UNIVERSITE D'AIX-MARSEILLE
ECOLE DOCTORALE ED352
Physicque et Sciences de la Matière
Centre de Physique Théorique, CNRS-UMR 7332

THESE DE DOCTORAT

Pour obtenir le grade de
Docteur d'Aix-Marseille Université

Specialité: Physique théorique et mathématiques

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# Gravité quantique à boucles et géométrie discrète

Soutenue le 21 juillet 2014

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Mingyi ZHANG

# Loop Quantum Gravity and Discrete Geometry

献给我的父母

# Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other University. This thesis is the result of my own work and includes the work done in collaboration.

This thesis is based on the following papers:

1. S. Speziale, **M. Zhang** (2014): *Null twisted geometries*, Physical Review D **89**, 084070.

2. M. Han, **M. Zhang** (2013): *Asymptotics of Spinfoam Amplitude on Simplicial Manifold: Lorentzian Theory*, Class.Quant.Grav. **30**, 165012.
   ⋆ Selected for the Class.Quant.Grav.(CQG) Highlights of 2012 - 2013

3. M. Han, **M. Zhang** (2012): *Asymptotics of Spinfoam Amplitude on Simplicial Manifold: Euclidean Theory*, Class.Quant.Grav. **29**, 165004.

4. C. Rovelli, **M. Zhang** (2011): *Euclidean three-point function in loop and perturbative gravity*, Class.Quant.Grav. **28**, 175010.

Mingyi ZHANG
2014

# Acknowledgements

Thanks to my supervisors Carlo Rovelli and Simone Speziale for their teachings and advises that lead me to the world of loop gravity. Thanks to Muxin Han for his collaboration and guidance, in Chinese, it is really helpful. Thanks the friends I met at the quantum group in Marseille for friendship, exploring and learning things together. I enjoy the study here very much!

Thanks to Prof.Fernando Barbero, Prof.Bianca Dittrich and Prof.Jerzy Lewandowski for being the members of jury of my dissertation. Thanks to Prof.Fernando Barbero and Prof.Michael Reisenberger for being the reporters of my thesis and their careful revising.

Thanks to the members of band Schengen Unlimited: Simone Speziale, Miklos Langvik and others. It is quite fun to jam with you guys.

Thanks to all my Chinese friends at Luminy and also at Marseille, who make the life in these four years very enjoyable.

Foremost, I thank my parents for their constant unconditional dedication, support and love, from which I gained the energy to achieve this PhD.

Thanks to my girlfriend Weiyi.

# Abstract

In this thesis, I will present how to extract discrete geometries of space-time from the covariant formulation of loop quantum gravity (LQG), which is called the spin foam formalism. LQG is a quantum theory of gravity that non-perturbative quantizes general relativity independent from a fix background. It predicts that the geometry of space is quantized, in which area and volume can only take discrete value. The kinematical Hilbert space is spanned by Penrose's spin network functions. The excitation of geometry can be neatly visualized as fuzzy polyhedra that glued through their facets. The spin foam defines the dynamics of LQG by a spin foam amplitude on a cellular complex, bounded by the spin network states. There are three main results in this thesis. First, the semiclassical limit of the spin foam amplitude on an arbitrary simplicial cellular complex with boundary is studied completely. The classical discrete geometry of space-time is reconstructed and classified by the critical configurations of the spin foam amplitude. Second, the three-point function from LQG is calculated. It coincides with the results from discrete gravity. Third, the description of discrete geometries of null hypersurfaces is explored in the context of LQG. In particular, the null geometry is described by a Euclidean singular structure on the two-dimensional spacelike surface defined by a foliation of space-time by null hypersurfaces. Its quantization is U(1) spin network states which are embedded nontrivially in the unitary irreducible representations of the Lorentz group.

# Résumé

Dans ce travail de thèse, je présente comment extraire les géométries discrètes de l'espace-temps de la formulation covariante de la gravitaté quantique à boucles ( LQG, "loop quantum gravity" en anglais ), qui est appelé le formalisme de la mousse de spin. LQG est une théorie quantique de la gravité qui non-perturbativement quantifie la relativité générale indépendante d'un fond fixe. Il prédit que la géométrie de l'espace est quantifiée, dans lequel l'aire et le volume ne peuvent prendre que la valeur discrète. L'espace de Hilbert cinématique est engendré par les fonctions du réseau de spin. L'excitation de la géométrie peut être parfaitement visualisée comme des polyèdres floue qui collées à travers leurs facettes. La mousse de spin définit la dynamique de la LQG par une amplitude de la mousse de spin sur un complexe cellulaire avec un état du réseau de spin comme la frontière. Cette thèse présente trois résultats principaux. Premièrement, la limite semi-classique de l'amplitude de la mousse de spin sur un complexe simplicial arbitraire avec une frontière est complètement étudiée. La géométrie discrète classique de l'espace-temps est reconstruite et classée par les configurations critiques de l'amplitude de la mousse de spin. Deuxièmement, la fonction de trois-point de LQG est calculé. Elle coïncide avec le résultat de la gravité discrète. Troisièmement, la description des géométries discrètes de hypersurfaces nulles est explorée dans le cadre de la LQG. En particulier, la géométrie nulle est décrit par une structure singulière euclidienne. Sa quantification est U(1) états du réseau de spin qui sont intégrés de façon non triviale dans les représentations unitaires irréductibles du groupe de Lorentz.

