k=n}^{i} u_k = \boldsymbol{u_n v_j} \cdot \prod_{\boldsymbol{k=n-1}}^{\boldsymbol{i}} \boldsymbol{u_k} - v_j u_n \cdot \prod_{k=n-1}^{i} u_k + \sum_{l=n-1}^{i} \Big(\prod_{k=n}^{l+1} u_k \cdot u_l v_j \cdot \prod_{k=l-1}^{i} u_k \Big) - \prod_{\boldsymbol{k=n}}^{\boldsymbol{n}} \boldsymbol{u_k \cdot v_j u_{n-1}} \cdot \prod_{\boldsymbol{k=n-2}}^{\boldsymbol{i}} \boldsymbol{u_k} \\
- \sum_{l=n-2}^{i} \Big(\prod_{k=n}^{l+1} u_k \cdot v_j u_l \cdot \prod_{k=l-1}^{i} u_k \Big), \\
= -v_j \cdot \prod_{k=n}^{i} u_k + \sum_{l=n-1}^{i} \Big(\prod_{k=n}^{l+1} u_k \cdot u_l v_j \cdot \prod_{k=l-1}^{i} u_k \Big) - \sum_{l=n-2}^{i} \Big(\prod_{k=n}^{l+1} u_k \cdot v_j u_l \cdot \prod_{k=l-1}^{i} u_k \Big), \\
= -v_j \cdot \prod_{k=n}^{i} u_k + \sum_{l=n-1}^{i+1} \Big(\prod_{k=n}^{l+1} u_k \cdot u_l v_j \cdot \prod_{k=l-1}^{i} u_k \Big) + \prod_{k=n}^{i+1} u_k \cdot u_i v_j - \sum_{l=n-2}^{i} \Big(\prod_{k=n}^{l+1} u_k \cdot v_j u_l \cdot \prod_{k=l-1}^{i} u_k \Big), \\
= \Big[\prod_{k=n}^{i} u_k, v_j \Big] + \sum_{l=n-1}^{i+1} \Big(\prod_{k=n}^{l+1} u_k \cdot u_l v_j \cdot \prod_{k=l-1}^{i} u_k \Big) - \sum_{l=n-2}^{i} \Big(\prod_{k=n}^{l+1} u_k \cdot v_j u_l \cdot \prod_{k=l-1}^{i} u_k \Big).
\end{aligned}
$$

Again the last two terms are the same so that we obtain,

$$
s_j \prod_{k=n}^{i} u_k = \Big[\prod_{k=n}^{i} u_k, v_j \Big], \qquad \text{for} j < i. \tag{3.21}
$$

With these two technical results it is now easy to obtain the action of the complete BRS operator on the dressing field $\prod\limits_{k=n}^{i} u_k$,

$$
\begin{aligned}
\bar{s}_n \Big(\prod_{k=n}^{i} u_k \Big) &= \sum_{j=0}^{n} s_j \Big(\prod_{k=n}^{i} u_k \Big) = \sum_{j=0}^{n} \Big(s_j \prod_{k=n}^{i} u_k \Big) = \sum_{j=0}^{i-1} \Big(s_j \prod_{k=n}^{i} u_k \Big) + \sum_{j=i}^{n} \Big(s_j \prod_{k=n}^{i} u_k \Big), \\
&= \sum_{j=0}^{i-1} \Big[\prod_{k=n}^{i} u_k, v_j \Big] - \sum_{j=i}^{n} \Big(v_j \prod_{k=n}^{i} u_k \Big), \qquad \text{by (3.21) and (3.20) respectively,} \\
&= \sum_{j=0}^{i-1} \Big(\prod_{k=n}^{i} u_k v_j \Big) - \sum_{j=0}^{i-1} \Big(v_j \prod_{k=n}^{i} u_k \Big) - \sum_{j=i}^{n} \Big(v_j \prod_{k=n}^{i} u_k \Big), \\
&= \Big(\prod_{k=n}^{i} u_k \Big) \Big(\sum_{j=0}^{i-1} v_j \Big) - \Big(\sum_{j=0}^{n} v_j \Big) \Big(\prod_{k=n}^{i} u_k \Big), \\
\bar{s}_n \Big(\prod_{k=n}^{i} u_k \Big) &= -\bar{v}_n \Big(\prod_{k=n}^{i} u_k \Big) + \Big(\prod_{k=n}^{i} u_k \Big) \bar{v}_{i-1}.
\end{aligned} \tag{3.22}
$$

It is now straightforward to obtain the composite ghost,

$$
\begin{aligned}
\widehat{v}_i : = \bar{v}_n^{\left(\prod_{k=n}^{i} u_k \right)} &= \Big(\prod_{k=n}^{i} u_k \Big)^{-1} \bar{v}_n \Big(\prod_{k=n}^{i} u_k \Big) + \Big(\prod_{k=n}^{i} u_k \Big)^{-1} \bar{s}_n \Big(\prod_{k=n}^{i} u_k \Big), \\
&= \Big(\prod_{k=n}^{i} u_k \Big)^{-1} \bar{v}_n \Big(\prod_{k=n}^{i} u_k \Big) + \Big(\prod_{k=n}^{i} u_k \Big)^{-1} \Big(- \bar{v}_n \Big(\prod_{k=n}^{i} u_k \Big) + \Big(\prod_{k=n}^{i} u_k \Big) \bar{v}_{i-1} \Big), \\
\widehat{v}_i &= \bar{v}_{i-1}.
\end{aligned} \tag{3.23}
$$

This means that $v_j = 0$ for $j \in [n, \ldots, i]$ so that,

$$
s_j \omega_i = 0, \qquad s_j \Omega_i = 0, \qquad \text{and} \qquad s_j \psi_i = 0, \qquad \text{we note} \quad BRS_{j,i} = 0 \quad \text{for} j \in [n, \ldots, i].
$$

And the non-trivial algebra is $\overline{BRS}_{i-1,i}$,

$$\bar{s}_{i-1}\omega_i = -D_i\bar{v}_{i-1}, \qquad \bar{s}_{i-1}\Omega_i = \left[\Omega_i, \bar{v}_{i-1}\right], \qquad \bar{s}_{i-1}\psi_i = -\rho_*(\bar{v}_{i-1})\psi_i, \quad \text{and} \qquad \bar{s}_{i-1}\bar{v}_{i-1} = -\tfrac{1}{2}\left[\bar{v}_{i-1}, \bar{v}_{i-1}\right],$$

where $D_i := d + \left[\omega_i,\ \right]$. And we have the subalgebras $BRS_{j,i}$ for $j \in [i-1, \dots, 0]$,

$$s_j\omega_i = -D_i v_j, \qquad s_j\Omega_i = \left[\Omega_i, v_j\right], \qquad s_j\psi_i = -\rho_*(v_j)\psi_i, \qquad \text{and} \qquad s_j v_j = -\tfrac{1}{2}\left[v_j, v_j\right].$$

This reflects the fact that the $K_j\big|_{j\geq i}$-invariant composite forms are genuine $\bar{K}_{i-1}$-gauge forms on $\mathcal{P}_{i-1}$. Schematically we have the reduction,

$$\mathcal{P}_n \xrightarrow{\prod_{k=n}^{i} u_k} \mathcal{P}_{i-1} \qquad\qquad \overline{BRS}_n \xrightarrow{\prod_{k=n}^{i} u_k} \overline{BRS}_{i-1,i}$$

From here, a new dressing operation would make sense. And the process of reduction could go on. Clearly enough if we have dressings down to $i = 0$ then (3.22) reduces to $\bar{s}_n\left(\prod_{k=n}^{0} u_k\right) = -\bar{v}_n\left(\prod_{k=n}^{0} u_k\right)$ and the composite ghost vanishes. It then provides a trivialized BRS algebra as in (3.11), which is in accordance with the fact that $\prod_{k=n}^{0} u_k$ reduces $\mathcal{P}_n$ to $\mathcal{M}$ and that the fields ω_0, Ω_0 and ψ_0 belong to the natural geometry of $\mathcal{M}$ and have no remaining gauge freedom.

A non-trivial final modified BRS algebra A stop is reached if the compatibility condition (3.18) does not hold for $j < i$ so that the $K_j\big|_{j\geq i}$-invariant composite forms ω_i, Ω_i and ψ_i won't behave as usual $\bar{K}_{i-1}$-gauge forms on $\mathcal{P}_{i-1}$ (that is as a connection, a curvature and a section respectively). The latter nevertheless display a $\bar{K}_{i-1}$-gauge freedom whose infinitesimal version is given by a **modified BRS algebra** ultimately depending on a **final composite ghost**. To find the expression of this final ghost we need to compute again the action of the full BRS operator on the dressing $\prod_{k=n}^{i} u_k$, remembering this time that since the compatibility condition (3.18) does not hold for $j < i$ the result (3.21) does not hold either. Then,

$$\begin{aligned}
\bar{s}_n\Big(\prod_{k=n}^{i} u_k\Big) &= \sum_{j=0}^{n} s_j\Big(\prod_{k=n}^{i} u_k\Big) = \sum_{j=0}^{i-1} s_j\Big(\prod_{k=n}^{i} u_k\Big) + \sum_{j=i}^{n}\Big(s_j\prod_{k=n}^{i} u_k\Big),\\
&= \sum_{j=0}^{i-1} s_j\Big(\prod_{k=n}^{i} u_k\Big) - \sum_{j=i}^{n}\Big(v_j\prod_{k=n}^{i} u_k\Big), \qquad \text{by (3.20)},\\
&= \sum_{j=0}^{i-1} s_j\Big(\prod_{k=n}^{i} u_k\Big) - \Big(\sum_{j=i}^{n} v_j\Big)\Big(\prod_{k=n}^{i} u_k\Big),\\
\bar{s}_n\Big(\prod_{k=n}^{i} u_k\Big) &= \bar{s}_{i-1}\Big(\prod_{k=n}^{i} u_k\Big) - \Big(\sum_{j=i}^{n} v_j\Big)\Big(\prod_{k=n}^{i} u_k\Big).
\end{aligned}$$

The **final composite ghost** is then,

$$
\begin{aligned}
\widehat{v}_i := \bar{v}_n^{\left(\prod_{k=n}^{i} u_k\right)} &= \Big(\prod_{k=n}^{i} u_k\Big)^{-1} \bar{v}_n \Big(\prod_{k=n}^{i} u_k\Big) + \Big(\prod_{k=n}^{i} u_k\Big)^{-1} \bar{s}_n \Big(\prod_{k=n}^{i} u_k\Big), \\
&= \Big(\prod_{k=n}^{i} u_k\Big)^{-1} \sum_{j=0}^{i-1} v_j \Big(\prod_{k=n}^{i} u_k\Big) + \boldsymbol{\Big(\prod_{k=n}^{i} u_k\Big)^{-1} \sum_{j=i}^{n} v_j \Big(\prod_{k=n}^{i} u_k\Big)} + \Big(\prod_{k=n}^{i} u_k\Big)^{-1} \bar{s}_{i-1} \Big(\prod_{k=n}^{i} u_k\Big) \\
&\qquad - \boldsymbol{\Big(\prod_{k=n}^{i} u_k\Big)^{-1} \sum_{j=i}^{n} v_j \Big(\prod_{k=n}^{i} u_k\Big)}, \\
\widehat{v}_i &= \Big(\prod_{k=n}^{i} u_k\Big)^{-1} \bar{v}_{i-1} \Big(\prod_{k=n}^{i} u_k\Big) + \Big(\prod_{k=n}^{i} u_k\Big)^{-1} \bar{s}_{i-1} \Big(\prod_{k=n}^{i} u_k\Big).
\end{aligned} \tag{3.24}
$$

This expression, analogue to (3.13), is important and we will soon consider an explicite application to Cartan-Möbius geometry. We see that again $v_j = 0$ for $j \in [n, \dots, i]$ so that

$$
s_j \omega_i = 0, \qquad s_j \Omega_i = 0, \qquad \text{and} \qquad s_j \psi_i = 0, \qquad \text{and note} \quad BRS_{j,i} = 0 \quad \text{for } j \in [n, \dots, i].
$$

The **final modified BRS algebra** $\overline{BRS}^{f}_{i-1,i}$, generalization of (3.14), reads

$$
\bar{s}_{i-1}\omega_i = -D_i \widehat{v}_i, \qquad \bar{s}_{i-1}\Omega_i = \big[\Omega_i, \widehat{v}_i\big], \qquad \bar{s}_{i-1}\psi_i = -\rho_*(\widehat{v}_i)\psi_i, \quad \text{and} \qquad \bar{s}_{i-1}\widehat{v}_i = -\tfrac{1}{2}\big[\widehat{v}_i, \widehat{v}_i\big], \tag{3.25}
$$

where $D_i := d + \big[\omega_i, \ \big]$. If we also define $\widetilde{v}_i := \Big(\prod\limits_{k=n}^{i} u_k\Big)^{-1} v_{i-1} \Big(\prod\limits_{k=n}^{i} u_k\Big) + \Big(\prod\limits_{k=n}^{i} u_k\Big)^{-1} s_{i-1} \Big(\prod\limits_{k=n}^{i} u_k\Big)$, we have the **final subalgebras** $BRS^{f}_{j,i}$ for $j \in [i-1, \dots, 0]$,

$$
s_j \omega_i = -D_i \widetilde{v}_j, \qquad s_j \Omega_i = \big[\Omega_i, \widetilde{v}_j\big], \qquad s_j \psi_i = -\rho_*(\widetilde{v}_j)\psi_i, \qquad \text{and} \qquad s_j \widetilde{v}_j = -\tfrac{1}{2}\big[\widetilde{v}_j, \widetilde{v}_j\big].
$$

Schematically this is the reduction,

$$
\mathcal{P}_n \xrightarrow{\prod_{k=n}^{i} u_k} \mathcal{P}_{i-1} \qquad\qquad \overline{BRS}_n \xrightarrow{\prod_{k=n}^{i} u_k} \overline{BRS}^{f}_{i-1,i}
$$

In the next section we consider two applications. One, trivial, to General Relativity. The second is the BRS version of the example of Cartan-Möbius geometry treated in 2.4.2. We will also see how the latter connects to recent litterature on the Weyl anomaly.

3.3 Applications

3.3.1 General Relativity

We here consider the BRS treatment of GR. We briefly recall the setup given in section 2.3.2. We work with a Cartan geometry modeled on the Klein pair (G, H) where G is the Poincaré group, $H = SO$ the Lorentz group and the associated homogeneous space is the n-dimensional Minkowski space. The principal bundle of this Cartan geometry is the first-order structure $\mathcal{P}(\mathcal{M}, SO)$ where live the Cartan connection and its curvature whose pull-backs are,

$$
\varpi = A + \theta = \begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix}, \qquad \Omega = R + \Theta = \begin{pmatrix} R & \Theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} dA + A \wedge A & d\theta + A \wedge \theta \\ 0 & 0 \end{pmatrix},
$$

where A is the Lorentz/spin connection, θ is the vielbein 1-form. Our dressing field was $u \sim e : \mathcal{M} \to GL$ and we had in matrix form the composite fields,

$$
\widehat{\varpi} = u^{-1}\varpi u + u^{-1}du = \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} d \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-1}Ae + e^{-1}de & e^{-1}\theta \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} \Gamma & dx \\ 0 & 0 \end{pmatrix},
$$

where we established that Γ is a metric linear connection, and

$$\widehat{\Omega} = u^{-1}\Omega u = \widehat{D}\widehat{\varpi} = d\widehat{\varpi} + \widehat{\varpi}\wedge\widehat{\varpi} = \begin{pmatrix} d\Gamma + \Gamma\wedge\Gamma & \Gamma\wedge dx \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{R} & T \\ 0 & 0 \end{pmatrix},$$

where $\widehat{R}$ is the Riemann curvature and T is the torsion, both written with space-time indices.

The initial BRS algebra Given the finite gauge element $\gamma = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}$ with $S \in SO$, we have the initial ghost $v = \begin{pmatrix} v_L & 0 \\ 0 & 0 \end{pmatrix}$ with $v_L \in \mathfrak{so}$, the subscript L standing for 'Lorentz'. The initial BRS algebra of General Relativity is,

$$s\varpi = -Dv, \qquad s\Omega = \big[\Omega, v\big] \qquad \text{and} \qquad sv = \tfrac{1}{2}\big[v, v\big] = -v^2.$$

Here the bracket is the matrix bracket, this is why the bracket of v with itself can be written as a square. Explicitely this gives,

$$\begin{pmatrix} sA & s\theta \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} dv_L & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_L & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} v_L & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -dv_L - \big[A, v_L\big] & -v_L\theta \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} sR & s\Theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R & \Theta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_L & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} v_L & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R & \Theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \big[R, v_L\big] & -v_L\Theta \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} sv_L & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}v_L^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

The modified BRS algebra The key element of the new algebra is of course the composite ghost. To find its expression we only need to find how the dressing field u transforms under the action of the (initial) BRS operator. As a dressing field its finite gauge transformation is given by,

$$u^\gamma = \gamma^{-1}u \rightarrow \begin{pmatrix} e^S & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} S^{-1}e & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{hence the BRS transformation,} \quad su = -vu \rightarrow \begin{pmatrix} se & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -v_L e & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore the **composite ghost** is,

$$\widehat{v} = u^{-1}vu + u^{-1}su = u^{-1}vu + u^{-1}(-vu) = 0, \tag{3.26}$$

and we have the **trivial modified BRS algebra**,

$$s\widehat{\varpi} = \begin{pmatrix} s\Gamma & sdx \\ 0 & 0 \end{pmatrix} = 0, \qquad s\widehat{\Omega} = \begin{pmatrix} s\widehat{R} & sT \\ 0 & 0 \end{pmatrix} = 0. \tag{3.27}$$

This expresses the invariance of the coordinate chart under gauge transformation, which is obvious, and the SO-invariance of Γ, $\widehat{R}$ and T. A fact that we knew already from our finite analysis of the last chapter. The fields $\widehat{\omega}$ and $\widehat{\Omega}$ belongs to the natural geometry of $\mathcal{M}$ and are blind to any gauge structure, the latter being neutralized by the dressing field u, as the vanishing of the composite ghost $\widehat{v}$ testifies. Lets now turn to a less trivial example.

3.3.2 Cartan-Möbius geometry

Let us recall the setup of this geometry. The underlying bundle is a 2^{nd}-order G-structure $\mathcal{P}(\mathcal{M}, H)$ whose structure group can be decomposed as

$$H = K_0K_1 = W\cdot SO\cdot K_1 = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & r & \frac{1}{2}rr^t \\ 0 & \mathbb{1} & r^t \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; z \in \mathbb{R}^*,\ S \in SO(r,s),\ r \in \mathbb{R}^{(r,s)*} \right\}.$$

Its Lie algebra is graded and can be decomposed as $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{h}_1 = \mathfrak{co}(r,s) + \mathbb{R}^{(r,s)*} = \mathbb{R} + \mathfrak{so}(r,s) + \mathbb{R}^{(r,s)*}$. In matrix form we get,

$$\mathfrak{h} = \left\{ \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\epsilon \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \iota & 0 \\ 0 & 0 & \iota^t \\ 0 & 0 & 0 \end{pmatrix} \middle| \, \epsilon \in \mathbb{R}^* \; s \in \mathfrak{so}(r,s), \; \iota \in \mathbb{R}^{(r,s)*} \right\}$$

The graded Lie algebra of the principal group in the Klein pair (G,H) modeling the Cartan-Möbius geometry is $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 = \mathbb{R}^{(r,s)} + \mathfrak{h}$. The normal Cartan connection and its curvature are then,

$$\varpi = \begin{pmatrix} a & \alpha & 0 \\ \theta & A & \alpha^t \\ 0 & \theta^t & -a \end{pmatrix}, \quad \text{and} \quad \Omega = \begin{pmatrix} 0 & \Pi & 0 \\ 0 & F & \Pi^t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d\alpha + \alpha(A - a\mathbb{1}) & 0 \\ 0 & dA + A^2 + \theta\alpha + \alpha^t\theta^t & d\alpha^t + (A + a\mathbb{1})\alpha^t \\ 0 & 0 & 0 \end{pmatrix}.$$

We restrict our considerations to the normal geometry for it is naturally equivalent to a conformal class of metrics on $\mathcal{M}$. The initial total algebra $\overline{BRS}_1$ of the geometry is,

$$s\omega = -Dv, \qquad s\Omega = \left[\Omega, v\right], \qquad \text{and} \qquad sv = -\tfrac{1}{2}\left[v, v\right],$$

whose explicit form is not needed for now. The initial ghost decomposes according to the grading of $\mathfrak{h}$,

$$v = v_W + v_L + v_1 = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\epsilon \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & v_L & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \iota & 0 \\ 0 & 0 & \iota^t \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.28}$$

so that we can write $s = s_0 + s_1 = s_W + s_L + s_1$ and get three subalgebras, BRS_W, BRS_L and BRS_1 corresponding to each ghost. We will now follow the scheme of the last section and reduce the BRS algebra step by step.

First reduction

We know that we have a dressing field $u_1 : U \to K_1$, to which we give the matrix expression $u_1 = \begin{pmatrix} 1 & q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix}$.

We then form the composite fields,

$$\varpi_1 := \varpi^{u_1} = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \theta & A_1 & \alpha_1^t \\ 0 & \theta^t & 0 \end{pmatrix}, \quad \text{and} \quad \Omega_1 := \Omega^{u_1} = \begin{pmatrix} 0 & \Pi_1 & 0 \\ 0 & F_1 & \Pi_1^t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d\alpha_1 + \alpha_1 A_1 & 0 \\ 0 & dA_1 + A_1^2 + \theta\alpha_1 + \alpha_1^t\theta^t & d\alpha_1^t + A_1\alpha_1^t \\ 0 & 0 & 0 \end{pmatrix}.$$

In the last chapter we identified A_1 with the Lorentz/spin connection. Furthermore since we are in the normal case $F_1 = F$ is the Weyl curvature 2-form, α_1 is the Schouten 1-form and Π_1 is the Cotton 2-form. As to the first composite ghost it reads,

$$\widehat{v}_1 := v^{u_1} = u_1^{-1}vu_1 + u_1^{-1}su_1 = u_1^{-1}\big(v_W + v_L + v_1\big)u_1 + u_1^{-1}\big(s_W + s_L + s_1\big)u_1.$$

We thus need to know the action of the various BRS operators on the dressing field u_1. Here the **compatibility conditions** enter the game. Indeed on account of (2.35) and (2.36) we have,

$$s_1 u_1 = -v_1 u_1 \qquad \text{and} \qquad s_L u_1 = \left[u_1, v_L\right], \tag{3.29}$$

which are nothing but instances of (3.17) and (3.18). The first composite ghost is then,

$$\begin{aligned} \widehat{v}_1 &= u_1^{-1}v_W u_1 + u_1^{-1}v_L u_1 + u_1^{-1}v_1 u_1 + u_1^{-1}s_W u_1 + u_1^{-1}\left[u_1, v_L\right] - u_1^{-1}v_1 u_1, \\ \widehat{v}_1 &= u_1^{-1}v_W u_1 + u_1^{-1}s_W u_1 + v_L. \end{aligned}$$

We can already see that the ghost v_1 has been killed by the dressing u_1 so that the subalgebra corresponding to s_1 is now trivial,

$$s_1\omega_1 = 0 \quad \text{and} \quad s_1\Omega_1 = 0. \qquad \text{We note} \quad BRS_{1,1} = 0. \tag{3.30}$$

This expresses the K_1-invariance of the composite fields ϖ_1 and Ω_1 which live on $\mathcal{P}_0 = \mathcal{P}(\mathcal{M}, K_0) \simeq \mathcal{P}_1/K_1$. The BRS operator $s = s_0 + s_1 = s_W + s_L + s_1$ has been reduced to $s_0 = s_W + s_L$. The latter handles the infinitesimal gauge freedom on $\mathcal{P}_0$ as we now see.

There remains to find the transformation of the dressing u_1 under s_W to get an explicit matrix expression for the composite ghost. To do this, recalling that $q = a \cdot e^{-1}$, we just need the first relation of the subalgebra BRS_W which is $s_W\varpi = -Dv_W$. Explicitely,

$$s_W \begin{pmatrix} a & \alpha & 0 \\ \theta & A & \alpha^t \\ 0 & \theta^t & -a \end{pmatrix} = \begin{pmatrix} -d\epsilon & -\epsilon\alpha & 0 \\ -\theta\epsilon & 0 & \alpha^t\epsilon \\ 0 & \epsilon\theta^t & d\epsilon \end{pmatrix}.$$

From this we find,

$$\begin{aligned} s_W a = -d\epsilon \quad &\rightarrow \quad s_W a \cdot dx = -dx \cdot \partial\epsilon \quad \rightarrow \quad s_W a = \partial\epsilon, \\ s_W\theta = -\theta\epsilon \quad &\rightarrow \quad s_W e \cdot dx = -e \cdot dx\, \epsilon \quad \rightarrow \quad s_W e = e\, \epsilon. \end{aligned}$$

Be careful with the notation $a = a \cdot dx$ which stands for $a = a_\mu dx^\mu$, and recall that ϵ is a ghost which anticommutes with odd forms. Compute now,

$$s_W q = s_W(a \cdot e^{-1}) = s_W a \cdot e^{-1} + a \cdot s_W e^{-1} = \partial\epsilon \cdot e^{-1} - \epsilon a \cdot e^{-1} = \partial\epsilon \cdot e^{-1} - \epsilon q. \tag{3.31}$$

Similar calculations give,

$$s_W q^t = -\epsilon q^t + \eta^{-1}(e^{-1})^T \cdot \partial\epsilon, \qquad \text{and} \qquad s_W\left(\tfrac{1}{2}qq^t\right) = -\epsilon q q^t + \partial\epsilon \cdot e^{-1}q^t \tag{3.32}$$

It is now easy to find,

$$\begin{aligned} u_1^{-1}s_W u_1 &= \begin{pmatrix} 1 & -q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & -q^t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & s_W q & s_W\left(\frac{1}{2}qq^t\right) \\ 0 & 0 & s_W q^t \\ 0 & 0 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} 1 & -q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & -q^t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -\epsilon q + \partial\epsilon \cdot e^{-1} & -\epsilon qq^t + \partial\epsilon \cdot e^{-1}q^t \\ 0 & 0 & -\epsilon q^t + \eta^{-1}(e^{-1})^T \cdot \partial\epsilon \\ 0 & 0 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} 0 & -\epsilon q + \partial\epsilon \cdot e^{-1} & 0 \\ 0 & 0 & -\epsilon q^t + \eta^{-1}(e^{-1})^T \cdot \partial\epsilon \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Find also,

$$u_1^{-1}v_W u_1 = \begin{pmatrix} 1 & -q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & -q^t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\epsilon \end{pmatrix}\begin{pmatrix} 1 & q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \epsilon & 0 & -\frac{1}{2}qq^t\epsilon \\ 0 & 0 & q^t\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix}\begin{pmatrix} 1 & q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \epsilon & \epsilon q & 0 \\ 0 & 0 & q^t\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix}.$$

Finally we obtain the **first composite ghost**,

$$\widehat{v}_1 = \begin{pmatrix} \epsilon & \partial\epsilon \cdot e^{-1} & 0 \\ 0 & v_L & \eta^{-1}(e^{-1})^T \cdot \partial\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix}. \tag{3.33}$$

From the form of this ghost we see that the subalgebra corresponding to v_L is unchanged,

$$s_L\varpi_1 = -D_1 v_L \quad \text{and} \quad s_L\Omega_1 = \left[\Omega_1, v_L\right],$$

where here $D_1 = d + \left[\varpi_1,\ \ \right]$. This means that ϖ_1 and Ω_1 still behave as connection and curvature under the group SO, as warranted by the second compatibility condition in (3.29). So a new dressing operation to neutralize it would make sense.

Second reduction and final BRS algebra

We know that we have a dressing $u_0 : U \to GL \supset SO$, to which we give the matrix expression $u_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

We then form the composite fields

$$\varpi_0 := \varpi_1^{u_0} = \begin{pmatrix} 0 & P & 0 \\ dx & \Gamma & g^{-1}P^T \\ 0 & dx^T \cdot g & 0 \end{pmatrix} \qquad \text{and} \qquad \Omega_0 := \Omega_1^{u_0} = \begin{pmatrix} 0 & C & 0 \\ 0 & W & g^{-1}C^T \\ 0 & 0 & 0 \end{pmatrix}.$$

Since we are in the normal case Γ is the Levi-Civita connection, P is the Schouten tensor, W is the Weyl tensor and C is the Cotton tensor. As to the second composite ghost it reads,

$$\widehat{v}_0 = u_0^{-1}\widehat{v}_1 u_0 + u_0^{-1}s_0 u_0 = u_0^{-1}\left(u_1^{-1}v_W u_1 + u_1^{-1}s_W u_1\right)u_0 + u_0^{-1}v_L u_0 + u_0^{-1}s_W u_0 + u_0^{-1}s_L u_0.$$

So we need to know the transformation of u_0 under s_W and s_L. The first is already known from the sub-algebra BRS_W and the second should be the BRS version of a dressing transformation law and is then just an instance of the compatibility condition (3.17). So we have,

$$s_W e = e\epsilon \quad \to \quad s_W u_0 = \widetilde{\epsilon} u_0 \quad \to \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & s_W e & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_L u_0 = -v_L u_0 \tag{3.34}$$

The **final composite ghost** is then,

$$\begin{aligned}
\widehat{v}_0 &= u_0^{-1}\left(u_1^{-1}v_W u_1 + u_1^{-1}s_W u_1\right)u_0 + u_0^{-1}v_L u_0 + u_0^{-1}\widetilde{\epsilon} u_0 + u_0^{-1}\left(-v_L u_0\right), \\
&= u_0^{-1}\left(u_1^{-1}v_W u_1 + u_1^{-1}s_W u_1\right)u_0 + \widetilde{\epsilon} u_0^{-1}u_0, \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \epsilon & \partial\epsilon \cdot e^{-1} & 0 \\ 0 & 0 & \eta^{-1}(e^{-1})^T \cdot \partial\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon\delta & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

$$\widehat{v}_0 = \begin{pmatrix} \epsilon & \partial\epsilon & 0 \\ 0 & \epsilon\delta & g^{-1}\partial\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix} =: \widehat{v}_W. \tag{3.35}$$

Comparing with the first composite ghost (3.33) we see that the ghost v_L has been killed by the second dressing u_0 so that the subalgebra corresponding to s_L is now trivial,

$$s_L\varpi_0 = 0 \quad \text{and} \quad s_L\Omega_0 = 0. \qquad \text{We note} \quad BRS_{L,0} = 0. \tag{3.36}$$

Furthermore the dressing u_0 satisfies the compatibility condition $s_1 u_0 = 0$, which is an instance of (3.19) and the infinitesimal version of (2.39), so that we also have

$$s_1\varpi_0 = 0 \quad \text{and} \quad s_1\Omega_0 = 0. \qquad \text{We note} \quad BRS_{1,0} = 0. \tag{3.37}$$

The triviality of these two subalgebras expresses the K_1- and SO-invariance of the composite fields ϖ_0 and Ω_0 which live on the **Weyl bundle** $\mathcal{P}_W := \mathcal{P}(\mathcal{M}, W)$. Their **infinitesimal residual Weyl gauge freedom** is given by the **final reduced BRS algebra** $BRS_{W,0}$,

$$s_W \varpi_0 = -D_0 \widehat{v}_W, \qquad s_W \Omega_0 = \left[\Omega_0, \widehat{v}_W\right], \qquad \text{and} \qquad s_W \widehat{v}_W = -\widehat{v}_W^2, \tag{3.38}$$

with a final ghost which depends only on the Weyl ghost ϵ.

Two steps in one

Perhaps going through the two steps is confusing or cumbersome. Hopefully we can reduce the inital algebra $\overline{BRS}_1$, $s = s_W + s_L + s_1$, to the final residual algebra $BRS_{W,0}$, s_W, in a single step thanks to the dressing field $u_1 u_0$. Let us do it again, at the risk of some repetitions. We already know that the map $u_1 u_0 : U \to SO \cdot K_1$ allows to reduce the initial 2^{nd}-order structure of the Cartan-Möbius geometry $\mathcal{P}_1$ to the first-order Weyl bundle $\mathcal{P}_W$, and allows to define the composite forms

$$\varpi_0 := \varpi^{u_1 u_0} = \begin{pmatrix} 0 & P & 0 \\ dx & \Gamma & g^{-1}P^T \\ 0 & dx^T \cdot g & 0 \end{pmatrix}, \qquad \text{and} \qquad \Omega_0 := \Omega^{u_1 u_0} = \begin{pmatrix} 0 & C & 0 \\ 0 & W & g^{-1}C^T \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.39}$$

We still consider the normal geometry. As to the composite ghost it reads,

$$\begin{aligned} \widehat{v} &= (u_1 u_0)^{-1}\, v\, (u_1 u_0) + (u_1 u_0)^{-1}\, s(u_1 u_0), \\ &= (u_1 u_0)^{-1}\left(v_W + v_L + v_1\right)(u_1 u_0) + (u_1 u_0)^{-1}\left(s_W + s_L + s_1\right)(u_1 u_0) \end{aligned}$$

Now the two dressing fields u_1 and u_0 satisfy compatibility conditions that we collect here,

$$s_L u_1 = \left[u_1, v_L\right], \qquad s_1 u_1 = -v_1 u_1, \qquad \text{and} \qquad s_1 u_0 = 0, \qquad s_L u_0 = -v_L u_0.$$

From this conditions follows, first

$$s_1(u_1 u_0) = s_1 u_1\, u_0 + u_1\, s_1 u_0 = -v_1(u_1 u_0), \tag{3.40}$$

meaning that $u_1 u_0$ is a dressing under K_1. And then,

$$s_L(u_1 u_0) = s_L u_1\, u_0 + u_1\, s_L u_0 = \left[u_1, v_L\right] u_0 - u_1 v_L u_0 = -v_L(u_1 u_0), \tag{3.41}$$

meaning that $u_1 u_0$ is a dressing under the Lorentz group SO. From (3.40) and (3.41), which are instances of (3.20), we have the composite ghost,

$$\begin{aligned} \widehat{v} &= (u_1 u_0)^{-1}\left(v_W + v_L + v_1\right)(u_1 u_0) + (u_1 u_0)^{-1}\left(s_W + s_L + s_1\right)(u_1 u_0), \\ &= (u_1 u_0)^{-1} v_W\, (u_1 u_0) + (u_1 u_0)^{-1}\, s_W (u_1 u_0) =: \widehat{v}_W. \end{aligned} \tag{3.42}$$

This is an instance of (3.24). We see right away that v_L and v_1 have been killed by the dressing $u_1 u_0$ so that the corresponding subalgebras are trivial,

$$\begin{aligned} &s_1 \varpi_0 = 0, \qquad \text{and} s_1 \Omega_0 = 0. \qquad \text{Noted} \quad BRS_{1,0} = 0, \\ &s_L \varpi_0 = 0, \qquad \text{and} s_L \Omega_0 = 0. \qquad \text{Noted} \quad BRS_{L,0} = 0. \end{aligned}$$

The triviality of these two subalgebra means that the composite forms ϖ_0 and Ω_0 live on the Weyl bundle $\mathcal{P}_W$, and their residual Weyl gauge freedom is handled by the **residual Weyl BRS algebra** $BRS_{W,0}$,

$$s_W \varpi_0 = -D_0 \widehat{v}_W, \qquad s_W \Omega_0 = \left[\Omega_0, \widehat{v}_W\right], \qquad \text{and} \qquad s_W \widehat{v}_W = -\widehat{v}_W^2. \tag{3.43}$$

The residual Weyl BRS algebra: explicit results

The matrix calculations are easy enough to be given here entirely. The infinitesimal residual Weyl gauge freedom of the dressed normal Cartan connection is,

$$
\begin{aligned}
s_W\varpi_0 &= -d\widehat{v}_W - \varpi_0\widehat{v}_W - \widehat{v}_W\varpi_0,\\
&= \begin{pmatrix} -d\epsilon & -d(\partial\epsilon) & 0\\ 0 & -d\epsilon\delta & -d(g^{-1}\partial\epsilon)\\ 0 & 0 & d\epsilon\end{pmatrix} - \begin{pmatrix} 0 & P & 0\\ dx & \Gamma & g^{-1}P^T\\ 0 & dx^T\cdot g & 0\end{pmatrix}\begin{pmatrix}\epsilon & \partial\epsilon & 0\\ 0 & \epsilon\delta & g^{-1}\partial\epsilon\\ 0 & 0 & -\epsilon\end{pmatrix}\\
&\qquad - \begin{pmatrix}\epsilon & \partial\epsilon & 0\\ 0 & \epsilon\delta & g^{-1}\partial\epsilon\\ 0 & 0 & -\epsilon\end{pmatrix}\begin{pmatrix} 0 & P & 0\\ dx & \Gamma & g^{-1}P^T\\ 0 & dx^T\cdot g & 0\end{pmatrix},\\
&= \begin{pmatrix} -d\epsilon & -d(\partial\epsilon) & 0\\ 0 & -d\epsilon\delta & -d(g^{-1}\partial\epsilon)\\ 0 & 0 & d\epsilon\end{pmatrix} - \begin{pmatrix} 0 & P\epsilon\delta & Pg^{-1}\partial\epsilon\\ dx\epsilon & dx\partial\epsilon + \Gamma\epsilon\delta & \Gamma g^{-1}\partial\epsilon - g^{-1}P\epsilon\\ 0 & dx^T\cdot g\,\epsilon\delta & dx^T\cdot g\, g^{-1}\partial\epsilon\end{pmatrix}\\
&\qquad - \begin{pmatrix}\partial\epsilon dx & \epsilon P + \partial\epsilon\Gamma & \partial\epsilon g^{-1}P\\ \epsilon\delta dx & \epsilon\delta\Gamma + g^{-1}\partial\epsilon\, dx^T\cdot g & \epsilon\delta g^{-1}\cdot P\\ 0 & -\epsilon dx^T\cdot g & 0\end{pmatrix}.
\end{aligned}
$$

Remembering that ϵ anticommutes with odd forms, that $d = dx\cdot\partial$ and using $\nabla g^{-1} = dg^{-1} + g^{-1}\Gamma^T + \Gamma g = 0$ in the computation of entry (2, 3), we obtain

$$
\begin{pmatrix} 0 & s_W P & 0\\ s_W dx & s_W\Gamma & s_W\left(g^{-1}P^T\right)\\ 0 & s_W\left(g\cdot dx\right) & 0\end{pmatrix} = \begin{pmatrix} 0 & -d(\partial\epsilon) - \partial\epsilon\Gamma & 0\\ 0 & -d\epsilon\delta - dx\partial\epsilon - g^{-1}\partial\epsilon\, dx^T\cdot g & -g^{-1}\left(d(\partial\epsilon) - \Gamma^T\partial\epsilon\right) - 2\epsilon g^{-1}\cdot P^T\\ 0 & 2\epsilon dx^T\cdot g & 0\end{pmatrix}.
$$

Let us write each entry in components. Entry (2, 1) is $s_W dx^\mu = 0$ and expresses the invariance of the coordinate chart under gauge transformation, here the Weyl rescaling. This is of constant use for the other entries. Entry (3, 2) is then,

$$
s_W g_{\mu\nu} = 2\epsilon g_{\mu\nu}, \tag{3.44}
$$

which gives the infinitesimal **Weyl rescaling of the metric** tensor. Entry (2, 2) is,

$$
s_W\Gamma^\rho{}_{\mu\nu} = \delta^\rho{}_\nu\partial_\mu\epsilon \;+\; \delta^\rho{}_\mu\partial_\nu\epsilon \;+\; g^{\rho\lambda}\partial_\lambda\epsilon\, g_{\mu\nu}, \tag{3.45}
$$

which is the infinitesimal **tranformation of the Christoffel symbols**, of the Levi-Civita connection, under Weyl rescaling. Entry (1, 2) is,

$$
s_W P_{\mu\nu} = \partial_\mu(\partial_\nu\epsilon) - \partial_\lambda\epsilon\Gamma^\lambda{}_{\mu\nu} = \nabla_\mu(\partial_\nu\epsilon), \tag{3.46}
$$

which is the infinitesimal **transformation of the Schouten tensor** under Weyl rescaling. Finally, the entry (2, 3) is,

$$
s_W\left(g^{\rho\lambda}P_{\lambda\mu}\right) = -2\epsilon g^{\rho\lambda}P_{\lambda\mu} + g^{\rho\lambda}\left(\partial_\mu(\partial_\lambda\epsilon) - \Gamma_{\mu\lambda}{}^\alpha\partial_\alpha\epsilon\right). \tag{3.47}
$$

This is redundant with (3.44) and (3.46). Comparing with the finite transformations given in 2.4.2 we see that the residual BRS algebra gives very easily the complete infinitesimal counterpart. Except for the Schouten tensor because the latter has a transformation which includes terms of order two in the Weyl parameter. These terms are of course out of reach for the linear scope of the BRS machinery.

The infinitesimal residual Weyl gauge freedom of the dressed normal curvarure is,

$$\begin{aligned}
s_W\Omega_0 &= \Omega_0\widehat{v}_W - \widehat{v}_W\Omega_0, \\
&= \begin{pmatrix} 0 & C & 0 \\ 0 & W & g^{-1}C^T \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \epsilon & \partial\epsilon & 0 \\ 0 & \epsilon\delta & g^{-1}\partial\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix} - \begin{pmatrix} \epsilon & \partial\epsilon & 0 \\ 0 & \epsilon\delta & g^{-1}\partial\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix}\begin{pmatrix} 0 & C & 0 \\ 0 & W & g^{-1}C^T \\ 0 & 0 & 0 \end{pmatrix}, \\
&= \begin{pmatrix} 0 & C\,\epsilon\delta & C\,g^{-1}\partial\epsilon \\ 0 & W\,\epsilon\delta & W\,g^{-1}\partial\epsilon - g^{-1}C^T\epsilon \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \epsilon C - \partial\epsilon\, W & \partial\epsilon\, g^{-1}C^T \\ 0 & \epsilon\delta W & \epsilon\delta g^{-1}C^T \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Remembering this time that ϵ commutes with even forms and using $Wg^{-1} = -g^{-1}W^T$, we obtain

$$s_W\Omega_0 = \begin{pmatrix} 0 & s_WC & 0 \\ 0 & s_WW & s_W\left(g^{-1}C^T\right) \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\partial\epsilon\cdot W & 0 \\ 0 & 0 & -g^{-1}W^T\partial\epsilon - 2\epsilon g^{-1}C \\ 0 & 0 & 0 \end{pmatrix}.$$

Using again the fact that $s_W dx^\mu = 0$ we can write the entries in components. Entry (1, 2) gives,

$$s_WC_{\nu,\mu\sigma} = -\partial_\lambda\epsilon W^\lambda{}_{\nu,\mu\sigma}, \tag{3.48}$$

which is the infinitesimal **transformation of the Cotton tensor** under Weyl rescaling. Entry (2, 3) is,

$$s_W\left(g^{\rho\lambda}C_{\lambda,\mu\sigma}\right) = -2\epsilon g^{\rho\lambda}C_{\lambda,\mu\sigma} - g^{\rho\lambda}W_{\lambda,\mu\sigma}{}^\alpha\partial_\alpha\epsilon. \tag{3.49}$$

This is redudant with (3.48) and (3.44). Finally entry (2, 2) gives,

$$s_WW^\rho{}_{\nu,\mu\sigma} = 0. \tag{3.50}$$

This is the **invariance of the Weyl tensor** under Weyl rescaling. Again we appreciate how easily the residual BRS algebra provides the complete infinitesimal counterpart of the finite transformations derived in 2.4.2.
At last the identity satisfied by the final composite ghost is,

$$\begin{aligned}
s_W\widehat{v} &= -\widehat{v}_W^2 \\
&= -\begin{pmatrix} \epsilon & \partial\epsilon & 0 \\ 0 & \epsilon\delta & g^{-1}\partial\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix}^2 = \begin{pmatrix} \epsilon^2 & \epsilon\partial\epsilon + \partial\epsilon\;\epsilon & \partial\epsilon g^{-1}\partial\epsilon \\ 0 & \epsilon^2\delta & \epsilon\delta g^{-1}\partial\epsilon + g^{-1}\partial\epsilon\;\epsilon \\ 0 & 0 & \epsilon^2 \end{pmatrix}.
\end{aligned} \tag{3.51}$$

Recalling that ϵ anticommutes with itself, we obtain

$$\begin{pmatrix} s_W\epsilon & s_W(\partial\epsilon) & 0 \\ 0 & s_W\epsilon\delta & s_W\left(g^{-1}\partial\epsilon\right) \\ 0 & 0 & -s_W\epsilon \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\epsilon g^{-1}\partial\epsilon \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.52}$$

This just gives again the Weyl rescaling of the (inverse) metric $s_Wg^{-1} = -2\epsilon g^{-1}$ which is redundant with (3.44), but also

$$s_W\epsilon = 0, \tag{3.53}$$

which expresses the fact that the residual Weyl gauge group is abelian.

Remark: The dressed **algebraic connection** $\widetilde{\varpi}_0$ on $\mathcal{P}_W \times \mathcal{W}$ is,

$$\widetilde{\varpi}_0 = \varpi_0 + \widehat{v}_W = \begin{pmatrix} \epsilon & P + \partial\epsilon & 0 \\ dx & \Gamma + \epsilon\,\delta & g^{-1}\left(P^T + \partial\epsilon\right) \\ 0 & dx^T \cdot g & -\epsilon \end{pmatrix}. \tag{3.54}$$

It turns out that this is the geometrical object that underlies the results obtained in (Boulanger, 2007a) by an entirely different approach. In this paper on the Weyl anomaly, the entries of $\widetilde{\varpi}_0$ are found as fields (called the *generalized connections*) belonging to a space of variables identified though cohomological techniques. It is satisfying to have a clear geometrical picture supporting this result. Moreover the table I in the paper is entirely given by $BRS_{W,0}$. We postpone the discussion on this to Appendix A.3. The latter should be read only after the next section which considers the inclusion of the infinitesimal diffeomorphisms in the present formalism.

3.4 Extended BRS algebra: infinitesimal diffeomorphisms.

3.4.1 Translations and local diffeomorphisms

Until now we've worked with principal bundles $\mathcal{P}(\mathcal{M},H)$ and their Cartan connection $\varpi \in \Lambda^1(U,\mathfrak{g})$. The gauge group of the bundle is $\mathcal{H}$ and accordingly the infinitesimal gauge freedom is handled by a BRS algebra whose ghost $v_\mathfrak{h}$ takes values in $\mathfrak{h}$. In the case of a **reductive Cartan geometry** the principal group G has a Lie algebra that splits as $\mathfrak{g} = \mathfrak{p} + \mathfrak{h}$, with $\mathfrak{p} \simeq \mathbb{R}^n$. Suppose one wants to consider a ghost

$$v_\mathfrak{g} = v_\mathfrak{p} + v_\mathfrak{h} \qquad \text{with values in } \mathfrak{g}. \tag{3.55}$$

What would this means? Simply that we consider the infinitesimal version of the gauge group $\mathcal{G}$, and the associated BRS algebra would express the infinitesimal $\mathcal{G}$-gauge transformations of the Cartan connection and its curvature. This could suggest to consider the Cartan connection ϖ as a usual Ehresmann connection on a bundle[4] $\mathcal{P}'(\mathcal{M},G)$ where G includes the group of translations $\mathbb{R}^n$. Such a situation would be the starting point of various gauge approaches to gravitation that go by the names of 'gauge affine gravity' or 'Poincaré gauge gravity', a move initiated as early as 1955-56 by Utiyama whose paper can be found in (O'Raifeartaigh, 1997).

One may we feel the usual unease with the idea of the translation group $\mathbb{R}^n$ being comprised in the structure group, the latter describing some 'internal' degrees of freedom. Therefore we stick to the viewpoint that ϖ is a Cartan connection on $\mathcal{P}(\mathcal{M},H)$, so that the 'internal' symmetry is H only. Even in this case the group of translations cannot be seen as 'external' either, it is not a symmetry of the base manifold $\mathcal{M}$ (except in the trivial case of a null Riemann cuvature, $R = 0$). Nevertheless this is not what we asked for. Indeed we wanted to consider a ghost $v_\mathfrak{p}$ with values in $\mathfrak{p} = \mathbb{R}^n$, that is *infinitesimal translations*, and this makes sense *locally*.

Remember that the ghost $v_\mathfrak{h}$ is a symbolical place holder for the Maurer-Cartan form, $\omega_\mathcal{H}$ of the gauge group. The latter being isomorphic to the group of vertical automorphisms of the bundle, $\mathcal{H} \simeq \mathrm{Aut}_v(\mathcal{P})$. Consider now the full group of automophisms $\mathrm{Aut}(\mathcal{P})$, including those which project as diffeomorphisms of $\mathcal{M}$. We have the exact sequence,

$$\mathrm{Aut}_v(\mathcal{P}) \sim \mathcal{H} \xrightarrow{\ \iota\ } \mathrm{Aut}(\mathcal{P}) \xrightarrow{\ \pi\ } \mathrm{Diff}(\mathcal{M})$$

The Lie algebra of the group of automorphism is isomorphic to the vector fields on $\mathcal{P}$, $\mathfrak{aut}(\mathcal{P}) \sim \Gamma(T\mathcal{P})$, so that the Lie algebra of the group of vertical automorphisms is isomorphic to the vertical vector fields, $\mathfrak{aut}_v(\mathcal{P}) \sim$

[4] Actually there is a rigorous way to view a Cartan connection as an Ehresmann connection on an enlarged bundle. We mention the fact that $\omega_{(p,g)} := Ad_g \pi_P^* \varpi + \pi_G^* \omega_G$ defines an Ehresmann connection on the bundle $\mathcal{P}' = \mathcal{P} \times_H G$ associated to $\mathcal{P}$. See Appendix A of (Sharpe, 1996). Notice that this quite resembles the form of the 'algebraic Cartan connection' $\widetilde{\varpi} = \varpi + \omega_\mathcal{H}$ on $\mathcal{P} \times \mathcal{H}$, so that the former could well be the geometrical base of the latter.

$\Gamma(V\mathcal{P})$. Thus the infinitesimal version of the above exact sequence involves the infinite dimensional Lie algebras,

$$\begin{tikzcd} \text{Lie}\mathcal{H} \arrow[r, "\iota"] & \Gamma(T\mathcal{P}) \arrow[r, "\pi_*"] \arrow[l, bend left, "\omega"] & \Gamma(T\mathcal{M}) \end{tikzcd}$$

This exact sequence[5] is split by the Ehresmann connection ω which is a $\mathfrak{h}$-part of the Cartan connection $\varpi = \omega + \theta$. So we can write, $\mathfrak{aut}(\mathcal{P}) = \Gamma(T\mathcal{M}) \oplus \text{Lie}\mathcal{H}$, and any form with values in $\mathfrak{aut}(\mathcal{P})$ splits accordingly. In particular the Maurer-Cartan form on the group $\text{Aut}(\mathcal{P})$ splits as,

$$\omega_{\text{Aut}(\mathcal{P})} = \omega_{\text{Diff}(\mathcal{M})} + \omega_{\mathcal{H}}$$

Compare with (3.55). We see that our requirement to consider a $\mathfrak{g}$-valued ghost amounts to take into account the infinitesimal diffeomorphisms. There remains a little caveat though. Indeed $v_{\mathfrak{p}}$ is $\mathbb{R}^n$-valued while $\omega_{\text{Diff}(\mathcal{M})}$ takes values in $\Gamma(T\mathcal{M})$. The brigde between the two target spaces is the soldering form, which is the $\mathfrak{p}$-part of the Cartan connection,

$$\begin{aligned} &\theta : \Gamma(TM) \to \mathbb{R}^n, && \text{let us write} \quad v_{\mathfrak{p}} = \theta \circ \omega_{\text{Diff}(\mathcal{M})}, \\ &\xi \mapsto \theta(\xi) =: \tau, && \text{in components} \quad e^a_\mu dx^u \left(\xi^\nu \partial_\nu\right) = e^a_\mu \xi^\mu =: \tau^a. \end{aligned} \tag{3.56}$$

Of course since the Lie algebra $\mathbb{R}^n$ is trivial while $\Gamma(T\mathcal{M})$ is not, θ is an isomorphism of vector spaces but not a Lie algebra morphism. It is the case only for a flat $\mathcal{M}$, $R = 0$. But this should not bother us. As the examples considered ahead will show, the dressing of such an extended ghost will allow to include the infinitesimal diffeomorphisms in the modified BRS framework.

3.4.2 Examples

Here we consider the inclusion of the ghost of translations, that is of infinitesimal diffeomorphisms, in the normal Cartan-Möbius geometry first, for it is the most complex example. We end the section and the chapter with the easier example of General Relativity. In both examples we will see that the dressing field method provides a form for the composite ghost inducing a *shifted BRS algebra* à la Langouche-Schücker-Stora.

The infinitesimal diffeomorphisms in Cartan-Möbius geometry

The normal Cartan connection and its curvature are $\mathfrak{g}$-valued forms, where $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 = \mathfrak{g}_{-1} + \mathfrak{h}$. After dressing, we obtained ω_0 and Ω_0 whose infinitesimal residual Weyl gauge freedom was given by the $BRS_{W,0}$. The associated composite ghost $\widehat{v}_W$ was the dressing of the initial ghost (3.28),

$$v_{\mathfrak{h}} = v_W + v_L + v_1 = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\epsilon \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & v_L & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \iota & 0 \\ 0 & 0 & \iota^t \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us add the ghost corresponding to the $\mathfrak{g}_{-1}$ sector to obtain the initial ghost,

$$v_{\mathfrak{g}} = v_{-1} + v_{\mathfrak{h}} = v_{-1} + v_W + v_L + v_1 = \begin{pmatrix} 0 & 0 & 0 \\ \tau & 0 & 0 \\ 0 & \tau^t & 0 \end{pmatrix} + \begin{pmatrix} \epsilon & \iota & 0 \\ 0 & v_L & \iota^t \\ 0 & 0 & -\epsilon \end{pmatrix}. \tag{3.57}$$

The initial BRS operator decomposes accordingly as $s = s_{-1} + s_W + s_L + s_1$. The dressing operation on the normal Cartan connection and its curvature remains unchanged. We only have to work out the new composite ghost. Once more, we will do so in two steps. Then we will write the associated BRS algebra.

[5]Which is a recurring feature of gauge theories and their generalizations from non-commutative geometry to Lie-algebroïds. See (François et al., 2014) and references therein.

The new composite ghost The first step is to obtain the new first composite ghost thanks to the dressing field u_1. We have,

$$\begin{aligned}
v_{\mathfrak{g}}^{u_1} &= u_1^{-1} v_{\mathfrak{g}} u_1 + u_1^{-1} s u_1, \\
&= u_1^{-1} v_{-1} u_1 + u_1^{-1} v_W u_1 + u_1^{-1} v_L u_1 + u_1^{-1} v_1 u_1 + u_1^{-1} s_{-1} u_1 + u_1^{-1} s_W u_1 + u_1^{-1} s_L u_1 + u_1^{-1} s_1 u_1, \\
&= u_1^{-1} v_{-1} u_1 + u_1^{-1} v_W u_1 + \boldsymbol{u_1^{-1} v_L u_1} + \boldsymbol{u_1^{-1} v_1 u_1} + u_1^{-1} s_{-1} u_1 + u_1^{-1} s_W u_1 + \boldsymbol{u_1^{-1}\big[u_1, v_L\big]} + \boldsymbol{u_1^{-1}(-v_1 u_1)}, \\
&= \underbrace{u_1^{-1} v_{-1} u_1 + u_1^{-1} s_{-1} u_1}_{= \mathrm{A}} + \underbrace{u_1^{-1} v_W u_1 + u_1^{-1} s_W u_1 + \boldsymbol{v_L}}_{= \mathrm{B}}.
\end{aligned}$$

The term B is already know, it is nothing but the first composite ghost (3.33),

$$B = \begin{pmatrix} \epsilon & \partial\epsilon \cdot e^{-1} & 0 \\ 0 & v_L & \eta^{-1}(e^{-1})^T \cdot \partial\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix}. \tag{3.58}$$

To find the term A we need the action of s_{-1} on u_1, that is on $q = a \cdot e^{-1}$. This will be provided by the first relation in the initial subalgebra BRS_{-1} which is,

$$s_{-1}\varpi = -Dv_{-1},$$

$$\begin{pmatrix} s_{-1}a & s_{-1}\alpha & 0 \\ s_{-1}\theta & s_{-1}A & s_{-1}\alpha^t \\ 0 & s_{-1}\theta^t & -s_{-1}a \end{pmatrix} = \begin{pmatrix} -\alpha\tau & 0 & 0 \\ -d\tau - (A - a\mathbb{1})\tau & -\alpha^t\tau^t - \tau\alpha & 0 \\ 0 & -d\tau^t - \tau^t(A + a\mathbb{1}) & \tau^t\alpha^t \end{pmatrix}.$$

We extract, on the one hand

$$s_{-1}a = -\alpha\tau \quad \rightarrow \quad s_{-1}a_\mu = \alpha_{\mu a}\tau^a, \quad \text{in index free notation}, \quad s_{-1}a = \alpha\tau.$$

On the other hand,

$$s_{-1}\theta = -d\tau - (A - a\mathbb{1})\tau \quad \rightarrow \quad s_{-1}e^a{}_\mu = \partial_\mu\tau^a + (A^a{}_{\mu b} - a_\mu\delta^a{}_b)\tau^b,$$

$$\text{in index free notation}, \quad s_{-1}e = \partial\tau + (A - a\mathbb{1})\tau.$$

As always, be careful with the index free notation. Confusions should be avoided from the context, often by checking the total degree, ghost + form. With these two resuts we can compute,

$$\begin{aligned}
s_{-1}q = s_{-1}(a \cdot e^{-1}) &= s_{-1}a \cdot e^{-1} + a s_{-1}e^{-1} = \alpha\tau \cdot e^{-1} - a \cdot e^{-1}(s_{-1}e)e^{-1}, \\
&= \alpha\tau \cdot e^{-1} - q\Big(\partial\tau + (A - a\mathbb{1})\tau\Big)e^{-1}, \\
&= \Big(\alpha - qA\Big)\tau e^{-1} + q\tau a \cdot e^{-1} - q\partial\tau \cdot e^{-1}, \\
&= \Big(\alpha - qA\Big)\tau e^{-1} + q\tau q - \partial(q\tau)e^{-1} + \partial q\tau e^{-1}, \\
s_{-1-}q &= \Big(\alpha - qA + \partial q\Big)\tau e^{-1} + q\tau q - \partial(q\tau)e^{-1}.
\end{aligned} \tag{3.59}$$

A similar calculation gives,

$$s_{-1}q^t = (e^{-1})^t\tau^t\Big(\alpha^t + Aq^t + \partial q^t\Big) + q^t\tau^t q^t - (e)^t\partial(\tau^t q^t). \tag{3.60}$$

We are now ready to write the term A,

$$
\begin{aligned}
A &= u_1^{-1}v_{-1}u_1 + u_1^{-1}s_{-1}u_1,\\
&= \begin{pmatrix} 1 & -q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & -q^t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ \tau & 0 & 0 \\ 0 & \tau^t & 0 \end{pmatrix}\begin{pmatrix} 1 & q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & -q^t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & s_{-1}q & s_{-1}\left(\frac{1}{2}qq^t\right) \\ 0 & 0 & s_{-1}q^t \\ 0 & 0 & 0 \end{pmatrix},\\
&= \begin{pmatrix} -q\tau & -q\tau q + \frac{1}{2}qq^t\tau^t & 0 \\ \tau & tq - q^t\tau^t & \tau\frac{1}{2}qq^t - q^t\tau^t q^t \\ 0 & \tau^t & \tau^t q^t \end{pmatrix} + \begin{pmatrix} 0 & s_{-1}q & 0 \\ 0 & 0 & s_{-1}q^t \\ 0 & 0 & 0 \end{pmatrix},\\
A &= \begin{pmatrix} -q\tau & \left(\alpha - qA + \partial q\right)\tau e^{-1} - \partial(q\tau)e^{-1} + \frac{1}{2}qq^t\tau^t & 0 \\ \tau & \tau q - q^t\tau^t & (e^{-1})^t\tau^t\left(\alpha^t + Aq^t + \partial q^t\right) - (e^{-1})^t\partial(\tau^t q^t) + \frac{1}{2}\tau qq^t \\ 0 & \tau^t & \tau^t q^t \end{pmatrix}.
\end{aligned}
$$

Finally the new first composite ghost is,

$$
v_g^{u_1} = \begin{pmatrix} \epsilon^{u_1} & \iota^{u_1} & 0 \\ \tau^{u_1} & v_L^{u_1} & (\iota^t)^{u_1} \\ 0 & (\tau^t)^{u_1} & -\epsilon^{u_1} \end{pmatrix} := u_1^{-1}v_g u_1 + u_1^{-1}su_1 = A + B = \tag{3.61}
$$

$$
\begin{pmatrix} \epsilon - q\tau & \partial(\epsilon - q\tau)e^{-1} + \left(\alpha - qA + \partial q\right)\tau e^{-1} + \frac{1}{2}qq^t\tau^t & 0 \\ \tau & v_L + \tau q - q^t\tau^t & (e^{-1})^t\partial(\epsilon - \tau^t q^t) + (e^{-1})^t\tau^t\left(\alpha^t + Aq^t + \partial q^t\right) + \frac{1}{2}\tau qq^t \\ 0 & \tau^t & -\epsilon + \tau^t q^t \end{pmatrix}. \tag{3.62}
$$

This ghost is associated to a BRS algebra for $\varpi_1 := \varpi^{u_1}$ and $\Omega_1 := \Omega^{u_1}$ that we won't write. Instead we go on with the second step.

The new second, and final, composite ghost is the dressing of $v_g^{u_1}$ with u_0. We have,

$$
\begin{aligned}
v_g^{u_1u_0} &:= u_0^{-1}v_g^{u_1}u_0 + u_0^{-1}su_0,\\
&= u_0^{-1}v_g^{u_1}u_0 + u_0^{-1}(s_{-1} + s_W + s_L + s_1)u_0,
\end{aligned}
$$

But u_0 satisfies the compatibility condition $s_1u_0 = 0$ so,

$$
v_g^{u_1u_0} = u_0^{-1}v_g^{u_1}u_0 + u_0^{-1}\left(s_{-1} + s_W + s_L\right)u_0. \tag{3.63}
$$

If we develop the calculation we get,

$$
\begin{aligned}
v_g^{u_1u_0} &= u_0^{-1}\left(u_1^{-1}v_{-1}u_1 + u_1^{-1}s_{-1}u_1 \;+\; u_1^{-1}v_Wu_1 + u_1^{-1}s_Wu_1 \;+ v_L\right)u_0 + u_0^{-1}\left(s_{-1} + s_W\right)u_0 + u_0^{-1}s_Lu_0,\\
&= u_0^{-1}u_1^{-1}(v_{-1} + v_W)u_1u_0 + u_0^{-1}u_1^{-1}(s_{-1} + s_W)u_1u_0 + \boldsymbol{u_0^{-1}v_Lu_0} + u_0^{-1}\left(s_{-1} + s_W\right)u_0 + \boldsymbol{u_0^{-1}(-v_Lu_0)},
\end{aligned}
$$

$$
v_g^{u_1u_0} = (u_1u_0)^{-1}\left(v_{-1} + v_W\right)(u_1u_0) + (u_1u_0)^{-1}\left(s_{-1} + s_W\right)(u_1u_0). \tag{3.64}
$$

This last expression, that could have been obtained in a single step with the dressing u_1u_0, clearly expresses the invariance of the composite fields ϖ_0 and Ω_0 under (SO, s_L) and (K_1, s_1) and the fact that their residual infinitesimal gauge freedom depends on the Weyl sector and on the infinitesimal translations.

Starting with (3.63) we have the matrix form,

$$
\begin{aligned}
v_{\mathrm{g}}^{u_1u_0} &= u_0^{-1}v_{\mathrm{g}}^{u_1}u_0 + u_0^{-1}\Big(s_{-1}+s_W+s_L\Big)u_0,\\
&= \begin{pmatrix}1&0&0\\0&e^{-1}&0\\0&0&1\end{pmatrix}\begin{pmatrix}\epsilon^{u_1}&\iota^{u_1}&0\\ \tau^{u_1}&v_L^{u_1}&(\iota^t)^{u_1}\\0&(\tau^t)^{u_1}&-\epsilon^{u_1}\end{pmatrix}\begin{pmatrix}1&0&0\\0&e&0\\0&0&1\end{pmatrix}+\begin{pmatrix}1&0&0\\0&e^{-1}&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&0\\0&\Big(s_{-1}+s_W+s_L\Big)e&0\\0&0&0\end{pmatrix},\\
&= \begin{pmatrix}\epsilon^{u_1}&\iota^{u_1}e&0\\ e^{-1}\tau& e^{-1}v_L^{u_1}e+e^{-1}(s_{-1}+s_W+s_L)e & e^{-1}(\iota^t)^{u_1}\\ 0&\tau^t e&-\epsilon^{u_1}\end{pmatrix}.
\end{aligned}
$$

Let us calculate each entry. We just need to remember that $\tau = e\xi$. Entries (1, 1) and (3, 3) are then just,

$$
\epsilon^{u_1} = \epsilon - q\tau = \epsilon - a\cdot e^{-1}\, e\xi = \epsilon - a\cdot\xi =: \widehat{\epsilon}, \qquad \text{in components} \qquad \widehat{\epsilon} = \epsilon - a_\mu\xi^\mu. \tag{3.65}
$$

Unexpectedly the infinitesimal diffeomorphisms redefine the Weyl rescaling parameter. We go on with entries (2, 1) and (3, 2) which are,

$$
e^{-1}\tau = \xi = \xi^\mu, \qquad \text{and} \qquad \tau^t e = \tau^T\eta e = \xi e^T\eta e = \xi g = \xi^\lambda g_{\lambda\nu}. \tag{3.66}
$$

So we see that after dressing we recover the infinitesimal diffeomorphisms of $\mathcal{M}$, not mere translations. Then, entries (1, 2) is,

$$
\begin{aligned}
\iota^{u_1}e &= \partial(\epsilon - q\tau) + \Big(\alpha - qA + \partial q\Big)\tau + \tfrac{1}{2}qq^t\tau^t e,\\
&= \partial\widehat{\epsilon} + \Big(\alpha - qA + t\frac{1}{2}qq^t e^t + \partial q\Big)e\xi,\\
&= \partial\widehat{\epsilon} + \alpha_1 e\,\xi, \qquad \text{where } \alpha_1 \text{ is the index free notation of } (\alpha_1)_{\mu b}.\ \text{See the expression of } \varpi_1.\\
&= \partial\widehat{\epsilon} + P\xi = \partial_\mu\widehat{\epsilon} + P_{\nu\lambda}\xi^\lambda.
\end{aligned} \tag{3.67}
$$

In the same way entry (2, 3) is,

$$
\begin{aligned}
e^{-1}(\iota^t)^{u_1} &= e^{-1}(e^{-1})^t\partial(\epsilon - \tau^t q^t) + e^{-1}(e^{-1})^t\tau^t\Big(\alpha^t + Aq^t + \partial q_b^t ig\Big) + e^{-1}\tfrac{1}{2}\tau qq^t,\\
&= e^{-1}\eta^{-1}(e^{-1})^T\partial\widehat{\epsilon} + e^{-1}\xi\Big(\alpha^t + Aq^t + \tfrac{1}{2}eqq^t + \partial q^t\Big),\\
&= g^{-1}\partial\widehat{\epsilon} + \xi e^{-1}\alpha_1^t, \qquad \text{where } \alpha_1^t \text{ is the index free notation of } (\alpha_1^t)_{b\mu}.\ \text{See the expression of } \varpi_1.\\
&= g^{-1}\partial\widehat{\epsilon} + \xi g^{-1}P^T = g^{\rho\alpha}\Big(\partial_\alpha\widehat{\epsilon} + \xi^\lambda P_{\lambda\alpha}\Big).
\end{aligned} \tag{3.68}
$$

Finally entry (2, 2) is,

$$
\begin{aligned}
e^{-1}v_L^{u_1}e + e^{-1}(s_{-1} +^{-1} s_W +^{-1} s_L)e &= \boldsymbol{e^{-1}v_L e} + e^{-1}\Big(\tau q - q^t\tau^t\Big)e + e^{-1}\Big(\partial\tau + (A - a\mathbb{1})\tau\Big) + \boldsymbol{e^{-1}(-v_L e)} + e^{-1}(\epsilon e),\\
&= e^{-1}\Big(e\xi q - q^t\xi e^t\Big)e + e^{-1}\partial e\,\xi + \partial\xi + e^{-1}Ae\xi - e^{-1}a\mathbb{1}e\,\xi + \epsilon\delta,\\
&= \partial\xi + \left[e\Big(A + eq - q^t e^t\Big)e + e^{-1}\partial e\right]\xi + (\epsilon - a\xi)\delta,\\
&= \partial\xi + \Gamma\xi + \widehat{\epsilon}\delta = \nabla\xi + \widehat{\epsilon}\delta, \qquad \text{where } \Gamma \text{ stands for } \Gamma^\rho{}_{\nu\lambda}, \qquad (3.69)\\
&= \partial_\nu\xi^\rho + \Gamma^\rho{}_{\nu\lambda}\xi^\lambda \ + \ \widehat{\epsilon}\delta^\rho_\nu = \nabla_\nu\xi^\rho + \widehat{\epsilon}\delta^\rho_\nu. \qquad (3.70)
\end{aligned}
$$

Write the matrix form of the **new final composite ghost**, which we now write simply $\widehat{v}$, as

$$
\widehat{v} := v_{\mathrm{g}}^{u_1u_0} = \begin{pmatrix}\widehat{\epsilon}&\partial\widehat{\epsilon}+P\xi&0\\ \xi&\widehat{\epsilon}\delta+\nabla\xi&g^{-1}\Big(\partial\widehat{\epsilon}+\xi P^T\Big)\\ 0&\xi g&-\widehat{\epsilon}\end{pmatrix} = \begin{pmatrix}\widehat{\epsilon}&\partial_\nu\widehat{\epsilon}+P_{\nu\lambda}\xi^\lambda&0\\ \xi^\rho&\widehat{\epsilon}\delta^\rho_\nu \ + \ \partial_\nu\xi^\rho+\Gamma^\rho{}_{\nu\lambda}\xi^\lambda & g^{\rho\alpha}\Big(\partial_\alpha\widehat{\epsilon}+\xi^\lambda P_{\lambda\alpha}\Big)\\ 0&\xi^\lambda g_{\lambda\nu}&-\widehat{\epsilon}\end{pmatrix} \tag{3.71}
$$

This result is already interesting in itself for it gives a geometrical interpretation to the cohomological results obtained by (Boulanger, 2005) and used in (Boulanger, 2007a). We refer to appendix A.3 for a closer look at this.

The new residual BRS algebra The final composite ghost depends only on the Weyl rescaling parameter and on the infinitesimal diffeomorphisms, as expected, so that these are the residual freedom of the composite fields ϖ_0 and Ω_0. One is a gauge freedom, the Weyl rescaling, the other is a space-time freedom, the diffeomorphisms, both are nevertheless handled by a BRS algebra that we now write as a proposition.

Proposition 1 (BRS algebra Weyl gauge + diff). *Notice that the final ghost admits the decomposition,*

$$\widehat{v} = \widehat{v}_W + v_\xi + i_\xi \varpi_0 = \begin{pmatrix} \widehat{\epsilon} & \partial\widehat{\epsilon} & 0 \\ 0 & \widehat{\epsilon}\delta & g^{-1}\partial\widehat{\epsilon} \\ 0 & 0 & -\widehat{\epsilon} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial\xi & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & P\xi & 0 \\ \xi & \Gamma\xi & g^{-1}\xi P^T \\ 0 & \xi g & 0 \end{pmatrix}. \tag{3.72}$$

Recall the expression of the dressed normal Cartan connection and its curvature,

$$\varpi_0 = \begin{pmatrix} 0 & P & 0 \\ dx & \Gamma & g^{-1}P^T \\ 0 & dx^T \cdot g & 0 \end{pmatrix}, \qquad \textit{and} \qquad \Omega_0 = \begin{pmatrix} 0 & C & 0 \\ 0 & W & g^{-1}C^T \\ 0 & 0 & 0 \end{pmatrix}.$$

Write $\widehat{s} = s_W + s_{-1}$ *the associated BRS operator. The algebra* $BRS_{(Weyl+Diff),0}$ *is,*

$$\widehat{s}\varpi_0 = \left(s_W + \mathcal{L}_\xi\right)\varpi_0 - dv_\xi - i_\xi\Omega_0, \tag{3.73}$$

$$\widehat{s}\Omega_0 = \left(s_W + \mathcal{L}_\xi\right)\Omega_0 + d\left(i_\xi\Omega_0\right) + \left[\varpi_0, i_\xi\Omega_0\right], \tag{3.74}$$

$$\widehat{s}\widehat{v} = \left(s_W + \mathcal{L}_\xi\right)\widehat{v} - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0 - i_\xi dv_\xi - \tfrac{1}{2}i_\xi i_\xi\Omega_0, \tag{3.75}$$

Proof. The beginning of the proof is easy.

$$\begin{aligned}\widehat{s}\varpi_0 &= -d\widehat{v} + \left[\varpi_0, \widehat{v}\right], \\ &= \underbrace{-d\widehat{v}_W - \left[\varpi_0, \widehat{v}_W\right]}_{s_W\varpi_0} - dv_\xi - d(i_\xi\varpi_0) - \left[\varpi_0, v_\xi\right] - \left[\varpi_0, i_\xi\varpi_0\right].\end{aligned}$$

Of course we recover already the result of the last section. Define the Lie derivative of r-forms, $L_\xi = i_\xi d - d i_\xi$,[6] and find,

$$\begin{aligned}\widehat{s}\varpi_0 &= s_W\varpi_0 - dv_\xi - i_\xi d\varpi_0 + L_\xi\varpi_0 - \left[\varpi_0, v_\xi\right] - \left[\varpi_0, i_\xi\varpi_0\right], \\ &= s_W\varpi_0 + \mathcal{L}_\xi\varpi_0 - dv_\xi - i_\xi\Omega_0,\end{aligned}$$

where $\mathcal{L}_\xi = L_\xi - \left[v_\xi, \ \right]$ is the Lie derivative of tensor-valued r-forms.[7] This proves the first relation. Then,

$$\begin{aligned}\widehat{s}\Omega_0 &= \left[\Omega_0, \widehat{v}\right], \\ &= \underbrace{\left[\Omega_0, \widehat{v}_W\right]}_{s_W\Omega_0} + \left[\Omega_0, v_\xi\right] + \left[\Omega_0, i_\xi\varpi_0\right].\end{aligned}$$

Now use the Bianchi identity, $d\Omega_0 + \left[\varpi_0, \Omega_0\right] = 0$, and $L_\xi = i_\xi d - d i_\xi$ to find,

$$\begin{aligned} i_\xi\left(d\Omega_0 + \left[\varpi_0, \Omega_0\right]\right) &= i_\xi d\Omega_0 + \left[i_\xi\varpi_0, \Omega_0\right] + \left[\varpi_0, i_\xi\Omega_0\right] = 0, \\ \rightarrow \qquad L_\xi\Omega_0 + d\left(i_\xi\Omega_0\right) + \left[\varpi_0, i_\xi\Omega_0\right] &= \left[\Omega_0, i_\xi\varpi_0\right].\end{aligned}$$

[6] The minus sign comes from the anticommuting nature of the ghost ξ, see (Bertlmann, 1996) equation (12.505-506) p.522-523.
[7] Ibid. Equation (2.362) p.87 and (12.504) p.522.

So that,

$$\begin{aligned}\widehat{s}\Omega_0 &= s_W\Omega_0 + \Big[\Omega_0, v_\xi\Big] + L_\xi\Omega_0 + d\Big(i_\xi\Omega_0\Big) + \Big[\varpi_0, i_\xi\Omega_0\Big],\\ &= s_W\Omega_0 + \mathcal{L}_\xi\Omega_0 + d\Big(i_\xi\Omega_0\Big) + \Big[\varpi_0, i_\xi\Omega_0\Big].\end{aligned}$$

Which proves the second relation. For the last one, we proceed backward,

$$\begin{aligned}\widehat{s\widehat{v}} &= s_W\widehat{v} + \mathcal{L}_\xi\widehat{v} - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0 - i_\xi dv_\xi - \tfrac{1}{2}i_\xi i_\xi\Omega_0,\\
&= \Big(s_Wv_W + s_Wv_\xi + s_W(i_\xi\varpi_0)\Big) + \Big(\mathcal{L}_\xi v_W + \mathcal{L}_\xi v_\xi + \mathcal{L}_\xi(i_\xi\varpi_0)\Big) - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0 - i_\xi dv_\xi - \tfrac{1}{2}i_\xi i_\xi\Big(d\varpi_0 + \varpi_0^2\Big),\\
&= \Big(-v_W^2 + 0 + i_\xi s_W\varpi_0\Big) + \Big(L_\xi v_W - \Big[v_\xi, v_W\Big] + L_\xi v_\xi - \tfrac{1}{2}\Big[v_\xi, v_\xi\Big] + L_\xi(i_\xi\varpi_0 - \Big[v_\xi, i_\xi\varpi_0\Big]\Big)\\
&\quad - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0 - i_\xi dv_\xi - \tfrac{1}{2}i_\xi i_\xi d\varpi_0 - i_\xi\varpi_0 i_\xi\varpi_0,\\
&= \Big(-v_W^2 \boldsymbol{-i_\xi dv_W} - i_\xi\varpi_0 v_W - v_W i_\xi\varpi_0\Big) + \Big(\boldsymbol{i_\xi dv_W} - \Big[v_\xi, v_W\Big] + \boldsymbol{i_\xi dv_\xi} - \tfrac{1}{2}\Big[v_\xi, v_\xi\Big] + i_\xi d i_\xi\varpi_0 - \Big[v_\xi, i_\xi\varpi_0\Big]\Big)\\
&\quad - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0 \boldsymbol{- i_\xi dv_\xi} - \tfrac{1}{2}i_\xi i_\xi d\varpi_0 - i_\xi\varpi_0 i_\xi\varpi_0,\\
&= \underbrace{-v_W^2 - \Big[i_\xi\varpi 0, v_W\Big] - \Big[v_\xi, v_W\Big] - \tfrac{1}{2}\Big[v_\xi, v_\xi\Big] - \Big[v_\xi, i_\xi\varpi_0\Big] - i_\xi\varpi_0 i_\xi\varpi_0}_{=(v_W+v_\xi+i_\xi\varpi_0)^2=-\widehat{v}^2} + i_\xi d i_\xi\varpi_0 - \tfrac{1}{2}i_\xi i_\xi d\varpi_0 - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0,\\
&= -\widehat{v}^2 + + i_\xi d i_\xi\varpi_0 - \tfrac{1}{2}i_\xi(L_\xi + d i_\xi)\varpi_0 - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0,\\
&= -\widehat{v}^2 + \tfrac{1}{2}i_\xi d i_\xi\varpi_0 - \tfrac{1}{2}i_\xi L_\xi\varpi_0 - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0,\\
&= -\widehat{v}^2 + \tfrac{1}{2}\Big(L_\xi i_\xi - i_\xi L_\xi\Big)\varpi_0 - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0,\\
&= -\widehat{v}^2 + \tfrac{1}{2}\Big[L_\xi, i_\xi\Big]\varpi_0 - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0,\\
&= -\widehat{v}^2 + \tfrac{1}{2}i_{[\xi,\xi]}\varpi_0 - i_{\frac{1}{2}[\xi,\xi]}\varpi_0,\\
&= -\widehat{v}^2\end{aligned}$$

□

Once written entry by entry, the above algebra provides the transformation under Weyl rescaling *and* diffeomorphisms of the metric tensor, the Christoffel symbols, the Schouten, Cotton and Weyl tensors, as well as the transformations of the Weyl and diffeomorphism ghosts themselves. On may appreciate to obtain all these well known results in such an economical way. One may want to see the subalgebra of diffeomorphisms as the infinitesimal active counterpart of the passive coordinate changes freedom discussed in appendix C.2.1. But the two cannot be confused. If one notes symbolically the operator of infinitesimal change of coordinates δ_c and s_ξ the BRS operator of infinitesimal diffeomorphisms, then as a rule we have $s_\xi = L_\xi + \delta_c$. See section 12.1.5 of (Bertlmann, 1996).