# Contents

# Chapter 1

# Introduction

## 1.1 The fundamental constants

Nature gives us three fundamental constants: $G$(Newton constant), $c$(the speed of light) and $\hbar$(Planck constant). They provide us a natural system of units. They control the domains of validity of our physics theories.

Not long ago, we still used some king's feet to measure lengths; even nowadays we still use a metal prototype in Paris to measure masses, use a period of a radiation from an atom to measure time. Einstein first recognised that with the speed of light $c$, we no longer need separate units for length and time. Following this idea, physicists realized that nature has already prepared us for free a universal system of units, which is called the *natural units*, given by $G$, $c$ and $\hbar$.

To see how it works, we need three great principles: the principle of invariant light speed, the uncertainty principle and the Newton law of gravity. The uncertainty principle tells us that $\hbar$ divided by the momentum $Mc$ is a length. Comparing the energy $mc^2$ of a particle of mass $m$ in a gravitational potential with its potential energy $-GMm/r$ and cancelling off $m$, we see that the combination $GM/c^2$ is also a length. Equating the two lengths $\hbar/Mc$ and $GM/c^2$, we realise that the combination $\hbar c/G$ is a squared mass. It means that with three fundamental constant $G$, $c$ and $\hbar$, we can define a mass, which is known as the Planck mass

$$M_p = \sqrt{\frac{\hbar c}{G}} \tag{1.1}$$

By using the uncertainty principle, a Planck length can be defined

$$l_p = \frac{\hbar}{M_p c} = \sqrt{\frac{\hbar G}{c^3}}. \tag{1.2}$$

By using the principle of invariant light speed, a Planck time can be defined

$$t_p = \frac{l_p}{c} = \sqrt{\frac{\hbar G}{c^5}}. \tag{1.3}$$

With these three natural units we can measure space, time and energy(mass). When we want to measure the universe or communicate with another civilization in our universe, we no longer have to invent some units. Nature tells us that we can measure mass in units of $M_p$, length in units of $l_p$ and time in units of $t_p$. Furthermore, later in the next section, I will present that these natural units are actually the fundamental scales of the nature. But before going to this point, I would like to discuss the role of the fundamental constants in physical theories.

The modern view is that any physical theory should have a domain of validity. The physics that we ignore beyond it needs some more fundamental theories to describe. The three fundamental constants $G$, $c^{-1}$ and $\hbar$ are the switches turning on the lights toward new world of physics. Starting from Newtonian mechanics, when $G$, $c^{-1}$ or $\hbar$ is turned on separately, we will get Newtonian gravity, special relativity or quantum mechanics, respectively. If the first two of them are turned on, physics moves to general relativity(GR), while if $c^{-1}$ and $\hbar$ are not zero, quantum field theory is obtained. When we want to go to the wonderland of quantum theory of gravity, which is the main context of the thesis, all three fundamental constants have to be turned on. To explore a proper quantum gravity theory is one of the ultimate dreams of physicists. But why do we say that quantum gravity(QG) comes into the physics playground when we turned on all these constants?

## 1.2 The search of quantum gravity

In the 1930s, after Heisenberg and Pauli quantized the electromagnetic field, most of the physicist believed that the gravitational field can be quantized as easily as the quantization of the electromagnetic field. Of course even quantizing the electromagnetic field was not that easy. The quantum electrodynamics was suffering with the infinities and various inconsistencies until 1940s, when Schwinger, Feynman and

Tomonaga introduced the technologies known as the *renormalization*[1]. But in 1935, a brilliant Russian physicist Matvei Bronstein first noticed that the quantization of gravitational field is intrinsically different from the quantization of the electromagnetic field. It is because the existence of the *gravitational radius* of massive objects (see e.g.[2]).

The quantum mechanics tells us that the quantum radius $r_Q$ of a particle with mass $m$ is of order

$$r_Q \sim \frac{\hbar}{mc} \tag{1.4}$$

which is the typical wave length of the particle. The more mass it contains, the smaller it is. But the gravitational theory tells us that for a particle of mass $m$ there is a gravitational radius

$$r_G \sim \frac{Gm}{c^2} \tag{1.5}$$

If the mass condensed inside of the region $r < r_G$, the particle forms a black hole. The more mass it contains, the larger it is, which is exactly an opposite behavior of the quantum radius. When we eliminate the mass $m$ of the particle, magically we obtain that

$$l_p \sim \sqrt{r_Q r_G} \tag{1.6}$$

The Planck length is the geometric mean of the quantum radius and the gravitational radius. The QG must happens at the scale of order the Planck length. When the mass of a particle becomes bigger and bigger, it condensed in a smaller and smaller region. At some point, the quantum radius hits the gravitational radius

$$r_Q = r_G \sim l_p \tag{1.7}$$

It cannot go beyond this scale, otherwise we cannot detect the particle any more.