Remark that the BRS algebra of Proposition 1 is close to what has been proposed by Langouche, Shücker and Stora in (Langouche et al., 1984) to study gravitational anomalies of the Adler-Bardeen type. See also (Bertlmann, 1996), chapter 12, on the same subject. In their case, the retained gauge symmetry is the Lorentz one rather than the Weyl symmetry, as here, but the corrective terms due to the inclusion of the ghost of diffeomorphisms are alike. Notice that to obtain their BRS algebra they suggest a shift of the (Lorentz) gauge-ghost by $i_\xi\omega$ where ω is the Lorentz connection. We here obtain such a **shifted ghost** (3.71)-(3.72), by $i_\xi\varpi_0$, the dressed normal Cartan connection, as a **result of the dressing field method**. The form of the shifted BRS algebra follows.

The infinitesimal diffeomorphisms in the BRS algebra of General Relativity

Recall that in this example the dressing field, the dressed Cartan connection and its curvature are given by,

$$u = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}, \qquad \widehat{\varpi} = \begin{pmatrix} \Gamma & dx \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \widehat{\Omega} = \begin{pmatrix} \widehat{R} & T \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d\Gamma + \Gamma \wedge \Gamma & \Gamma \wedge dx \\ 0 & 0 \end{pmatrix}.$$

In section 3.3.1 we initially considered the Lorentz ghost whose corresponding composite ghost vanished as a sign of the SO-gauge invariance of $\widehat{\varpi}$ and $\widehat{\Omega}$. Let us now include the ghost of translations, so that the full initial ghost corresponding to the BRS operator $s = s_{-1} + s_L$ is,

$$v = \begin{pmatrix} v_L & \tau \\ 0 & 0 \end{pmatrix}.$$

This does not amount to do Poincaré gauge theory for the translational symmetry is not considered as 'internal' but truly reflects 'external', space-time, symmetry. Since the new composite ghost is $\widehat{v} := u^{-1}vu + u^{-1}su$, we need to know the action of the new BRS operator on e. This is given by the first relation of the initial BRS algebra which is,

$$s\varpi = -Dv, \quad \rightarrow \quad \begin{pmatrix} sA & s\theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -dv_L + Av_L - v_L A & -d\tau - A\tau - v_L\theta \\ 0 & 0 \end{pmatrix}.$$

From this we find,

$$s(e \cdot dx) = -dx \cdot \partial\tau - A \cdot dx\, \tau - v_L e \cdot dx \quad \rightarrow \quad se = \partial\tau + A\tau - v_L e = D\tau - v_L e.$$

Then the new composite ghost is,

$$\begin{aligned} \widehat{v} &= \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_L & \tau \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} se & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{-1}ve + e^{-1}se & e^{-1}\tau \\ 0 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} e^{-1}ve + e^{-1}D\tau - e^{-1}v_L e & e^{-1}\tau \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{-1}D\tau & e^{-1}\tau \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

If we now recall that $\tau = e\xi$ we find that,

$$e^{-1}D\tau = e^{-1}\partial(e\xi) + e^{-1}Ae\xi = e^{-1}\partial e\, xi + \partial\xi + e^{-1}Ae\, \xi = \partial\xi + (e^{-1}Ae + e^{-1}\partial e)\xi = \partial\xi + \Gamma\xi,$$

where of course $\Gamma = \Gamma^\rho{}_{\lambda\nu}$. The ghost is then,

$$\widehat{v} = \begin{pmatrix} \partial\xi + \Gamma\xi & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \partial\xi & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Gamma\xi & \xi \\ 0 & 0 \end{pmatrix} =: v_\xi + i_\xi\widehat{\varpi}.$$

Again we notice that it admits a decomposition alike the ghost in the Cartan-Möbius example. As a consequence we can write the corresponding BRS algebra,

$$s_{-1}\widehat{\varpi} = \mathcal{L}_\xi\widehat{\varpi} - dv_\xi, \tag{3.76}$$

$$s_{-1}\widehat{\Omega} = \mathcal{L}_\xi\widehat{\Omega} + d(i_\xi\widehat{\Omega}) + \left[\widehat{\varpi}, i_\xi\widehat{\Omega}\right], \tag{3.77}$$

$$s_{-1}\widehat{v} = \mathcal{L}_\xi\widehat{v} - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\widehat{\varpi} - i_\xi dv_\xi - \tfrac{1}{2}i_\xi i_\xi\widehat{\Omega}. \tag{3.78}$$

This is again a shifted BRS algebra à la Langouch-Schücker-Stora (for a trivial gauge-ghost), induced by a shifted ghost obtained through the dressing field method. For the two first relations we have the explicit results,

$$\begin{pmatrix} s_{-1}\Gamma & s_{-1}dx \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{L}_\xi\Gamma - d(\partial\xi) & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{in components} \quad s_{-1}\Gamma^\rho{}_{\mu\nu} = \mathcal{L}_\xi\Gamma^\rho{}_{\mu\nu} + \partial_\mu\partial_\nu\xi^\rho.$$

This is indeed the transformation law of the Christoffel symbols under infinitesimal diffeomorphisms.

$$\begin{aligned}\begin{pmatrix} s_{-1}\widehat{R} & s_{-1}T \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} \mathcal{L}_\xi \widehat{R} & \mathcal{L}_\xi T \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d(i_\xi \widehat{R}) & d(i_\xi T) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Gamma & dx \\ 0 & 0 \end{pmatrix}\begin{pmatrix} i_\xi \widehat{R} & i_\xi T \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} i_\xi \widehat{R} & i_\xi T \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Gamma & dx \\ 0 & 0 \end{pmatrix}, \\ &= \begin{pmatrix} \mathcal{L}_\xi \widehat{R} & \mathcal{L}_\xi T \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d(i_\xi \widehat{R}) + \left[\Gamma, i_\xi \widehat{R}\right] & d(i_\xi T) + \Gamma\, i_\xi T \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

The term $i_\xi \widehat{R} dx = \widehat{R}^\rho_{\ \nu,\mu\sigma} \xi^\mu dx^\sigma dx^\nu$ vanishes by the Bianchi identity for the Riemann tensor, $\widehat{R}^\rho_{\ [\nu,\mu\sigma]} = 0$.

Here is an example of a totally geometrized gauge theory. The gauge structure is, as we know, neutralized so that $\widehat{\varpi}$ and $\widehat{\Omega}$ belong to the natural geometry of the base manifold $\mathcal{M}$. This neatly reflected by the above BRS algebra where the composite ghost depends entirely on the infinitesimal diffeomorphisms so that it is a ghost of purely 'external' symmetry, the 'internal' gauge symmetry being neutralized by the dressing field u.

Conclusion In this chapter we've seen that the dressing field method provides a way to reduce the BRS algebra of a gauge theory. The question of the generalization to higher-order G-structures has been solved. Examples of application to General Relativity and to Cartan-Möbius geometry have shown the effectiveness of the method. Especially in the latter case where several results of conformal geometry are obtained with little effort and sum-up in a compact BRS algebra. We've seen that the inclusion of infinitesimal diffeomorphisms in the formalism is possible. The dressing field method thus provides a composite ghost whose form induces a shifted BRS algebra quite alike the one proposed first in (Langouche et al., 1984).

We mentionned three papers to which the present works connects closely. Two of them are actually concerned with the question of gravitationnal anomalies. The cohomology of the BRS operator is widely used in the non-perturbative study of anomalies in Quantum Field Theory. It is then worthwhile, and natural, to address the question of the interaction of the dressing field method with the treatment of anomalies. Preliminary work on the matter is the object of our last chapter.

Chapter 4
Dressing field and anomalies

This short closing chapter goes in three parts. First we give a definition of the notion of anomaly in a Quantum Field Theory, its BRS characterization and its link with the topology of the principal bundle of the theory. In a second part, we consider how the dressing field finds a place among some standard tools used to study anomalies. In a last part, we take a look at the long standing issue of the Weyl (or conformal) anomaly.

4.1 Anomalies in Quantum Field Theory

This section, which just sketches the main lines of the subject, is based largely on (Bertlmann, 1996) to which I refer for all matter concerning anomalies. See also chapter 13 of (Nakahara, 2003), and chapter 10 of (Azcarraga and Izquierdo, 1995) for an exposition from the cohomological viewpoint. All three references give the details of the geometric tools presented in the second part of the section.

4.1.1 What is an anomaly?

The importance of symmetry considerations have qualitatively changed in the last century. From their mere usefulness in solving problems, they became *principles* constraining the very form of admissible theories.

Among the most known touchstones of this change of status are the Noether theorems of 1918.[1] Summarized in a single sentence, these theorems show that to each rigid symmetry of a Lagrangian theory corresponds a conserved quantity, and to each local symmetry corresponds a conserved current. The first statement being a particular case of the second.[2] In the case of gauge theories, given the action $S = \int L(\phi)$ with ϕ any field, performing an infinitesimal gauge variation λ of the action gives,

$$\delta S = \int \lambda DJ, \qquad \text{with the current} \qquad J = \frac{\delta L}{\delta(\partial\phi)}\delta\phi.$$

So that the gauge invariance of the Lagrangian form provides the conservation law, $DJ = 0$.

Now the path integral quantization of the Lagrangian theory is defined by the generating functional,

$$Z = \int d\phi d\bar{\phi}\, e^{iS}, \qquad \text{and the quantum action is} \qquad W = -i \ln Z.$$

The classical conservation law for the current implies the so-called Ward-Takahashi (in abelian gauge theories) or Taylor-Slavnov (non-abelian gauge theories) identity on the Green functions. These must hold for the quantum theory to be renormalizable. It turns out that these identities are equivalent to the gauge-invariance of the quantum action,

$$\delta W = \int \lambda D\langle J\rangle = 0.$$

It may happen that a symmetry enjoyed by a classical Lagrangian field theory is broken upon quantization,

$$\delta S = 0 \qquad \xrightarrow{\text{quantization}} \qquad \delta W = \int \mathcal{A}(\lambda) \quad \rightarrow \quad D\langle J\rangle \propto \mathcal{A}.$$

[1] Obtained while Emmy Noether was working with Hilbert and Klein at Göttingen on the question of energy in General Relativity.

[2] The fact that symmetries where related to conservation law was already noted in the XIX$^{\text{th}}$, but the proof of the generality of this fact is to be credited to Noether.

The term $\mathcal{A}$ is called an **anomaly**, and the conservation law is said anomalous.

If we are dealing with a 'fundamental' theory, that is a theory where the gauge field is quantized along with the matter fields, then the occurrence of an anomaly is disastrous since it means the loss of renormalizability. If on the other hand we are dealing with an effective theory, that is a theory where the gauge field is 'external', it is a classical field interacting with other quantum fields, then an anomaly may be a useful correcting term. The famous example is the decay width of the neutral pion in two photons, $\pi_0 \to \gamma\gamma$, which is entirely given by the Adler-Bardeen-Jackiw chiral anomaly. We are in such an 'effective' case when we are dealing with gravity interacting with quantum fields. Here gravitational anomalies may appear. At the end of the chapter we will say a word about the most famous of them, the Weyl or conformal anomaly.

If anomalies happen to be useful, it is necessary to have a mean to characterize and derive them easily. Here is a situation where the BRS formalism proves very effective.

4.1.2 BRS characterization of anomalies

As we know the BRS operator, s, reproduces infinitesimal gauge transformations with parameter given by the ghost, v. So the above breaking of gauge invariance can be written,

$$sS = 0 \qquad \xrightarrow{\text{quantization}} \qquad sW = \int \mathcal{A}(v).$$

The anomaly is then a 4-form depending linearly on the ghost v. Due to the nilpotency of the BRS operator we must have,

$$s^2 W = s \int \mathcal{A} = 0.$$

So the anomaly should be closed under the BRS operator, it should be an s-cocycle. For $\mathcal{A}$ to be a genuine anomaly it should not be a s-coboundary. Indeed if $\mathcal{A} = sC$ for some C, then

$$sW = \int sC, \qquad \to \qquad W' := W - \int C, \quad \text{and} \quad sW' = 0.$$

If $\mathcal{A}$ is an s-coboundary it just provides a counterterm C which redefines the quantum action so as to restore the gauge invariance. An anomaly is then s-closed but not s-exact, it then belong to the **cohomology of the BRS operator**, $H^*(s)$, which is actually as we know the cohomology of the Lie algebra of the gauge group in disguise.

If one works on an m-dimensional boundaryless manifold, or if one imposes suitable fall-off conditions of the fields at infinity, we can ask for quasi-invariance. Then the anomaly satisfies the so-called **Wess-Zumino consistency condition**,

$$s\mathcal{A} = dB^2_{m-1}, \tag{4.1}$$

where B^2_{m-1} is a $(m-1)$-form and depends quadratically on the ghost. This is so for the total degree, form+ghost, in both sides of the equality to match. An anomaly satisfying the WZ consistency condition is said **consistent**. Thus $\mathcal{A}$ is s-closed up to d-exact terms, but not s-exact. We are then dealing with the cohomology of s modulo d, and a **consistent anomaly** belongs to the cohomology of $s \equiv d$, $H^*(s,d)$.

Often one works with the *shifted BRS operator* defined as $\widetilde{s} = s + d$. The WZ consistency condition just reads $\widetilde{s}\mathcal{A} = 0$, and one is interested in the cohomology of this shifted operator, $H^*(\widetilde{s})$. There is indeed a one to one correspondance between $H^*(s,d)$ and $H^*(\widetilde{s})$.

4.1.3 The Stora-Zumino chain of descent equations and the consistent anomaly

A theorem by Adler and Bardeen (1969) states that anomalies are 1-loop effects and do not receive higher radiative corrections. In other words, anomalies are beyond perturbation theory. One can thus suspect that they might have a deeper origin. Their link with the cohomology of the Lie algebra of the gauge group has been alluded to above. They have also an indirect link with the topological invariants, know as *characteristic classes* , of the principal bundle underlying the gauge theory. This is seen either through the so called Stora-Zumino descent equations, or through the index theorems for differential operators.[3] For a technical account of the various approaches to anomalies, from Quantum Field Theory to index theorems, see (Bertlmann, 1996) and (Nakahara, 2003). For a historical review of the gradual understanding of the deep geometrical and topological meaning of anomalies, see (Fine and Fine, 1997). Here we are interested in the Stora-Zumino approach. To describe it we need some definitions and results.

Characteristic classes from the curvature and invariant polynomials

Let $\mathfrak{g} = \text{Lie}G$. An **invariant polynomial** on $\mathfrak{g}$, or *characteristic* polynomial, is a symmetric r-linear mapping $P : \mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{R}$ which is Ad_G-invariant,

$$P(\text{Ad}_g\lambda_1, \ldots, \text{Ad}_g\lambda_r) = P(\lambda_1, \ldots, \lambda_r), \qquad \lambda_i \in \mathfrak{g} \text{ and } g \in G.$$

The set of invariant polynomials of degree r we denote $I^r(G)$. The vector space $I^*(G) = \sum_{r=0}^{\infty} I^r(G)$ of invariant polynomials is a graded ring if we define the product,

$$(PP')(\lambda_1, \ldots, \lambda_{r+s}) = \frac{1}{(r+s)!} \sum_{\text{perm}\sigma} P(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(r)}) P'(\lambda_{\sigma(r+1)}, \ldots, \lambda_{\sigma(r+s)})$$

for $P \in I^r(G)$, $P' \in I^s(G)$ and $PP' \in I^{r+s}(G)$. Invariant polynomials are obviously important for Physics since they are used to construct Lagrangians. Usually they are constructed using the trace. A theorem by Weyl states that any invariant polynomial can be written as a polynomial of the trace. The theorem is mentionned and referenced in the proof of Theorem 2 of (Kobayashi and Ochiai, 1971).

Proposition 2. *Let ω be a connection on $\mathcal{P}(\mathcal{M}, G)$ and Ω its curvature. The $2n$-form on $\mathcal{P}$ defined by*

$$P^n(\Omega) := P(\underbrace{\Omega, \ldots, \Omega}_{n}) = P(\Omega^n).$$

is projectable, closed, and defines an element of the de Rham cohomology group $H^{2n}(\mathcal{M}, d)$ independent of the connection. It has a topologically invariant integral which is thus a characteristic number of the base manifold.

Proof. The projectability is easily seen. It stems from the fact that Ω is tensorial of type $(\text{Ad}_G, \mathfrak{g})$ and from the invariance of P.

$$\mathcal{R}_g^* P^n(\Omega) = P^n(\mathcal{R}_g^*\Omega) = P^n(\text{Ad}_{g^{-1}}\Omega) = P^n(\Omega).$$
$$P^n(\Omega)(X_1^v, X_2, \ldots, X_{2n}) = P\Big(\Omega(X_1^v, X_2), \ldots, \Omega(X_{2n-1}, X_{2n}\Big) = P\Big(0, \ldots, \Omega(X_{2n-1}, X_{2n}\Big) = 0.$$

$P^n(\Omega)$ projects to $P^n(F)$, with $F = \sigma^*\Omega$ or $\Omega = \pi^*F$.

It is closed due to the Bianchi identity $D\Omega = 0$ and to the fact that for any projectable r-form $\alpha = \pi^*a$ we have,

$$\begin{aligned} d\alpha(X_1, \ldots, X_r) &= d\pi^*a(X_1, \ldots, X_r) = da(\pi_*X_1, \ldots, \pi_*X_r) = da(\pi_*X_1^h, \ldots, \pi_*X_r^h), \\ &= \pi^*da(X_1^h, \ldots, X_r^h) = d\pi^*a(X_1^h, \ldots, X_r^h) = d\alpha(X_1^h, \ldots, X_r^h) =: D\alpha(X_1, \ldots, X_r). \end{aligned}$$

So,

$$dP^n(\Omega) = \sum_{i=0}^{n} P(\Omega, \ldots, d\Omega_i, \ldots, \Omega) = nP(d\Omega, \Omega^{n-1}) = nP(D\Omega, \Omega^{n-1}) = 0.$$

[3]The index of a differential operator being given by the integral of the suitable characteristic class.

We used the symmetry of the polynomial and the commutation of even forms. Upon projection we thus have $dP^n(F) = 0$. $P^n(F)$ is thus closed but not exact, it is an element of the de Rham cohomology group, $H^{2n}(\mathcal{M}, d)$, of the base manifold.

It remains to show the independance from the connection. Given two connections ω_0 and ω_1 on $\mathcal{P}$, the form $\beta = \omega_1 - \omega_0$ is tensorial of type $(\mathrm{Ad}_G, \mathfrak{g})$. Define the 1-parameter family of homotopic connections $\omega_t := \omega_0 + t\beta$, $t \in [0,1]$. Its curvature is,

$$\Omega_t = d\omega_t + \tfrac{1}{2}\big[\omega_t, \omega_t\big] = d\omega_0 + \tfrac{1}{2}\big[\omega_0, \omega_0\big] \;+\; td\beta + t\big[\omega_0, \beta\big] \;+\; \tfrac{t^2}{2}\big[\beta, \beta\big] = \Omega_0 \;+\; tD_0\beta \;+\; \tfrac{t^2}{2}\big[\beta, \beta\big].$$

Upon derivation with respect to the parameter t,

$$\frac{d}{dt}\Omega_t = D_0\beta + t\big[\beta, \beta\big] = d\beta + \big[\omega_0, \beta\big] + \big[t\beta, \beta\big] = d\beta + \big[\omega_0 + t\beta, \beta\big] = d\beta + \big[\omega_t, \beta\big] =: D_t\beta.$$

Remember that the covariant derivative of a tensorial form is a tensorial form of the same type. Now we have,

$$\begin{aligned}
P^n(\Omega_1) - P^n(\Omega_0) &= \int_0^1 dt\ \tfrac{d}{dt}P^n(\Omega_t) = n\int_0^1 dt\ P(\tfrac{d}{dt}\Omega_t, \Omega_t^{n-1}) = n\int_0^1 dt\ P(D_t\beta, \Omega_t^{n-1}),\\
&= n\int_0^1 dt\ D_tP(\beta, \Omega_t^{n-1}) = n\int_0^1 dt\ dP(\beta, \Omega_t^{n-1}),
\end{aligned}$$

$$P^n(\Omega_1) - P^n(\Omega_0) = d\left(n\int_0^1 dt\ P(\beta, \Omega_t^{n-1})\right) =: dQ_{2n-1}(\omega_1, \omega_0). \tag{4.2}$$

The form $Q_{2n-1}(\omega_1, \omega_0)$ is clearly projectable, so that we have upon projection,

$$P^n(F_1) - P^n(F_0) = d\left(n\int_0^1 dt\ P(A_1 - A_0, F_t^{n-1})\right) =: dQ_{2n-1}(A_1, A_0). \tag{4.3}$$

Both $P^n(F_1)$ and $P^n(F_0)$ have the same integral over a boundaryless base manifold, they define the same characteristic number. This proves that $P^n(F)$ defines a characteristic class independent of the connection. □

Remark: the mapping $I^*(G) \to H^{2n}(\mathcal{M}, d)$ which associates to each invariant polynomial P the de Rham cohomology class of $P(F)$ is called the *Weil homomorphism*. The product is $PP'(F) = P(F) \wedge P'(F)$.

Equations (4.2) and (4.3) are referred to as the *transgression formulas*, and the form

$$Q_{2n-1}(A_1, A_0) := n\int_0^1 dt\ P(A_1 - A_0, F_t^{n-1}) \tag{4.4}$$

is known as the **Chern-Simons form**. Locally, on $U \subset \mathcal{M}$, it is always possible to find a flat connection with a trivial representative, so we can write $A_1 = A$ and $A_0 = 0$. The homotopic connection it then just $A_t = tA$ and its curvature is $F_t = tdA + \tfrac{t^2}{2}\big[A, A\big]$. The transgression formula reduces to

$$P^n(F) = dQ_{2n-1}(A, 0), \qquad \text{with the Chern-Simons form} \quad Q_{2n-1}(A, 0) = n\int_0^1 dt\ P(A, F_t^{n-1}). \tag{4.5}$$

So locally the invariant polynomial is exact. If it vanishes identically, then the Chern-Simons form defines what is called the *Chern-Simons secondary class* which depends on the connection.

Equation (4.2) is a particular case of the so-called *extended Cartan homotopy formula*. We refer to (Mañes et al., 1985) for a description. It gives, as another special case, the direct generalization of (4.2) known as the **triangle formula**. Given three connections ω_0, ω_1 and ω_2, define $\beta_1 = \omega_1 - \omega_0$ and $\beta_2 = \omega_2 - \omega_0$. The homotopic connection is $\omega_{t_*} = \omega_0 + t_1\beta_1 + t_2\beta_2$ with curvature Ω_{t_*}. The triangle formula is,

$$Q_{2n-1}(\omega_0, \omega_1) + Q_{2n-1}(\omega_1, \omega_2) + Q_{2n-1}(\omega_2, \omega_0) = d\chi_{2n-2}(\omega_0, \omega_1, \omega_2), \tag{4.6}$$

$$\text{with} \qquad \chi_{2n-2}(\omega_0, \omega_1, \omega_2) = n(n-1)\int_0^1 dt_1 \int_0^{1-t_1} dt_2\ P(\beta_1, \beta_2, \Omega_{t_*}^{n-2}).$$

It will be of some use when we will come to the role of the dressing field.

The chain of descent equations

We are now ready to take the final step. It starts with the algebraic connection $\widetilde{A} := A + v$ and the shifted BRS operator $\widetilde{s} = s + d$. The BRS algebra implies what Stora nicknamed the "Russian formula",

$$\widetilde{F} = \widetilde{D}\widetilde{A} := \widetilde{s}\widetilde{A} + \tfrac{1}{2}\left[\widetilde{A}, \widetilde{A}\right] = F.$$

This implies $P^n(\widetilde{F}) = P^n(F)$. Calculations similar to what is done above provide the (trivial) *shifted transgression*,

$$P^n(\widetilde{F}) = \widetilde{s}Q_{2n-1}(\widetilde{A}, 0)$$

We expand the shifted Chern-Simons form in powers of the ghost,

$$\begin{aligned} Q_{2n-1}(A+v,0) &= \sum_{p=0}^{2n-1} Q^p_{2n-1-p}, \\ &= Q^0_{2n-1}(A,0) + Q^1_{2n-2}(A,v) + Q^2_{2n-3}(A,v) + \ldots + Q^{2n-1}_0(v). \end{aligned}$$

The shifted transgression is then,

$$\begin{aligned} P^n(F) &= P^n(\widetilde{F}), \\ &= \widetilde{s}Q_{2n-1}(\widetilde{A},0), \\ &= (s+d)Q^0_{2n-1}(A,0) + (s+d)Q^1_{2n-2}(A,v) + (s+d)Q^2_{2n-3}(A,v) + \ldots + (s+d)Q^{2n-1}_0(v). \end{aligned}$$

Sorting the equations according to the total degree gives the **Stora-Zumino chain of descent equations**,

$$\begin{aligned} P^n(F) &= dQ^0_{2n-1}(A,0), \\ sQ^0_{2n-1}(A,0) + dQ^1_{2n-2}(A,v) &= 0, \\ sQ^1_{2n-2}(A,v) + dQ^2_{2n-3}(A,v) &= 0, \\ &\vdots \\ sQ^{2n-2}_1(A,v) + dQ^{2n-1}_0(A,v) &= 0, \\ sQ^{2n-1}_0(v) &= 0. \end{aligned}$$

We find that the third descent equation, $sQ^1_{2n-2}(A,v) + dQ^2_{2n-3}(A,v) = 0$, is precisely the **Wess-Zumino consistency condition**. The chain term $Q^1_{2n-2}(A,v) = \mathcal{A}$ is then a **consistent anomaly** in even dimension $m = 2n-2$. This anomaly is obtained from the Chern-Simons form through the second descent equation by application of the BRS algebra.

The polynomial usually used in gauge theories is simply the trace, $P = \mathrm{Tr}$. In dimension $m = 2$ and $m = 4$ we need to start with a bilinear and trilinear invariant polynomial, $P^2(F)$ and $P^3(F)$ respectively. The chain terms are,

$$\begin{aligned} Q^0_3 &= \mathrm{Tr}\Big(AF - \tfrac{1}{3}A^3\Big) = \mathrm{Tr}\Big(AdA + \tfrac{2}{3}A^3\Big), \\ Q^1_2 &= \mathrm{Tr}\Big(vdA\Big), \\ Q^2_1 &= -\mathrm{Tr}\Big(v^2A\Big), \\ Q^3_0 &= -\tfrac{1}{3}\mathrm{Tr}\Big(v3\Big) \end{aligned}$$

$$\begin{aligned} Q^0_5 &= \mathrm{Tr}\left(A(dA)^2 + \tfrac{2}{3}A^3dA + \tfrac{3}{5}A^5\right), \\ Q^1_4 &= \mathrm{Tr}\left[vd\left(AdA + \tfrac{1}{2}A^3\right)\right], \\ Q^2_3 &= -\tfrac{1}{2}\mathrm{Tr}\left((vA + vAv + Av)dA + v^2A^3\right), \\ Q^3_2 &= \tfrac{1}{2}\mathrm{Tr}(-v^3dA + AvAv^2), \\ Q^4_1 &= \tfrac{1}{2}\mathrm{Tr}(v^4A), \\ Q^5_0 &= \tfrac{1}{10}\mathrm{Tr}(v^5). \end{aligned}$$

The chain term Q_4^1 is the well known non-abelian chiral anomaly. We give here the chain terms with some details for the sake of completeness but also to facilitate the comparison with results of the next two sections. First we will show that the dressing field together with the triangle formula provides an alternative, and anecdotal, way to find the anomaly. Second because we will give the chain terms for another polynomial when we will speak of the Weyl anomaly in dimension 4.

4.2 The place of the dressing field

In this section we give preliminary considerations, a first sketch, on how the dressing field method interacts with the tools presented above. Two criterion are drawn, each assuming two outcomes. This gives four possible cases. Two of them susceptible to be treated on general ground and allow to draw some results. The other two depending on the specific invariant polynomial chosen. Since we are ultimately interested in anomalies in fields theory we will always work locally, on the base manifold.

4.2.1 Four possibilities

In the tools described above we said that the invariance group of the polynomial P was also the structure group of the bundle $\mathcal{P}$. This is of course not necessary, the latter could merely be a subgroup of the former. We would then deal with an invariant polynomial $P \in I(G)$ and a principal bundle $\mathcal{P}(\mathcal{M}, H)$, with $G \supseteq H$. This is the case for example with the trace, $P = \text{Tr} \in I(GL)$ which is always used in Yang-Mills theory where the underlying bundle has a special unitary structure group. Two groups are then relevant, G and H.

If we now introduce a dressing field u, more exactly if we identify a dressing field in the theory, its **equivariance group** K' and **target group** K are new relevant data. By 'equivariance group' we mean the group constituted of elements k' such that $\mathcal{R}^*_{k'} u = k'^{-1} u$. The target group is simply defined by, $u : \mathcal{P} \to K$. Often we've seen that $K = K'$, remember the example of the electroweak sector of the standard model where $K = SU(2) = K'$, or the first dressing u_1 in the Cartan-Möbius geometry where $K = K_1 = K'$. It also happens that $K \neq K'$, as in General Relativity or with the second dressing u_0 in Cartan-Möbius geometry where $K = GL$ and $K' = SO$. The distinction between the equivariance group and the target group of the dressing should be made clear, here more than anywhere else, because it gives rise to two criterion. The first is,

$$\text{either} \qquad \text{(II)} \quad K' \supseteq H \qquad \text{or} \qquad \text{(II')} \quad H \supset K'.$$

We are now accustomed to these two possibilities which correspond respectively to a total and partial neutralization of the gauge symmetry, so to the absence or existence of a residual gauge freedom. The second criterion is,

$$\text{either} \qquad \text{(I)} \quad G \supseteq K \qquad \text{or} \qquad \text{(I')} \quad K \supset G.$$

This is a new consideration which positions the dressing fields with respect to the invariant polynomial. Two criterion each coming as two variants, this gives four possible cases. Only two can be treated on a general ground.

4.2.2 Cases (I) and (I')

Case (I) We assume that the target group is contained in the invariance group of the polynomial, $G \supseteq K$. The curvature F of a local connection $A = \sigma^*\omega \in \Lambda^1(U, \mathfrak{h})$ and its associated composite field are related by $F = u\widehat{F}u^{-1}$. If we evaluate an invariant polynomial $P \in I^*(G)$ on F we get,

$$P^n(F) = P^n(u\widehat{F}u^{-1}) = P^n(\widehat{F}), \qquad \text{due to} \quad u : \mathcal{P} \to K \subseteq G.$$

This means that we have the dressed version of the transgression formula (4.5),

$$P^n(\widehat{F}) = dQ_{2n-1}(\widehat{A}, 0), \quad \text{with the dressed Chern-Simons form} \quad Q_{2n-1}(\widehat{A}, 0) = n\int_0^1 dt\; P(\widehat{A}, \widehat{F}_t^{n-1}). \tag{4.7}$$

Nevertheless it turns out that the Chern-Simons forms $Q_{2n-1}(A,0)$ and $Q_{2n-1}(\widehat{A},0)$ are not cohomologous. To see this, notice first that the connection A and its associated composite field are related by, $A = u(\widehat{A} - u^{-1}du)u^{-1}$. So that,

$$A - udu^{-1} = u\widehat{A}u^{-1} - u(u^{-1}du)u^{-1} - udu^{-1} = u\widehat{A}u^{-1} - duu^{-1} - udu^{-1} = u\widehat{A}u^{-1} - d(uu^{-1}) = u\widehat{A}u^{-1} - 0.$$

Define the homotopic connection, $A_t = A_0 + t(A_1 - A_0) = udu^{-1} + t(u\widehat{A}u^{-1})$. Its curvature is,

$$\begin{aligned}
F_t(A_t) &= dA_t + \tfrac{1}{2}\left[A_t, A_t\right],\\
&= \boldsymbol{dudu^{-1}} + td(u\widehat{A}u^{-1}) + \tfrac{1}{2}\boldsymbol{\left[udu^{-1}, udu^{-1}\right]} + t\left[udu^{-1}, u\widehat{A}u^{-1}\right] + \tfrac{1}{2}\left[ut\widehat{A}u^{-1}, ut\widehat{A}u^{-1}\right],\\
&= t\left(du\widehat{A}u^{-1} + ud\widehat{A}u^{-1} - u\widehat{A}du^{-1}\right) + t\left[udu^{-1}, u\widehat{A}u^{-1}\right] + \tfrac{1}{2}u\left[t\widehat{A}, t\widehat{A}\right]u^{-1},\\
&= t\left(duu^{-1}\cdot u\widehat{A}u^{-1} - u\widehat{A}u^{-1}\cdot udu^{-1}\right) + t\left[udu^{-1}, u\widehat{A}u^{-1}\right] + ud(t\widehat{A})u^{-1} + \tfrac{1}{2}u\left[t\widehat{A}, t\widehat{A}\right]u^{-1},\\
&= \boldsymbol{t\left(-udu^{-1}\cdot u\widehat{A}u^{-1} - u\widehat{A}u^{-1}\cdot udu^{-1}\right) + t\left[udu^{-1}, u\widehat{A}u^{-1}\right]} + u\left(d(t\widehat{A}) + \tfrac{1}{2}\left[t\widehat{A}, t\widehat{A}\right]\right)u^{-1},\\
F_t(A_t) &= u\left(d(t\widehat{A}) + \tfrac{1}{2}\left[t\widehat{A}, t\widehat{A}\right]\right)u^{-1} = u\widehat{F}_t(t\widehat{A})u^{-1}.
\end{aligned}$$

The term udu^{-1} is a connection since u is a dressing field.[4] Applying equation (4.4) with $A_1 = A$ and $A_0 = udu^{-1}$,

$$\begin{aligned}
Q_{2n-1}(A, udu^{-1}) &= n\int_0^1 dt\, P(A - udu^{-1}, F_t^{n-1}) = n\int_0^1 dt\, P(u\widehat{A}u^{-1}, u\widehat{F}_t^{n-1}u^{-1}),\\
&= n\int_0^1 dt\, P(\widehat{A}, \widehat{F}_t^{n-1}) = Q_{2n-1}(\widehat{A}, 0).
\end{aligned} \tag{4.8}$$

This means that the Chern-Simons form $Q_{2n-1}(\widehat{\omega},0)$ is projectable to $Q_{2n-1}(\widehat{A},0)$. This is not the case of $Q_{2n-1}(\omega,0)$. Apply now the triangle formula to the triplet $A_0 = A$, $A_1 = udu^{-1}$ and $A_2 = 0$ and find,

$$\begin{aligned}
Q_{2n-1}(A, udu^{-1}) + Q_{2n-1}(udu^{-1}, 0) + Q_{2n-1}(0, A) &= d\chi_{2n-2}(A, udu^{-1}, 0),\\
Q_{2n-1}(\widehat{A}, 0) + Q_{2n-1}(udu^{-1}, 0) + Q_{2n-1}(0, A) &= d\chi_{2n-2}(A, udu^{-1}, 0),
\end{aligned}$$

$$Q_{2n-1}(\widehat{A}, 0) = Q_{2n-1}(A, 0) + Q_{2n-1}(0, udu^{-1}) + d\chi_{2n-2}(A, udu^{-1}, 0). \tag{4.9}$$

This shows that unless $Q_{2n-1}(0, udu^{-1})$ is identically vanishing, the dressed Chern-Simons form is not cohomologous to the original one. Applying the exterior derivative to equation (4.9) gives,

$$\begin{aligned}
dQ_{2n-1}(\widehat{A}, 0) &= dQ_{2n-1}(A, 0) + dQ_{2n-1}(0, udu^{-1}),\\
P^n(\widehat{F}) &= P^n(F) + dQ_{2n-1}(0, udu^{-1}).
\end{aligned}$$

Which implies $dQ_{2n-1}(0, udu^{-1}) = 0$, as can be checked by direct calculation. We can try to apply the BRS operator to equation (4.9), but the result would depend on the second criterion.

Case (I') If the dressing field has a target group $K \supset G$, obviously the results derived in case (I) do not hold. Nothing general can be said, any progress would depend on the precise properties enjoyed by the polynomial besides its symmetry and invariance. We will consider an example in the section dedicated to the Weyl anomaly.

[4] Note that the formal relation $s(udu^{-1}) = -d(usu^{-1}) - \left[udu^{-1}, usu^{-1}\right]$ always hold. If u is a dressing we have $su = -vu$, and this relation becomes $s(udu^{-1}) = -dv + \left[udu^{-1}, v\right]$. This is a genuine BRS transformation for a connection and the infinitesimal counterpart of $(udu^{-1})^\gamma = \gamma^{-1}(udu^{-1})\gamma + \gamma^{-1}d\gamma$, stemming from $u^\gamma = \gamma^{-1}u$.

4.2.3 Cases (II) and (II')

Case (I)+(II) We furthermore assume that $K' \supseteq H$, that is the composite fields are **gauge-invariant**. In this case applying s to (4.9) and using the initial BRS algebra gives,

$$0 = sQ_{2n-1}(A,0) + sQ_{2n-1}(0,udu^{-1}) + sd\chi_{2n-2},$$
$$0 = -dQ^1_{2n-2}(A,v) - dQ^1_{2n-2}(v,udu^{-1}) - ds\chi_{2n-2},$$

$$Q^1_{2n-2}(A,v) = Q^1_{2n-2}(udu^{-1},v) + s\chi_{2n-2} + d\alpha^1_{2n-3}, \qquad \text{up to } d\text{-closed terms.} \tag{4.10}$$

This is an alternative way to compute a consistent anomaly. Consider the example of dimension $m = 2$, that is $n = 2$, with $P = \text{Tr}$. We have,

$$Q_3(0,udu^{-1}) = \tfrac{1}{3}\text{Tr}(udu^{-1}dudu^{-1}),$$
$$\text{and using the BRS algebra,} \quad sQ_3(0,udu^{-1}) = -dQ^1_2(v,udu^{-1}) = -d\text{Tr}(vdudu^{-1}).$$

Moreover a direct calculation gives,

$$\chi_2(A,udu^{-1},0) = \text{Tr}(udu^{-1}A), \qquad \text{and} \qquad s\chi_2 = -\text{Tr}\Big[dv(A - udu^{-1})\Big].$$

Then,

$$\begin{aligned} Q^1_2(A,v) - d\alpha^1_1 &= Q^1_2(v,udu^{-1}) + s\chi_2 = \text{Tr}\Big[dvA - dvudu^{-1} + vdudu^{-1}\Big], \\ &= \text{Tr}\Big[dvA - d(vudu^{-1})\Big] = \text{Tr}\Big[vdA + d(vA) - d(vudu^{-1})\Big], \\ &= \text{Tr}(vdA) - d\text{Tr}\Big[-v(A - udu^{-1})\Big]. \end{aligned}$$

Compare with the chain terms given at the end of the previous section. This is however anecdotal since the calculations are at least as long as those needed in the descent equation.

The most relevant feature to be noted in this case is that all the composite fields being gauge invariant, in the resulting theory we have,

$$P^n(\widehat{F}) = dQ_{2n-1}(\widehat{A},0),$$
$$sQ_{2n-1}(\widehat{A},0) = 0$$

So there are **no descent equations** and **no consistent gauge anomaly**. This was expected since here we have a geometrized theory and no more gauge symmetry. The anomaly is defined as the breaking of the gauge invariance of the quantum action, $sW = \int \mathcal{A}$. If, as we assume here, the theory can be written with the dressed variables then the quantum action written in term of these variables is gauge invariant, $s\widehat{W} = 0$. So the theory after dressing cannot have an anomaly. And indeed the equation (4.10) seems to entail that the anomaly of the theory before dressing is s- and d-cohomologous to an anomaly written in term of a flat connection liable to be gauged away. This may be a hint that actually the gauge theory has no genuine anomaly, and that only after the dressing method is applied does this fact appear without ambiguity.

Let us say it again: if we are able to geometrize a theory by working with gauge invariant fields, then this **theory is anomaly free**. The dressing field method is then a probe of the anomalous content of a gauge theory. Since we've applied it to the electroweak sector of the Standard Model in Chapter 2 we can conclude that the model has no $SU(2)$-anomaly. This result is another net benefit of the method, besides the reinterpretation already discussed.

Case (I)+(II') Now we assume that $H \supset K'$, that is the composite fields have a residual gauge freedom. This residual gauge freedom is handled by the **modified BRS algebra** exposed in Lemma 3 of Chapter 3. Using it we obtain the Stora-Zumino chain of descent equations for the composite fields and composite ghost,

$$
\begin{aligned}
P^n(\widehat{F}) &= dQ^0_{2n-1}(\widehat{A},0),\\
sQ^0_{2n-1}(\widehat{A},0) + dQ^1_{2n-2}(\widehat{A},\widehat{v}) &= 0,\\
sQ^1_{2n-2}(\widehat{A},\widehat{v}) + dQ^2_{2n-3}(\widehat{A},\widehat{v}) &= 0,\\
&\vdots\\
sQ^{2n-2}_1(\widehat{A},\widehat{v}) + dQ^{2n-1}_0(\widehat{A},\widehat{v}) &= 0,\\
sQ^{2n-1}_0(\widehat{v}) &= 0.
\end{aligned}
$$

The solution for the chain terms are the same as for the initial theory. It suffices to replace the initial variables by the dressed ones. In particular the **residual anomaly** is $\mathcal{A} = Q^1_{2n-1}(\widehat{A},\widehat{v})$. It is seen to satisfy the third equation of the new descent which is a Wess-Zumino consistency condition.

By applying the BRS operator on (4.9) we obtain,

$$
\begin{aligned}
sQ_{2n-1}(\widehat{A},0) &= sQ_{2n-1}(A,0) + sQ_{2n-1}(0,udu^{-1}) + sd\chi_{2n-2}(A,udu^{-1},0),\\
-dQ^1_{2n-2}(\widehat{A},\widehat{v}) &= -dQ^1_{2n-2}(A,v) - dQ^1_{2n-2}(v,udu^{-1}) - ds\chi_{2n-2}(A,udu^{-1},0),
\end{aligned}
$$

$$
Q^1_{2n-2}(\widehat{A},\widehat{v}) = Q^1_{2n-2}(A,v) + Q^1_{2n-2}(v,udu^{-1}) + s\chi_{2n-2}(A,udu^{-1},0), \qquad \text{up to } d\text{-closed terms.} \tag{4.11}
$$

This shows that if either $Q_{2n-1}(0,udu^{-1})$ or $Q^1_{2n-2}(v,udu^{-1})$ vanishes, the residual anomaly is cohomologous to the initial one. We will see an instance of this situation in the section concerning the Weyl anomaly.

4.3 An attempt toward the Weyl anomaly

4.3.1 The Weyl anomaly

Consistent gauge anomalies in Yang-Mills and in gravitational theories are efficiently treated through BRS algebra and descent equations à la Stora-Zumino. See (Bertlmann, 1996). There's however a noticeable resisting case: the Weyl anomaly. It is associated to the quantum breaking of the Weyl rescaling symmetry, or local conformal symmetry,[5] enjoyed by classical theories of massless fields interacting with gravity. It is then a **gravitational anomaly**. It was first discovered on Christmas 1973 by M. Duff and D. M. Capper through field theoretic methods (dimensional regularization), and was shown to reflect the non-vanishing trace of the effective quantum stress-energy tensor, $\mathcal{A}_W \propto \langle T^\mu{}_\mu\rangle$.

Since we do not have a quantum theory of gravity, the gravitational field is to be treated as an external classical gauge field interacting with quantum fields. The occurrence of a gravitational anomaly in such effective theories may have many interesting information to provide. It is the case of the Weyl anomaly. It finds applications in black holes physics, cosmology, string theory etc... For instance in 1977 it was shown that the Hawking radiation of two dimensional black holes was entirely given by the Weyl anomaly. In the context of string theory, the worldsheet of strings is required to enjoy an unbroken Weyl symmetry. In 1981 A. Polyakov show that for the bosonic and fermionic string we have respectively, $\langle T^\mu{}_\mu\rangle = \frac{1}{24\pi}(d-26)\mathcal{A}_W$ and $\langle T^\mu{}_\mu\rangle = \frac{1}{16\pi}(d-10)\mathcal{A}_W$. So the vanishing of the Weyl anomaly, the preservation of the local conformal symmetry of the worldsheet of the strings, is related to the critical dimension of the theory. For a review of the history and applications of the Weyl anomaly see (Duff, 1994).

Early attempts for a deeper geometrical understanding of the Weyl anomaly began in the 80's. In their papers (Bonora et al., 1983) and (Bonora et al., 1986), using BRS cohomological arguments, gave explicit expressions for the Weyl anomalies in dimensions 4 and 6. Their structure was found to consist of two kind

[5] First introduced by H. Weyl in his work of 1918 which ended up in seeding the idea of local gauge symmetry as well as the very terminology.

of terms: $\epsilon \times e(\mathcal{M})$, where ϵ is the Weyl ghost and $e(\mathcal{M})$ is the Euler density of the manifold, and $\epsilon\times$Weyl-invariants. Yet we are warned: "Unfortunately, extending this method to higher dimensions does not seem technically easy [...] *nor does a general algorithm as in the case of the chiral anomaly seem to exist for Weyl anomalies.*" [6](my emphasis added).

Using field theoretic methods (Deser and Schwimmer, 1993) confirmed this structure for any even dimension and called the first kind *type A* anomaly, and the second kind *type B* anomaly. On the basis of field theoretic arguments still, they suggested that the type A anomaly could enjoy a "descent identity" like the chiral anomaly. But they confess "[...]we have not been able to find one"[7].

Using original BRS tools, (Boulanger, 2007a) proposed an approach to the type A anomaly that looks like a scheme à la Stora-Zumino and works in any even dimension. See appendix A.3, precisely A.3.2, for a brief discussion on how this work relates to the approach advocated in this thesis. In this appendix I propose some critical remarks, mainly related to the absence of a clear underlying geometrical picture. I therefore suggest that a genuine approach of the full Weyl anomaly through descent equations is still missing.

In A.3 still, I argue that the dressing field method applied to the Cartan-Möbius geometry provides precisely the right geometrical picture to start with. The important missing ingredient is an adequate invariant polynomial. We propose here a candidate and give preliminary results.

4.3.2 A candidate polynomial

The aforementioned works focused on the derivation of the type A Weyl anomaly which is proportional to the Euler density $e(\mathcal{M})$ of the base manifold. Its integral $\chi(\mathcal{M}) = \int_{\mathcal{M}} e(\mathcal{M})$ is the Euler characteristic, a topological invariant which is an obstruction to the possibility to find a global pseudo-riemannian (Lorentzian) metric on a manifold.[8] The Euler density is thus a characteristic class, as such it is given by an invariant polynomial: the Pfaffian.

Given a $2n\times 2n$ antisymmetric matrix $A = \{A_{ij}\}$, the Pfaffian of A is a polynomial of degree n in the matrix entries defined by,

$$\mathrm{Pf}(A) = \frac{1}{n!2^n} \sum_{\sigma} \mathrm{sign}(\sigma) \prod_{i=1}^{\ell} A_{\sigma(2i-1)\sigma(2i)},$$

where $\sigma \in \mathfrak{S}(2n)$ the group of permutation of $2n$ elements. The Pfaffian vanishes on $(2n+1)\times(2n+1)$ matrices and on symmetric ones. This is clearer when we write it as,

$$\mathrm{Pf}(A) = \frac{1}{n!2^n}\, \epsilon^{i_1\cdots i_{2n}} A_{i_1 i_2} \ldots A_{i_{2n-1} i_{2n}} = \frac{1}{n!} \sum_{\substack{i_1<i_2,\\ \ldots \\ i_{2n-1}<i_{2n}}} \epsilon^{i_1\cdots i_{2n}} A_{i_1 i_2} \ldots A_{i_{2n-1} i_{2n}},$$

where $\epsilon^{i_1\cdots i_{2n}}$ is the totally antisymmetric Levi-Civita symbol. The index summation is understood and all indices go from 1 to $2n$. Given a matrix $M = A + S$ with antisymmetric part A and symmetric part S, it is then easily seen that $\mathrm{Pf}(M) = \mathrm{Pf}(A)$. With the identity $\epsilon^{i_1\cdots i_{2n}} {M_{i_1}}^{j_1} \ldots {M_{i_{2n}}}^{j_{2n}} = \det(M)\, \epsilon^{j_1\cdots j_{2n}}$, it is easy to show,

$$\begin{aligned}
\mathrm{Pf}(M^T A M) &= \frac{1}{n!2^n}\, \epsilon^{i_1\cdots i_{2n}} {M_{i_1}}^{j_1} A_{j_1 j_2} {M^{j_2}}_{i_2} \ldots {M_{i_{2n-1}}}^{j_{2n-1}} A_{j_{2n-1} j_{2n}} {M^{j_{2n}}}_{i_{2n}},\\
&= \frac{1}{n!2^n}\, \det(M)\, \epsilon^{j_1\cdots j_{2n}} A_{j_1 j_2} \ldots A_{j_{2n-1} j_{2n}},\\
\mathrm{Pf}(M^T A M) &= \det(M)\, \mathrm{Pf}(A).
\end{aligned}$$

This shows that the Pfaffian is invariant under the adjoint action of $SO(n)$. Indeed since $R \in SO(n)$ means $R^T = R^{-1}$ we have,

$$\mathrm{Pf}(\mathrm{Ad}_{R^{-1}} A) = \mathrm{Pf}(R^{-1} A R) = \mathrm{Pf}(R^T A R) = \det(R)\, \mathrm{Pf}(A) = \mathrm{Pf}(A).$$

[6](Bonora et al., 1986), p.635.

[7](Deser and Schwimmer, 1993), p.8.

[8]Thus a $SO(r,s)$-reduction of the frame bundle $L\mathcal{M}$, thus the impossibility to define a Cartan-Möbius geometry with pseudo-euclidean signature.

These definitions and properties of the Pfaffian are well known. See the chapter on characteristic classes in (Nakahara, 2003) and (Azcarraga and Izquierdo, 1995).

Define now the polarization, or symmetrization, of the Pfaffian by,

$$\bar{P}(A^1,\dots,A^n) = \frac{1}{n!2^n}\,\epsilon^{i_1\cdots i_{2n}} A^1_{i_1 i_2}\dots A^n_{i_{2n-1} i_{2n}} = \frac{1}{n!} \sum_{\substack{i_1<i_2,\\ \cdots \\ i_{2n-1}<i_{2n}}} \epsilon^{i_1\cdots i_{2n}} A^1_{i_1 i_2}\dots A^n_{i_{2n-1} i_{2n}},$$

where $A^1,\dots A^n$ are n antisymmetric matrices. Clearly if any of the variables A^i is a symmetric matrix, the above polynomial vanishes. If a random set of even matrices $\left\{M^i = A^i + S^i\right\}_{i\in[1,\dots,n]}$ is plugged in $\bar{P}$, A^i and S^i being respectively the antisymmetric and symmetric parts, then

$$\bar{P}(M^1,\dots,M^n) = \bar{P}(A^1,\dots,A^n)$$

Owing to symmetry of the Levi-Civita symbol under exchange of two pairs of indices, the polynomial is symmetric,

$$\bar{P}(A^1,\dots,A^i,\dots,A^j,\dots,A^n) = \bar{P}(A^1,\dots,A^j,\dots,A^i,\dots,A^n).$$

Like the Pfaffian the polarization satisfies,

$$\bar{P}(M^T A^1 M,\dots,M^T A^n M) = \bar{P}(A^1,\dots,A^n).$$

The proof is completely analogue to the one given above. This means that the polynomial is $SO(n)$-invariant,

$$\bar{P}(\mathrm{Ad}_{R^{-1}}A^1,\dots,\mathrm{Ad}_{R^{-1}}A^n) = \bar{P}(A^1,\dots,A^n).$$

The Pfaffian is just the diagonal combination, $\bar{P}(A,\dots,A) := \bar{P}^n(A) = \mathrm{Pf}(A)$.

Is $\bar{P}$ the right candidate polynomial we were looking for? No, not as it stands. Indeed the relevant geometry underlying the Weyl anomaly, we argue, is the Cartan-Möbius geometry which is based on the Möbius group defined as a pseudo-orthogonal group $G = SO(d,2) = \left\{R \in GL(d+2) \mid R^T S R = S\right\}$. The polynomial $\bar{P}$ is invariant under $SO(n)$, not under $SO(d,2)$. Moreover the designated variables, the Cartan connection and its curvature, are $\mathfrak{g}$-valued so not antisymmetric matrices. This is not really a big problem, but this raises our suspicion that a slight modification of the polynomial is needed.

This modification is suggested by a simple observation. The defining equation for the Lie algebra $\mathfrak{g}$ is,

$$\mathfrak{g} = \left\{X \in GL(d+2) \mid X^T S + SX = 0 \rightarrow (SX)^T + SX = 0\right\},$$

since the group metric $S = \{-1,\eta,-1\}$ (second diagonal) satisfies, $S = S^T = S^{-1}$. This means that the map $S : X \mapsto SX$ allows to turn $\mathfrak{g}$-matrices into symmetric matrices.[9] Lets then define the polynomial,

$$P := \bar{P}\circ S, \qquad \rightarrow \qquad P(A^1,\dots,A^n) := \bar{P}(SA^1,\dots,SA^n). \tag{4.12}$$

P retains all the good properties of $\bar{P}$, symmetry, linearity, vanishing on symmetric or odd-rank variables. But it is furthermore invariant under $SO(d,2)$. Indeed since $R \in SO(d,2)$ means $R^{-1} = SR^T S$, we have

$$\begin{aligned}
P(\mathrm{Ad}_{R^{-1}}A^1,\dots,\mathrm{Ad}_{R^{-1}}A^n) &= P\left(R^{-1}A^1R,\dots,R^{-1}A^nR\right) = \bar{P}\left(S(R^{-1}A^1R),\dots,S(R^{-1}A^nR)\right),\\
&= \bar{P}\left(R^T(SA^1)R,\dots,R^T(SA^n)R\right) = \det(R)\ \bar{P}(SA^1,\dots,SA^n),\\
&= \det(R)\ P(A^1,\dots,A^n),\\
P(\mathrm{Ad}_{R^{-1}}A^1,\dots,\mathrm{Ad}_{R^{-1}}A^n) &= P(A^1,\dots,A^n).
\end{aligned} \tag{4.13}$$

[9] This is obviously not a Lie algebra morphism.

Notice an interesting feature of this polynomial: its degree is controlled by the size of the matrix variables. This is to be contrasted with the trace. In the calculations of the descent equations in the standard Yang-Mills case we started with $P^n(F) := \mathrm{Tr}(F^n)$. The degree is related to the number of variables. In our case, consider the diagonal combination $P^n(F) = \mathrm{Pf}(SF)$. It is of degree n if F is of size $2n$. This particularity of our polynomial implies a greater structural rigidity. Greater is then our appreciation of the following match.

For the descent equations to provide a consistent anomaly on a d-dimensional manifold, the degree of the homogeneous polynomial at the top of the descent should be $n = d/2 + 1$. Indeed starting with $P^n(F)$ we have the anomaly Q^1_{2n-2} which is a maximal form on $\mathcal{M}$, $2n - 2 = d$. The Cartan-Möbius geometry proposes variables, the Cartan connection ϖ and its curvature Ω, whose size is related to the dimension d of the base manifold by $2n = d + 2$. Evaluating our polynomial on Ω we have a polynomial $P^n(\Omega) = \mathrm{Pf}(S\Omega)$ of degree $n = d/2 + 1$. Therefore the corresponding anomaly is of the right de Rham degree. The match is perfect, and a priori unexpected.

Let us find a handy notation for the polynomial. We can write an antisymmetric matrix as a formal 2-form,

$$A = \frac{1}{2} A_{ij} e^i \cdot e^i, \qquad \text{where "$\cdot$" stands for a formal wedge product}$$

The formal wedge product of n such object is,

$$\begin{aligned} \frac{1}{n!}\, A^1 \cdot A^2 \cdot \ldots \cdot A^n &= \frac{1}{n! 2^n} A^1_{i_1 i_2} A^2_{i_3 i_4} \ldots A^n_{i_{2n-1} i_{2n}}\; e^{i_1} \cdot e^{i_2} \cdot \ldots \cdot e^{i_{2n}}, \\ &= \frac{1}{n! 2^n} A^1_{i_1 i_2} A^2_{i_3 i_4} \ldots A^n_{i_{2n-1} i_{2n}}\; \epsilon^{i_1 i_2 \cdots i_{2n}}\; e^1 e^2 \ldots e^{2n} = \bar{P}(A^1, \ldots, A^n)\ \mathrm{vol}_{2n}. \end{aligned}$$

Accordingly our polynomial can be written,

$$P(A^1, A^2, \ldots, A^n)\ \mathrm{vol}_{2n} = \frac{1}{n!} A^1 \bullet A^2 \bullet \ldots \bullet A^n =: \frac{1}{n!} SA^1 \cdot SA^2 \cdot \ldots \cdot SA^n = \bar{P}(SA^1, SA^2, \ldots, SA^n)\ \mathrm{vol}_{2n}. \tag{4.14}$$

As a genuine $SO(d,2)$-invariant polynomial, P can be used to perform all the constructions, from the transgression to the descent equations, described in the previous section. We are on the bundle $\mathcal{P}(\mathcal{M}, H)$ of the Cartan-Möbius geometry and consider the local Cartan connection $\varpi \in \Lambda^1(U, \mathfrak{g})$ and its curvature $\Omega \in \Lambda^2(U, \mathfrak{g})$. The ghost is simply $\mathfrak{h}$-valued and we use the initial BRS algebra involving the variables v, A, and F. We have a chain of descent equation which is,

$$\begin{aligned} P^n(\Omega)\ \mathrm{vol}_{2n} &= dQ^0_{2n-1}(\varpi, 0)\ \mathrm{vol}_{2n}, \\ sQ^0_{2n-1}(\varpi, 0)\ \mathrm{vol}_{2n} + dQ^1_{2n-2}(\varpi, v)\ \mathrm{vol}_{2n} &= 0, \\ sQ^1_{2n-2}(\varpi, v)\ \mathrm{vol}_{2n} + dQ^2_{2n-3}(\varpi, v)\ \mathrm{vol}_{2n} &= 0, \\ &\vdots \\ sQ^{2n-2}_1(\varpi, v)\ \mathrm{vol}_{2n} + dQ^{2n-1}_0(\varpi, v)\ \mathrm{vol}_{2n} &= 0, \\ sQ^{2n-1}_0(v)\ \mathrm{vol}_{2n} &= 0. \end{aligned}$$

Specializing to $\dim\mathcal{M} = d = 4$, that is for $n = 3$, we have the solution for the chain terms,

$$P_3 \,\mathrm{vol}_6 = \frac{1}{6}\, \Omega \bullet \Omega \bullet \Omega,$$

$$\begin{aligned} Q_5^0 \,\mathrm{vol}_6 &= \frac{1}{6}\left(\varpi \bullet \Omega \bullet \Omega - \frac{1}{2}\,\varpi \bullet \varpi^2 \bullet \Omega + \frac{1}{10}\,\varpi \bullet \varpi^2 \bullet \varpi^2\right) \\ &= \frac{1}{6}\left(\varpi \bullet d\varpi \bullet d\varpi + \frac{3}{2}\,\varpi \bullet \varpi^2 \bullet d\varpi + \frac{3}{5}\,\varpi \bullet \varpi^2 \bullet \varpi^2\right), \end{aligned}$$

$$Q_4^1 \,\mathrm{vol}_6 = \frac{1}{6}\, v \bullet d\left(\varpi \bullet d\varpi + \frac{1}{2}\,\varpi \bullet \varpi^2\right),$$

$$\begin{aligned} Q_3^2 \,\mathrm{vol}_6 &= -\frac{1}{12}\Big((v \bullet [\varpi, v] + \varpi \bullet v^2) \bullet d\varpi + v^2 \bullet \varpi \bullet \varpi^2\Big) \\ &= -\frac{1}{12}\Big(v \bullet [\varpi, v] \bullet d\varpi + v^2 \bullet \varpi \bullet \Omega\Big), \end{aligned}$$

$$\begin{aligned} Q_2^3 \,\mathrm{vol}_6 &= \frac{1}{12}\left(-v^2 \bullet v \bullet d\varpi + \frac{1}{2}\, v \bullet [\varpi, v] \bullet [\varpi, v] - \frac{1}{2}\,[\varpi, v] \bullet v^2 \bullet \varpi\right) \\ &= \frac{1}{12}\left(-v^2 \bullet v \bullet d\varpi + \frac{1}{4}\, v \bullet [\varpi, v] \bullet [\varpi, v] + \frac{1}{4}\,[v^2, \varpi] \bullet v^2 \bullet \varpi\right), \end{aligned}$$

$$Q_1^4 \,\mathrm{vol}_6 = \frac{1}{12}\, v^2 \bullet v^2 \bullet \varpi\,,$$

$$Q_0^5 \,\mathrm{vol}_6 = \frac{1}{60}\, v^2 \bullet v^2 \bullet v.$$

Compare with the solutions found for the trace, in the same dimension, in the previous section. The formal similarity of both sets of chain terms is clear. Actually the only properties necessary to derive the above results are the symmetry of the polynomial, its linearity and invariance. The latter translating as the identity,

$$\sum_{i=1}^{n} (-)^{|A^1| + \ldots + |A^{i-1}|} P\Big(A^1, \ldots, \big[\lambda, A^i\big], \ldots, A^n\Big) = 0, \qquad \lambda \in \mathfrak{g} \text{ and } |A^j| \text{ the degree of the } \mathfrak{g}\text{-valued form } A^j.$$

For our polynomial in the case $n = 3$ this specializes to,

$$\big[\lambda, A^1\big] \bullet A^2 \bullet A^3 (-)^{|A^1|} A^1 \bullet \big[\lambda, A^2\big] \bullet A^3 (-)^{|A^1| + |A^2|} A^1 \bullet A^2 \bullet \big[\lambda, A^3\big] = 0. \tag{4.15}$$

It is very usefull if one recalls that since we are using a matrix formalism, for a $\mathfrak{g}$-valued 1-form β we have,

$$\beta^2 := \beta \wedge \beta = \frac{1}{2}\big[\beta, \beta\big], \qquad \text{and the square of the ghost means} \quad v^2 = \frac{1}{2}\big[v, v\big].$$

Since symmetry, linearity and invariance are properties enjoyed by any characteristic polynomial, the sets of solutions for the chain terms of any of them will display a structural similarity. Of course the details of the explicit results nevertheless heavily depend on the specific polynomial chosen.

These formal results are already encouraging. We can pause for a moment and start to ask how the dressing fields in the Cartan-Möbius, identified in the previous chapters, find their places in this scheme. The general analysis given in the last section will help in getting closer to the Weyl anomaly.

4.3.3 The dressing field method and the Weyl anomaly

We identified two dressing fields in the Cartan-Möbius geometry. Each belongs to a different category according to the analysis of section 4.2. We then pay attention to the peculiarities of each situation.

The first dressing field

The first dressing field of the Cartan-Möbius geometry is u_1 and takes values in the group $K_1 \subset H \subset G$. Since this target group is also the equivariance group we are in the case (I)+(II) of section 4.2.3. We have thus the triangle formula applied to the triplet $\left\{\varpi_1, u_1 du_1^{-1}, 0\right\}$,

$$Q_{2n-1}(\varpi_1, 0) = Q_{2n-1}(\varpi, 0) + Q_{2n-1}(0, u_1 du_1^{-1}) + d\chi_{2n-2}(\varpi_1, u_1 du_1^{-1}, 0),$$

where $\varpi_1 := \varpi^{u_1}$ is the first K_1-invariant composite field. But the term $Q_{2n-1}(0, u_1 du_1^{-1})$ depends on $du_1 du_1^{-1}$ which is,

$$d\left(u_1 du_1^{-1}\right) = d\begin{pmatrix} 1 & q & \frac{1}{2}qq^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -dq & qdq^t \\ 0 & 0 & -dq^t \\ 0 & 0 & 0 \end{pmatrix} = d\begin{pmatrix} 0 & -dq & 0 \\ 0 & 0 & -dq^t \\ 0 & 0 & 0 \end{pmatrix} = 0,$$

So the Chern-Simons forms are d-cohomologous,

$$Q_{2n-1}(\varpi_1, 0) = Q_{2n-1}(\varpi, 0) + d\chi_{2n-2}(\varpi, u_1 du_1^{-1}, 0).$$

Applying the russian algebra, that is writing the same formula for the algebraic connection $\widetilde{\varpi} = \varpi + v$ and its u_1-dressed counterpart $\widetilde{\varpi}_1 = \varpi_1 + \widehat{v}_1$, where $\widehat{v}_1 := v^{u_1}$ is the first composite ghost,

$$Q_{2n-1}(\widetilde{\varpi}_1, 0) = Q_{2n-1}(\widetilde{\varpi}, 0) + \widetilde{s}\chi_{2n-2}(\widetilde{\varpi}, u_1\widetilde{s}u_1^{-1}, 0), \qquad \text{where of course } \widetilde{s} = d + s.$$

Developing according to the ghost degree and selecting the linear terms we have,

$$Q^1_{2n-2}(\varpi_1, \widehat{v}_1) = Q^1_{2n-2}(\varpi, v) + \widetilde{s}\chi(\widetilde{\varpi}, u_1\widetilde{s}u_1^{-1}, 0). \tag{4.16}$$

The **residual anomaly** $Q^1_{2n-2}(\varpi_1, \widehat{v}_1)$ is then **$\widetilde{s}$-cohomologous to the original one** $Q^1_{2n-2}(\varpi, v)$. We can thus focus our attention on this residual anomaly. In particular we will consider the case $n = 3$ since we are primarily interested in the dimension $d = 4$.

Until now we've derived strictly formal results. To see if these have any genuine content, that is to see if the residual anomaly $Q^1_4(\varpi_1, \widehat{v}_1)$ really gives the form of the full Weyl anomaly in space-time dimension 4, we need to better control the intricate combinatorial properties of the polynomial. This can be helped by the following alternative form,

$$\begin{aligned}
P\left(A^1, \ldots, A^n\right) &= \frac{1}{n!2^n}\,\epsilon^{i_1 i_2 \cdots i_{2n}}(SA^1)_{i_1 i_2}\cdots(SA^n)_{i_{2n-1}i_{2n}},\\
&= \frac{1}{n!2^{n-1}}\,(-)^{i_1+i_2+1}(SA^1)_{i_1 i_2}\,\epsilon^{i_3 i_4\cdots i_{2n}}(SA^2)_{i_3 i_4}\cdots(SA^n)_{i_{2n-1}i_{2n}},\\
&= \frac{1}{n}\,(-)^{i_1+i_2+1}(SA^1)_{i_1 i_2}\,\frac{1}{(n-1)!2^{n-1}}\,\epsilon^{i_3 i_4\cdots i_{2n}}(SA^2)_{i_3 i_4}\cdots(SA^n)_{i_{2n-1}i_{2n}},\\
P\left(A^1, \ldots, A^n\right) &= \frac{1}{n}\,(-)^{i_1+i_2+1}A^1_{i_1 i_2}\,P\left(\bar{A}^2, \ldots, \bar{A}^n\right).
\end{aligned} \tag{4.17}$$

Where $i_1 < i_2$ and $i_1 \neq i_2$ since only the antisymmetric part of A^1 survives, and $\bar{A}^i$ is the submatrix obtained from A^i by removing the i_1^{th} line and the i_2^{th} column. This is analogous to the Laplace formula for the determinant in terms of the cofactors. Using the notation introduced above,

$$\begin{aligned}
P\left(A^1, \ldots, A^n\right)\,\mathrm{vol}_{2n} &= \frac{1}{n}\,(-)^{i+j+1}(SA^1)_{ij}\,P\left(\bar{A}^2, \ldots, \bar{A}^n\right)\,\mathrm{vol}_{2n},\\
\frac{1}{n!}\,A^1 \bullet \ldots \bullet A^n &= \frac{1}{n!}\,(-)^{i+j+1}(SA^1)_{ij}\,\bar{A}^2 \bullet \ldots \bullet \bar{A}^n.
\end{aligned} \tag{4.18}$$

Specializing to the case $n = 3$ we get,

$$P\Big(A^1,A^2,A^3\Big) = \frac{1}{3}(-)^{i+j+1}(SA^1)_{ij}\, P\Big(\bar{A}^2,\bar{A}^3\Big), \qquad \text{or} \qquad \frac{1}{6}\, A^1 \bullet A^2 \bullet A^3 = \frac{1}{6}(-)^{i+j+1}(SA^1)_{ij}\, \bar{A}^2 \bullet \bar{A}^3. \tag{4.19}$$

Now write the residual anomaly $Q_4^1(\varpi_1,\widehat{v}_1)$ as,

$$\begin{aligned}
Q_4^1(\varpi_1,\widehat{v}_1)\ \mathrm{vol}_6 &= \frac{1}{6}\,\widehat{v}_1 \bullet d\Big(\varpi_1 \bullet d\varpi_1 + \tfrac{1}{2}\varpi_1 \bullet \varpi_1^2\Big) = \frac{1}{12}\,\widehat{v}_1 \bullet d\Big(\varpi_1 \bullet d\varpi_1 + \varpi_1 \bullet \Omega_1\Big),\\
&= \frac{1}{12}\,\widehat{v}_1 \bullet \Big(d\varpi_1 \bullet d\varpi_1 + d\varpi_1 \bullet \Omega_1 - \varpi_1 \bullet d\Omega_1\Big),\\
&= \frac{1}{12}\,\widehat{v}_1 \bullet \Big(d\varpi_1 \bullet d\varpi_1 + \Omega_1 \bullet \Omega_1 - \varpi_1^2 \bullet \Omega_1 - \varpi_1 \bullet d\Omega_1\Big), \qquad \text{since} \quad d\varpi_1 = \Omega_1 - \varpi_1^2,\\
&= \frac{1}{12}\,(-)^{i+j+1}(S\widehat{v}_1)_{ij}\,\Big(d\bar{\varpi}_1 \bullet d\bar{\varpi}_1 + \bar{\Omega}_1 \bullet \bar{\Omega}_1 - \overline{\varpi_1^2} \bullet \bar{\Omega}_1 - \bar{\varpi}_1 \bullet d\bar{\Omega}_1\Big), \qquad \text{using (4.19).}
\end{aligned}$$

We have,

$$S\widehat{v}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \epsilon & \partial\epsilon\cdot e^{-1} & 0 \\ 0 & v_L & \eta^{-1}(e^{-1})^T\cdot\partial\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix} = \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \eta v_L & (e^{-1})^T\cdot\partial\epsilon \\ -\epsilon & -\partial\epsilon\cdot e^{-1} & 0 \end{pmatrix}$$

We recall that the u_1-dressed normal Cartan connection and its curvature are,

$$\varpi_1 = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \theta & A_1 & \alpha_1^t \\ 0 & \theta^t & 0 \end{pmatrix}, \quad \text{and} \quad \Omega_1 = \begin{pmatrix} 0 & \Pi_1 & 0 \\ 0 & F & \Pi_1^t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d\alpha_1 + \alpha_1 A_1 & 0 \\ 0 & dA_1 + A_1^2 + \theta\alpha_1 + \alpha_1^t\theta^t & d\alpha_1^t + A_1\alpha_1^t \\ 0 & 0 & 0 \end{pmatrix},$$

where A_1 is the Lorentz/spin connection, α_1 is the Schouten 1-form, Π_1 it the Cotton 2-form and F is the Weyl 2-form. Focusing on the Weyl sector of the ghost we have,

$$Q_4^1(\varpi_1,\widehat{v}_1)\Big|_W\ \mathrm{vol}_6 = \frac{1}{12}\,\epsilon\,\Big(dA_1 \bullet dA_1 + F \bullet F - \Big(A_1^2 + \theta\alpha_1 + \alpha_1^t\theta^t\Big) \bullet F - A_1 \bullet dF\Big).$$

Now since $dA_1 = R_1 - A_1^2$ where R_1 is the Riemann 2-form, we have,

$$dA_1 \bullet dA_1 = R_1 \bullet R_1 - 2A_1^2 \bullet R_1 + A_1^2 \bullet A_1^2.$$

But the last term vanishes due to the invariance of the subpolynomial under $SO(r,s)$ which implies the identity,

$$\Big[\lambda,B^1\Big] \bullet B^2(-)^{|B^1|}B^1 \bullet \Big[\lambda,B^2\Big] = 0, \qquad \text{for } \lambda \in \mathfrak{so}(r,s) \text{ and } \beta\ \mathfrak{so}(r,s)\text{-valued.}$$

This means,

$$\frac{1}{2}\Big[A_1,A_1\Big] \bullet A_1^2 - \frac{1}{2}A_1 \bullet \Big[A_1,A_1^2\Big] = 0 \qquad \rightarrow \qquad A_1^2 \bullet A_1^2 = 0.$$

Then the Weyl ghost sector of the residual anomaly reads,

$$Q_4^1(\varpi_1,\widehat{v}_1)\Big|_W\ \mathrm{vol}_6 = \frac{1}{12}\,\epsilon\,\Big(R_1 \bullet R_1 + F \bullet F - 2A_1^2 \bullet R_1 - \Big(A_1^2 + \theta\alpha_1 + \alpha_1^t\theta^t\Big) \bullet F - A_1 \bullet dF\Big).$$

It is possible to clean up a bit this expression. Using the identity above we have,

$$\Big[A_1,A_1\Big] \bullet R_1 - A_1 \bullet \Big[A_1,R_1\Big] = 0 \qquad \rightarrow \qquad 2A_1^2 \bullet R_1 = A_1 \bullet \Big[A_1,R_1\Big],$$

so that,

$$-2A_1^2 \bullet R_1 - A_1 \bullet dF = -A_1 \bullet \Big[A_1, R_1\Big] - A_1 \bullet \bigg(dR_1 + d\Big(\theta\alpha + \alpha^t\theta^t\Big)\bigg) = -A_1 \bullet d\Big(\theta\alpha + \alpha^t\theta^t\Big),$$

due to the Bianchi identity for the Riemann 2-form, $dR_1 + \Big[A_1, R_1\Big] = 0$. So finally the **Weyl ghost sector of the residual anomaly** is,

$$Q_4^1(\varpi_1, \widehat{v}_1)\Big|_W \mathrm{vol}_6 = \frac{1}{12}\,\epsilon\,\bigg(R_1 \bullet R_1 + F \bullet F - \Big(A_1^2 + \theta\alpha_1 + \alpha_1^t\theta^t\Big) \bullet F - A_1 \bullet d\Big(\theta\alpha + \alpha^t\theta^t\Big)\bigg). \tag{4.20}$$

Proposition 3. *The first and second term in the residual anomaly* (4.20) *are respectively the type A Weyl anomaly and the type B Weyl anomaly in dimension $d = 4$.*

Proof. The first term reads,

$$\frac{1}{12}\,\epsilon\, R_1 \bullet R_1 := \frac{1}{12}\,\epsilon\, P^2(R_1)\ \mathrm{vol}_4,$$

by definition of our notation. Discard the formal volume form vol_4 we have,

$$\frac{1}{12}\,\epsilon\, P^2(R_1) = \frac{1}{12}\,\epsilon\, \mathrm{Pf}(\eta R_1) := \frac{1}{12}\,\epsilon\, e(\mathcal{M}), \tag{4.21}$$

since the diagonal combination of our polynomial is the Pfaffian. This is indeed the **type A weyl anomaly**, proportionnal to the **Euler density**, written in a compact form. Let us give the explicit expression.
We have $\eta R_1 = \eta_{il}R^l{}_j = R_{ij}$ which is antisymmetric indeed since R_1 is $\mathfrak{so}(r,s)$-valued. Moreover we can write, $R_{ij} = \frac{1}{2}R_{ij,pq}\,\theta^p \wedge \theta^q$. This is so because R_1 is a tensorial form and $\{\theta^i\}$ is a basis for tensorial forms.[10] Then,

$$\begin{aligned}
\frac{1}{12}\,\epsilon\, P^2(R_1) &= \frac{1}{12}\,\epsilon\, \mathrm{Pf}(\eta R_1) = \frac{1}{12}\,\epsilon\, \epsilon^{ijkl}R_{ij}\,R_{kl},\\
&= \frac{1}{12}\,\epsilon\,\frac{1}{4}\epsilon^{ijkl}R_{ij,pq}\,R_{kl,rs}\,\theta^p \wedge \theta^q \wedge \theta^r \wedge \theta^s = \frac{1}{12}\,\epsilon\,\frac{1}{4}\epsilon^{ijkl}R_{ij,pq}\,R_{kl,rs}\,\epsilon^{pqrs}\theta^1\theta^2\theta^3\theta^4,\\
&= \frac{1}{12}\,\epsilon\,\frac{1}{4}\epsilon^{ijkl}R_{ij}{}^{pq}\,R_{kl}{}^{rs}\epsilon_{pqrs}dV = \frac{1}{12}\,\epsilon\,\frac{1}{4}\delta^{ijkl}_{pqrs}\,R_{ij}{}^{pq}\,R_{kl}{}^{rs}dV.
\end{aligned}$$

Here δ^{ijkl}_{pqrs} is the generalized Kronecker delta. It is the determinant of a matrix whose entries are simple Kronecker delta's δ^i_j where i is a line index and j a column index. Using the properties of the Riemann tensor: $R_{ij,pq} = -R_{ji,pq} = -R_{ij,qp} = R_{pq,ij}$, the definition of the Ricci tensor $R_{ij} = R^l{}_{i,lj}$ and of the scalar curvature $R = R^l{}_{ili}$, we obtain,

$$\frac{1}{12}\,\epsilon\, P^2(R_1) = \frac{1}{12}\,\epsilon\, \mathrm{Pf}(\eta R_1) = \frac{1}{12}\,\epsilon\,\Big(R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2\Big)dV. \tag{4.22}$$

The second term in (4.20) reads,

$$\frac{1}{12}\,\epsilon\, F \bullet F := \frac{1}{12}\,\epsilon\, P^2(F)\ \mathrm{vol}_4, \qquad \rightarrow \qquad \frac{1}{12}\,\epsilon\, P^2(F) = \frac{1}{12}\,\epsilon\, \mathrm{Pf}(\eta F).$$

A calculation in all respect similar to what is done above gives,

$$\frac{1}{12}\,\epsilon\, P^2(F) = \frac{1}{12}\,\epsilon\, \mathrm{Pf}(\eta F) = \ldots = \frac{1}{12}\,\epsilon\,\frac{1}{4}\delta^{ijkl}_{pqrs}\,F_{ij}{}^{pq}\,F_{kl}{}^{rs}dV.$$

Taking into account the properties of the Weyl tensor: $F_{ij,pq} = -F_{ji,pq} = -F_{ij,qp} = F_{pq,ij}$ and $F^l{}_{i,lj} = 0 = F^l{}_{ili}$, we have

$$\frac{1}{12}\,\epsilon\, P^2(F) = \frac{1}{12}\,\epsilon\, \mathrm{Pf}(\eta F) = \frac{1}{12}\,\epsilon\, F_{ijkl}F^{ijkl}dV = \frac{1}{12}\,\epsilon\,\Big(R_{ijkl}R^{ijkl} - R_{ij}R^{ij} + \tfrac{1}{3}R^2\Big)dV. \tag{4.23}$$

This is indeed the **type B Weyl anomaly** which is proportional to the **square of the Weyl tensor**. The last equality stems from the expression of the Weyl tensor in term of the Riemann and Schouten tensors. □

[10] Said otherwise, we write the Riemann 2-form in the orthogonal frame so as to keep latin indices everywhere.

Expression (4.22) and (4.23) can be compared with the result found in (Bonora et al., 1983) by strictly cohomological considerations. We then obtain the full structure of the Weyl anomaly, both types, through descent equations à la Stora-Zumino. This is a good start but some obvious critics can be formulated. The first one concerns the meaning of the remaining terms in (4.20). The cleanest result would be that they can be expressed as s-exact or d-exact terms, so could be neglected. For now this expectation is vexed. In the same vein one could ask about the contribution of the other sectors of the ghost. Indeed until now we've focused our attention on the Weyl sector, ϵ, but what about the contribution of $\partial\epsilon \cdot e^{-1}$ and v_L. To start to address this latter critics, it is usefull to consider the impact of the second dressing field.

The second dressing field

Unfortunately this time the second dressing u_0 has a target group $K \supset G$. We are in the case (I') so that no general result can be used. In particular $u_0 du_0^{-1}$ is not $\mathfrak{g}$-valued so that the triangle formula cannot be applied to check that the residual anomaly $Q_4^1(\varpi_0, \widehat{v}_0)$ is cohomologous to $Q_4^1(\varpi_1, \widehat{v}_1)$.