One would say this discussion is too abstract, then could we design a certain experiment to see that in our world we can only measure the length bigger than $l_p$? So let us look at the following thought experiment[3]:

Consider a measuring device of size $L$ and mass $M$. To determine the length of something, we have to know the positions of the measuring device. One can proceeds to measure the position of the device at time 0 and at time $t$, take the difference $s \equiv x(t) - x(0)$, and see whether it can be made arbitrarily small, as shown in Fig.1.1

For simplicity let us assume that the measuring device is moving in a constant speed, i.e. a constant momentum $p$. The relevant Heisenberg operators are related by

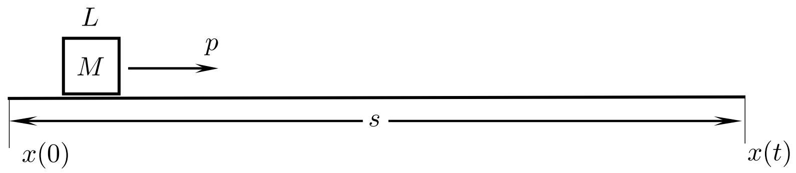


Fig. 1.1 The thought experiment of measuring the minimum length

$$\hat{x}(t) - \hat{x}(0) = \frac{\hat{p}}{M}t \tag{1.8}$$

Commuting it with $\hat{x}(0)$ and using the Robertson uncertainty relation

$$\sigma_A \sigma_B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|, \quad \text{where} \quad \sigma_A = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2} \tag{1.9}$$

we get

$$\sigma_{x(0)}\sigma_{x(t)} \geq \frac{\hbar t}{2M} \tag{1.10}$$

It means that if one tries to get the uncertainty in the measurement of $x(0)$ down, the uncertainty in the measurement of $x(t)$ must goes up. The best one can make is to let $\sigma_{x(0)} = \sigma_{x(t)} = \sqrt{\hbar t/2M}$. Then there is a limitation in the measurement of $s$ which is the uncertainty of $s$

$$s \geq \sigma_s \geq \sqrt{\frac{\hbar t}{M}} \tag{1.11}$$

Now if we are in a world without gravity and special relativity, one can make $\sigma_s$ as small as we like. Just to move the device sufficiently fast and make the device massive enough. In other words, make $t$ as small as possible and $M$ as large as possible.

But as soon as we turn on $c^{-1}$, since we can not move the device faster than light, then there is a minimum time $t$ is needed

$$t > \frac{s}{c} \geq \frac{L}{c} \tag{1.12}$$

The second $\geq$ means that the scale of the device should be smaller than the scale that we want to measure, otherwise the definition of the "position" of the device is invalid. When we turn on $G$, general relativity tells us that the device would form a black hole if it is too "heavy". If the device is a black hole, we will not receive any measurement.

So the mass of the device $M$ has an upper bound

$$M < \frac{c^2 L}{G} \tag{1.13}$$

We can thus conclude that

$$s \geq \sigma_s \geq \sqrt{\frac{\hbar t}{M}} > \sqrt{\frac{\hbar L}{Mc}} > \sqrt{\frac{\hbar G}{c^3}} = l_p \tag{1.14}$$

The Planck length is indeed the smallest distance one can measure.

The appearance of the smallest distance means that a quantum theory of gravity, no matter what it is, will definitely not a quantum field theory. It is because that quantum field theory is defined based on local observables, the fields at each *point* of space-time. But at the scale of Planck length because that we cannot go beyond it, we cannot even tell where particle locates. Just as what Bronstein mentioned in his 1935 paper that "a radical reconstruction of the theory ... perhaps also the rejection of our ordinary concepts of space and time, replacing them by some much deeper and nonevident concepts."

Nowadays in the research of the QG, physicists inherit and carry forward his idea. Many candidate theories have been developed. String theory from the perturbative perspective suggests that the elementary building blocks of our world are strings. The existing particles are the vibrations of the strings in a fixed ten or eleven background. The renormalization-group equations of the theory imply the Einstein equations of the background metric coupled to a dilaton and some fermions and bosons. Gravity is not a fundamental field but only an effective field. On the other hand, the non-perturbative approach of QG suggests to straightforward quantize general relativity independent from a fixed background by using the symmetries it already has: the general coordinate transformation and the local Lorentz transformation. Loop quantum gravity is a particular theory realizing this idea, and it is the main subject of this thesis.

Then what is loop quantum gravity?

## 1.3 A glance at loop quantum gravity

Loop quantum gravity(LQG) is a straightforward non-perturbative quantization of GR independent from a fix background. It predicts that the geometry of space is quantized, in which area and volume can only take discrete value[4]. When applied to cosmology, LQG naturally removes the cosmological singularity for the homogeneous isotropic cosmology model[5, 6]. When applied to black holes, LQG gives a microscopic statistical origin of the black hole entropy, which coincides with Bekenstein-Hawking entropy at the leading order[7, 8].