It is nevertheless possible to write a nice feature of the polynomial when the second dressing enters the game. Define,

$$G = u_0^T S u_0, \qquad \rightarrow \qquad \begin{pmatrix} 0 & 0 & -1 \\ 0 & g & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^T & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & e^T \eta e & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Clearly, $\det(u_0) = \det(e)$ and $\det(G) = \det(g) = |g|$. Consider then the diagonal combination,

$$P^n\big(\Omega_1\big) = P^n\big(u_0\Omega_0 u_0^{-1}\big) := \mathrm{Pf}\big(Su_0\Omega_0u_0^{-1}\big) = \mathrm{Pf}\big((u_0^{-1})^T G\Omega_0 u_0^{-1}\big) = \det(u_0^{-1})\ \mathrm{Pf}(G\Omega_0).$$

We verify that $G\Omega_0$ is indeed antisymmetric,

$$\begin{aligned} G\Omega_0 &= u_0^T S u_0\, u_0^{-1}\Omega_1 u_0 = u_0^T S\Omega_1 u_0, \qquad \text{but} \quad S\Omega_1 + \Omega_1^T S = 0, \\ &= -u_0^T\Omega_1^T S u_0 = -u_0^T\Omega_1^T u_0^T G = -\Omega_0^T G. \end{aligned}$$

We have furthermore,

$$P^n\big(\Omega_1\big) = \det(u_0^{-1})\ \mathrm{Pf}\big(G\Omega_0\big) = \det(u_0^{-1})\ \mathrm{Pf}\big(G\Omega_0G^{-1}G\big) = \det(u_0^{-1})\det(G)\ \mathrm{Pf}\big(\Omega_0G^{-1}\big) = \sqrt{|g|}\ \mathrm{Pf}\big(\Omega_0G^{-1}\big).$$

It is easy to show that Ω_0G^{-1} is symmetric. The above property holds for the polarization,

$$P\big(A_{1,1},\ldots,A_{1,n}\big) := \bar{P}\big(SA_{1,1},\ldots,SA_{1,n}\big) = \sqrt{|g|}\ \bar{P}\big(A_{0,1}G^{-1},\ldots,A_{0,n}G^{-1}\big), \tag{4.24}$$

where $A_{0,i} = A_{1,i}^{u_0} = u_0^{-1}A_{1,i}u_0$. For the type A and B Weyl anomaly in proposition 3 this gives right away,

$$\begin{aligned} \frac{1}{12}\,\epsilon\, P^2(R_1) &= \frac{1}{12}\,\epsilon\,\mathrm{Pf}(\eta R_1) = \frac{1}{12}\,\epsilon\,\sqrt{|g|}\ \mathrm{Pf}(Rg^{-1}) = \frac{1}{12}\,\epsilon\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)\sqrt{|g|}d^4x. \\ \frac{1}{12}\,\epsilon\, P^2(F) &= \frac{1}{12}\,\epsilon\,\mathrm{Pf}(\eta F) = \frac{1}{12}\,\epsilon\,\sqrt{|g|}\ \mathrm{Pf}(Wg^{-1}) = \frac{1}{12}\,\epsilon\, W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}\sqrt{|g|}d^4x, \\ &= \frac{1}{12}\,\epsilon\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2\right)\sqrt{|g|}d^4x. \end{aligned}$$

Here $d^4x = dx^1dx^2dx^3dx^4$, and we used $Rg^{-1} = R^\mu{}_\lambda g^{\lambda\nu} = R^{\mu\nu} = \frac{1}{2}R^{\mu\nu}{}_{\rho\sigma}dx^\rho \wedge dx^\sigma$. Idem for the Weyl tensor Wg^{-1}.[11]

Another net advantage of going through the second dressing is that the final Weyl ghost is,

$$\widehat{v}_0 = \begin{pmatrix} \epsilon & \partial\epsilon & 0 \\ 0 & \epsilon\mathbb{1} & g^{-1}\partial\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix}, \qquad \text{so that} \qquad G\widehat{v}_0 = \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & \epsilon g & \partial\epsilon \\ -\epsilon & -\partial\epsilon & 0 \end{pmatrix}. \tag{4.25}$$

[11]Remember that we used the notation $W = u_0^{-1}Fu_0$ for the Weyl tensor, and $R = u_0^{-1}R_1u_0$ for the Riemann tensor.

We see that the Lorentz sector v_L is replaced by ϵg which is symmetric and then does not contribute to the residual anomaly $Q^1_4(\varpi_0, \widehat{v}_0)$. This answer part of the critics mentionned above.

Nevertheless all the manipulations based on the invariance of the polynomial performed in the previous section, which aimed at simplifying the expression of the term Q^1_4, are now forbidden. So even if we would still be able to derive the type A and B Weyl anomalies above, the number of remaining terms would be more important and their meaning still not clear.

Conclusion If we were able to find the full Weyl anomaly in dimension $d = 4$, the result is not so clean. There remain junk terms and to neglect them is permissible but arbitrary. In higher dimension it seems always possible to find the type A and B Weyl anomaly in the way presented here, but the number of such junk terms grows quickly. Perhaps further work will give satisfaction, perhaps another polynomial is waiting to be found. This attempt may inspire cleverer constructions.

The real point of this section was to illustrate briefly how the dressing field could interact with anomalies. Some simple general features were described in the second section and the application to the Weyl anomaly, especially after the first dressing, provides a concrete example showing how a residual anomaly can be cohomologous to the 'original' one. It was also argued that in the case of a full neutralization of the gauge symmetry by the dressing field method, no anomaly could appear. This provide an obvious criterion for an anomaly-free gauge theory. This latter situation is especially relevant, given our treatment of the electroweak sector of the Standard Model by the dressing field method.

Conclusion

It was soon recognized that what physicists called a gauge theory was in fact based on the geometrical framework of Ehresman connections on principal fibered bundles.

Chapter 1 ended on the observation that the advent of Gauge Theories came with some problems related precisely to the gauge symmetry: masslessness of the mediating bosons, divergence of the path integral. There we distinguished three general approaches to handle these problems: the gauge fixing, the mechanisms of spontaneous symmetry breaking and finally the bundles reductions theorem.

Gauge fixing is the explicit breaking of the gauge symmetry by the adding of a constraint equation, often directly in the Lagrangian of the theory. This is done in non-abelian gauge theories by the Fadeev-Popov method, (Faddeev and Popov, 1967). The only consistency condition that needs to be imposed is that physical outcomes of the gauge-fixed theory shouldn't depend on the choice of gauge.

The spontaneous symmetry mechanism, was first imported from solid state physics to particle physics in the early 60's, (Brout and Englert, 1964), (Higgs, 1964), (Guralnik et al., 1964). The idea is that the high energy phase of a theory could have a larger gauge symmetry than its low energy phase. The spontaneous $SU(2)$-symmetry breaking of the electroweak sector of the Standard Model is usually held responsible for the mass of the weak bosons.

The bundles reduction theorem is a mathematical result stating the conditions for a principal bundle to be reduced to a subbundle with smaller structure group. That is, when a gauge theory can be reduced to a theory with smaller gauge symmetry. Several authors recast the SSBM in the language of the bundle reductions theorem, see (Trautman, 1979), (Westenholz, 1980) and (Sternberg, 1994). Nevertheless we stressed that while there is a dynamical viewpoint in the former, there is none in the latter. So if certain formal similarities are undeniable, the interpretive baggage of each shouldn't be confused.

We then entered the core of our subject by advertising a fourth way to handle gauge symmetry which is the **dressing field method**.

Chapter 2 introduces the notion of **dressing field**, $\bar{u}$, and showed how it was possible to construct projectable forms out of the standard gauge fields of a theory, that is the connection of a principal bundle, its curvature and the sections of associated bundles.

$$\widehat{\omega} = \omega^{\bar{u}} := \bar{u}^{-1}\omega\bar{u} + \bar{u}^{-1}d\bar{u}, \qquad \widehat{\Omega} = \Omega^{\bar{u}} := \bar{u}^{-1}\Omega\bar{u}, \qquad \widehat{\Psi} = \Psi^{\bar{u}} := \rho(\bar{u}^{-1})\Psi.$$

Since the above composite forms resemble closely mere gauge transformations, it was necessary to emphasize that they are not. Indeed the dressing field do not belong to the gauge group of the theory and, for example, $\widehat{\omega}$ does not belong to the space of connections anymore. According to the equivariance group of the dressing field these composite forms project either on the base manifold, the **gauge symmetry is then fully neutralized**, or on a subbundle in which case the composite forms display a **residual gauge symmetry**. This was defined in Lemma 1 and Lemma 2.

We made clear the differences between the dressing field method and the three general strategies mentioned above. Our method has obviously nothing to do with spontaneous symmetry breaking. Nor is it a gauge fixing, but it is a perfect substitute to it since it allows to work with gauge-equivalent classes. It coincides with the bundles reduction theorem only in the case of a structure group which is a direct product.

It was argued that this geometrical construction is the foundation of the notion of **Dirac variables**, (Dirac, 1955), (Dirac, 1958), and their generalization. I gave examples of some papers gathered in the hadronic physics literature, which are clearly related to the present work and which I analyzed in the Appendix A.

We then worked out two main physical applications. One to General Relativity, where we recast the formulation in term of the adequate Cartan geometry and used an operative matrix formalism to handle the calculations. We saw that the transition from the Palatini/tetrad/gauge formulation of the theory to the Einstein-Hilbert formulation is an instance of the method where the dressing field is the tetrad itself. The other is the electroweak sector of the Standard Model, where a dressing field is extracted from the scalar field φ. With it, composite fields (or generalized Dirac variables) are constructed and the $SU(2)$-gauge symmetry is neutralized. With the method comes an interpretive shift. Indeed it is proved that the $SU(2)$-invariant composite fields were present in the theory from the start, that is in both phases of the theory. Then the 'breaking' of the $SU(2)$ symmetry, if mandatory for the weak bosons to have mass, is not immediately correlated to this mass attribution as is the case in the usual viewpoint. Moreover the ground state of the Higgs field η is unique here, so that even the usual terminology '*spontaneous* symmetry breaking' is challenged.

A significant generalization of the method to **higher-order G-structures** was then proposed. There the possibility to compose the dressing operations is conditioned by **compatibility conditions** on the various dressing fields. As an illustration of this generalization, the 2^{nd}-order conformal structure, or **Cartan-Möbius geometry**, is studied. There the isotropy group H of the Möbius group G, which is the structure group of the bundle under study, is reduced through two successive dressing fields to the 1-dimensional abelian group of Weyl rescaling. This is the residual gauge symmetry of the fully **dressed normal Cartan connection** ϖ_0 and its curvature Ω_0, which contains respectively, the metric, the Christoffel symbols, the Schouten tensor and the Cotton and Weyl tensors. In other words we find in a single move the conformal transformations of all these important objects of the conformal geometry of the base manifold. And we do so a priori without supposing that all the tensors should be functions of the metric tensor. By the way, the composite field (or generalized Dirac variable) ϖ_0 is the so-called **Riemannian parameterization of the normal conformal Cartan connection** as found in (Ogiue, 1967).

Chapter 3 explains how the dressing field method modifies the BRS algebra of a gauge theory. In particular, the notion of **composite ghost** is introduced and shown to represent the infinitesimal residual gauge freedom (if there's any). The latter is handled precisely by the modified or **residual BRS algebra**. Again a generalization to higher-order G-structure is proposed and the BRS version of the compatibility conditions are derived. As an illustration we applied the scheme to Cartan-Möbius geometry. The residual BRS algebra involving the dressed Cartan connection, its curvature and the final Weyl ghost is shown to provide the infinitesimal Weyl rescaling of the various tensors mentioned above. Again this is performed through a handy matrix formalism and the results are obtained in a even easier way than in the finite case.

Finally the chapter ends with the incorporation of the **infinitesimal diffeomorphisms** of the base manifold as a residual symmetry. Application to General Relativity and to Cartan-Möbius geometry allow us see that the dressing field method provides a shift of the composite ghost which implies a form for the corresponding residual BRS algebra very similar to what is derived in (Langouche et al., 1984). We stressed that the form of the final composite ghost in this case give a geometrical interpretation of the cohomological results obtain by (Boulanger, 2005) and (Boulanger, 2007a)

Chapter 4 touches upon the question of the interaction of the dressing field with anomalies in Quantum Field Theory. The latter being often studied with the help of BRS machinery, the question was unavoidable. In particular we saw under which condition an anomaly written in terms of the initial gauge variable is cohomologous to the **residual anomaly** written in terms of the composite fields. We were thus able to draw a quite obvious yet interesting conclusion: **if a theory can be geometrized through the dressing field method, then it is anomaly-free**. Indeed, an anomaly being the signature of the breaking of the gauge symmetry, if a full neutralization of the latter can be achieved with an adequate dressing, then obviously there should be no gauge-anomaly. This is especially relevant when we think about our treatment of the electroweak sector of the Standard Model. The mere fact that this treatment was possible indicates that the model cannot display an $SU(2)$-gauge anomaly. The dressing field method can thus be considered as a **probe of the anomalous content of a gauge theory**.

Finally an attempt toward the **Weyl anomaly** illustrates the general discussion. Being a famous exception to a treatment through descent equations, an anomaly among the anomalies, this is of course not final.

We are nevertheless emboldened by the conviction that with the Cartan-Möbius geometry and the dressing field method, we have the right geometrical context to ground a serious understanding of the Weyl anomaly (which is a quantum breaking of the conformal symmetry). The main proposition is a candidate **characteristic polynomial**, the essential ingredient for genuine descent equations à la Stora-Zumino. An ingredient missing to this day. It turns out that we were able to find the full structure of the Weyl anomaly, but the result is not so clean because of junk terms whose number grows quickly with the dimension of space-time. Perhaps further work will give better satisfaction. Perhaps the true solution will come from the identification of the right characteristic polynomial.

As a matter of perspective, it is possible to propose at least two directions. First, the dressing field method allowed to derive classical results about the 2^{nd}-order conformal structure, usually approached through the jet formalism as in (Kobayashi, 1972) and (Ogiue, 1967). I have no doubt that the method could be applied to another classical geometry: the 2^{nd}-order projective structure. An example easier to treat, by the way. This time again, we will describe an alternative way toward the results that (Kobayashi and Nagano, 1964) obtained through jet formalism.

A second direction, most directly tied to Physics, is to investigate the obvious question of quantization. In the line of the last chapter, further studies of the interaction of the dressing field method with anomalies in Quantum Field Theory may be worth pursuing. One could be encouraged in this by finding hitherto unnoticed examples of the method scattered in the literature on the subject. For example in the third section of the paper "Algebraic study of chiral anomaly" by Mañes, Stora and Zumino (Mañes et al., 1985), appears the element g supposedly belonging to the gauge group. But its BRS transformation is defined as $sg = -vg$ (equation (37) of the paper). Further in the text we have the chiral splitting, $sg_R = -v_R g_R$ and $sg_L = -v_L g_L$ (equation (82)). We clearly recognize the BRS transformation law of a dressing field. Furthermore their reconstruction of the Wess-Zumino-Witten action (equation (41)) rests on writting a Chern-Simons form $\omega^0_{2n-1}(A^{g_t}_t, A_0)$ where A_0 is a background field, gauge-invariant by definition, and $A^{g_t}_t$ is a 1-parameter family interpolating between A_0 and A^g (their notations). Clearly since g is a dressing field, A^g is an invariant composite field, instance of (2.6) in Chapter 2, that we write in our notation $\widehat{A} := A^u$. Likewise, in the first chapter of the thesis of Mañes, "Anomalies in Quantum Field Theory and Differential Geometry", one finds in the reconstruction of the Wess-Zumino effective action (equation (1.28)) the quantity $T(g)\omega^0_{2n-1}(A)$, where g has the finite gauge transformation $g \to h^{-1}g$. There again g is just a dressing field u, and the quantity $T(g)\omega^0_{2n-1}(A)$ is what we wrote $Q_{2n-1}(\widehat{A}, 0)$ in Chapter 4.

In the same vein, one can ask if the dressing field method facilitates the quantization of a theory, and how. This ironically closes the loop: the dressing field method is, as we saw, at the basis of the notion of Dirac variables. Variables that Dirac introduces precisely in order to quantize electrodynamics ((Dirac, 1955), (Dirac, 1958)). In this regard, the literature about Dirac quantization (e.g (Pervushin, 2001), (Lantsman, 2009)) and generalized Dirac variables is a good place to start.

Appendix A
Papers analysis

This appendix is devoted to a brief glimpse on how the dressing field method presented in this thesis underlies some constructions found in selected papers about the physics of gauge theories, and how the present work is sometimes bound to challenge the interpretations associated to these constructions.

A.1 A paper by M. Lavelle & D. McMullan

In their paper (Lavelle and McMullan, 1997) Martin Lavelle and David McMullan developed a program of reconstruction of the constituent quark model, quite successful in explaining the hadronic spectroscopy, based on the formalism of Quantum ChromoDynamics (QCD). As mentioned all too briefly in section 2.3.2 their approach rests on the founding idea of (Dirac, 1955). Not so surprisingly then, we find that the dressing field approach developed in chapter 2 underlies some crucial part of their work.

Among interesting considerations, they themselves stress the importance of two connected most salient points. First they aim at showing that **dressings** can be expressed as functions of the gauge potential A, and are thus in close relationship with **gauge fixing**. Second, since observable quarks must be dressed quarks, as they argue convincingly, and since dressings are related to gauge fixing, the **Gribov ambiguity**[1] may be a reason for the impossibility to have a well defined observable asymptotic quark field. This, in the end, would be the explanation for the phenomenon of **confinement** and the fact that in non-perturbative regime one can only speak of colourless hadrons.

We've seen in 2.2.1 that (2.5) are not gauge transformations. In particular the 1-form $\widehat{\omega}$ is projectable thus is not a connection on $\mathcal{P}$. It is not a representative of the gauge orbit of the connection ω, even if there's a bijective correspondance between $\widehat{\omega}$ and and the orbit of ω. This follows from the fact that $\bar{u}$ does not belong to the gauge group $\mathcal{H}$. All this is then true for the local versions (2.6). The composite field $\widehat{A}$ is not a representative of the gauge orbit of the gauge potential A, and the dressing field u is not in the gauge group $\mathcal{H}_{loc}$.

This seems a knock-down argument against the the idea that dressings are related to gauge fixing. Worse, at first sight is seems that this would invalidate the appealing suggestion that the Gribov ambiguity is the geometrical reason for physical confinement of quarks. Yet the construction of Lavelle and McMullan seems sound and I deem that it deserves careful analysis. An analysis that will confirm that the fate of the first point is settled, but also that the most important and appealing second point might be left untouched. In the following I then expose the logical outlines of their paper before focusing on the discussion of the two mentioned salient points.

The structure of the paper

In section 2 of their paper, Lavelle and McMullan recall the successes of the constituent quark model in explaining hadronic spectroscopy. They recall that the effective masses of each of the constituent quarks (u, d and s) are much more important than the masses of their alter ego quarks of QCD, and furthermore different when considered in mesons or baryons. This, they argue, points toward the idea that the bare quarks of QCD are somehow dressed with a sea of quarks/gluons that contribute to a major part of the effective masses and that this dressing may indeed change in different bound states, hence the different effective masses for the constituent quarks.[2]

[1] See (Gribov, 1978), (Singer, 1978).

[2] This dressing being different from the one acknowledged in non-relativistic, effective models describing confinement in terms of phenomenological potentials. There the dressed quarks have masses close to those of bare quarks.

Section 3 is devoted to a brief field theoretic review of gauge theories. They warn that while A and ψ in the Lagrangian "are conventionally identified with the gluons and the quarks [...] this identification is, at best, misleading [...] we shall refine this terminology and refer to these fields as the Lagrangian quarks and gluons".[3] The main result in this section is the proof that colour charge is well defined only for gauge invariant states, ruling out A and ψ as genuine colour charge bearers. Hence the urge to "look for a *gauge invariant generalization* of the Lagrangian fermion ψ"', and they announce "[...] our aim being to show to what extend such charged states can be constructed in QCD, [...] That there should be an *obstruction* of fully carrying this programme is, we will argue, *simply a restatement of confinement*."[4] (my emphasis added). Emphasis is also put on the expected non-locality (and the loss of manifest Lorentz-covariance) of such gauge invariant charged states. Another instance of the trade, now familiar, between gauge invariance and non-locality. See Chapter 2.

Section 4 deals with a perturbative extension of Dirac's idea to QCD. Their equations (4.1) or (4.5) are analogues of equation [16] in (Dirac, 1955), that is, instances of our $\widehat{\varphi}_i$ in (2.6) where the dressing field is expressed as a non-local function of the gauge potential A. These equations are interpreted along the lines of Dirac: "[...] we have dressed the quarks above in such a way that they are perturbatively gauge invariant, we have dressed them with gluons [...]".[5] The section ends on a fair concern: "[...] can we systematise the derivation of these dressed quarks fields and are we able to *dress quarks (and gluons) non-perturbatively*?"[6] (my emphasis added). They also raise the question of the tractability of a quantum theory of these dressed fields. The former point is the object of the section 5, the latter is studied in the rest of the paper.

Section 5 deals with non perturbative dressings. Equation (5.1) is the equivariance law for an H-valued dressing field h (see 2.2.1), equation (5.2) and (5.5) are instances of $\widehat{\varphi}_i$ and $\widehat{\mathscr{A}}$ in (2.6), as already stated in 2.3.2. Equation (5.6) is even the local version of the demonstration of the global gauge invariance of $\widehat{\omega}$ performed in the proof of our Lemma 1. Then one reads: "We have thus reduced the problem of finding gauge invariant, dressed quarks, and gluons to that of constructing such a field h."[7] In the answer Lavelle and McMullan give to this problem lies my main, and only, disagreement. Indeed they write: "we now want to demonstrate that the existence of such a dressing field is equivalent to finding a gauge fixing condition." The remaining of their section deals with this task.

Section 6 to 9 are preliminary investigation of the quantum theory of the non-local, non-covariant, dressed fields produced. My lack of expertise prevents me to fully appreciate and/or criticize these sections. Notice however that the whole section 8 investigates the behavior of the dressing fields under Lorentz transformations. As alluded to in 2.3.2, our differential geometric construction of (2.5) and of (2.6) is intrinsic, that is, naturally generally covariant.

Being the very reason for this whole discussion, I analyze in the following the argument presented in section 5.

Dressing field, gauge fixing, Gribov ambiguity and confinement

We will follow step by step the argument put forward in section 5 of (Lavelle and McMullan, 1997). We left them on the statement: "[...] we now want to demonstrate that the existence of a dressing field is equivalent to finding a gauge fixing condition."[8]

Before this statement, was given an example of a perturbatively A-dependant dressed quark, $\psi_c^{g^2} = h_c^{-1}\psi$, with $h_c = e^{-v_c} + O(g^3)$, g the coupling constant. The phase is $v_c = f(\partial_i A_i)$, with f a (complicated) function depending on g and g^2, given by their equation (5.3). Just after the above statement, a warm-up argument is given: "That there is a connection between dressing and gauge fixing should not come as too much of a surprise. Indeed, from the specific example (5.3) we see that, at least to this low order in g, the Coulomb gauge $\partial_i A_i = 0$ removes the dressing." (f is indeed such that $f(0) = 0$). But what does this observation *really* mean?

[3](Lavelle and McMullan, 1997) "Constituent quarks from QCD", p.9.
[4]Ibid, p.20.
[5]Ibid, p.21.
[6]Ibid, p.22.
[7]Ibid, p.23
[8]Ibid, p.23

In our notation, this just means that given the gauge invariant field $\widehat{\varphi} = u^{-1}\varphi$, it may be possible to find a gauge transformation $\gamma \in \mathcal{H}_{loc}$ such that $\varphi^\gamma = \widehat{\varphi}$, or in other words, such that $\gamma = u$. But these equalities are obviously not gauge invariant owing to the different transformation properties of the objects on both sides. So in my view, this observation is so far a misleading one.

Before studying the non-perturbative version of the above argument, it is worth noting something that might cast further doubts on the claimed equivalence dressing/gauge fixing. One reads "[...] the QCD Lagrangian must contain a gauge fixing term [...]. Dressing the quark will involve a further gauge fixing which *is not* this Lagrangian gauge fixing",[9] hence the terminology they employ: "dressing gauge fixing". This should sound strange. A gauge fixing is a cross-section in $\mathcal{A}$ (or $\mathcal{F}$) that selects a single representative in each gauge orbit, as the authors explain latter. If the term in the Lagrangian already does this adequately, that is without leaving some residual gauge freedom, what meaning are we to ascribe to a *second* gauge fixing? Further differences between the 'two gauge fixing' are exposed, but the above remark should be sufficient for us.

The authors then want "to show that [...] a gauge fixing condition $\chi(A) = 0$ can be used to construct the dressing h." Before doing so for the non-abelian case, they propose to illustrate the principle with the abelian case of QED and with the Coulomb gauge $\chi(A) = \partial_i A_i = 0$. Lets sum up their argument.

The example of QED For A, a $U(1)$-gauge potential, the gauge orbit is $\mathcal{O}_A = \{A_i - i\gamma^{-1}\partial_i\gamma\} = \{A_i + \partial_i\lambda\}$ for $\gamma = e^{i\lambda} \in \mathcal{H}_{loc} = \mathcal{U}(1)_{loc}$. The Coulomb gauge picks out a single representative in each orbit, so that there must exists a unique scalar field v such that $\partial_i(A_i + \partial_i v) = 0$. This Gives $v(A) = -\partial_i A_i/\nabla^2$, an A-dependant gauge tranformation. Then we are told that for another potential along the same orbit $A_i^\gamma = A_i + \partial_i\lambda$ the scalar field transform as $v^\gamma(A) := v(A^\gamma) = -\partial_i(A_i + \partial_i\lambda)/\nabla^2 = v - \lambda$. So that for $h = e^{iv}$ one has $h^\gamma = \gamma^{-1}h$, the law for a dressing field. Then from a gauge fixing condition they have constructed a dressing field, as claimed. Moreover the field $A_i^h = A_i - ih^{-1}\partial_i h = A_i + \partial_i v$ is gauge invariant.

How is it that it happened? This plainly contradicts our Lemma 1 and our discussion in 2.2.1. Many things should disconcert us. First, if the Coulomb gauge selects a representative in $\mathcal{O}_A$, then $A_i + \partial_i v$ is a gauge potential and cannot be gauge invariant for the gauge group is supposed to act freely on $\mathcal{O}_A$. So $h = e^{iv}$ must be in the gauge group. What is the transformation law for an element in $\mathcal{U}(1)_{loc}$? Given $\alpha, \gamma \in \mathcal{U}(1)_{loc}$ we have (recall equation (1.2) in section 1.1.1), $\gamma^\alpha = \alpha^{-1}\gamma\alpha = \gamma$ owing to the abelian nature of the gauge group. Then we should have $v^\gamma = v$! Clearly not a dressing law of transformation. This is an inescapable conclusion, nonetheless clearly contradicted by the above construction. How are we to understand the situation?

Well, the fact is that a subtle tacit assumption is made when writing $v^\gamma(A) := v(A^\gamma)$ the way they did. This amounts to ask for the equation $\chi(A^h) = 0$ *to be invariant under gauge transformations*, which it has a priori no reason to be. Let us show in the QED case how this requirement entails the result of Lavelle and McMullan.

Given A, the scalar field v/h is constructed as the unique solution to $\chi(A^h) = \partial_i(A_i + \partial_i v) = 0$. Now require the invariance of this functional constraint $\chi(A^h) = \chi(A^h)^\gamma := \chi\big((A^\gamma)^{h^\gamma}\big) = 0$. This means that there must be a unique v^γ/h^γ such that $\chi\big((A^\gamma)^{h^\gamma}\big) = \partial_i(A_i^\gamma + \partial_i v^\gamma) = 0$. Now $\chi\big((A^\gamma)^{h^\gamma}\big) = \partial_i(A_i + \partial_i\lambda + \partial_i v^\gamma) = 0$, so that we find $v^\gamma = -(\partial_i A_i + \nabla^2\lambda)/\nabla^2 = v - \lambda$. This is the infinitesimal version of the dressing law of transformation $h^\gamma = \gamma^{-1}h$. So it is the tacit admission of the gauge invariance of the functional condition $\chi(A^h) = 0$ that makes h a dressing, which cannot be in $\mathcal{U}(1)_{loc}$ by definition. And the field $A_i^h = A_i - \frac{i}{e}h^{-1}\partial_i h$ is then indeed gauge invariant and cannot be a gauge potential in $\mathcal{O}_A$. Exactly the same reasoning applies to their treatment of the non-abelian case and of arbitrary 'gauge fixing' $\chi(A^h)$.

The general argument A short preparation preceeds their general argument. It is a description of how one is to understand gauge fixing in gauge theories, similar to what we have seen in section 1.3. Lets then elaborate on this by using a mix of both their notations and ours.[10]

[9] Ibid, p.23.

[10] We correct in passing some imprecisions.

The space of (local) gauge potentials $\mathcal{A}$ is seen as a bundle, $\mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{H}_{loc}$, over the (open set of the) moduli space $\mathcal{A}/\mathcal{H}_{loc}$.[11] An orbit $O_A = \left\{A^\gamma \mid \gamma \in \mathcal{H}_{loc}\right\}$ of a gauge potential $A \in \mathcal{A}$ is a fiber over $\pi(A) = \widehat{A} \in \mathcal{A}/\mathcal{H}_{loc}$. The gauge group $\mathcal{H}_{loc}$ is supposed to act freely on $\mathcal{A}$ so that $\mathcal{H}_{loc} \simeq O_A$. A gauge fixing, $\chi(A) = 0$, is a slice in $\mathcal{A}$ which cuts through each orbit O_A once. Given $A \in O_A$ we can define $h(A) \in \mathcal{H}_{loc}$ that sends A to the point in O_A where the gauge fixing condition holds, i.e such that $\chi(A^{h(A)}) = 0$. This could be seen as a section $\sigma_\chi : \mathcal{A}/\mathcal{H}_{Loc} \to \mathcal{A}$ given by $\sigma_\chi(\widehat{A}) = \left\{A \in O_A \mid \chi(A) = 0\right\}$. This section induces a canonical trivialization $i : \mathcal{A}/\mathcal{H}_{loc} \times \mathcal{H}_{loc} \to \mathcal{A}$ given *symbolically* by $i(\widehat{A}, e) := \sigma_\chi(\widehat{A})$.

So far so good. But then we are presented with equation (5.16) on the product space: $(\widehat{A}, \gamma_1)^{\gamma_2} := (\widehat{A}, \gamma_2\gamma_2)$. Call it (5.16a). Despite its apparent reasonableness it is actually wrong. Indeed it describes the right-action of the gauge group on the product space: $\left(\mathcal{A}/\mathcal{H}_{loc} \times \mathcal{H}_{loc}\right) \times \mathcal{H}_{loc} \to \left(\mathcal{A}/\mathcal{H}_{loc} \times (\mathcal{H}_{loc} \times \mathcal{H}_{loc})\right)$, where of course the group acts on the second factor only. But then (5.16a) seems to implies that the action of the gauge group on itself is a mere right-action. But it is not. Remember again equation (1.2) in section 1.1.1, the true action of the gauge group on itself is $(\gamma_1, \gamma_2) := \gamma_1^{\gamma_2} = \gamma_2^{-1}\gamma_1\gamma_2$. So that equation (5.16) should be: $(\widehat{A}, \gamma_1)^{\gamma_2} = (\widehat{A}, \gamma_1^{\gamma_2}) = (\widehat{A}, \gamma_2^{-1}\gamma_1\gamma_2)$. Call it (5.16b).

Where did the trouble came from? Simply from mistaking the action of the gauge group on the space of gauge potential (which is indeed a right action) for its action on itself (which is not), and from taking too seriously the, in this case, purely *symbolical* notation of the trivialization i in their equation (5.15): $i(\widehat{A}, \gamma) = \sigma_\chi(\widehat{A})^\gamma$. Indeed, taking $\sigma_\chi(\widehat{A}) = A \in O_A$ (i.e $\chi(A) = 0$) we can send it to another point in O_A by a gauge transformation $\gamma \in \mathcal{H}_{loc}$. We would then write $A^\gamma = \sigma_\chi(\widehat{A})^\gamma =: i(\widehat{A}, e)^\gamma = i(\widehat{A}, e^\gamma) \neq i(\widehat{A}, \gamma)$, since $e^\gamma = e$. So (5.15) must be strictly symbolical. If we reverse the logic and make (5.15) a *definition* then we would have $\left(\sigma_\chi(\widehat{A})^{\gamma_1}\right)^{\gamma_2} = \sigma_\chi(\widehat{A})^{\gamma_1\gamma_2}$, the right-action of $\mathcal{H}_{loc}$ on $\mathcal{A}$, which would entails the symbolical notation $i(\widehat{A}, \gamma_1)^{\gamma_2} = i(\widehat{A}, \gamma_1\gamma_2)$. This equation is quite analogous, but not equivalent, to (5.16a) misleading anyway.

Lets follow the final steps of the argument. A bundle chart, $j : \mathcal{A} \to \mathcal{A}/\mathcal{H}_{loc} \times \mathcal{H}_{loc}$, associated to the section σ_χ is defined as $j(A) = \left(\pi(A), h^{-1}(A)\right) = \left(\widehat{A}, h^{-1}(A)\right)$ (equation (5.17)), for any $A \in O_A$. Then a gauge transformation is performed on (5.17): $j(A^\gamma) = \left(\widehat{A}, h^{-1}(A^\gamma)\right) = \left(\widehat{A}, h^{-1}(A)^\gamma\right) = \left(\widehat{A}, h^{-1}(A)\right)^\gamma$, and a final appeal to (5.16), that is (5.16a), gives $\left(\widehat{A}, h^{-1}(A)\right)^\gamma = \left(\widehat{A}, h^{-1}(A)\gamma\right)$. Hence they conclude: "Thus we must have $h(A)^\gamma = \gamma^{-1}h(A)$, and we have recovered the dressing transformation [...]."[12]

Notice first that this transformation for $h(A)$ is in contradiction with it being an element if $\mathcal{H}_{loc}$. This should raise our suspicion. Furthermore we have seen that (5.16a) could not be trusted and that (5.16b) should be preferred. In this case in the last step we would have $h(A)^\gamma = \gamma^{-1}h(A)\gamma$ which is, this time, perfectly consistent with $h(A)$ being in $\mathcal{H}_{loc}$.

In this general argument I would like to show that, again, this is the tacit assumption of the invariance of the 'gauge fixing condition' $\chi = 0$ which produces the transformation law of a dressing for $h(A)$. Or to state it differently but equivalently, this is the tacit assumption that the section σ_χ remains unaffected by the action of $\mathcal{H}_{loc}$ that produces the transformation law of a dressing for $h(A)$.

To see this let us restate their demonstration slightly differenty. The bundle chart j associated to σ_χ is defined as $j(A) = \left(\widehat{A}, h^{-1}(A)\right)$, where $h^{-1}(A)$ is the unique element in $\mathcal{H}_{loc}$ such that $\sigma_\chi(\widehat{A})^{h^{-1}(A)} = A$, that is, $A^{h(A)} = \sigma_\chi(\widehat{A}) \Leftrightarrow \chi(A^{h(A)}) = 0$, by definition. Under gauge transformation we have, $j(A^\gamma) = \left(\widehat{A}, h^{-1}(A^\gamma)\right)$, where $h^{-1}(A^\gamma)$ is the only element in $\mathcal{H}_{loc}$ such that $\sigma_\chi(\widehat{A})^{h^{-1}(A^\gamma)} = A^\gamma$, that is, $(A^\gamma)^{h(A^\gamma)} = A^{\gamma h(A^\gamma)} = \sigma_\chi(\widehat{A}) \Leftrightarrow \chi(A^{\gamma h(A^\gamma)}) = 0$, by definition. From this we want to conclude that $h(A^\gamma) := h(A)^\gamma = \gamma^{-1}h(A)$, a dressing transformation.

We are now convinced that this conclusion cannot be correct. Where is the oversight? The crucial point is when we write $\sigma_\chi(\widehat{A})^{h^{-1}(A^\gamma)} = A^\gamma$. This equation assumes that while A is moved by γ, the section $\sigma_\chi(\widehat{A})$ is not. But this is out of question for $\mathcal{H}_{loc}$ acts freely on $\mathcal{A}$. The section must move as well. So the correct relation

[11] Of course everything here is local, the true bundle of gauge potentials is a gluing of all the trivial bundles $\mathcal{A}$.

[12] Ibid, p.28.

must rather be $\left(\sigma_\chi(\widehat{A})^\gamma\right)^{h^{-1}(A^\gamma)} = \sigma_\chi(\widehat{A})^{\gamma h^{-1}(A^\gamma)} = A^\gamma$. Reversing the relation we have $\sigma_\chi(\widehat{A}) = A^{\gamma h(A^\gamma)\gamma^{-1}} \Leftrightarrow \chi(A^{\gamma h(A^\gamma)\gamma^{-1}}) = 0$, by definition. By comparison with $\chi(A^{h(A)}) = 0$ we conclude $h(A^\gamma) := h(A)^\gamma = \gamma^{-1}h(A)\gamma$, which is indeed the transformation law for an element $h(A) \in \mathcal{H}_{loc}$.

Conclusion To sum up, we've seen that in the construction of Lavelle and McMullan, the tacit assumption of the gauge invariance of the 'gauge fixing condition' $\chi(A^h) = \chi\left((A^\gamma)^{h^\gamma}\right)$ is at the root of their derivation of a dressing field transformation for h out of it. Clearly an invariant gauge fixing is no gauge fixing at all, thinking of it in terms of a section, for the gauge group acts freely on $\mathcal{A}$. Actually requiring $\chi(A^h) = \chi\left((A^\gamma)^{h^\gamma}\right)$ is precisely asking for A^h to be an invariant composite field, preventing it from being in $\mathcal{A}$ anymore,[13] thus asking for h to have a dressing transformation law, preventing it from being in $\mathcal{H}_{loc}$ by its very definition. All in agreement with our views of chapter 2.

If the condition $\chi(A^h)$ is not a gauge fixing, as we were suspicious of for it being said a 'second gauge fixing', what is the meaning of imposing it in the first place. The answer for me is clear: this is just a functional equation destined to obtain an explicit construction for the dressing field as a function of the gauge potential A. Such an explicit expression must however be compatible with the transformation law of the dressing, hence the necessity of the invariance of the functionnal equation under gauge transformation, $\chi^\gamma = \chi$, the highlighted tacit assumption in the above argument.

Gribov ambiguity and confinement The original second idea of Lavelle and McMullan is that, due to the Gribov ambiguity, a globally well defined section in the bundle of gauge potential, or a global gauge fixing $\chi = 0$, cannot be defined. Thus a globally well defined dressing cannot exist either. Thus a globally well defined dressed quark cannot be defined either. Hence the breakdown of the quark picture in a non-perturbative regime and the phenomenon of confinement.

Our exclusion of $\chi(A^h) = 0$ as a genuine gauge fixing (once its invariance imposed) may not detract the above argument from its strength. Indeed if the Gribov ambiguity can be seen as an analytical problem of uniqueness in the solution of a functional equation, it does not matter that this equation cannot be interpreted as a gauge fixing. So it would remain true that a globally well defined A-dependant dressing field cannot be constructed. And the conclusion of Lavelle and McMullan, "Given that physical quarks would need to be dressed, this shows that it is not possible to construct a non-perturbative asymptotic quark field. This, we propose, is a direct proof of quark confinement",[14] would keep all its appeal.

In that respect a brief remark they make in their conclusion is worth noting: "The weak interaction poses a different problem: we observe, e.g., the W and Z bosons and yet their interaction come from a non-abelian gauge theory. The Gribov ambiguity can, however be sidestepped in theories with spontaneous symmetry breaking by fixing the gauge in the scalar (Higg) sector of the theory. [...] This enables us to construct gauge invariant physical fields in such theories [...]."[15] This should sound familiar. Taking that 'fixing the gauge' must be translated by 'constructing a dressing', we recognize our example treating the electroweak sector of the Standard Model in section 2.3.2 in chapter 2.

Final remark For completeness I would like to say a final word about the question of the transformation of the dressed quark $\psi_{phys} = h^{-1}\psi$ under a *rigid* gauge transformation U. The authors ask for ψ_{phys} to transform like the bare quarks, $\widetilde{\psi} := \psi^U = U^{-1}\psi$, a requirement necessary for their good behavior as actors of the constituent quarks model. This is the case, that is $\widetilde{\psi}_{phys} := \psi^U_{phys} = U^{-1}\psi_{phys}$, only if $\widetilde{h} := h^U = U^{-1}hU$. They claim to demonstrate this: "We know that A^h is a gauge potential, so it transforms under a rigid transformation as $A^h \to \widetilde{A^h} = U^{-1}A^hU$. We now need to see that $\widetilde{A^h}$ is in the same orbit as $\widetilde{A} = U^{-1}AU$: this may be restated as there being a field dependent $\widetilde{h} \in \mathcal{H}_{loc}$ such that $\widetilde{A}^{\widetilde{h}} = \widetilde{A^h}$. It is now simple to show that $\widetilde{h} = U^{-1}hU$ has

[13] The composite field A^h is really in bijection with the orbits O_A, like $\widehat{A} = \pi(O_A) \in \mathcal{A}/\mathcal{H}_{loc}$, so that $A^h \sim \widehat{A}$. Hence this choice of notation for the element of $\mathcal{A}/\mathcal{H}_{loc}$.

[14] Ibid, p.30.

[15] Ibid, p.58.

the required properties (recall that U is rigid)."[16]

We have to disagree on the premise that A^h is a gauge potential. We've seen that it is not. It is an invariant composite field that does not belong to $\mathcal{A}$ anymore. It is still possible of course to obtain the law $\widetilde{h} = U^{-1}hU$ under rigid transformation, but this comes out by *imposing* $\widetilde{A^h} = \widetilde{A}^{\widetilde{h}} = U^{-1}A^hU$, or $\widetilde{\psi}_{phys} = U^{-1}\psi_{phys}$. Both implying each other anyway. The result is then obtained in a less natural way than if A^h had been a gauge potential, hopefully it is still relatively consistent.[17]

[16]Ibid, p.28.

[17]We overlook the fact that the rigid transformations should be a subgroup of the gauge group, so one would expect that $\widetilde{h} = U^{-1}h$. The authors warned us that in their description rigid transformations are not to be seen as subgroup of the gauge group, see ibid p.13.

A.2 A paper by C. Lorcé

In his paper (Lorcé, 2013b), Cédric Lorcé deals with the question of the proton spin decomposition. What is the issue? He explains that experiments carried out by the European Muon Collaboration in 1988-89 showed that, contrary to what would be expected by the naive model of the *static* constituent quarks, the quarks contribute for only a fraction of the proton spin. He gives the state of the art experimental results (2005-07) on this problem: it turns out that the quarks contribute for about a third of the proton spin, the gluon spin contribution is very small and it seems that the substantial missing part is due to the Orbital Angular Momentum (OAM). He further tells us that, in the 90's, essentially two decompositions for the proton spin were proposed. The Jaffe-Monohar decomposition, on the one hand, which has a simple partonic interpretation providing contributions identified as the spin and OAM of the quarks and gluons, but which is not gauge-invariant. The Ji decomposition, on the other hand, which is gauge-invariant but has no clear partonic interpretation (as far as the gluons are concerned). Recently (2008-09) Chen & al. proposed a trick that separates the gauge potential in 'pure-gauge' and 'physical' part and provides a gauge-invariant decomposition that reduces to the Jaffe-Monohar one in a specific gauge. We refer to Lorcé's paper for all references.

The aim of the paper is to adress some of the criticisms the Chen & al. trick received, in particular regarding the question of the Lorentz covariance of the construction. Most of the paper enters the details of the controversy and the various decompositions of the proton spin and their relations, and this is beyond our interest. Nevertheless the first part of the paper, 'II. Gauge Invariance', aims at giving a geometrical interpretation to the Chen & al. trick. It turns out that this whole section is underlied by the dressing field method, as I want to now show.

The construction of Lorcé The section II of (Lorcé, 2013b) starts with a short field theoretic review of gauge theories. Lorcé uses a convention different from ours for gauge transformations, the inverse one. I noticed a slight incoherence between his convention and his equation (15). Since it would be intricate to keep track of the consistency throughout the section, I choose to restate his whole construction, step by step, in our convention but with a mix of both his notations and ours so as to facilitate comparison with the original paper. Moreover I use the language of differential forms, which will facilitate all the calculations as well as the comparison with our own approach.

Lorcé remarks that the gauge symmetry is the source of many theoretical difficulties (as we know) and that to avoid the latters one could remove the gauge freedom by gauge fixing. This is interpreted as if working with the physical degrees of freedom only. Then he says that Chen & al. proposed a *different approach*[18] whose logical starting point is the decomposition of the gauge potential A in what is called a 'pure gauge' part and a 'physical part': $A = A_{\text{phys}} + A_{\text{pure}}$ (equation (12) of the paper). The pure gauge part is defined by two statements: it transforms as a gauge potential, that is as a connection, and it does not contribute to the field strength. Explicitely, for $\gamma \in \mathcal{H}_{loc}$,

$$\begin{aligned} A^{\gamma}_{\text{pure}} &= \gamma^{-1} A_{\text{pure}} \gamma + \gamma^{-1} d\gamma, && \text{(equation (10))}, \\ F_{\text{pure}} &= dA_{\text{pure}} + A^2_{\text{pure}} = 0, && \text{(equation (11))}. \end{aligned}$$

Then, given A^{γ} and by the very definition of A^{γ}_{pure}, we have a tensorial transformation for the physical part, $A^{\gamma}_{\text{phys}} = \gamma^{-1} A_{\text{phys}} \gamma$ (equation (14)).[19] We are asked to notice that of course $F \neq dA_{\text{phys}} + A^2_{\text{phys}}$ (equation (13)).

So far the construction was a matter of definitions. Now we enter the important step. We read: "Since a physical gauge condition removes all gauge freedom, there should exist a gauge transformation such that $\widetilde{A} = \widetilde{A}_{\text{phys}}$ and therefore $\widetilde{A}_{\text{pure}} = 0$."[20] The '$\sim$' denoting the gauge transformation. Noting U_{pure} the supposed gauge transformation, he then writes A_{pure} as a pure-gauge term, $A_{\text{pure}} = U_{\text{pure}} dU^{-1}_{\text{pure}}$ (equation (15)). He then states his interpretation: "From a geometrical point a view, the Chen & al. approach amounts to assume that

[18] So the proposed construction is indeed considered as distinct from a gauge fixing, as we agree for one will recognize the dressing approach. Yet as we're about to see, we will be bound to dispute Lorcé's geometrical interpretation.

[19] Lets thus remark that this makes the subscript 'phys' quite inadequate: the 'physical' gauge potential is not gauge invariant. A footnote p.3 attests that the author is aware of this.

[20] (Lorcé, 2013b), p.3.

there exists some *privileged or 'natural' basis* in each copy of the internal space. In the following, we will denote the components of any internal tensor in this natural basis with a hat."[21] (my emphasis added). In other words, U_{pure} is seen a a gauge transformation that sends any field in a specific (privileged/natural) point in its gauge orbit. Next, he writes the equation relating ψ to its 'natural' representative in its gauge orbit, $\widehat{\psi} = U_{\text{pure}}^{-1}\psi$ (equation (16)). Then he assumes that $\widehat{\psi}$ is gauge-invariant, $\widehat{\psi}^\gamma = \widehat{\psi}$ (equation (17)) and deduces the transformation law for U_{pure}, $U_{\text{pure}}^\gamma = \gamma^{-1}U_{\text{pure}}$ (equation (18). We here recognize the transformation law of a dressing field.

We should pause for some remarks. The most evident one is that if U_{pure} is an element of the gauge group, the final transformation obtained in equation (18) is not possible. So what is the assumption that leads to (18)? It is the requirement of the gauge-invariance of $\widehat{\psi}$. By the way if $\widehat{\psi}$ is a representative in the gauge orbit of ψ it has no reason to be gauge-invariant, quite the reverse on account of the free action of $\mathcal{H}_{loc}$. If this assumption is relaxed, $\widehat{\psi}^\gamma = \gamma^{-1}\widehat{\psi}$, we have $\widehat{\psi}^\gamma := (U_{\text{pure}}^\gamma)^{-1}\psi^\gamma = (U_{\text{pure}}^\gamma)^{-1}\gamma^{-1}\psi$ on the one hand, and $\gamma^{-1}\widehat{\psi} = \gamma^{-1}U_{\text{pure}}^{-1}\psi$ on the other. So that we find $U_{\text{pure}}^\gamma = \gamma^{-1}U_{\text{pure}}\gamma$, a genuine transformation law for an element of the gauge group $\mathcal{H}_{loc}$. Remark that this latter law of transformation for a gauge element would prevent A_{pure} as defined in equation (15) to transform as a connection. The dressing law for U_{pure}, on the contrary, is precisely what is required for A_{pure} (15) to transform as a connection. We will return to this important observation latter.

Let us carry on with Lorcé's construction. He defines the 'natural variation' of ψ as the quantity $U_{\text{pure}}d\widehat{\psi} = U_{\text{pure}}d(U_{\text{pure}}^{-1}\psi) = U_{\text{pure}}dU_{\text{pure}}^{-1}\psi + d\psi = (d+A_{\text{pure}})\psi := D_{\text{pure}}\psi$, which he calls 'pure gauge covariant derivative'. Since an easy, and classic, calculation show that $D_{\text{pure}}^2\psi = F_{\text{pure}}\psi = 0$, he inteprets that "in this approach the internal space is not considered as curved."[22] Next he asserts that since F and A_{phys} are gauge-tensorial their 'natural representatives' are obtained by,

$$\begin{aligned}\widehat{F} &= U_{\text{pure}}^{-1}FU_{\text{pure}}, \qquad \text{equation (20)},\\ \widehat{A}_{\text{phys}} &= U_{\text{pure}}^{-1}A_{\text{phys}}U_{\text{pure}}. \qquad \text{equation (21)}.\end{aligned}$$

He stresses the 'welcome feature', $\widehat{F} = d\widehat{A}_{\text{phys}} + \widehat{A}_{\text{phys}}^2$ (equation (22)). Finally, the variation of the 'natural representant' of the physical part of the gauge potential is defined by (equation 23),

$$\begin{aligned}U_{\text{pure}}d\widehat{A}_{\text{phys}}U_{\text{phys}}^{-1} &= U_{\text{pure}}d\left(U_{\text{pure}}^{-1}A_{\text{phys}}U_{\text{pure}}\right)U_{\text{phys}}^{-1} = U_{\text{pure}}dU_{\text{pure}}^{-1}A_{\text{phys}} + dA_{\text{phys}} - A_{\text{phys}}dU_{\text{pure}}U_{\text{pure}}^{-1},\\ &= A_{\text{pure}}A_{\text{phys}} + dA_{\text{phys}} + A_{\text{phys}}A_{\text{pure}},\\ &= dA_{\text{phys}} + [A_{\text{pure}}, A_{\text{phys}}] =: D_{\text{pure}}A_{\text{phys}}.\end{aligned}$$

Here we used the graded commutator of forms. This he calls 'adjoint representation pure gauge covariant derivative'. Thanks to this new derivarive he relates the field strength to the physical part of the gauge potential through, $F = D_{\text{pure}}A_{\text{phys}} + A_{\text{phys}}^2$ (equation (24)).

Now, we are not interested in how precisely the author uses these objects within the issue of the decomposition of the proton spin. This is beyond our interest and beyond my expertise anyway. On the other hand I would like to show how it is possible to fully recover the presented objects by the dressing field method, plus the ansatz of Chen & al. for which I would like to propose a plausible geometrical justification.

Global reconstruction through the prism of the dressing field method Maybe enough has been said above about the geometric interpretation of the construction of Lorcé. Clearly it cannot be given the claimed meaning: U_{pure} does not belong to the gauge group $\mathcal{H}_{loc}$, thus it is not a 'natural or preferred' basis in the internal space, the 'hat' quantities are not gauge transformations of the original fields, the formers do not belong to the latters gauge orbits. U_{pure} is rather a dressing field and the 'hat' quantities are gauge-invariant

[21] Ibid, p.3. This choice of notation is another funny coincidence.

[22] Ibid, p.3. Strange statement for the curvature is related to the integrability of the horizontal distribution in the bundle, not to the 'curvature' of the fiber.

composite fields/Dirac variables, instances of equations (2.6) of chapter 2. It is possible to see the whole construction a local version of something.

Starting from the beginning, let us recall a remark made in section 1.1.2 about connections: the space of connections $\mathcal{A}_{\mathcal{P}}$ is an affine space, so that the sum of two connection is not a connection.[23] Nevertheless the sum of a connection ω and of an $(\mathfrak{h}, \mathrm{Ad})$-tensorial 1-form α is still a connection. Lets then define the connection ω' on the bundle $\mathcal{P}$ over $\mathcal{M}$ by,

$$\omega' = \omega + \alpha. \tag{A.1}$$

The action of the gauge group $\mathcal{H}$ is given by,

$$\begin{aligned} \omega'^{\gamma} &= \quad \omega^{\gamma} \quad + \quad \alpha^{\gamma}, \\ &= \gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma \quad + \quad \gamma^{-1}\alpha\gamma. \end{aligned} \tag{A.2}$$

Let us require the flatness of ω, that is $\Omega = d\omega + \omega^2 = 0$, which is the global version of Lorcé's equation (11). Then we have the curvature of ω',

$$\begin{aligned} \Omega' = d\omega' + \omega'^2 &= d\omega + d\alpha + \omega^2 + \alpha^2 + \omega\alpha + \alpha\omega, \\ &= d\alpha + [\omega, \alpha] + \alpha^2, \end{aligned} \tag{A.3}$$

where the graded commutator is used. We perceive clearly that (A.1) is the global version on $\mathcal{P}$ of the Chen & al. local ansatz on $\mathcal{M}$, equation (12) of Lorcé. We see that indeed (A.2) is the global version of equation (10) and (14). Finally (A.3) seems to be the global version of equation (24) and shows trivially that the so-called 'adjoint representation pure-gauge covariant derivative', equation (23), is just a bit of the curvature Ω'. Furthermore the trivial observation that $\Omega' \neq d\alpha + \alpha^2$ is a global restatement of equation (13).

Now there is a very important proposition known as the *Fundamental Theorem of Non-Abelian Calculus* which goes as follows (see (Sharpe, 1996) p.124 Theorem 7.14),

Theorem 1. *(Fundamental Theorem of Non-Abelian Calculus). Let H be a Lie group with Lie algebra $\mathfrak{h}$. The Maurer-Cartan form is ω_H. Let M be a smooth, connected manifold and let ω be an $\mathfrak{h}$-valued 1-form on M.*

If M is 1-connected and if ω satisfies the structure equation $d\omega + \omega^2 = 0$, then ω is the Darboux derivative of some map $\bar{u} : M \to H$. In other words we have: $\omega = \bar{u}^*\omega_H = \bar{u}d\bar{u}^{-1}$.[24] This map $\bar{u}$ is unique up to right multiplication by a constant element of H.

Applied to the case of the total space of a principal bundle, that is taking $M = \mathcal{P}$ and ω being a connection 1-form, it is not hard to adapt the proof and to show that the map $\bar{u}$ is endowed with the equivariance property $\mathcal{R}_h^*\bar{u} = h^{-1}\bar{u}$. This is our dressing field. We know from Lemma 2 of chapter 2 that the global existence of an H-valued dressing field implies the triviality of the bundle. It happens that the condition of the above theorem, 1-connectedness of $\mathcal{P}$ and flatness of ω, implies the triviality of the bundle indeed, as shown in Corollary 9.2 p.92 of (Kobayashi and Nomizu, 1963). Both theorems are then consistent with each other.

So we have a map $\bar{u} : \mathcal{P} \to H$ such that $\omega = \bar{u}^*\omega_H = \bar{u}d\bar{u}^{-1}$. The global version of the familiar form of a pure gauge potential, equation (15). Notice that the gauge transformation of this map is $\bar{u}^{\gamma} = \gamma^{-1}\bar{u}$, the lifted version of equation (18), so that indeed $\omega^{\gamma} = \gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma$, as it should. Now we can rewrite (A.1) as,

$$\omega' = \omega + \alpha = \bar{u}d\bar{u}^{-1} + \alpha. \tag{A.4}$$

Now (A.4) is definitely the global twin of equations (12) and (15), (A.3) is definitely the global version of (24), and equation (23) is just a piece of it. The covariant derivative associated to ω' is then just,

$$D\psi = d\psi + (\omega + \alpha)\psi = d\psi + (\bar{u}d\bar{u}^{-1})\psi + \alpha\psi. \tag{A.5}$$

Notice that the piece $d\psi + (\bar{u}d\bar{u}^{-1})\psi$ is the global version of the so-called 'pure-gauge covariant derivative' defined by equation (19). Appreciate how easily we recover all these results, with the same assumptions made

[23] Except for one case where the coefficients of the linear combination are $(1, 1 - \lambda)$.

[24] Contrary to Sharpe, I here choose the right instead of left convention for the Maurer-Cartan form ω_H.

by Lorcé, but lifted to the (trivial) bundle $\mathcal{P}$.

Let us now apply Lemma 1 of chapter 2 and create the gauge-invariant quantities (2.5). Starting with ω' and its curvature Ω' we have,

$$\widehat{\omega}' := \bar{u}^{-1}\omega'\bar{u} + \bar{u}^{-1}d\bar{u} = \bar{u}^{-1}(\bar{u}d\bar{u}^{-1} + \alpha)\bar{u} + \bar{u}^{-1}d\bar{u} = \bar{u}^{-1}\alpha\bar{u} =: \widehat{\alpha}, \tag{A.6}$$
$$\widehat{\Omega}' := \bar{u}^{-1}\Omega\bar{u}. \tag{A.7}$$

These are the *global invariants*, or generalized Dirac variables, that correspond to equation (21) and (20) respectively. Notice that our automatic relation $\widehat{\Omega}' = d\widehat{\omega}' + \widehat{\omega}'^2 = d\widehat{\alpha} + \widehat{\alpha}^2$ is the global counterpart of the the praised "welcome feature" of equation (22). Let us now dress ψ and its covariant derivative $D\psi$,

$$\widehat{\psi} := \bar{u}^{-1}\psi, \tag{A.8}$$
$$\widehat{D\psi} := \widehat{D}\widehat{\psi} = d\widehat{\psi} + \widehat{\omega}'\widehat{\psi} = \bar{u}^{-1}D\psi. \tag{A.9}$$

Clearly (A.8) together with its by-definition-gauge-invariance are the global matching of equations (16) and (17). Notice that the first bit of the reversed (A.9): $\bar{u}\widehat{D}\widehat{\psi} = \bar{u}d\widehat{\psi} + \bar{u}\widehat{\omega}'\psi$ would give $d\psi + \bar{u}d\bar{u}^{-1}\psi$, which is equation (19), while the second bit would give $\alpha\psi$, so that taken together they obviously give $D\psi$.

For the sake of completeness let us write locally (A.1)-(A.8) and use Lorcé's notation. Let $\sigma : \mathcal{M} \to \mathcal{P}$ be a (global) section. We then have,

$$\sigma^*\bar{u} := U_{\text{pure}} \quad \text{so that} \quad U^{\gamma}_{\text{pure}} = \gamma^{-1}U_{\text{pure}}, \qquad \text{equation (18)}, \tag{A.10}$$
$$\sigma^*\omega' := A = \sigma^*\omega + \sigma^*\alpha := A_{\text{pure}} + A_{\text{phys}} = U_{\text{pure}}dU^{-1}_{\text{pure}} + A_{\text{phys}}, \qquad \text{equations (12) and (15)}, \tag{A.11}$$
$$\sigma^*\omega'^{\gamma} := A^{\gamma} = A^{\gamma}_{\text{pure}} + A^{\gamma}_{\text{phys}} = \gamma^{-1}A_{\text{pure}}\gamma + \gamma^{-1}d\gamma \quad + \quad \gamma^{-1}A_{\text{phys}}\gamma, \quad \text{equations (10) and (14)}, \tag{A.12}$$
$$\sigma^*\Omega = F_{\text{pure}} = dA_{\text{pure}} + A^2_{\text{pure}} = 0, \qquad \text{equation (11)}, \tag{A.13}$$
$$\sigma^*\Omega' = F = dA_{\text{phys}} + [A_{\text{pure}}, A_{\text{phys}}] + A^2_{\text{phys}}, \qquad \text{equations (13), (23) and (24)}, \tag{A.14}$$
$$\sigma^*D\psi := d\psi + (U_{\text{pure}}dU^{-1}_{\text{pure}})\psi + A_{\text{phys}}\psi, \qquad \text{equation (19)}, \tag{A.15}$$
$$\sigma^*\widehat{\omega}' := \widehat{A} = \sigma^*\widehat{\alpha} := \widehat{A}_{\text{phys}} = U^{-1}_{\text{pure}}A_{\text{phys}}U_{\text{pure}}, \quad \text{equation (21)}, \tag{A.16}$$
$$\sigma^*\widehat{\Omega}' := \widehat{F} = U^{-1}_{\text{pure}}FU_{\text{pure}}, \quad \text{equation (20)}, \tag{A.17}$$
$$\sigma^*\widehat{\Omega}' = \sigma^*(d\widehat{\alpha} + \widehat{\alpha}^2) \to \widehat{F} = d\widehat{A}_{\text{phys}} + \widehat{A}^2_{\text{phys}}, \quad \text{equation (22)}, \tag{A.18}$$
$$\sigma^*\widehat{\psi} = \sigma^*(\bar{u}^{-1}\psi) \to \widehat{\psi} = U^{-1}_{\text{pure}}\psi, \quad \text{equation (16)}, \tag{A.19}$$

where I kept ψ for $\sigma^*\psi$ and γ for $\sigma^*\gamma$.

I would like to make some final comments. First, if one doesn't want to work on a trivial bundle the whole construction still applies locally on $\pi^{-1}(U) \subset \mathcal{P}$ over $U \subset \mathcal{M}$.

Second, notice that even if it is possible to find a gauge transformation $\gamma = U_{\text{pure}}$ such that $A^{\gamma} = \widehat{A} = \widehat{A}_{\text{phys}}$, this would not be a gauge-invariant equalities. We cannot confuse the dressing field with a gauge transformation, nor the invariant field $\widehat{A}$ with a representative in the orbit of A.

Third, the section III of (Lorcé, 2013b) is fully devoted to the investigation of the Lorentz covariance (real or not, manifest or not) of the construction. In our formulation such a question never arises for we use the language of differential forms and differential geometry which secures general covariance.

Finally, at the end of his section II, Lorcé comments briefly on what he calls the 'Stueckelberg gauge symmetry'. Viewing U_{pure} as an element of the gauge group $\mathcal{H}_{loc}$, he interprets that "From a geometrical point of view, the Stueckelberg symmetry corresponds to a change of natural basis without changing the actual basis used in the internal space [...]."[25] It is thus argued that this symmetry, noted U_0, does not affect ψ

[25] Ibid, p.4.

so that $\psi = U_{\text{pure}} U_0^{-1} U_0 \widehat{\psi}$. The Stueckelberg transformation law for U_{pure} is then deduced, $U^g_{\text{pure}} = U_{\text{pure}} U_0^{-1}$. This transformation for U_{pure} implies a transformation for A_{pure}, $A^g_{\text{pure}} = A_{\text{pure}} + U_{\text{pure}} U_0^{-1} dU_0 U^{-1}_{\text{pure}}$. But again, given the mentioned interpretation, this symmetry cannot affect A. Hence the transformation law for A_{phys}, $A^g_{\text{phys}} = A_{\text{phys}} - U_{\text{pure}} U_0^{-1} dU_0 U^{-1}_{\text{pure}}$. From there, the transformation of U_{pure} induces transformations for $\widehat{A}$ and $\widehat{F}$. Some remarks then elaborate on the result.

We've seen that the interpretation of U_{pure} as a 'natural basis' is not possible. So, if the trick of inserting the identity $U_0^{-1} U_0$ in between U_{pure} and $\widehat{\psi}$ in ψ is formally permissible, it has no content whatsoever. Indeed having nothing to do with a 'natural basis', there is no reason to split this insertion as two transformations, one for $\widehat{\psi}$ and one for U_{pure}. Then without this transformation for U_{pure}, there is no transformation on A_{pure}. And even if we were to admit this splitting and the ensuing transformation for A_{pure}, we have no reason to assume the reversed-sign transformation for A_{phys}. No reason other than the ad hoc requirement to have A unchanged. A requirement which is, in any case, a completely trivial formal symmetry: to add 0.

A.3 Papers by N. Boulanger

Contrary to what I did for the previous appendix, in this one I do not really analyze the papers. My aim this time is simply to show how the results described in chapter 3 make contact with these quite recent works using the BRS machinery. To do that I will briefly sum-up the aim of the papers and highlight the results which, I argue, are geometrically interpreted by the dressing field method. All along I will use slightly different notations than the author, simplified ones.

A.3.1 First paper

In his publication "A Weyl covariant tensor calculus", (Boulanger, 2005) wants to construct the space of tensors of a manifold that transform covariantly under Weyl rescaling of the metric. By covariant is meant that the transformation displays only the Weyl rescaling parameter ϵ or its first derivative $\partial\epsilon$, no higher derivatives. This space is relevant for Weyl (conformal) theories of gravity. He does so by using BRS techniques. The paper is divided in three parts.

In the first part Boulanger states the setup. He considers the metric tensor $g_{\mu\nu}$ as the only classical field of the theory[26] and takes the Weyl rescaling and the (infinitesimal) diffeomorphisms as the gauge symmetries. The BRS operator he needs is $s = s_W + s_{\text{diff}}$. But as it is often done, he actually considers the extended operator $\widetilde{s} = s + d$. The reason being that one is often (if not always) interested in integrated local functionals in the fields, so that the relevant cohomology is that of s modulo d. And it turns out that there is a bijective correspondence between the cohomology of $\widetilde{s}$ and the cohomology of $s \equiv d$.

In the second part he constructs his space of variables, that is of Weyl covariant tensors. Since one is interested in the functional in the fields and their derivatives, one usually works on the jet space of these fields, where the fields and their derivatives are local jet coordinates. For his construction Boulanger relies on a result found in (Brandt, 97), known as *contracting homotopy* (in the jet space), which is stated as follows.

Lemma 4 (Contracting homotopy)**.** *Suppose your initial space of variables is the set of local jet coordinates $J = \left\{[\varphi_i], x^\mu, dx^\mu\right\}$ where 'i' labels the various fields and the bracket [ ] their derivatives. Suppose furthermore that there is a local invertible change of jet coordinates from J to $B = (\mathcal{U}, \mathcal{V}, \mathcal{W})$ with,*

$$\widetilde{s}\,\mathcal{U} = \mathcal{V}, \qquad \textit{and} \qquad \widetilde{s}\,\mathcal{W} = \mathcal{R}(\mathcal{W}),$$

where $\mathcal{R}(\mathcal{W})$ is a functional on the space $\mathcal{W}$. Then the local jet coordinates $\mathcal{U}$ and $\mathcal{V}$, called trival pair, can be eliminated from the $\widetilde{s}$-cohomology and the latter depends only on the space $\mathcal{W}$.

Applied to his case, $\mathcal{W}$ is the space Boulanger is interested in. He then states the following,

Proposition 4. *The initial space of variables is the jet space $J = \left\{[g_{\mu\nu}], [\epsilon], [\xi^\mu], x^\mu, dx^\mu\right\}$. The shifted BRS operator is $\widetilde{s} = s_W + s_{diff} + d$, and acts on J according to,*

$$\begin{aligned}
&s_W g_{\mu\nu} = 2\epsilon g_{\mu\nu}, \quad s_W\epsilon = 0, \qquad s_W\xi^\mu = 0,\\
&s_{diff}\, g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}, \quad s_{diff}\,\epsilon = \mathcal{L}_\xi\epsilon = \xi^\lambda\partial_\lambda\epsilon, \quad s_{diff}\,\xi^\mu = \mathcal{L}_\xi\xi^\mu = \xi^\lambda\partial_\lambda\xi^\mu.
\end{aligned}$$

Then the non-trivial space $\mathcal{W}$ is,

$$\mathcal{W} = \left\{T_i, \widetilde{C}_i\right\} \qquad \textit{where,}$$

$$\left\{T_i\right\} = \left\{g_{\mu\nu},\ \mathcal{D}_{(\alpha_1}\dots\mathcal{D}_{\alpha_k}W^\rho{}_{\nu,\mu)\sigma}\right\}, \quad \textit{with } k \textit{ arbitrary,}$$

$$\left\{\widetilde{C}_i\right\} = \left\{2\epsilon,\ \widetilde{\xi}^\rho,\ \widetilde{C}^\rho_\nu, \partial_\nu\widetilde{\epsilon}\right\}, \quad \textit{with} \quad \widetilde{\xi}^\rho := \xi^\rho + dx^\rho, \quad \widetilde{C}^\rho_\nu := \partial_\nu\xi^\rho + \Gamma^\rho{}_{\lambda\nu}\widetilde{\xi}^\lambda, \quad \partial_\nu\widetilde{\epsilon} := \partial_\nu\epsilon + P_{\lambda\nu}\widetilde{\xi}^\lambda.$$

[26]This means that he assumes all other fields constructed from it, in particular the connection which is then Levi-Civita. The manifold is thus torsion free. This is why the relevant underlying geometry is the normal Cartan-Möbius geometry.

Boulanger gives explicitly the trivial pair $\mathcal{U}$ and $\mathcal{V}$, but this is not needed here. He calls the operator $\mathcal{D}$ 'Weyl covariant derivative' and of course $W^\rho{}_{\nu,\mu\sigma}$ is the Weyl tensor.

Let us take a closer look at the subspace $\left\{\widetilde{C}_i\right\}$. It is of total degree one and called 'generalized connection'. It is further decomposed according to the ghost and form degree as,

$$\left\{\widetilde{C}_i\right\} = \left\{\widehat{C}_i + \mathcal{A}_i\right\} \quad \text{with,}$$

$$\left\{\widehat{C}_i\right\} = \left\{2\epsilon,\ \xi^\rho,\ \nabla_\nu \xi^\rho := \partial_\nu \xi^\rho + \Gamma^\rho{}_{\lambda\nu}\xi^\lambda,\ \partial_\nu \epsilon + P_{\lambda\nu}\xi^\lambda\right\},$$

$$\left\{\mathcal{A}_i\right\} = \left\{\delta^\rho_\mu dx^\mu,\ \Gamma^\rho{}_{\mu\nu}dx^\mu,\ P_{\mu\nu}dx^\mu\right\}.$$

Following the terminology of (Brandt, 97) he calls the $\mathcal{A}_i$'s connection 1-forms, and the $\widehat{C}_i$'s the covariant ghost.

One cannot fail to recognize in $\left\{\mathcal{A}_i\right\}$ the matrix entries of the **dressed normal Cartan connection** of the Cartan-Möbius geometry, derived in 2.4.2 equation (2.44),

$$\varpi_0 = \begin{pmatrix} 0 & P_{\mu\nu} & 0 \\ \delta^\rho_\mu & \Gamma^\rho{}_{\mu\nu} & g^{\rho\lambda}P_{\lambda\mu} \\ 0 & g_{\mu\nu} & 0 \end{pmatrix} dx^\mu, \quad \text{with} \quad P_{\mu\nu} = \frac{-1}{(m-2)}\left(R_{\mu\nu} - \frac{R}{2(m-1)}g_{\mu\nu}\right).$$

Nor can one fail to recognize in $\left\{\widehat{C}_i\right\}$ the matrix entries of the **composite ghost** of the extended BRS algebra of the Cartan-Möbius geometry derived in 3.4.2 equation (3.71),

$$\widehat{v} := v_{\mathrm{g}}^{u_1 u_0} = \begin{pmatrix} \widehat{\epsilon} & \partial\widehat{\epsilon} + P\xi & 0 \\ \xi & \widehat{\epsilon}\delta + \nabla\xi & g^{-1}\left(\partial\widehat{\epsilon} + \xi P^T\right) \\ 0 & \xi g & -\widehat{\epsilon} \end{pmatrix} = \begin{pmatrix} \widehat{\epsilon} & \partial_\nu\widehat{\epsilon} + P_{\nu\lambda}\xi^\lambda & 0 \\ \xi^\rho & \widehat{\epsilon}\delta^\rho_\nu \ + \ \partial_\nu\xi^\rho + \Gamma^\rho{}_{\nu\lambda}\xi^\lambda & g^{\rho\alpha}\left(\partial_\alpha\widehat{\epsilon} + \xi^\lambda P_{\lambda\alpha}\right) \\ 0 & \xi^\lambda g_{\lambda\nu} & -\widehat{\epsilon} \end{pmatrix}.$$

It then appears that the non-trivial space of fields $\mathcal{W} = \left\{T_i, \widehat{C}_i, \mathcal{A}_i\right\}$ that Boulanger obtains by cohomological methods have an underlying clear geometrical origin: they come from the application of the dressing field method to the normal Cartan-Möbius geometry and its associated BRS algebra.[27]

Interestingly, we obviously see that 'generalized connection' $\left\{\widetilde{C}_i\right\}$ is symply given by $\widetilde{\varpi}_0 = \varpi_0 + \widehat{v}$ which is the dressed *algebraic connection.* This fact gives to the terminological similarity a deeper meaning.

In the third and last part of the paper Boulanger gives the transformation of the element of $\mathcal{W}$ under $\widetilde{s}$. This of course is equivalent to our algebra $BRS_{(\text{Weyl+Diff}),0}$ found in 3.4.2 (ignoring the corrective terms),

$$\widehat{s}\varpi_0 = \left(s_W + \mathcal{L}_\xi\right)\varpi_0 - dv_\xi - i_\xi\Omega_0,$$

$$\widehat{s}\Omega_0 = \left(s_W + \mathcal{L}_\xi\right)\Omega_0 + d\left(i_\xi\Omega_0\right) + \left[\varpi_0, i_\xi\Omega_0\right],$$

$$\widehat{s\widehat{v}} = \left(s_W + \mathcal{L}_\xi\right)\widehat{v} - i_{\frac{1}{2}\mathcal{L}_\xi\xi}\varpi_0 - i_\xi dv_\xi - \tfrac{1}{2}i_\xi i_\xi\Omega_0,$$

To complete the equivalence we need the action of the exterior derivative d on $\widetilde{\varpi}_0$, which is easy to write.

In his introduction the author explains that the generalized connection, that we found to be the dressed algebraic connection, plays no role in the construction of Weyl invariants. Indeed these are constructed from the subspace of tensors $\left\{T_i\right\}$. But he explains that the generalized connection is nevertheless of "prime importance" in many other issues, especially for the classification of the consistent Weyl anomalies. A question he proposes to tackle in a subsequent paper.

[27]The metric $g_{\mu\nu}$ and the Weyl tensor $W^\rho{}_{\nu,\mu\sigma}$ that Boulanger stores in the subspace T_i are respectively in the dressed normal Cartan connection ϖ_0 and its curvature Ω_0.

A.3.2 Second paper

Again there is here no detailed analysis, only an highlighting of the geometrical origin of some results of the paper. In his publication 'Algebraic Classification of Weyl Anomalies in Arbitrary Dimensions' (Boulanger, 2007a) proposes a treatment of the Weyl anomalies through descent equations à la Stora-Zumino. All consistent gauge anomalies (sometimes referred to as of 'Alder-Bardeen type') were treated through this approach. Only the Weyl anomaly resisted all attempts to fit it in this framework. Perhaps one of the main reason is that a crucial ingredient of the Stora-Zumino approach is a so-called *invariant polynomial.* Invariant meaning invariant under the gauge group. Evaluation of the polynomial on the curvature of a connection on the underlying bundle of a gauge theory is at the beginning of the descent. For the usual Yang-Mills anomalies, the polynomlial is the trace. The trace is inadequate in the case of the Weyl anomaly and no substitute has been found until now. Boulanger does not propose a candidate invariant polynomial, and makes no mention of any underlying bundle. So strictly speaking his proposition is not a Stora-Zumino approach, but displays some resemblances.

In their papers (Bonora et al., 1983) and (Bonora et al., 1986), using cohomological arguments, gave explicit expressions for the Weyl anomalies in dimensions 4 and 6. Their structure was found to consist of two kinds of terms: $\epsilon \times e(\mathcal{M})$, where $e(\mathcal{M})$ is the Euler density of the manifold, and $\epsilon\times$Weyl-invariants. Using field theoretic methods (Deser and Schwimmer, 1993) confirmed this structure for any $2n$ dimensions and called the first kind *type A* anomaly and the second kind *type B* anomaly. On the basis of field theoretic arguments still, they suggested that the type A anomaly could enjoy a "descent identity" like the chiral anomaly. But they confess "[...]we have not been able to find one", as no one else since then. In his paper Boulanger then proposes a descent for the type A anomaly, the type B being considered as trivial.

His work essentially rests on two steps. First he finds the space of field variables that he needs. Then he chooses some of these variables to construct a kind of cochain that he will submit to a descent procedure. The space of fields is obtained by the same BRS technics used in his previous papers, commented above. The operator considered is $\widetilde{s} = s_W + d$ and the space of fields $\mathcal{W}$ is then,

$$\mathcal{W} = \left\{T_i, \widetilde{C}_i\right\} \qquad \text{where,}$$

$$\left\{T_i\right\} = \left\{g_{\mu\nu},\ \mathcal{D}_{(\alpha_1}\ldots\mathcal{D}_{\alpha_k}W^{\rho}{}_{\nu,\mu)\sigma}\right\}, \quad \text{with } k \text{ arbitrary,}$$

$$\left\{\widetilde{C}_i\right\} = \left\{2\epsilon,\ dx^{\rho},\ \Gamma^{\rho}{}_{\mu\nu}dx^{\mu},\ \widetilde{\epsilon}_{\nu}\ \right\}, \quad \text{with} \quad \widetilde{\epsilon}_{\nu} := \epsilon_{\nu} + P_{\mu\nu}dx^{\mu}, \quad \text{and} \quad \epsilon_{\nu} := \partial_{\nu}\epsilon.$$

The generalized connection decomposes according to the ghost and form degree as,

$$\left\{\widetilde{C}_i\right\} = \left\{\widehat{C}_i + \mathcal{A}_i\right\} \quad \text{with,}$$

$$\left\{\widehat{C}_i\right\} = \left\{2\epsilon,\ \partial_{\nu}\epsilon\right\},$$

$$\left\{\mathcal{A}_i\right\} = \left\{\delta^{\rho}_{\mu}dx^{\mu},\ \Gamma^{\rho}{}_{\mu\nu}dx^{\mu},\ P_{\mu\nu}dx^{\mu}\right\}.$$

Again one does not fail to recognize the dressed normal Cartan connection on the one hand, and the **composite Weyl ghost** (3.35) derived in 3.3.2 on the other hand,

$$\widehat{v}_W = \begin{pmatrix} \epsilon & \partial\epsilon & 0 \\ 0 & \epsilon\delta & g^{-1}\partial\epsilon \\ 0 & 0 & -\epsilon \end{pmatrix}.$$

This shows anew that the dressing field method applied to Cartan-Möbius geometry underlies these cohomological results.

Boulanger gives a special role to the quantity $\widetilde{\epsilon}_{\nu} := \partial_{\nu}\epsilon + P_{\mu\nu}dx^{\mu}$. He decomposes the shifted BRS operator as $\widetilde{s}_W = \widetilde{s}_{\flat} + \widetilde{s}_{\natural} + \widetilde{s}_{\sharp}$, where each piece respectively decreases, does not change and raises the $\widetilde{\epsilon}_{\nu}$-degree when acting on $\mathcal{W}$. This is summarized in Table 1 of the paper. This table is of course equivalent to our algebra $BRS_{W,0}$

$$s_W\varpi_0 = -D_0\widehat{v}_W, \qquad s_W\Omega_0 = \left[\Omega_0, \widehat{v}_W\right], \qquad \text{and} \qquad s_W\widehat{v}_W = -\widehat{v}_W^2.$$

From $\widetilde{\epsilon}_\nu$, dx^μ and the Weyl tensor, Boulanger then constructs the cochains $\Phi_r^{[n-r]}$, where $n = 2m$, $r \in [0,m]$ and n =dim$\mathcal{M}$. These cochains are then submitted to the action of the operators $\widetilde{s}_\flat$, $\widetilde{s}_\natural$ and $\widetilde{s}_\sharp$ and shown to satisfy the 'exotic' descent equations,

$$\widetilde{s}_\flat\Phi_r^{[n-r]} + \widetilde{s}_\natural\Phi_{r-1}^{[n-r+1]} = 0, \qquad \widetilde{s}_\sharp\Phi_r^{[n-r]} = 0, \qquad \text{and} \qquad \widetilde{s}_\flat\Phi_1^{[n-1]} = 0 = \widetilde{s}_W\Phi_0^{[n]}.$$

He defines,

$$\alpha := \sum_{r=1}^{m}\Phi_r^{[n-r]}, \qquad \beta := \Phi_0^{[n]}.$$

He then states that the top form degree a_1^n of α satisfies the Wess-Zumino consistency condition for the Weyl anomaly and is part of a non-trivial descent. The anomaly β, which is of B type, is seen to have a trivial descent. It is then proved that the top form degree of $\alpha + \beta$ is the type A anomaly, $\epsilon \times e(\mathcal{M})$.