LQG comes in three versions. The historically first of which provides a canonical quantisation of general relativity, and seeks to solve the Wheeler-DeWitt equation, a quantum version of Einstein equation[9]. The rest, we call them spin foam gravity[10] and Group Field Theory (GFT)[11] respectively, propose the covariant path-integral formulation. All approaches share the kinematical structure of LQG: the Hilbert space with observables representing, for example, discrete areas and volumes (because in general any geometric quantity is an observable and it is coded in the quantum states). The kinematical Hilbert space is spanned by Penrose's spin network functions. The excitation of geometry can be neatly visualized as fuzzy polyhedra that glued through their facets[12, 13]. However, the three versions differ concerning their description of the quantum dynamics of the theory. The canonical LQG follows the Dirac's quantization. The Wheeler-DeWitt equation is rigorously defined as the Hamiltonian constraint operator on the kinematical Hilbert space[14]. The quantum dynamics of LQG can be extracted once the Hamiltonian constraint is solved and the physical Hilbert space is constructed. The spin foam gravity defines the dynamics of LQG by a spin foam amplitude on a cellular complex, bounded by the spin network states[15]. Using the technique of quantum group, the amplitude is finite, and its low energy limit gives the discrete Einstein gravity with a positive cosmological constant[16, 17]. For capturing the infinite number of degrees of freedom in GR, the spin foam gravity should take a continuum limit. It comes into two strategies: the first strategy rests on a lattice gauge theory interpretation of spin foam formalism, refining the cellular complex to estimate the continuum geometry of space-time; while the second one rests on the 2nd quantization reformulation of LQG by summing over all possible complexes with the same boundary. The GFT is a quantum field theory (QFT) sitting on a Lie group manifold, which closely relates to canonical LQG and spin foam formalism. It is a QFT or 2nd quantization version of the LQG formalism[18]. GFT provides a prescription for summing over the spin foam complexes, in which the complexes arise as Feynman diagrams of GFT with the given spin foam amplitude as Feynman amplitude[19]. The continuum dynamics of quantum gravity is expected to be recovered after summing over all spin foams and analysing the renormalization of GFT[20].

In this thesis I am focusing on the spin foam formalism of LQG. I will discuss in detail how to reconstruct the classical discrete geometry from the spin foam amplitude.

# Chapter 2

# Spin foam formalism

The spin foam formalism adapts the covariant path integral approach of quantum gravity into the LQG framework. In the traditional path integral approach of quantum gravity by Misner, Gibbons, Hawking and Hartle[21–23], the dynamics of quantum gravity is encoded in a quantum gravity amplitude, which is defined by a formal path integral

$$Z[M;h^{\mathrm{in}},h^{\mathrm{out}}]:=\int_{h_{ab}^{\mathrm{in}}}^{h_{ab}^{\mathrm{out}}}[\mathcal{D}g_{\mu\nu}]\ \mathrm{e}^{\frac{\mathrm{i}}{l_p^2}\int_M \mathrm{d}^4x\sqrt{-g}R+\cdots} \tag{2.1}$$

where $\int_M \mathrm{d}^4x\sqrt{-g}R$ is the Einstein-Hilbert action of gravity on a four-dimensional smooth manifold $M$, and $\cdots$ includes the boundary terms as well as the high curvature corrections[1]. $\mathcal{D}g_{\mu\nu}$ is a formal integral measure on the space of four dimensional metric on $M$, whose boundary data are fixed by three-dimensional metric $h_{ab}^{\mathrm{out}}, h_{ab}^{\mathrm{in}}$ on $\partial M$ (the boundary of $M$).

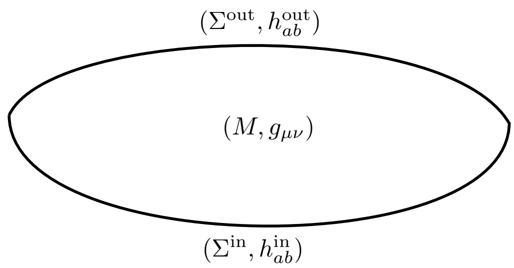


Fig. 2.1 Four-metric as a history of three-metrics

The situation is illustrated in Fig.2.1, where the four-metric $g_{\mu\nu}$ on $M$ can be viewed

[1]The high curvature terms include the terms of $o(R^2)$ and higher. The high curvature terms have to be included in order to make the quantum theory perturbatively renormalizable or finite, as suggested by perturbative QG [24] and string theory [25].

as a history of three-metrics evolving from $h_{ab}^{\text{in}}$ on $\Sigma^{\text{in}}$ to $h_{ab}^{\text{out}}$ on $\Sigma^{\text{out}}$, $\partial M = \Sigma^{\text{in}} \cup \Sigma^{\text{out}}$. The path integral Eq.(2.1), as a sum over $g_{\mu\nu}$, can be viewed as a sum over the histories of three geometries with boundary data $h_{ab}^{\text{in}}$ and $h_{ab}^{\text{out}}$, weighted by $\mathrm{e}^{\frac{\mathrm{i}}{l_p^2}\int_M \mathrm{d}^4x\sqrt{-g}R+\cdots}$.