We refer to the original paper for the precise definitions and to (Boulanger, 2007b) for the detailed proofs. This scheme gives indeed a purely algebraic way to find the type A anomaly in arbitrary dimensions and displays some resemblance with the descent equations à la Stora-Zumino. Nevertheless the differences with the canonical Stora-Zumino approach are worth stressing.

First, the split of $\widetilde{s}_W$ according to the $\widetilde{\epsilon}_\nu$-degree is quite esoteric. Usually the grading coming from the original ghost is enough to provide a descent containing the consistent anomaly. The deep reason for the privileged role played by the quantity $\widetilde{\epsilon}_\nu$, if any, should be better understood. Secondly there is no mention of an underlying geometry, a principal bundle associated to a Weyl-gauge theory. The last remark concerns the mysterious origin of the proposed cochains Φ. Usually the cochains are constructed from an invariant polynomial. At the beginning of the descent is the homogeneous polynomial evaluated on the curvature 2-form of the bundle. In the approach of Boulanger this is not so.

These reasons may be enough to convince oneself that Boulanger's work, if encouraging, may not be the final word on the matter. The fact that we showed here that his cohomological results can be given a clear geometrical meaning could give rise to a renewed hope for a genuine Stora-Zumino approach to the Weyl anomaly, perhaps giving both type A and B at once. This ultimately rests on the discovery of the right invariant polynomial. See the last chapter of this essay for our own proposal and preliminary analysis.

Appendix B
A minimal dressing field for General Relativity

The question boils down to the decomposition of the GL-valued dressing field e as a product of two matrices, one of which being a SO matrix. Lets consider $e = e^a{}_\mu$ as a set of m vectors $\mathbf{v} = \{(e_\mu)\}_{\mu=(1,\cdots,m)}$, with components $(e^a)_\mu$, forming a basis for $\mathbb{R}^m$. The problem is to go from the basis $\mathbf{v}$ to a pseudo-orthonormal basis $\mathbf{w}$.

Were we doing euclidean gravity, we could have used either of the two most known methods which are the Gram-Schmidt and the Schweinler-Wigner procedures.[1] Each would have provided us a decomposition $\mathbf{v} = \mathbf{w}_{1/2}S_{1/2}$ where the vectors of $\mathbf{w}_{1/2}$ are the columns of an $SO(m)$-valued matrix $\widetilde{e}_{1/2}$ which is thus our minimal dressing field. The Gram-Schmidt basis $\mathbf{w}_1$ and the Schweinler-Wigner basis $\mathbf{w}_2$ are related by an $SO(m)$-matrix (whose precise definition can be found in (Chaturvedi et al., 1998)), but this is gauge and is bound to be neutralized anyway. The matrix $S_{1/2}$ is the residual field since it is easily seen that $S^T_{1/2}\mathrm{id}\, S_{1/2} = e^T\mathrm{id}\, e = g$. The basis $\mathbf{w}_2$ has the property of making extremal a quartic functional on the manifold of orthonormal basis of $\mathbb{R}^m$. See (Schweinler and Wigner, 1970). One could see this Schweinler-Wigner functional as a ‘potential’ for the auxliary field $\theta \sim e \sim \mathbf{v}$, whose ground state selects the minimal dressing associated to $\mathbf{w}_2$.

Gram-Schmidt doesn’t work for pseudo-euclidean signature, hopefully the Schweinler-Wigner procedure can be adapted to such a case and goes as follows.

Given the bilinear form, $(\, , \,)$, associated to the pseudo-euclidean metric η of signature (r, s), form the Gram matrix of the basis $\mathbf{v} = \{(e_\mu)\}_{\mu=(1,\cdots,m)}$, $\mathbf{G}_{\mu\nu} = (e_\mu, e_\nu)$. Or in index free notation, $\mathbf{G} = \mathbf{v}^T\eta\mathbf{v}$. In our case, this Gram matrix is nothing but the metric, $g = e^T\eta e$, by definition. So $\mathbf{G}$ is symmetric, as such it is diagonalized by an orthogonal matrix $R \in SO(m)$, $R^TR = \mathrm{id}_m$. We have then $\mathbf{G} = RDR^{-1}$, where $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_m)$ is the diagonal matrix of eigenvalues of $\mathbf{G} = g$, which are all nonvanishing since g is invertible, and the columns of R are the corresponding eigenvectors. By an adequate choice of R (that is a choice of order for its columns) it is possible to arrange the eigenvalues so as to place the r positive ones first and then the remaining s negative. If we now define $\mathbf{w} = \mathbf{v}R|D|^{-\frac{1}{2}}$ we easily check that $\mathbf{w}$ is indeed the wanted pseudo-orthogonal basis,

$$\mathbf{w}^T\eta\,\mathbf{w} = |D|^{-\frac{1}{2}}R^{-1}\mathbf{v}^T\eta\,\mathbf{v}R|D|^{-\frac{1}{2}} = |D|^{-\frac{1}{2}}R^{-1}\cdot G\cdot R|D|^{-\frac{1}{2}} = |D|^{-\frac{1}{2}}R^{-1}\cdot RDR^{-1}\cdot R|D|^{-\frac{1}{2}} = |D|^{-\frac{1}{2}}D|D|^{-\frac{1}{2}} = \eta.$$

This basis also displays the property of making extremal the Schweinler-Wigner quartic functional on a certain compact submanifold of the the manifold of pseudo-orthonormal bases of $\mathbb{R}^m$ (that is the non-compact group manifold $SO(r, s)$). This functional could then again be seen as a‘potential’ for the auxiliary field $\theta \sim e \sim \mathbf{v}$, whose ground state selects the minimal dressing associated to $\mathbf{w}$. I refer to (Chaturvedi et al., 1998) and (Simon et al., 1999) for details.

Finally we have $\mathbf{v} = \mathbf{w}|D|^{\frac{1}{2}}R^{-1} := \mathbf{w}S$. Or, coming back from vector set notation to matrix notation, $e = \widetilde{e}S$, where the columns of the $SO(r, s)$-matrix $\widetilde{e}$ are the vectors of the basis $\mathbf{w}$. Thus we’ve found our minimal SO-valued dressing field $\widetilde{e}$, and we identify S as the residual field. Indeed we see explicitly that S carries $n - n(n-1)/2 = n(n+1)/2$ degrees of freedom exactly as g, and again it is easily seen that $S^T\eta S = g$.

[1] Actually Schweinler and Wigner did not introduce the basis which bears their name and never claimed to do so, quite the contrary. Their result in (Schweinler and Wigner, 1970) was to show that the basis extremizes a functional on the manifold of orthonormal basis of a finite dimensional Hilbert space.

Moreover, being constructed out of D and R, that is $\mathbf{G} = g$ which is invariant, S is an SO-invariant field. Writing the minimal dressing $\widetilde{u}$ in matrix form, we find the dressed Cartan connection,

$$\widehat{\widetilde{\varpi}} = \widetilde{u}^{-1}\varpi\widetilde{u} + \widetilde{u}^{-1}d\widetilde{u} = \begin{pmatrix} \widetilde{e}^{-1}A\widetilde{e} + \widetilde{e}^{-1}d\widetilde{e} & \widetilde{e}^{-1}\theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{\Gamma} & Sdx \\ 0 & 0 \end{pmatrix},$$

where $\widetilde{\Gamma} = \widetilde{\Gamma^a}_b$ is an SO-valued SO-gauge invariant field, and we see the invariant residual field written as the 1-form $Sdx = S^a{}_\mu dx^\mu$. The associated curvature is,

$$\widehat{\widetilde{\Omega}} = \widetilde{u}^{-1}\Omega\widetilde{u} = \begin{pmatrix} \widetilde{e}^{-1}R\widetilde{e} & \widetilde{e}^{-1}\Theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widehat{\widetilde{R}} & \widetilde{T} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d\widetilde{\Gamma} + \widetilde{\Gamma} \wedge \widetilde{\Gamma} & d(Sdx) + \widetilde{\Gamma} \wedge Sdx \\ 0 & 0 \end{pmatrix}.$$

It is easily verified that the change of variables in the Lagrangian gives $L_{pal}(A,\theta) \to L'_{pal}(\widetilde{\Gamma}, S)$ with,

$$L'_{pal}(\widetilde{\Gamma}, S) = \frac{-1}{32\pi G}\mathrm{Tr}\Big(\widehat{\widetilde{R}} \wedge *(Sdx \wedge S^t dx)\Big).$$

From L'_{Pal} one would extract Einstein equations that are *SO-invariant*, and not *SO-covariant* as those found from L_{Pal}. Einstein equations found from both L'_{Pal} and L_{EH} are nevertheless GL-covariant, as they should, since the GL-symmetry associated to the natural geometry of $\mathcal{M}$ is the only non-trivial symmetry left after the gauge symmetry reduction achieved through the dressing field method.

Now there's two problems with this construction. The first is the arbitrariness in the ordering of r positive and s negative eigenvalues in D. The whole construction is only well defined up to a permutation matrix $P_{r,s} \in \mathcal{S}_r \times \mathcal{S}_s \subset \mathcal{S}_m$ which acts as, $R \to RP_{r,s}$, $D \to P_{r,s}DP_{r,s}$, $s \to P_{r,s}S$ and $\widetilde{e} \to \widetilde{e}P_{r,s}$. But one choice can be propagated from one open $U \subset \mathcal{M}$ to the other, for a continuous map from a compact domain to a discrete codomain is constant. The major shortcoming of this construction (and of the euclidean counterpart) is its obviously bad behavior under coordinate changes which allows no operative transformation law for the fields $\widetilde{\Gamma}$ and S. So if searching a minimal dressing in this case was a nice exercise, it is more tractable to work with the usual non-minimal dressing e.

Appendix C
Some Detailed Calculations

C.1 Calculation of the entries of Ω_0

The curvature of ϖ_0 is

$$\Omega_0 := \Omega_1^{u_0} = u_0^{-1}\Omega_1 u_0 = \begin{pmatrix} f_1 & C & 0 \\ T & W & C^t \\ 0 & T^t & -f_1 \end{pmatrix} := \begin{pmatrix} f_1 & \Pi_1 e & 0 \\ e^{-1}\Theta & e^{-1}F_1 e & e^{-1}\Pi^t \\ 0 & \Theta^t e & -f_1 \end{pmatrix}.$$

Using (2.37), which we recall to be

$$\Omega_1 = d\varpi_1 + \varpi_1 \wedge \varpi_1 = \begin{pmatrix} \alpha_1\theta & d\alpha_1 + \alpha_1 A_1 & 0 \\ d\theta + A_1\theta & dA_1 + A_1^2 + \theta\alpha_1 + \alpha_1^t\theta^t & d\alpha_1^t + A_1\alpha_1^t \\ 0 & d\theta^t + \theta^t A_1 & \theta^t\alpha_1^t \end{pmatrix} = \begin{pmatrix} f_1 & \Pi_1 & 0 \\ \Theta & F_1 & \Pi_1^t \\ 0 & \Theta^t & -f_1 \end{pmatrix},$$

we can perform the explicit calculation for each entry. Entry (1, 1) gives,

$$f_1 = \alpha_1 \wedge \theta = \alpha_1 \wedge e \cdot dx = P \wedge dx, \quad \text{in components} \quad \frac{1}{2}(f_1)_{\mu\sigma}\, dx^\mu \wedge dx^\sigma = \frac{1}{2}P_{[\mu\sigma]}dx^\mu \wedge dx^\sigma. \tag{C.1}$$

In the same way, the entry (3, 3) gives,

$$\begin{aligned} -f_1 &= \theta^t \wedge \alpha_1^t = \theta^T\eta \wedge \eta^{-1}\alpha_1^T = dx^T \cdot e^T\eta \wedge \eta^{-1}\alpha_1^T, \\ &= dx^T \wedge (\alpha_1 e)^T = dx^T \wedge P^T, \quad \text{in components} \quad -\frac{1}{2}(f_1)_{\mu\sigma}\, dx^\mu \wedge dx^\sigma = \frac{1}{2}P_{[\mu\sigma]}dx^\sigma \wedge dx^\mu. \end{aligned} \tag{C.2}$$

Entry (2, 1) gives,

$$\begin{aligned} T &= e^{-1}\Theta = e^{-1}\Big(d\theta + A_1 \wedge \theta\Big) = e^{-1}d\theta + e^{-1}A_1 \wedge \theta, \\ &= e^{-1}d(e \cdot dx) + e^{-1}A_1 \wedge e \cdot dx = e^{-1}de \wedge dx + e^{-1}A_1 e \wedge dx, = \Big(e^{-1}A_1 e + e^{-1}de\Big) \wedge dx, \\ T &= \Gamma \wedge dx, \quad \text{in components} \quad \frac{1}{2}T^\rho{}_{\mu\sigma}\, dx^\mu \wedge dx^\sigma = \frac{1}{2}\Gamma^\rho{}_{[\mu\sigma]}dx^\mu \wedge dx^\sigma. \end{aligned} \tag{C.3}$$

The entry (3, 2) is,

$$\begin{aligned} T^t &= \Theta^t e = \Big(d\theta^t + \theta^t \wedge A_1\Big)e = d\Big(\theta^t e\Big) - \theta^t \wedge de + \theta^t \wedge A_1 e, \\ &= d\Big(\theta^T\eta e\Big) + \theta^t e \wedge e^{-1}de + \theta^t e \wedge e^{-1}A_1 e = d\Big(dx^T \cdot e^T\eta e\Big) + \theta^t e \wedge \Big(e^{-1}A_1 e + e^{-1}de\Big), \\ &= -dx^T \wedge dg + dx^T \cdot e^T\eta e \wedge \Gamma = -dx^T \wedge \Big(dg - g\Gamma\Big). \end{aligned}$$

Now due to $\nabla g = dg - \Gamma^T g - g\Gamma = 0$, we have

$$T^t = -dx^T \wedge \Big(\nabla g + \Gamma^T g\Big), \quad \text{in components} \quad \frac{1}{2}T_{\nu,\mu\sigma}\, dx^\mu \wedge dx^\sigma = \frac{1}{2}\Big(\nabla_{[\mu}g_{\sigma]\nu} + \Gamma_{[\mu\sigma]}{}^\lambda g_{\lambda\nu}\Big)dx^\mu \wedge dx^\sigma. \tag{C.4}$$

The entry (1, 2) is,

$$
\begin{aligned}
C &= \Pi_1 e = \Big(d\alpha_1 + \alpha_1 \wedge A_1\Big)e = d(\alpha_1 e) + \alpha_1 \wedge de + \alpha_1 \wedge A_1 e,\\
&= d(\alpha_1 e) + \alpha_1 e \wedge e^{-1}de + \alpha_1 e \wedge e^{-1}A_1 e = d(\alpha_1 e) + \alpha_1 e \wedge \Big(e^{-1}A_1 e + e^{-1}de\Big),\\
C &= dP + P\wedge\Gamma, \quad \text{in components} \quad \frac{1}{2}C_{\nu,\mu\sigma}\, dx^\mu \wedge dx^\sigma = \frac{1}{2}\Big(\partial_{[\mu}P_{\sigma]\nu} + P_{[\mu\lambda}\Gamma^\lambda{}_{\sigma]\nu}\Big)\, dx^\mu\wedge dx^\sigma.
\end{aligned}
\tag{C.5}
$$

The entry (2, 3) is,

$$
\begin{aligned}
C^t &= e^{-1}\Pi_1^t = e^{-1}\Big(d\alpha_1^t + A_1\wedge\alpha_1^t\Big) = d(e^{-1}\alpha_1^t) - de^{-1}\wedge\alpha_1^t + e^{-1}A_1 e\wedge e^{-1}\alpha_1^t,\\
&= d(e^{-1}\alpha_1^t) - de^{-1}e^{-1}\wedge e^{-1}\alpha_1^t + e^{-1}A_1 e\wedge e^{-1}\alpha_1^t,\\
&= d(e^{-1}\alpha_1^t) + \Big(e^{-1}A_1 e + e^{-1}de\Big)\wedge e^{-1}\alpha_1^t,\\
C^t &= d(g^{-1}P^T) + \Gamma\wedge g^{-1}P^T = dg^{-1}\wedge P^T + g^{-1}dP^T + \Gamma g^{-1}\wedge P^T,\\
&= \Big(dg^{-1} + \Gamma g^{-1}\Big)\wedge P^T + g^{-1}dP^T,\\
&= \Big(\nabla g^{-1} - g^{-1}\Gamma^T\Big)\wedge P^T + g^{-1}dP^T, \quad \text{due to} \quad \nabla g^{-1} = dg^{-1} + g^{-1}\Gamma^T + \Gamma g^{-1} = 0,\\
&= \nabla g^{-1}\wedge P^T + g^{-1}\Big(dP^T + \Gamma^T\wedge P^T\Big),\\
C^t &= \nabla g^{-1}\wedge P^T + g^{-1}C^T, \text{ in components } \frac{1}{2}C^\rho{}_{\mu\sigma}\, dx^\mu\wedge dx^\sigma = \frac{1}{2}\Big(\nabla_{[\mu}g^{\rho\lambda}P_{\lambda\sigma]} + g^{\rho\lambda}C_{\lambda,\mu\sigma}\Big)dx^\mu\wedge dx^\sigma.
\end{aligned}
\tag{C.6}
$$

At last, the entry (2, 2) is,

$$
\begin{aligned}
W &= e^{-1}F_e = e^{-1}\Big(dA_1 + A_1\wedge A_1 + \theta\wedge\alpha_1 + \alpha_1\wedge\theta^t\Big)e,\\
&= d\Big(e^{-1}A_1 e + e^{-1}de\Big) - de^{-1}\wedge A_1 e + e^{-1}A_1\wedge de - de^{-1}\wedge de + e^{-1}A_1 e\wedge e^{-1}A_1 e + e^{-1}\theta\wedge\alpha_1 e + e^{-1}\alpha_1^t\wedge\theta^t e,\\
&= d\Big(e^{-1}A_1 e + e^{-1}de\Big) + e^{-1}de\wedge e^{-1}A_1 e + e^{-1}A_1 e\wedge e^{-1}de + e^{-1}de\wedge e^{-1}de + e^{-1}A_1 e\wedge e^{-1}A_1 e\\
&\qquad + e^{-1}\theta\wedge\alpha_1 e + e^{-1}\alpha_1^t\wedge\theta^t e,\\
&= d\Big(e^{-1}A_1 e + e^{-1}de\Big) + \Big(e^{-1}A_1 e + e^{-1}de\Big)^2 + e^{-1}\theta\wedge\alpha_1 e + e^{-1}\alpha_1^t\wedge\theta^t e,\\
&= d\Gamma + \Gamma\wedge\Gamma + e^{-1}e\cdot dx\wedge P + e^{-1}\eta^{-1}\alpha_1^T\wedge\theta^T\eta e,\\
&= d\Gamma + \Gamma\wedge\Gamma + \delta\cdot dx\wedge P + g^{-1}e^T\alpha_1^T\wedge dx^T\cdot e^T\eta e, \quad \text{due to} \quad g^{-1} = e^{-1}\eta^{-1}(g^{-1})^T,\\
&= d\Gamma + \Gamma\wedge\Gamma + \delta\cdot dx\wedge P + g^{-1}(\alpha_1 e)^T\wedge dx^T\cdot g, \quad \text{due to} \quad g = e^T\eta e.
\end{aligned}
$$

Finally we have,

$$
W = R + \delta\cdot dx\wedge P + g^{-1}P^T\wedge dx^T\cdot g, \quad \text{where } R = d\Gamma + \Gamma\wedge\Gamma \quad \text{is the Riemann curvature,} \tag{C.7}
$$

$$
\text{in components} \quad \frac{1}{2}W^\rho{}_{\nu,\mu\sigma}\, dx^\mu\wedge dx^\sigma = \frac{1}{2}\Big(R^\rho{}_{\nu,\mu\sigma} + \delta^\rho_{[\mu}P_{\sigma]\nu} + g^{\rho\lambda}P_{\lambda[\mu}g_{\sigma]\nu}\Big)dx^\mu\wedge dx^\sigma. \tag{C.8}
$$

These justify the matrix form (2.42) and (2.43) for the final composite field Ω_0 in section 2.4.2.

The **normal** case is easily obtained for it means,

$$
\begin{aligned}
&f_1 = 0 \quad \to \quad P_{\mu\sigma} = P_{\sigma\mu}\\
&T = 0 \quad \to \quad \Gamma^\rho{}_{\mu\sigma} = \Gamma^\rho{}_{\sigma\mu} \quad \text{so, since } \nabla g = 0, \quad \Gamma \text{ is the } \textbf{Levi-civita connection}.\\
&\mathrm{Ric}(W) = W^\lambda{}_{\nu,\lambda\sigma} = 0 \quad \to \quad P_{\mu\sigma} = \frac{-1}{(m-2)}\Big(R_{\mu\sigma} - \frac{R}{2(m-1)}g_{\mu\sigma}\Big) \;\text{ is the } \textbf{Schouten tensor}.
\end{aligned}
$$

The latter line implies that $W^\rho{}_{\nu,\mu\sigma}$ is the **Weyl tensor**, and $C_{\nu,\mu\sigma} = \nabla_\mu P_{\sigma\nu}$ is the **Cotton tensor**.
To sum-up, in the normal case we have,

$$\varpi_0 = \begin{pmatrix} 0 & P_{\mu\nu} & 0 \\ \delta^\rho_\mu & \Gamma^\rho{}_{\mu\nu} & g^{\rho\lambda}P_{\lambda\mu} \\ 0 & g_{\mu\nu} & 0 \end{pmatrix} dx^\mu, \quad \Omega_0 = \frac{1}{2}\begin{pmatrix} 0 & C_{\nu,\mu\sigma} & 0 \\ 0 & W^\rho{}_{\nu,\mu\sigma} & g^{\rho\lambda}C_{\lambda,\mu\sigma} \\ 0 & 0 & 0 \end{pmatrix} dx^\mu \wedge dx^\sigma. \tag{C.9}$$

All this is fully equivalent to Propositions 26, 27 and equation (24) p.221-223 in (Ogiue, 1967), and (C.9) is the so-called **Riemannian parameterization of the normal conformal Cartan connection**.

C.2 The remaining symmetries of the final composite fields

C.2.1 Coordinate changes for ϖ_0 and Ω_0

By definition $\varpi_0 := \varpi_1^{u_0} = (\varpi^{u_1})^{u_0}$. We can ask the question of the transformation of this composite field under coordinate change. We first observe that ϖ being a 1-form, it is invariant under coordinate changes. As for the first dressing field $u_1 \sim q$ it is invariant too. Indeed q is the solution of the constrain $a - q\theta = 0$, which is an equation for 1-forms, thus invariant. Explicitely, given the coordinate change $dy^\mu = \frac{\partial y^\mu}{\partial x^\nu} dx^\nu = G^\mu{}_\nu dx^\nu$, we have $a = a_\nu dx^\nu = a'_\mu dy^\mu$ so that $a'_\mu = a_\nu (G^{-1})^\nu{}_\mu$, and $\theta^a = e^a{}_\nu dx^\nu = e'^a{}_\mu dy^\mu$ so that $e'^a{}_\mu = e^a{}_\nu (G^{-1})^\nu{}_\mu$. Finally, $q'_a := a'_\mu \cdot (e'^{-1})^\mu{}_a = a_\nu (G^{-1})^\nu{}_\mu \cdot G^\mu{}_\rho (e^{-1})^\rho{}_a = a_\nu \cdot (e^{-1})^\nu{}_a = q_a$. Hence the invariance of u_1 under coordinate changes.

Thus if any susceptibility to coordinate change is to be found it is then in the second dressing field $u_0 \sim e$. From above we have $e'^a{}_\mu = e^a{}_\nu (G^{-1})^\nu{}_\mu$, or in index free notation $e' = eG^{-1}$, so that

$$u'_0 = u_0 \cdot G^{-1} \quad \text{in matrix form} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & e' & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & G^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The dressing u_0 is associated to the coordinate chart $\{x^\alpha\}$, while u'_0 is associated to the coordinate chart $\{y^\mu\}$. Had we used the latter to dress ϖ_1 we would have had,

$$\begin{aligned} \varpi'_0 &= u_0'^{-1} \cdot \varpi_1 \cdot u'_0 + u_0'^{-1} du'_0 = G u_0^{-1} \cdot \varpi_1 \cdot u_0 G^{-1} + G \cdot u_0 du_0 \cdot G^{-1} + G dG^{-1}, \\ \varpi'_0 &= G \cdot \varpi_0 G^{-1} + G dG^{-1}. \end{aligned} \tag{C.10}$$

The latter equation has the matrix form,

$$\begin{pmatrix} 0 & P' & 0 \\ dy & \Gamma' & g'^{-1}P'^T \\ 0 & dy^T \cdot g' & 0 \end{pmatrix} = \begin{pmatrix} 0 & PG^{-1} & 0 \\ Gdx & G\Gamma G^{-1} + GdG^{-1} & Gg^{-1}P^T \\ 0 & dx^T \cdot gG^{-1} & 0 \end{pmatrix}. \tag{C.11}$$

Entry (2, 1) of this matrix equation just expresses the coordinate change in an index free way. If we use this in entry (3, 2) we have,

$$dy^T \cdot g' = dx^T \cdot gG^{-1} = dy^T (G^{-1})^T \cdot gG^{-1} \quad \rightarrow \quad g' = (G^{-1})^T g G^{-1}, \quad \text{with indices} \quad g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu} g_{\alpha\delta} \frac{\partial x^\delta}{\partial y^\nu}.$$

Doing the same in entry (1, 2),

$$P' := P'dy = PG^{-1} := PdxG^{-1} = PG^{-1}dyG^{-1} \quad \rightarrow \quad P' = PG^{-1}G^{-1}, \quad \text{with indices} \quad P'_{\mu\nu} = P_{\alpha\delta} \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\delta}{\partial y^\nu}.$$

Entry (2, 3) is redundant with entry entries (3, 2) and (1, 2),

$$\begin{gathered} g'^{-1}P'^T = Gg^{-1}P^T \rightarrow g'^{-1}(P'dy)^T = Gg^{-1}(Pdx)^T \rightarrow g'^{-1}dy^T P'^T = Gg^{-1}dx^T P^T = Gg^{-1}dy^T (G^{-1})^T P^T, \\ \text{from which we have} \quad g'^{-1} = Gg^{-1}G^T \quad \text{and} \quad P'^T = (G^{-1})^T (G^{-1})^T P^T. \end{gathered}$$

At last, the entry (2, 2) reads,

$$\Gamma' := \Gamma' dy = G\Gamma G^{-1} + GdG^{-1} := G\Gamma dx G^{-1} + GdG^{-1} = G\Gamma G^{-1} dy G^{-1} + Gdy\, \partial G^{-1},$$
$$\to \qquad \Gamma' = G\Gamma G^{-1}G^{-1} + G\partial G^{-1}, \quad \text{with indices} \quad {\Gamma'^{\rho}}_{\mu\nu} = \frac{\partial y^\rho}{\partial x^\gamma} {\Gamma^\gamma}_{\alpha\delta} \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\delta}{\partial y^\nu} + \frac{\partial y^\rho}{\partial x^\gamma}\frac{\partial}{\partial y^\mu}\frac{\partial x^\gamma}{\partial y^\nu}.$$

These are the coordinate transformations of the symbols of a linear connection . Thus Γ is the linear connection of the base manifold $\mathcal{M}$.

In the same way exactly, for the dressed Ω_1 we have,

$$\begin{aligned} \Omega_0' &= {u_0'}^{-1}\Omega_1 u_0' = G u_0^{-1}\Omega_1 u_0 G^{-1}, \\ \Omega_0' &= G\Omega_0 G^{-1}. \end{aligned} \tag{C.12}$$

The latter equation has the matrix form,

$$\begin{pmatrix} f_1' & C' & 0 \\ T' & W' & C'^t \\ 0 & T'^t & -f_1' \end{pmatrix} = \begin{pmatrix} f_1 & CG^{-1} & 0 \\ GT & GWG^{-1} & GC^t \\ 0 & T^tG^{-1} & -f_1 \end{pmatrix}, \tag{C.13}$$

which expresses the coordinate changes of Ω_0 in an index free notation. The composite field Ω_0' is associated to the chart $\{y^\mu\}$, while Ω_0 is associated to the chart $\{x^\alpha\}$. Let us write each entry explicitely. The entry (1, 1) is,

$$f_1' := f_1'\, dy\wedge dy = f_1 := f_1\, dx\wedge dx = f_1\, G^{-1}dy \wedge G^{-1}dy,$$
$$\to \qquad f_1' = f_1 G^{-1}G^{-1} \quad \text{with indices} \quad (f_1')_{\mu\sigma} = (f_1)_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\sigma}.$$

The same for entry (3, 3). The entry (2, 1) is,

$$T' := T'\, dy\wedge dy = GT := GT\, dx\wedge dx = GTG^{-1}dy \wedge G^{-1}dy,$$
$$\to \qquad T' = GTG^{-1}G^{-1} \quad \text{with indices} \quad {T'^\rho}_{\mu\sigma} = \frac{\partial y^\rho}{\partial x^\gamma}{T^\gamma}_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\sigma}.$$

The entry (3, 2) is,

$$T'^t := T'^t\, dy\wedge dy = T^tG^{-1} := T^tG^{-1}\, dx\wedge dx = T^tG^{-1}\, G^{-1}dy \wedge G^{-1}dy,$$
$$\to \qquad T'^t = T^tG^{-1}G^{-1}G^{-1} \quad \text{with indices} \quad T'_{\nu,\mu\sigma} = T_{\delta,\alpha\beta}\frac{\partial x^\delta}{\partial y^\nu}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\sigma}.$$

An alternative calculation taking into account the fact that $T'^t = T'^T g'$ is possible, but it is redundant with the known transformation for T and g. It gives, $T'^T \cdot g' = (G^{-1})^T(G^{-1})^T T^T G^T \cdot (G^{-1})^T g G^{-1}$.
The entry (1, 2) is,

$$C' := C'\, dy\wedge dy = CG^{-1} := CG^{-1}\, dx\wedge dx = CG^{-1}\, G^{-1}dy \wedge G^{-1}dy,$$
$$\to \qquad C' = CG^{-1}G^{-1}G^{-1} \quad \text{with indices} \quad C'_{\nu,\mu\sigma} = C_{\delta,\alpha\beta}\frac{\partial x^\delta}{\partial y^\nu}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\sigma}.$$

The entry (2, 3) is,

$$C'^t := C'^t\, dy\wedge dy = GC^t := GC^t\, dx\wedge dx = GC^t\, G^{-1}dy \wedge G^{-1}dy,$$
$$\to \qquad C'^t = GC^tG^{-1}G^{-1} \quad \text{with indices} \quad {C'^\rho}_{\mu\sigma} = \frac{\partial y^\rho}{\partial x^\gamma}{C^\gamma}_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\sigma}.$$

An alternative calculation taking into account the fact that $C'^t = g'^{-1}C'^T$ is possible, but it is redundant with the known transformation for g^{-1} and C. It gives, $g'^{-1} \cdot C'^T = Gg^{-1}G^T \cdot (G^{-1})^T(G^{-1})^T(G^{-1})^T C^T$.
At last, the entry (2, 2) is,

$$W' := W'\, dy \wedge dy = GWG^{-1} := GWG^{-1}\, dx \wedge dx = GWG^{-1}\, G^{-1}dy \wedge G^{-1}dy,$$
$$\rightarrow \quad W' = GWG^{-1}G^{-1}G^{-1} \quad \text{with indices} \quad {W'^{\rho}}_{\nu,\mu\sigma} = \frac{\partial y^{\rho}}{\partial x^{\gamma}} {W^{\gamma}}_{\delta,\alpha\beta} \frac{\partial x^{\delta}}{\partial y^{\nu}} \frac{\partial x^{\alpha}}{\partial y^{\mu}} \frac{\partial x^{\beta}}{\partial y^{\sigma}}.$$

Remark: Since ϖ_0 and Ω_0 are not invariant under coordinates changes, they are not strictly speaking differential forms anymore. This is an instance of the warning mentionned in paragraph 'What this construction is...' at the end of section 2.2.1. The difference here is that ϖ_0 and Ω_0 do not exactly belong to the natural geometry of the base manifold $\mathcal{M}$ yet. Indeed they belong to the principal bundle $\mathcal{P}_W := \mathcal{P}(\mathcal{M}, W)$ whose structure group is the Weyl group $W = \left\{z \in \mathbb{R}^*\right\}$, so that they both present a residual Weyl rescaling gauge symmetry.

C.2.2 Residual Weyl gauge symmetry of ϖ_0 and Ω_0

Here I detail the calculations mentionned in the final subsection of 2.4.2, and lead to the residual gauge transformations of the final composite fields ϖ_0 and Ω_0 living on the Weyl bundle $\mathcal{P}_W(\mathcal{M}, W)$.

Calculation of ϖ_1^W and ϖ_0^W

Entries of ϖ_1^W : We recall that we have,

$$\varpi_1^W = \begin{pmatrix} a_1^W & \alpha_1^W & 0 \\ \theta_1^W & A_1^W & {\alpha_1^t}^W \\ 0 & {\theta_1^t}^W & -a_1^W \end{pmatrix},$$

ϖ_1 being defined in the main text of section 2.4.2. Lets calculate each entry since they are needed for the final goal. We need the transformations of the Cartan connection and of the dressing field u_1 under the Weyl group, given by (2.47) and (2.49) respectively.
Entry (1, 1) and (3, 3) were set to zero in the main text. Why? We have by definition $a_1 =: \chi\left(\varpi^{u_1}\right) = 0$. We know that this gauge-like condition is preserved by K_1 and by $S \simeq SO(r,s)$, we must show that it is also preserved by the Weyl group $W = \mathbb{R}^*$.

$$\begin{aligned} a_1^W &= \left(a - q\theta\right)^W = a^W - q^W\theta^W = (a - z^{-1}dz) - z^{-1}(q + z^{-1}\partial z \cdot e^{-1})z\theta, \\ &= a + z^{-1}dz - q\theta - z^{-1}\partial z \cdot e^{-1}e \cdot dx = a + z^{-1}dz - q\theta - z^{-1}\partial z \cdot dx = a - q\theta, \\ a_1^W &= a_1 = 0. \end{aligned} \tag{C.14}$$

Entry (2, 1) and (3, 2) are easily obtained since the sector $\mathfrak{g}_{-1}$ of the Cartan connection is unaffected by the action of K_1,

$$\theta_1^W = \theta^W = z\theta, \qquad \text{and} \qquad {\theta_1^t}^W = {\theta^t}^W = z\theta^t = z\theta^T\eta. \tag{C.15}$$

Entry (2, 2) gives,

$$\begin{aligned} A_1^W &= \left(A + \theta q - q^t\theta^t\right)^W = A^W + \theta^W q^W - {q^t}^W {\theta^t}^W, \\ &= A \;+\; z\theta\, z^{-1}\left(q + z^{-1}\partial z \cdot e^{-1}\right) \;-\; z^{-1}\left(q^t + \eta^{-1}(e^{-1})^T \cdot z^{-1}\partial z\right) z\theta^t, \\ &= \boldsymbol{A + \theta q - q^t\theta^t} \;+\; \theta\, z^{-1}\partial z \cdot e^{-1} \;-\; \eta^{-1}(e^{-1})^T \cdot z^{-1}\partial z\, \theta^t, \\ A_1^W &= \boldsymbol{A_1} \;+\; \theta\, z^{-1}\partial z \cdot e^{-1} \;-\; \eta^{-1}(e^{-1})^T \cdot z^{-1}\partial z\, \theta^t \end{aligned} \tag{C.16}$$

This is the transformation of the Lorentz/spin-connection under Weyl rescaling.
Entry (1, 2) gives,

$$\begin{aligned}
\alpha_1^W &= \left(\alpha - qA + \frac{1}{2}qq^t\theta^t + dq\right)^W = \alpha^W - q^W A^W + \frac{1}{2}q^W q^{t\,W}\theta^{t\,W} + dq^W,\\
&= z^{-1}\alpha \; - \; z^{-1}\left(q + z^{-1}\partial z\cdot e^{-1}\right)A \; + \; \frac{1}{2}z^{-2}\left(qq^t + 2z^{-1}\partial z\cdot e^{-1}q^t + z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\right)z\theta^t\\
&\quad + \; d\left(z^{-1}\left(q + z^{-1}\partial z\cdot e^{-1}\right)\right),\\
&= \boldsymbol{z^{-1}\alpha} \; - \; \boldsymbol{z^{-1}qA} \; - \; z^{-1}(z^{-1}\partial z\cdot e^{-1})A \; + \; \boldsymbol{\frac{1}{2}z^{-1}qq^t\theta^t} \; + \; \frac{1}{2}\left(2z^{-1}\partial z\cdot e^{-1}q^t + z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\right)\theta^t\\
&\quad + dz^{-1}\left(q + z^{-1}\partial z\cdot e^{-1}\right) \; + \; \boldsymbol{z^{-1}dq} \; + \; z^{-1}d\left(z^{-1}\partial z\cdot e^{-1}\right),\\
\alpha_1^W &= \boldsymbol{z^{-1}\alpha_1} \; - \; z^{-1}(z^{-1}\partial z\cdot e^{-1})A \; + \; z^{-1}(z^{-1}\partial z\cdot e^{-1})q^t\theta^t \; + \; \frac{1}{2}z^{-1}\left(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\right)\theta^t\\
&\quad + \; dz^{-1}\left(q + z^{-1}\partial z\cdot e^{-1}\right) \; + \; z^{-1}d(z^{-1}\partial z\cdot e^{-1}).
\end{aligned} \tag{C.17}$$

Entry (2, 3) gives,

$$\begin{aligned}
\alpha_1^{t\,W} &= \left(\alpha^t + Aq^t + \frac{1}{2}\theta qq^t + dq^t\right)^W = \alpha^{t\,W} + A^W q^{t\,W} + \frac{1}{2}\theta^W q^W q^{t\,W} + dq^{t\,W},\\
&= \alpha^t z \; + \; A\,z^{-1}\left(q^t + \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\right) \; + \; \frac{1}{2}z\theta\,z^{-2}\left(qq^t + 2\,z^{-1}\partial z\cdot e^{-1}q^t + z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\right)\\
&\quad + \; d\left(z^{-1}\left(q^t + \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\right)\right),\\
&= \boldsymbol{\alpha^t z} \; + \; \boldsymbol{z^{-1}Aq^t} \; + \; z^{-1}A\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z \; + \; \boldsymbol{z^{-1}\frac{1}{2}\theta qq^t} \; + \; z^{-1}\theta(z^{-1}\partial z\cdot e^{-1}q^t)\\
&\quad + \; z^{-1}\frac{1}{2}\theta\left(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\right) + dz^{-1}\left(q^t + \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\right) \; + \; \boldsymbol{z^{-1}dq^t} \; + \; z^{-1}d\left(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\right),\\
\alpha_1^{t\,W} &= \boldsymbol{z^{-1}\alpha_1^t} \; + \; z^{-1}A\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z \; + \; z^{-1}\theta\left(q\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\right) \; + \; z^{-1}\frac{1}{2}\theta\left(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\right)\\
&= dz^{-1}\left(q^t + \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\right) \; + \; z^{-1}d\left(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\right).
\end{aligned} \tag{C.18}$$

In the last step the equality $z^{-1}\partial z\cdot e^{-1}q^t = q\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z$ is used.