## 2.1 Quantum three-geometry: spin-networks

When we adapt the above construction to the LQG framework, the classical notion of three-geometry, the three-metric $h_{ab}$ should be properly replaced by the notion of quantum three-geometry in LQG. LQG has a clean and beautiful description of quantum three-geometry in the kinematical framework. The description is unique in terms of the representation theory of holonomy-flux algebra[26, 27]. In the LQG description of three-geometry, the quantum three-geometry is represented by the spin-network state $S = (\Gamma, j_l, i_n)$(proposed by Rovelli and Smolin[28]) in the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$, i.e. in LQG

$$\text{Quantum three-geometry} = \text{Spin-network state} \tag{2.2}$$

Let us explain briefly the notion of spin-networks. A spin-network state $S = (\Gamma, j_l, i_n)$ is a triple of three types of data: a graph $\Gamma$, some spins $j_l$ and some intertwiners $i_n$ (see Fig.2.2)

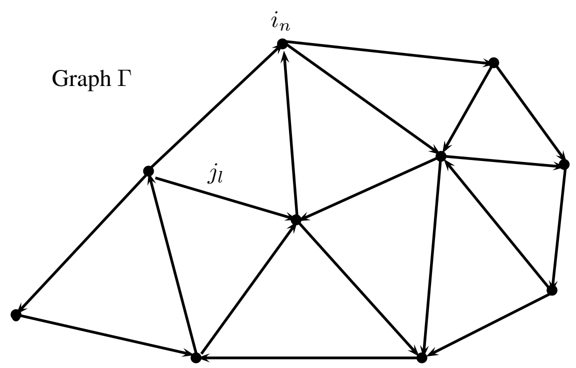


Fig. 2.2 A spin-network $S = (\Gamma, j_l, i_n)$

- A graph $\Gamma$ consists a number of oriented links $l$ and a number of nodes $n$. The links are analytic curvatures if the graph is embedded in a three-manifold $\Sigma$. The uni-valent node is excluded by the gauge invariance.

- Each link $l$ is colored by a unitary irreducible representation (labelled by a spin)

$$j_l \in \mathrm{Irrep}[\mathrm{SU}(2)] \tag{2.3}$$

- Each node $n$ is colored by an invariant tensor (an intertwiner)

$$i_n \in \mathrm{Inv}\left(\bigotimes_{k(\text{outgoing})\in n} V_{j_k} \otimes \bigotimes_{k(\text{incoming})\in n} V^*_{j_k}\right) \tag{2.4}$$

  where $V_{j_k}$ ($V^*_{j_k}$) is the SU(2) irreducible representation space (dual space) associated with a link $k$ outgoing (incoming) adjacent to the node $n$.

Clearly each spin-network $S$ with $L$ links and $N$ nodes associates a function $F_S(h_l)$ in $L^2\left(\mathrm{SU}(2)^L/\mathrm{SU}(2)^N\right) \equiv \mathcal{H}_\Gamma$ by

$$F_S(h_l) := \mathrm{tr}\left(\bigotimes_n i_n \bigotimes_l \sqrt{2j_l+1} D^{j_l}(h_l)\right), \quad h_l \in \mathrm{SU}(2), \tag{2.5}$$

where $D^{j_l}(h_l)$ is the SU(2) unitary irreducible representation matrix with spin $j_l$ and tr denotes the contractions of tensor indices according to the graph $\Gamma$. The LQG kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ is a union of $\mathcal{H}_\Gamma$ over all graphs modulo some equivalence relations[10, 29, 30]

$$\mathcal{H}_{\text{kin}} := \bigcup_\Gamma \mathcal{H}_\Gamma / \sim \tag{2.6}$$

The geometric interpretation of spin-networks is clarified by the geometrical operators defined on $\mathcal{H}_{\text{kin}}$, e.g. the area operator and volume operator [31–33]. It turns out that the spin-network states diagonalize the area and volume operators and give discrete spectra. Given a spin-network $S$, each link $l$ carries quantum number $j_l$, which labels the quanta of area on a two-surface transverse to the link $l$. The spectrum of area operator is given by $A = 8\pi\gamma G\hbar\sqrt{j(j+1)}$ in the simplest case ($\gamma$ is the Barbero-Immirzi parameter). Each node $n$ carries the quantum number $i_n$, which labels the quanta of spatial volume occupied by the node $n$. The volume spectrum is more complicated, whose computations and results are proposed in e.g. [34, 35]. Each node of the spin-network is associated naturally with a chunk of three-space. The chunks of space may be represented by a three-dimensional polytope, whose face areas and volumes relate to the quantum numbers $j_l$ and $i_n$.

## 2.2 Quantum four-geometry: Spin foam

Recall Fig.2.1, where the classical four-metric $g_{\mu\nu}$ is understood as a history of classical three-geometry. In the context of LQG, the three-geometry is quantized to be the spin-network state. Thus the quantum analogy of a four-metric is then a history of quantum three-geometry, i.e. a hisotry of spin-networks, which we call a *spin foam*(proposed by Reisenberger and Rovelli[36–38]):

$$\text{Quantum four-geometry} \equiv \text{History of spin-networks} \equiv \text{Spin foam} \tag{2.7}$$

An example of spin foam is illustrated in Fig.2.3, as an evolution history of spin-networks. A link in the spin-network evolves and creates a surface in the spin foam, and a node in the spin-network evolves and creates an edge in the spin foam. If we imagine the spin foam is embedded in a four-manifold, any hypersurface transverse to the spin foam edges intersects the spin foam and gives a spin-network as the intersection.

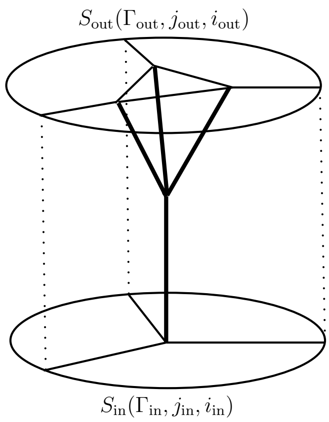


Fig. 2.3 Spin foam = history of spin-networks

As an analogy of the traditional path integral approach Eq.(2.1), the sum of spin foams with given boundary data defines a transition amplitude $Z(\mathcal{K}, S_{\text{Boundary}})$ (spin foam amplitude) between quantum three-geometries. Here $S_{\text{Boundary}}$ is the boundary spin-network (boundary quantum three-geometry) which serves as the boundary data in analogy with $h^{\text{in}}$ and $h^{\text{out}}$ in Eq.(2.1). $\mathcal{K}$ is a 2-complex (definition is given in the next section) as an analogy of the smooth four-manifold $M$ in Eq.(2.1).

## 2.3 Formal definition of spin foam amplitude

In this section I give a formal definition of the spin foam amplitude $Z(\mathcal{K}, S_{\text{Boundary}})$. The definition follows the framework presented in [39].

First of all, a spin foam is a triple of data

$$(\mathcal{K}, j_f, i_e) \tag{2.8}$$

where the three types of data are explained in the following:

- $\mathcal{K}$ denotes a 2-complex (or cellular complex), or namely a “foam”, which consists a number of oriented faces $f$, oriented edges $e$ and vertices $v$

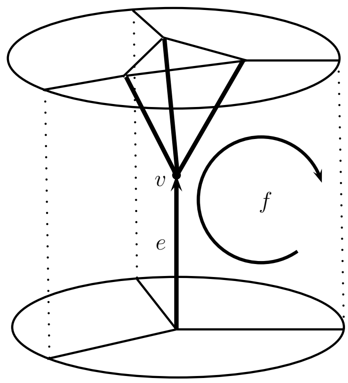


Fig. 2.4 A two-complex and orientations

- $j_f$ assigns to each oriented face $f$ an SU(2) spin-$j_f$ unitary irreducible representation.

- $i_e$ assigns to each oriented edge $e$ an SU(2) intertwiner $i_e \in \mathrm{Inv}(V_{j_{f_1}} \otimes \cdots \otimes V^*_{j_{f_k}})$, where $f_1, \cdots, f_k$ are the faces sharing the edge $e$. Taking $V_{j_f}$ or $V^*_{j_f}$ in the definition of $i_e$ depends on whether the orientation of the face $f$ is consistent or opposite to the orientation of $e$.

An amplitude can associated with each object of the two-complex: given a vertex $v$ shared by a number of edges, it associates a vertex amplitude $A_v(j_f, i_e)$ as a complex-valued function of the intertwiners $i_e$ of the adjacent edges and the spins $j_f$ of the adjacent faces. Given an edge $e$ shared by a number of faces, it associates an edge amplitude $A_e(j_f, i_e)$ as a complex-valued function of the intertwiner $i_e$ of the edge itself and the spins $j_f$ of the adjacent faces. Given a face $f$, it associates a face amplitude $A_f(j_f)$ as a complex-valued function of the spin $j_f$ of the face itself.

A spin foam amplitude is constructed by a product of all the amplitudes associated with vertices, edges and faces, followed by a sum over the data $j_f$ and $i_e$

$$Z(\mathcal{K}, S_{\text{Boundary}}) := \sum_{j_f, i_e} \prod_f A_f(j_f) \prod_e A_e(j_f, i_e) \prod_v A_v(j_f, i_e) \tag{2.9}$$

A concrete construction of spin foam amplitude is present here by following the construction by Engle, Pereira, Rovelli[40] and Livine[41], Freidel and Krasnov[42], Livine and Speziale[43]: the spin foam vertex amplitude $A_v$ is defined by a contraction of the $\text{SL}(2,\mathbb{C})$ intertwiners $I_e$ associated with the oriented edges $e$ joining at the vertex $v$:

$$A_v(j_f, i_e) := \operatorname{tr}\left(\bigotimes_{\text{incoming } e} I_e \bigotimes_{\text{outgoing } e} I_e^*\right) \tag{2.10}$$

Here each $\text{SL}(2,\mathbb{C})$ intertwiner $I_e$ is "evolved" from the SU(2) intertwiner $i_e$ by the following "propagation": We define a map $Y$ from SU(2) spin-$j$ unitary irreducible representations to the $\text{SL}(2,\mathbb{C})$ unitary irreducible representation labelled by $(\rho, k)$, where $\rho \in \mathbb{R}$ and $m \in \mathbb{Z}/2$, requiring $\rho = \gamma j$ and $k = j$

$$Y : |j, m\rangle \mapsto |(\gamma j, j); j, m\rangle \tag{2.11}$$

where $|(\gamma j, j); j, m\rangle$ is the canonical basis in the of the $\text{SL}(2,\mathbb{C})$ unitary irreducible representation $(\gamma j, j)$, with $\gamma \in \mathbb{R}$ being the Barbero-Immirzi parameter. The $\text{SL}(2,\mathbb{C})$ intertwiner $I_e$ is then defined as

$$I_e(j_f, i_e) = P^{\text{inv}}_{\text{SL}(2,\mathbb{C})} \circ Y^{\otimes k}(i_e) = \int_{\text{SL}(2,\mathbb{C})} \mathrm{d}g \prod_{i=1}^{k} D^{(\gamma j_i, j_i)}_{j_i m_i', j_i, m_i}(g)\; i_e^{m_1 \cdots m_k} \tag{2.12}$$

where $k$ is the valence of the intertwiner $i_e$ and $P^{\text{inv}}_{\text{SL}(2,\mathbb{C})}$ is a projector into the space of $k$-valent $\text{SL}(2,\mathbb{C})$ intertwiners.