Entries of $\boldsymbol{\varpi_0^W}$: Now the Weyl gauge transformation of ϖ_0 is given by,

$$\varpi_0^W := \begin{pmatrix} 0 & P^W & 0\\ dx^W & \Gamma^W & (g^{-1}P^T)^W\\ 0 & (dx^T\cdot g)^W & 0\end{pmatrix} = \begin{pmatrix} 0 & \alpha_1^W ze & 0\\ e^{-1}z^{-1}\theta^W & e^{-1}A_1^W e + e^{-1}e\,z^{-1}dz + e^{-1}de & e^{-1}z^{-1}\alpha_1^{t\,W}\\ 0 & z\theta^{t\,W}e & 0\end{pmatrix}.$$

Let us calculate the entries. Entry (2, 1) gives,

$$dx^W = e^{-1}z^{-1}\theta_1^W = e^{-1}z^{-1}z\theta = e^{-1}e\cdot dx = dx, \quad \text{in components} \quad \left(dx^\mu\right)^W = dx^\mu. \tag{C.19}$$

This expresses the obvious invariance of the coordinate system under Weyl rescaling.
Entry (3, 2) gives,

$$\begin{aligned}
&\left(dx^T\cdot g\right)^W = z\theta_1^{t\,W}e = z^2\theta^t e = z^2\theta^T\eta e = z^2 dx^T\cdot e^T\eta e = dx^T\cdot z^2 g\\
&\text{in components} \quad g_{\mu\nu}dx^\mu = z^2 g_{\mu\nu}dx^\mu.
\end{aligned} \tag{C.20}$$

This expresses the **Weyl rescaling of the metric**. Entry (2, 2) gives,

$$\begin{aligned}
\Gamma^W &= e^{-1}A_1^W e + e^{-1}de \; + \; e^{-1}e\,z^{-1}dz,\\
&= \boldsymbol{e^{-1}A_1 e + e^{-1}de} \; + \; e^{-1}\theta\,z^{-1}\partial z \; - \; e^{-1}\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\,\theta^t e \; + \; \delta z^{-1}dz,\\
\Gamma^W &= \boldsymbol{\Gamma} \; + \; \delta z^{-1}dz \; + \; \delta\cdot dx\,z^{-1}\partial z \; - \; g^{-1}\cdot z^{-1}\partial z\,dx^T\cdot g.
\end{aligned} \tag{C.21}$$

The latter reads in components,

$$\Big(\Gamma^\rho{}_{\mu\nu}\Big)^W dx^\mu = \Big(\,\Gamma^\rho{}_{\mu\nu} \;+\; \delta^\rho_\nu z^{-1}\partial_\mu z \;+\; \delta^\rho_\mu z^{-1}\partial_\nu z \;-\; g^{\rho\lambda} z^{-1}\partial_\lambda z\, g_{\mu\nu}\,\Big) dx^\mu,$$

$$\text{or}\qquad \Big(\Gamma^\rho{}_{\mu\nu}\Big)^W = \Gamma^\rho{}_{\mu\nu} \;+\; \delta^\rho_\nu \Gamma_\mu \;+\; \delta^\rho_\mu \gamma_\nu \;-\; g^{\rho\lambda}\gamma_\lambda\, g_{\mu\nu}, \tag{C.22}$$

where $\gamma_\mu := z^{-1}\partial_\mu z$. This looks like the familiar transformations of the Christoffel symbols under Weyl rescaling, but actually this result is more general. Indeed even if the metricity condition $\nabla g = 0$ is automatically satisfied, the above calculation is performed without the assumption that $\Gamma^\rho{}_{\mu\nu}$ is symmetric in its two lower indices, that is without the assumption that it is a function of the metric tensor $g_{\mu\nu}$. The fact that the transformation under Weyl rescaling tuns out to be the same implies that the anti-symmetric component of Γ, that is the torsion, is Weyl invariant. This will be confirmed by the calculation of Ω_0^W in the next section. See also the discussion in the main text.

Entry (1, 2) gives,

$$\begin{aligned}
P^W &= \alpha_1^W z e = \alpha_1 e \;-\; (z^{-1}\partial z\cdot e^{-1})Ae \;+\; (z^{-1}\partial z\cdot e^{-1})q^t\theta^t e \;+\; \frac{1}{2}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\theta^t e\\
&\quad -\; z^{-1}dz\Big(q + z^{-1}\partial z\cdot e^{-1}\Big)e \;+\; d\Big(z^{-1}\partial z\cdot e^{-1}\Big)e,\\
&= \alpha_1 e \;-\; \boldsymbol{z^{-1}\partial z\cdot e^{-1}Ae} \;\boldsymbol{+}\; \boldsymbol{z^{-1}\partial z\cdot e^{-1}q^t\theta^t e} \;+\; \frac{1}{2}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\theta^T\eta e\\
&\quad \boldsymbol{-\,z^{-1}dz\, qe} \;-\; z^{-1}dz\, z^{-1}\partial z \;+\; \boldsymbol{d(z^{-1}\partial z)} \;\boldsymbol{-}\; \boldsymbol{z^{-1}\partial z\cdot e^{-1}de},\\
&= \alpha_1 e + \left\{\boldsymbol{d(z^{-1}\partial z) - z^{-1}\partial z\cdot \Big(e^{-1}\Big(A + \theta q - q^t\theta^t\Big)e + e^{-1}de\Big)}\right\}\\
&\quad -\; z^{-1}dz\, z^{-1}\partial z \;+\; \frac{1}{2}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)dx^T\cdot g,
\end{aligned}$$

where in this last equality we used $z^{-1}dz\; qe = z^{-1}\partial z\cdot dx\; qe = z^{-1}\partial z\cdot e^{-1}\theta\; qe$.

$$\begin{aligned}
P^W &= \alpha_1 e \;+\; \left\{d(z^{-1}\partial z) - z^{-1}\partial z\cdot \Big(\boldsymbol{e^{-1}A_1 e + e^{-1}de}\Big)\right\} \;-\; z^{-1}dz\, z^{-1}\partial z \;+\; \frac{1}{2}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)dx^T\cdot g,\\
&= P \;+\; \left\{d(z^{-1}\partial z) - z^{-1}\partial z\cdot \boldsymbol{\Gamma}\right\} \;-\; z^{-1}dz\, z^{-1}\partial z \;+\; \frac{1}{2}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)dx^T\cdot g,\\
P^W &= P \;+\; \nabla(z^{-1}\partial z) -\; z^{-1}dz\, z^{-1}\partial z \;+\; \frac{1}{2}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)dx^T\cdot g.
\end{aligned}\tag{C.23}$$

The latter reads in components,

$$\Big(P_{\mu\nu}\Big)^W dx^\mu = \Big(\,P_{\mu\nu} \;+\; \nabla_\mu(z^{-1}\partial_\nu z) \;-\; z^{-1}\partial_\mu z\, z^{-1}\partial_\nu z \;+\; \frac{1}{2} z^{-1}\partial_\lambda z g^{\lambda\alpha} z^{-1}\partial_\alpha z\, g_{\mu\nu}\,\Big)dx^\mu,$$

$$\text{or,}\qquad \Big(P_{\mu\nu}\Big)^W \;=\; P_{\mu\nu} \;+\; \nabla_\mu(\gamma_\nu) \;-\; \gamma_\mu\gamma_\nu \;+\; \frac{1}{2}\gamma_\lambda\gamma^\lambda\, g_{\mu\nu}, \tag{C.24}$$

with again $\gamma_\mu := z^{-1}\partial_\mu z$ and $\gamma_\lambda := g^{\lambda\alpha}\gamma_\alpha$. This looks like the familiar transformation of the Schouten tensor under Weyl rescaling. But again, this result is more general for we do not assume that $P_{\mu\nu}$ is symmetric and that it is the Schouten tensor, expressed as a function of the metric through the Levi-Civita connexion Γ and through the Ricci and Scalar curvature. From (C.24) we can deduce the transformation under Weyl rescaling of both the symmetric and anti-symmetric part of the tensor $P_{\mu\nu}$. See the discussion in the main text.

Entry (2, 3) gives,

$$
\begin{aligned}
P^{t\,W} &= (g^{-1}P^T)^W = z^{-1}e^{-1}\alpha_1^{t\,W},\\
&= z^{-2}e^{-1}\alpha_1^t \;+\; z^{-2}e^{-1}A\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z \;+\; z^{-2}e^{-1}\theta\Big(q\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big) \;+\; z^{-2}\frac{1}{2}z^{-1}\theta\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\\
&\quad z^{-1}e^{-1}dz^{-1}\Big(q^t+\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big) \;+\; z^{-2}e^{-1}d\Big(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big),\\
&= z^{-2}\Big\{\, e^{-1}\alpha_1^t \;+\; e^{-1}A\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z \;+\; e^{-1}\theta q\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z \;+\; \frac{1}{2}e^{-1}\theta\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\\
&\quad -\; z^{-1}dz\, e^{-1}q^t \;-\; z^{-1}dz\, e^{-1}\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z \;+\; e^{-1}\eta^{-1}d(e^{-1})^T\cdot z^{-1}\partial z \;+\; e^{-1}\eta^{-1}(e^{-1})^T d(z^{-1}\partial z)\Big\}
\end{aligned}
$$

Now $A\in\mathfrak{so}(r,s)$ so, $A^T\eta+\eta A=0 \;\to\; A\eta^{-1}=-\eta^{-1}A^T$. And since $g^{-1}=e^{-1}\eta^{-1}(e^{-1})^T$, we have

$$
\begin{aligned}
P^{t\,W} &= (g^{-1}P^T)^W,\\
&= z^{-2}\Big\{\, e^{-1}\alpha_1^t \;\boldsymbol{-\,g^{-1}e^T A^T(e^{-1})^T\cdot z^{-1}\partial z} \;+\; e^{-1}\theta q\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z \;+\; \frac{1}{2}e^{-1}\theta\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\\
&\quad -\; z^{-1}dz\, e^{-1}q^t \;-\; z^{-1}dz\, g^{-1}\cdot z^{-1}\partial z \;\boldsymbol{+\,g^{-1}e^T d(e^{-1})^T\cdot z^{-1}\partial z \;+\; g^{-1}d(z^{-1}\partial z)}\Big\},\\
P^{t\,W} &= z^{-2}\Big\{\, e^{-1}\alpha_1^t \;+\; \boldsymbol{g^{-1}\Big\{d(z^{-1}\partial z) - \Big(e^T A^T(e^{-1})^T + de^T(e^{-1})^T\Big)\cdot z^{-1}\partial z\Big\}},\\
&\quad +\; e^{-1}\theta q\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z \;-\; z^{-1}dz\, e^{-1}q^t \;-\; z^{-1}dz\, g^{-1}\cdot z^{-1}\partial z \;+\; \frac{1}{2}e^{-1}\theta\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\Big\}
\end{aligned}
\tag{C.25}
$$

Now, $\theta q\in\mathfrak{so}(r,s)$ so that: $(\theta q)^T\eta+\eta(\theta q)=0 \;\to\; \theta q\eta^{-1}=-\eta^{-1}(\theta q)^T$. Then,

$$
e^{-1}\theta q\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z = -e^{-1}\eta^{-1}(\theta q)^T(e^{-1})^T\cdot z^{-1}\partial z = -g^{-1}e^T(\theta q)^T(e^{-1})^T\cdot z^{-1}\partial z. \tag{C.26}
$$

Again we used $g^{-1}=e^{-1}\eta^{-1}(e^{-1})^T$. Moreover,

$$
(z^{-1}dz\; qe)^t = e^t q^t\, z^{-1}dz = ge^{-1}q^t\, z^{-1}dz,
$$

since $g=e^T\eta e=e^t e$. So,

$$
\begin{aligned}
e^{-1}q^t\, z^{-1}dz &= g^{-1}\Big(z^{-1}dz\; qe\Big)^t = g^{-1}\Big(z^{-1}\partial z\cdot dx\; qe\Big)^t = g^{-1}\Big(z^{-1}\partial z\cdot e^{-1}\theta\; qe\Big)^t,\\
&= g^{-1}\Big(e^t(\theta q)^t(e^{-1})^t\cdot z^{-1}\partial z\Big) = g^{-1}\Big(e^T\eta(\theta q)^t\eta^{-1}(e^{-1})^T\Big).
\end{aligned}
$$

Notice also that $(\theta q)^t\in\mathfrak{so}(r,s)$ so that: $\eta(\theta q)^t\eta^{-1}=-[(\theta q)^t]^T=-(q^t\theta^t)^T$. Then the above reads,

$$
e^{-1}q^t\, z^{-1}dz = g^{-1}e^T(q^t\theta^t)^T(e^{-1})^T\cdot z^{-1}\partial z \tag{C.27}
$$

Finally replace (C.26) and (C.27) in (C.25) and obtain,

$$
\begin{aligned}
P^{t\,W} &= z^{-2}\Big\{\, e^{-1}\alpha_1^t \;+\; \boldsymbol{g^{-1}\Big\{d(z^{-1}\partial z) - \Big(e^T\Big(A^T + (\theta q)^T - (q^t\theta^t)^T\Big)(e^{-1})^T + de^T(e^{-1})^T\Big)\cdot z^{-1}\partial z\Big\}},\\
&\qquad -\; z^{-1}dz\, g^{-1}\cdot z^{-1}\partial z \;+\; \frac{1}{2}e^{-1}\theta\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\Big\},\\
&= z^{-2}\Big\{\, P^t \;+\; \boldsymbol{g^{-1}\Big\{d(z^{-1}\partial z) - \Big(e^T A_1^T(e^{-1})^T + de^T(e^{-1})^T\Big)\cdot z^{-1}\partial z\Big\}}\\
&\qquad -\; z^{-1}dz\, g^{-1}\cdot z^{-1}\partial z \;+\; \frac{1}{2}g^{-1}e^T\eta\theta\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\Big\},\\
&= z^{-2}g^{-1}\Big\{\, P^T \;+\; \boldsymbol{\Big\{d(z^{-1}\partial z) - \Gamma^T\cdot z^{-1}\partial z\Big\}} \;-\; z^{-1}dz\, z^{-1}\partial z \;+\; \frac{1}{2}g\cdot dx\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\Big\},\\
(g^{-1}P^T)^W &= z^{-2}g^{-1}\Big\{\, P^T \;+\; \nabla(z^{-1}\partial z)^T \;-\; z^{-1}dz\, z^{-1}\partial z \;+\; \frac{1}{2}g\cdot dx\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\Big\}
\end{aligned}
\tag{C.28}
$$

In components this reads,

$$\left(g^{\rho\lambda}P_{\lambda\mu}\right)^W dx^\mu = \left(z^{-2}g^{\rho\lambda}\left\{P_{\lambda\mu} + \nabla_\mu(z^{-1}\partial_\lambda z) - z^{-1}\partial_\lambda z\, z^{-1}\partial_\mu z + \frac{1}{2}g_{\lambda\mu}z^{-1}\partial_\alpha g^{\alpha\beta}z^{-1}\partial_\beta z\right\}\right)dx^\mu,$$

$$\text{or,}\qquad \left(g^{\rho\lambda}P_{\lambda\mu}\right)^W = z^{-2}g^{\rho\lambda}\left\{P_{\lambda\mu} + \nabla_\mu(\gamma_\lambda) - \gamma_\lambda\gamma_\mu + \frac{1}{2}g_{\lambda\mu}\gamma_\alpha\gamma^\alpha\right\}. \tag{C.29}$$

This is of course redundant with (C.20) and (C.24).

Calculation of the entries of Ω_1^W and Ω_0^W

Entries of Ω_1^W: We recall that we have,

$$\Omega_1^W = \begin{pmatrix} f_1^W & \Pi_1^W & 0 \\ \Theta^W & F_1^W & \Pi_1^{t\,W} \\ 0 & \Theta^{t\,W} & -f_1^W \end{pmatrix}.$$

$\Omega_1 = u_1^{-1}\Omega u_1$ being defined in the main text of section 2.4.2. Let us calculate each entry. For this we need the transformation of the dressing field u_1 under the Weyl group, given by (2.49) , together with

$$\Omega^W = \begin{pmatrix} f & z^{-1}\Pi & 0 \\ z\Theta & F & z^{-1}\Pi^t \\ 0 & z\Theta^t & 0 \end{pmatrix}.$$

Entries (1, 1) and (3, 3) give,

$$f_1^W := \left(f - q\Theta\right)^W = f^W - q^W\Theta^W = f - (q + z^{-1}\partial z \cdot e^{-1})z\Theta = f - q\Theta - z^{-1}\partial z \cdot e^{-1}\Theta = f_1 - z^{-1}\partial z \cdot e^{-1}\Theta.$$

Now according to $\Omega_1 = d\varpi_1 + \varpi_1 \wedge \varpi_1$, we have also $f_1 = \alpha_1\theta$. So,

$$(\alpha_1\theta)^W = \alpha\theta - z^{-1}\partial z \cdot e^{-1}\Theta. \tag{C.30}$$

Entry (2, 1) and (3, 2) need no further effort. They are the same as Ω^W since the group K_1 to which u_1 belongs does not act on the $\mathfrak{g}_{-1}$ sector. We have,

$$\Theta^W = z\Theta, \qquad \text{and} \qquad \Theta^{t\,W} = z\Theta^t. \tag{C.31}$$

Entry (2, 2) gives,

$$\begin{aligned} F_1^W &= \left(F + \Theta q - q^t\Theta^t\right)^W = F^W + \Theta^W q^W - q^{t\,W}\Theta^{t\,W}, \\ &= F + z\Theta\, z^{-1}\left(q + z^{-1}\partial z \cdot e^{-1}\right) - z^{-1}\left(q^t + \eta^{-1}(e^{-1})^T \cdot z^{-1}\partial z\right) z\Theta^t, \\ &= \boldsymbol{F} + \boldsymbol{\Theta q} + \Theta\, z^{-1}\partial z \cdot e^{-1} \boldsymbol{- q^t\Theta^t} - \eta^{-1}(e^{-1})^T \cdot z^{-1}\partial z\, \Theta^t, \\ F_1^W &= \boldsymbol{F_1} + \Theta\, z^{-1}\partial z \cdot e^{-1} - \eta^{-1}(e^{-1})^T \cdot z^{-1}\partial z\, \Theta^t. \end{aligned} \tag{C.32}$$

Entry (1, 2) gives,

$$\begin{aligned}
\Pi_1^W &= \Big(\Pi - qF + fq - \frac{1}{2}qq^t\Theta^t\Big)^W = \Pi^W - q^W F^W + f^W q^W - \frac{1}{2}q^W q^{tW}\Theta^{tW},\\
&= z^{-1}\Pi \; - \; z^{-1}\Big(q + z^{-1}\partial z\cdot e^{-1}\Big)\Big(F_1 + \Theta\, z^{-1}\partial z\cdot e^{-1} - \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\,\Theta^t\Big) \; + \; f\, z^{-1}\Big(q + z^{-1}\partial z\cdot e^{-1}\Big)\\
&\quad - \; \frac{1}{2}z^{-2}\Big(qq^t + 2\, z^{-1}\partial z\cdot e^{-1} + z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\, z\Theta^t,\\
&= \boldsymbol{z^{-1}\Pi} \; \boldsymbol{-} \; \boldsymbol{z^{-1}qF_1} \; - \; z^{-1}(z^{-1}\partial z\cdot e^{-1})F_1 \; - \; z^{-1}\Big(q + z^{-1}\partial z\cdot e^{-1}\Big)\Big(\Theta\, z^{-1}\partial z\cdot e^{-1} - \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\,\Theta^t\Big)\\
&\quad + \; \boldsymbol{z^{-1}fq} \; + \; z^{-1}f\, z^{-1}\partial z\cdot e^{-1} \; \boldsymbol{-\frac{1}{2}z^{-1}qq^t\Theta^t} \; - z^{-1}\partial z\cdot e^{-1}\, q^t\Theta^t \; - \; \frac{1}{2}z^{-1}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\Theta^t,\\
&= \boldsymbol{z^{-1}\Pi_1} \; - \; z^{-1}(z^{-1}\partial z\cdot e^{-1})\Big(F_1 - f\mathbb{1}\Big)\\
&\quad - \; z^{-1}q\Theta\, z^{-1}\partial z\cdot e^{-1} \; - \; z^{-1}(z^{-1}\partial z\cdot e^{-1})\Theta\, z^{-1}\partial z\cdot e^{-1} \; + \; z^{-1}q\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\,\Theta^t\\
&\quad + \; z^{-1}(z^{-1}\partial z\cdot e^{-1})\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\,\Theta^t \; - \; z^{-1}(z^{-1}\partial z\cdot e^{-1})q^t\Theta^t \; - \; \frac{1}{2}z^{-1}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\Theta^t.
\end{aligned}$$

Observing that $q\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z = z^{-1}\partial z\cdot e^{-1}q^t$, one obtains finally

$$\begin{aligned}
\Pi_1^W &= z^{-1}\Pi_1 \; - \; z^{-1}(z^{-1}\partial z\cdot e^{-1})\Big(F_1 - f\mathbb{1}\Big)\\
&\quad + \; z^{-1}\Big\{\frac{1}{2}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\Theta^t - q\Theta\, z^{-1}\partial z\cdot e^{-1} - (z^{-1}\partial z\cdot e^{-1})\Theta(z^{-1}\partial z\cdot e^{-1})\Big\}
\end{aligned} \tag{C.33}$$

Entry (2, 3) gives,

$$\begin{aligned}
\Pi_1^{tW} &= \Big(\Pi^t - F_1^t q^t + q^t f - \frac{1}{2}\Theta qq^t\Big)^W = \Pi^{tW} + F_1^W q^{tW} + q^{tW} f^W - \frac{1}{2}\Theta^W q^W q^{tW},\\
&= z^{-1}\Pi^t \; + \; \Big(f_1 + \Theta\, z^{-1}\partial z\cdot e^{-1} - \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\,\Theta^t\Big)z^{-1}\Big(q^t + \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big)\\
&\quad + \; z^{-1}\Big(q^t + \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big)f \; - z\Theta\,\frac{1}{2}\Big(qq^t + 2\, z^{-1}\partial z\cdot e^{-1}q^t + \; z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big),\\
&= \boldsymbol{z^{-1}\Pi^t} \; \boldsymbol{+} \; \boldsymbol{z^{-1}F_1q^t} + z^{-1}F_1\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z \; + \; z^{-1}\Theta\, z^{-1}\partial z\cdot e^{-1}q^t\\
&\quad + \; z^{-1}\Theta\, z^{-1}\partial z\cdot e^{-1}\Big(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big) \; - \; z^{-1}\Big(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big)\Theta^t q^t\\
&\quad - \; z^{-1}\Big(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big)\Theta^t\Big(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big) \; + \; \boldsymbol{z^{-1}q^t f} \; + \; z^{-1}\Big(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big)f\\
&\quad \boldsymbol{- z^{-1}\Theta\frac{1}{2}qq^t} \; - z^{-1}\Theta\, z^{-1}\partial z\cdot e^{-1}q^t \; - \; \frac{1}{2}z^{-1}\Theta\Big(z^{-1}\partial z\cdot g^{-1}z^{-1}\partial z\Big),\\
&= \boldsymbol{\Pi_1^t} \; + \; z^{-1}\Big(F_1 + f\mathbb{1}\Big)\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\\
&\quad + \; z^{-1}\Big\{\frac{1}{2}\Theta\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big) \; - \; \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\,\Theta^t q^t \; - \; \Big(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big)\Theta^t\Big(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big)\Big\}
\end{aligned}$$

Finally one gets,

$$\begin{aligned}
\Pi_1^{tW} &= \Pi_1^t \; - \; z^{-1}\Big(F_1^t - f\mathbb{1}\Big)\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\\
&\quad + \; z^{-1}\Big\{\frac{1}{2}\Theta\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big) \; - \; \eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\,\Theta^t q^t \; - \; \Big(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big)\Theta^t\Big(\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\Big)\Big\}
\end{aligned} \tag{C.34}$$

Entries of Ω_0^W : Now the Weyl gauge transformation of Ω_0 is given by,

$$\Omega_0^W = \begin{pmatrix} f_1^W & C^W & 0 \\ T^W & W^W & C^{t\,W} \\ 0 & T^{t\,W} & -f_1^W \end{pmatrix} = \begin{pmatrix} f_1^W & z\Pi_1 e & 0 \\ e^{-1}z^{-1}\Theta^W & e^{-1}F_1 e & e^{-1}z^{-1}\Pi^{t\,W} \\ 0 & z\Theta^{t\,W}e & -f_1^W \end{pmatrix},$$

Let us calculate the entries.
Entries (1, 1) and (3, 3) are unchanged, but since $f_1 = \alpha_1\theta = \alpha_1 e\cdot dx := P\cdot dx$, we now rewrite the result as

$$f_1^W = f_1 - z^{-1}\partial z\cdot e^{-1}\Theta \quad \to \quad \Big(P\cdot dx\Big)^W = P\cdot dx - z^{-1}\partial z\cdot T, \tag{C.35}$$

which reads in components,

$$\begin{aligned} &\frac{1}{2}\Big(P_{[\mu\sigma]}\Big)^W dx^\mu\wedge dx^\sigma = \frac{1}{2}\Big(P_{[\mu\sigma]} - z^{-1}\partial_\lambda z\, T^\lambda{}_{\mu\sigma}\Big)dx^\mu\wedge dx^\sigma, \\ &\text{or} \qquad P^W_{[\mu\sigma]} = P_{[\mu\sigma]} - \gamma_\lambda \Gamma^\lambda{}_{[\mu\sigma]}, \end{aligned} \tag{C.36}$$

since dx is invariant and $T^\lambda{}_{\mu\sigma} = \Gamma^\lambda{}_{[\mu\sigma]}$. This is actually redundant with equation (C.24) for it is its anti-symmetric part. See the discussion in the main text.
Entry (2, 1) gives,

$$T^W = e^{-1}z^{-1}\Theta^W = e^{-1}z^{-1}z\Theta = e^{-1}\Theta = T. \tag{C.37}$$

This is the **invariance of the torsion under Weyl rescaling**. Actually this is subtly redundant with (C.22). Indeed the latter equation gives the transformation of $\Gamma^\lambda{}_{\mu\sigma}$ under Weyl rescaling. It is identical with the well known transformation of the Christoffel symbols which are the symmetric part of $\Gamma^\lambda{}_{\mu\sigma}$. From this we draw the conclusion that the anti-symmetric part is invariant. But the anti-symmetric part is precisely the torsion, as we've just noted.
Entry (3, 2) gives,

$$T^{t\,W} := (T^T g)^W = z\Theta^{t\,W}e = z^2\Theta^t e = z^2 T^t = T^T\, z^2 g \tag{C.38}$$

This is again the invariance of the torsion, and the Weyl rescaling of the metric. So far then no new informations are provided.
Entry (2, 2) gives,

$$\begin{aligned} W^W &= eF_1^W e = e^{-1}F_1 e \;+\; e^{-1}\Theta\, z^{-1}\partial z \;-\; g^{-1}\cdot z^{-1}\partial z\,\Theta^t e, \\ W^W &= W \;+\; T\, z^{-1}\partial z \;-\; g^{-1}\cdot z^{-1}\partial z\, T^T g \end{aligned} \tag{C.39}$$

In components, this reads

$$\begin{aligned} &\frac{1}{2}\Big(W^\rho{}_{\nu,\mu\sigma}\Big)^W dx^\mu\wedge dx^\sigma = \frac{1}{2}\Big(W^\rho{}_{\nu,\mu\sigma} \;+\; T^\rho{}_{\mu\sigma}\, z^{-1}\partial_\nu z \;-\; g^{\rho\lambda}z^{-1}\partial_\lambda z\, T_{\mu\sigma}{}^\alpha g_{\alpha\nu}\Big)dx^\mu\wedge dx^\sigma, \\ &\text{or,} \quad \Big(W^\rho{}_{\nu,\mu\sigma}\Big)^W = W^\rho{}_{\nu,\mu\sigma} \;+\; T^\rho{}_{\mu\sigma}\,\gamma_\nu \;-\; g^{\rho\lambda}\gamma_\lambda\, T_{\mu\sigma}{}^\alpha g_{\alpha\nu} \end{aligned} \tag{C.40}$$

Here is a new relation, the Weyl rescaling of the $(1,3)$-tensor W.
Entry (1, 2) gives,

$$\begin{aligned} C^W &= z\Pi_1^W e, \\ &= \Pi_1 e \;-\; z^{-1}\partial z\cdot e^{-1}\Big(F_1 - f\mathbb{1}\Big)e \;+\; \frac{1}{2}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)\Theta^t e \;-\; q\Theta\, z^{-1}\partial z \;-\; z^{-1}\partial z\cdot e^{-1}\Theta\, z^{-1}\partial z, \\ C^W &= C \;-\; z^{-1}\partial z\cdot W \;+\; z^{-1}\partial z\cdot\delta f \;+\; \frac{1}{2}\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big)T^T g \;-\; \Big(a + z^{-1}\partial z\Big)\cdot T\, z^{-1}\partial z. \end{aligned} \tag{C.41}$$

In components this reads,

$$\frac{1}{2}\Big(C_{\nu,\mu\sigma}\Big)^W dx^\mu \wedge dx^\sigma = \frac{1}{2}\bigg(C_{\nu,\mu\sigma} \ - \ z^{-1}\partial_\lambda z\, W^\lambda{}_{\nu,\mu\sigma} \ + \ z^{-1}\partial_\nu z\, f_{\mu\sigma}$$
$$+ \ \frac{1}{2}\Big(z^{-1}\partial_\alpha z\, g^{\alpha\beta}\, z^{-1}\partial_\beta z\Big)T_{\mu\sigma}{}^\lambda g_{\lambda\nu} \ - \ \Big(a_\lambda + z^{-1}\partial_\lambda z\Big)T^\lambda{}_{\mu\sigma}\, z^{-1}\partial_\nu z\bigg)dx^\mu \wedge dx^\sigma,$$

$$\text{or} \quad \Big(C_{\nu,\mu\sigma}\Big)^W = C_{\nu,\mu\sigma} - \gamma_\lambda\, W^\lambda{}_{\nu,\mu\sigma} \ + \ \gamma_\nu\, f_{\mu\sigma} + \ \frac{1}{2}(\gamma_\alpha g^{\alpha\beta}\gamma_\beta)T_{\mu\sigma}{}^\lambda g_{\lambda\nu} \ - \ (a_\lambda + \gamma_\lambda)T^\lambda{}_{\mu\sigma}\,\gamma_\nu. \tag{C.42}$$

At last, entry (2, 3) gives,

$$\begin{aligned}
C^{t\,W} &= \ e^{-1}z^{-1}\Pi_1^{t\,W}, \\
&= z^{-2}\bigg\{e^{-1}\Pi_1^t \ + \ e^{-1}\Big(F_1^t + f\mathbb{1}\Big)\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z \ + \ \frac{1}{2}e^{-1}\Theta\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big) \\
&\quad - \ e^{-1}\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\,\Theta^t q^t \ - \ e^{-1}\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\,\Theta^t\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\bigg\}, \\
&= z^{-2}\bigg\{C^t \ + \ e^{-1}F_1 e g^{-1}\cdot z^{-1}\partial z \ + \ f\delta g^{-1}\cdot z^{-1}\partial z \ + \ \frac{1}{2}T\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big) \\
&\quad - \ g^{-1}\cdot z^{-1}\partial z\,\Theta^T\eta\eta^{-1}(e^{-1})^T\cdot a \ - \ g^{-1}\cdot z^{-1}\partial z\,\Theta^T\eta\eta^{-1}(e^{-1})^T\cdot z^{-1}\partial z\bigg\}, \\
&= z^{-2}\bigg\{g^{-1}C^T \ + \ Wg^{-1}\cdot z^{-1}\partial z \ + \ f\delta g^{-1}\cdot z^{-1}\partial z \ + \ \frac{1}{2}T\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big) \\
&\quad - \ g^{-1}\cdot z^{-1}\partial z\,\Theta^T(e^{-1})^T\cdot\Big(a + z^{-1}\partial z\Big)\bigg\}, \\
\Big(g^{-1}C^T\Big)^W &= \ z^{-2}g^{-1}\bigg\{C^T \ - \ W^T\cdot z^{-1}\partial z \ + \ f\delta\cdot z^{-1}\partial z \ + \ \frac{1}{2}gT\Big(z^{-1}\partial z\cdot g^{-1}\cdot z^{-1}\partial z\Big) \\
&\quad - \ z^{-1}\partial z\, T^T\cdot\Big(a + z^{-1}\partial z\Big)\bigg\},
\end{aligned} \tag{C.43}$$

where in the last equality we used,

$$\begin{aligned}
Wg^{-1} &= e^{-1}F_1 e g^{-1} = e^{-1}F_1 e\, e^{-1}\eta^{-1}(e^{-1})^T = e^{-1}F_1\eta^{-1}(e^{-1})^T, \quad \text{but} \quad F_1^T\eta + \eta F_1 = 0 \to F_1\eta^{-1} = -\eta^{-1}F_1^T, \\
&= \ -e^{-1}\eta^{-1}F_1^T(e^{-1})^T = -g^{-1}e^T F_1^T(e^{-1})^T, \quad \text{since} \quad g^{-1} = e^{-1}\eta^{-1}(e^{-1})^T, \\
Wg^{-1} &= \ -g^{-1}W^T.
\end{aligned}$$

Equation (C.43) reads in components,

$$\frac{1}{2}\Big(g^{\rho\lambda}C_{\lambda,\mu\sigma}\Big)^W dx^\mu \wedge dx^\sigma = \ \frac{1}{2}z^{-2}g^{\rho\lambda}\bigg\{C_{\lambda,\mu\sigma} \ - \ W_\lambda{}^\alpha{}_{\mu\sigma}z^{-1}\partial_\alpha z \ + \ f_{\mu\sigma}z^{-1}\partial_\lambda z$$
$$+ \ \frac{1}{2}g_{\lambda\alpha}T^\alpha{}_{\mu\sigma}\Big(z^{-1}\partial_\beta z\, g^{\beta\delta}\, z^{-1}\partial_\delta z\Big) \ - \ z^{-1}\partial_\lambda z\, T_{\mu\sigma}{}^\alpha\Big(a_\alpha + z^{-1}\partial_\alpha z\Big)\bigg\}dx^\mu \wedge dx^\sigma,$$

or,

$$\Big(g^{\rho\lambda}C_{\lambda,\mu\sigma}\Big)^W = \ z^{-2}g^{\rho\lambda}\bigg\{C_{\lambda,\mu\sigma} \ - W_\lambda{}^\alpha{}_{\mu\sigma}\gamma_\alpha \ + f_{\mu\sigma}\,\gamma_\lambda \ + \frac{1}{2}g_{\lambda\alpha}T^\alpha{}_{\mu\sigma}\Big(\gamma_\beta\, g^{\beta\delta}\,\gamma_\delta\Big) \ - \ \gamma_\lambda\, T_{\mu\sigma}{}^\alpha\Big(a_\alpha + \gamma_\alpha\Big)\bigg\}. \tag{C.44}$$

This is of course redundant with (C.42) and (C.20).

Residual Weyl gauge transformation of ϖ_0 and Ω_0 in the normal case. The normality conditions for ϖ are: $f = 0$, $\Theta = 0$ and $\text{Ric}(F) = 0$. These are preserved by the successive dressing operations so that ϖ_0 is such that: $f_1 = 0$, $T = 0$ and $\text{Ric}(W) = 0$. In that case Γ is the **Levi-Civita connection**, P is the **Schouten tensor**, $C = \nabla P$ is thus the **Cotton tensor** and finally W is the **Weyl tensor**. We have then,

$$
\begin{aligned}
\varpi_0^W &= \begin{pmatrix} 0 & P_{\nu\mu} & 0 \\ \delta^\rho_\mu & \Gamma^\rho{}_{\mu\nu} & g^{\rho\lambda}P_{\lambda\mu} \\ 0 & g_{\mu\nu} & 0 \end{pmatrix}^W dx^\mu, \\
&= \begin{pmatrix} 0 & P_{\mu\nu} + \nabla_\mu(\gamma_\nu) - \gamma_\mu\gamma_\nu + \tfrac{1}{2}\gamma_\lambda\gamma^\lambda\, g_{\mu\nu} & 0 \\ \delta^\rho_\mu & \Gamma^\rho{}_{\mu\nu} + \delta^\rho_\nu\Gamma_\mu + \delta^\rho_\mu\gamma_\nu - g^{\rho\lambda}\gamma_\lambda\, g_{\mu\nu} & z^{-2}g^{\rho\lambda}\left\{P_{\lambda\mu} + \nabla_\mu(\gamma_\lambda) - \gamma_\lambda\gamma_\mu + \tfrac{1}{2}g_{\lambda\mu}\gamma_\alpha\gamma^\alpha\right\} \\ 0 & z^2 g_{\mu\nu} & 0 \end{pmatrix} dx^\mu,
\end{aligned}
$$

where $\gamma_\mu := z^{-1}\partial_\mu z$. As for the associated curvature,

$$
\begin{aligned}
\Omega_0^W &= \frac{1}{2}\begin{pmatrix} 0 & C_{\nu,\mu\sigma} & 0 \\ 0 & W^\rho{}_{\nu,\mu\sigma} & g^{\rho\lambda}C_{\lambda,\mu\sigma} \\ 0 & 0 & 0 \end{pmatrix}^W dx^\mu \wedge dx^\sigma, \\
&= \frac{1}{2}\begin{pmatrix} 0 & C_{\nu,\mu\sigma} - \gamma_\lambda\, W^\lambda{}_{\nu,\mu\sigma} & 0 \\ 0 & W^\rho{}_{\nu,\mu\sigma} & z^{-2}g^{\rho\lambda}\left\{C_{\lambda,\mu\sigma} - W_\lambda{}^\alpha{}_{\mu\sigma}\gamma_\alpha\right\} \\ 0 & 0 & \end{pmatrix} dx^\mu \wedge dx^\sigma.
\end{aligned}
$$

These are well known transformations under Weyl rescaling of the metric on $\mathcal{M}$, here seen as a residual gauge symmetry of the final composite fields.

Conclusion: As far as I know, these transformations for the *Riemann parameterization of the normal conformal Cartan connection* and its curvature are not performed in the jet formalism. Appreciate how easily we obtain it. Remember we mentioned the fact that the computational complexity of the jet formalism grows very quickly with the order, and was already intricate at order two. The matrix formalism clearly reduces this complexity.

Appendix D
Published papers/ in preparation

D.1 Gauge invariant composite fields out of connections with examples

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Gauge invariant composite fields out of connections, with examples

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In this paper, we put forward a systematic and unifying approach to construct gauge invariant composite fields out of connections. It relies on the existence in the theory of a group-valued field with a prescribed gauge transformation. As an illustration, we detail some examples. Two of them are based on known results: the first one provides a reinterpretation of the symmetry breaking mechanism of the electroweak part of the Standard Model of particle physics; the second one is an application to Einstein's theory of gravity described as a gauge theory in terms of Cartan connections. The last example depicts a new situation: starting with a gauge field theory on Atiyah Lie algebroids, the gauge invariant composite fields describe massive vector fields. Some mathematical and physical discussions illustrate and highlight the relevance and the generality of this approach.

Keywords: Gauge field theories; dressing field; Higgs field.

Mathematics Subject Classification 2010: 53A99, 53Z05, 53C05, 53C80, 83C05

PACS: 02.40.Hw, 11.15.Ex, 12.15.-y, 04.20.Cv

D.2 Gauge field theories: various mathematical approaches

Gauge field theories: various mathematical approaches

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Mathematical Structures of the Universe
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Abstract

This paper presents relevant modern mathematical formulations for (classical) gauge field theories, namely, ordinary differential geometry, noncommutative geometry, and transitive Lie algebroids. They provide rigorous frameworks to describe Yang-Mills-Higgs theories or gravitation theories, and each of them improves the paradigm of gauge field theories. A brief comparison between them is carried out, essentially due to the various notions of connection. However they reveal a compelling common mathematical pattern on which the paper concludes.

D.3 Weyl residual gauge freedom out of conformal geometry, with a new BRS tool

Weyl residual gauge freedom out of conformal geometry, with a new BRS tool

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Abstract

Here is shown, in a first part, how the *dressing field technique* presented in [1] can be applied to the second-order conformal structure where it provides the so-called *Riemannian parametrization* of the normal Cartan connection and its curvature. The Weyl rescaling transformation is seen as the residual gauge freedom of these objects after neutralization of most of the gauge conformal group $SO(d,2)$. In a second part of the paper the aforementioned technique is adapted to the BRS technology and the notion of "dressed ghost" handling the infinitesimal gauge freedom is described. This new tool is applied to the conformal structure and shown to provide in a handy way the linearized version of the Weyl rescaling transformation derived in the first part of the paper. Finally the inclusion of the infinitesimal diffeomorphisms in the example is considered.

In preparation.

D.4 Nucleon spin decomposition and differential geometry

Nucleon spin decomposition and differential geometry

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Abstract

In the last few years, the so-called *Chen et al. approach* of the nucleon spin decomposition has been widely used, discussed and elaborated on. In this letter we propose a genuine differential geometric understanding of this approach. We furthermore show that it is linked to the "dressing field method" we advocated in [1].

In preparation.

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