Inserting the $\text{SL}(2,\mathbb{C})$ intertwiner defined by Eq.(2.12) to Eq.(2.10), a spin foam vertex amplitude $A_v$ is obtained concretely, which is often referred as the *EPRL/FK vertex amplitude* in the literature.

Moreover we choose the face amplitude $A_f(j_f) = 2j_f + 1$ for the reason of consistency that if a two-complex can be decomposed into two, then its spin foam amplitude should be the multiplication of the amplitudes of the two[44]. The edge amplitude is chosen to be $A_e = 1$ for simplicity. The resulting spin foam amplitude

$$Z(\mathcal{K}, S_{\text{boundary}}) = \sum_{j_f} \sum_{i_e} \prod_f (2j_f + 1) \prod_v \text{tr} \left( \bigotimes_{\text{incoming } e} I_e \bigotimes_{\text{outgoing } e} I_e^* \right) \tag{2.13}$$

is often referred as the *EPRL/FK spin foam amplitude.*

There are a few important properties of the EPRL/FK spin foam amplitude:

- for a generic two-complex $\mathcal{K}$, the summand of the spin-sum $\sum_{j_f}$ in the EPRL/FK amplitude Eq.(2.13) is *finite* after removing an $\mathrm{SL}(2,\mathbb{C})$ gauge redundancy for each vertex[45, 46]. Therefore the only possible divergence in the spin foam amplitude comes from the summation on the spins $\sum_{j_f}$. See e.g. [47, 48] for computation of the degree of divergence on certain two-complex.

- The EPRL/FK spin foam amplitude is Lorentz invariant in the bulk and Lorentz covariant near the boundary [49]: Although the construction of the EPRL/FK vertex amplitude depends on specifying an SU(2) subgroup in $\mathrm{SL}(2,\mathbb{C})$, or a "time-gauge" $x_e$ (time-like Minkowski four-vector) for each edge $e$, the amplitude $Z(\mathcal{K}, S_{\text{boundary}})$ is independent from the choice of $x_e$ for an internal edge $e$, and transforms covariantly as $x_e$ of boundary edges transform under the Lorentz transformation.

- The above construction of EPRL/FK spin foam amplitude is an analogy of the Feynman diagram construction of quantum field theory scattering amplitude[19, 50]. The representation in Eq.(2.13) factorized the spin foam amplitude in terms of vertices in $\mathcal{K}$. Indeed such a representation of spin foam amplitude can be generated from a quantum field theory on group manifold by the corresponding Feymann diagrams [48].

## 2.4 Other representations of spin foam amplitude

The spin foam amplitude, as the central object in the spin foam formulation of LQG, has several other remarkable representations in addition to the above definition. I review these representations briefly in the follows. Some of the representations are the equivalent formulations of the above EPRL/FK amplitude while others admit certain extensions or completions in some sense.

**Face Amplitude and Charaters:** Instead of factorizing the EPRL/FK spin foam amplitude $Z$ in terms of vertices as Eq.(2.13), $Z$ can be factorized in terms of faces (see [51] for a set of Feynman rules):

$$Z = \sum_{j_f} \int_{\mathrm{SL}(2,\mathbb{C})} \mathrm{d}g_{ve} \int_{\mathrm{SU}(2)} \mathrm{d}h_{ef} \prod_f \dim(j_f)^2 \, \chi^{(\gamma j_f, j_f)} \left( \prod_{(e,f)} (g_{e,s(e)} h_{ef} g_{e,t(e)}^{-1})^{\varepsilon_{ef}} \right) \prod_{(e,f)} \chi^{j_f}\left(h_{ef}\right) \tag{2.14}$$

where the factor corresponding to each $f$ is called a face amplitude (which should not be confused with the face amplitude $A_f$ in the previous representation). $\chi^{\gamma j_f, j_f}$ and $\chi^{j_f}$ are respectively the characters of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SU}(2)$ unitary irreducible representations $\varepsilon_{ef} = \pm 1$ depends on whether the orientations of $e$ and $f$ agree or not.

**Edge Projector:** The spin foam amplitude can also be factorized in terms of edges, which leads to the following representation [52, 53]:

$$Z = \sum_{j_f} \prod_f \dim(j_f) \ \mathrm{Tr} \left( \prod_e P_e^{inv} \right) \tag{2.15}$$

where $P_e^{inv}$ is a certain projector onto a subspace of $\mathrm{SL}(2,\mathbb{C})$ intertwiners. It is remarkable that, as the factorization in terms of vertex amplitudes, this representation is a general structure valid for all spin foam models, including e.g. the Barrett-Crane model [54, 55], Ponzano-Regge model [56], Ooguri model [57], etc.

**Holonomy Spin foam:** The spin foam amplitude (Euclidean EPRL/FK, Barrett-Crane, Ooguri, etc) can be expressed as an analogy of lattice gauge theory by performing the spin-sum in the first place [58]:

$$Z = \int_{\mathrm{Spin}(4)} \mathrm{d}g_{ev} \int_{\mathrm{Spin}(4)} \mathrm{d}g_{ef} \prod_f \omega(g_f) \prod_{e \subset f} E(g_{ef}) \tag{2.16}$$

where $\omega$ and $E$ are certain distributions on the group manifold. Such a representation is useful in semiclassical analysis and a coarse graining procedure in spin foam formulation [59–62].

**Group Field Theory (GFT):** Each spin foam model can associates a GFT, as a certain quantum field theory on group manifold. GFT generates the spin foam amplitudes via the Feynman perturbative expansion, and in addition, generates the sum of spin foam amplitudes over a class of two-complexes. I will not go into details of the GFT formulation but rather refer to the literature e.g. [18]. For the GFT corresponding to the EPRL/FK spin foam model, see [48].

**Coherent State Path Integral:** By using coherent states on Lie group [63], there is a useful representation of EPRL/FK spin foam amplitude as a coherent state path integral [16, 64–68]:

$$Z = \sum_{j_f} \prod_f \dim(j_f) \int_{\mathrm{SL}(2,\mathbb{C})} \mathrm{d}g_{ve} \int_{\mathbb{C}P^1} \mathrm{d}z_{vf}\ e^{S[j_f, g_{ve}, z_{vf}]} \tag{2.17}$$

where $S[j_f, g_{ve}, z_{vf}]$ is a "spin foam action". This path integral representation is in particularly useful in the semiclassical analysis in spin foam formulation and is one of the main context of the thesis. I will come back to this representation later in the follows.

**Spinor and Twistor:** With spinor or twistor, the holonomy-flux phase space of LQG can be reparametrized and has a very clear geometric interpretation, which is known as twisted geometry first introduced by Freidel and Speziale[12, 13]. In this reparametrization, the EPRL transition amplitude can be derived as a path integral in twistor space, by using the quantized LQG twistorial phase space and a discretization of the BF action which is bilinear in the spinors[69].

In this thesis, I am mainly focusing on the coherent state path integral formulation of the spin foam amplitude and the spinor/twistor reparametrization of the LQG. In the next chapter, I will present the semiclassical analysis of the spin foam amplitude based on its coherent state path integral formulation, and show how to reconstruct the classical discrete geometry by using the spin foam critical configurations. In chapter 4, how to describe the geometry of the null hypersurface will be discussed within the twistorial reparametrization of the LQG phase space.

# Chapter 3

# Semiclassical behavior of spin foam amplitude

## 3.1 Motivations and outlines

Every physical theory has its domain of validity which is controlled by the three fundamental constants $G$, $c^{-1}$ and $\hbar$, as mentioned in the chapter of introduction. LQG, as a quantum theory of gravity, is in the domain where all of the three fundamental constants are turned on. However since the theory of LQG, especially in the construction of the spin foam amplitude presented in the previous chapter, seems coming out from nowhere, people will ask immediately "How can you tell that the spin foam amplitude gives a quantum theory of gravity?" or "How this mathematical theory relates the theory of gravity?" To answer these questions, let me remind the correspondence principle firstly formulated by Niels Bohr in 1920 [70], which states that the behavior of systems described by the theory of quantum mechanics reproduces classical physics in the limit of large quantum number. The conditions under which quantum and classical physics agree are referred to as the *semiclassical limit*. So in order to clarify the relation between spin foam amplitude and the classical gravity, we only need to perform the semiclassical limit to get the semiclassical behavior of the theory.

The semiclassical behavior of spin foam model is currently understood in terms of the *large-$j$ asymptotics* of the spin foam amplitude, i.e. if we consider a spin foam model as

$$A(\mathcal{K}) = \sum_{j_f} \mu(j_f) A_{j_f}(\mathcal{K}) \tag{3.1}$$

where $\mu(j_f)$ is a measure, we are investigating the asymptotic behavior of the (partial-

)amplitude $A_{j_f}$ as all the spins $j_f$ are taken to be large uniformly. The area spectrum in LQG is given approximately by $A_f = \gamma j_f \ell_p^2$, so the semiclassical limit of spin foam models is argued to be achieved by taking $\ell_p^2 \to 0$ while keeping the area $A_f$ fixed, which results in $j_f \to \infty$ uniformly as $\gamma$ is a fixed Barbero-Immirzi parameter. There is another argument relating the large-$j$ asymptotic of the spin foam amplitude to the semiclassical limit, by imposing the semiclassical boundary state to the vertex amplitude [71]. Mathematically the asymptotic problem is posed by making a uniform scaling for the spins $j_f \mapsto \lambda j_f$, and studying the asymptotic behavior of the amplitude $A_{\lambda j_f}(\mathcal{K})$ as $\lambda \to \infty$.

There were various investigations for the large-$j$ asymptotics of the spin foam models. The asymptotics of the Barrett-Crane vertex amplitude (10j-symbol) was studied in [72, 73], which showed that the degenerate configurations in Barrett-Crane model were nonoscillatory, but dominant. The large-$j$ asymptotics of the FK model was studied in [64], concerning the nondegenerate Riemanian geometry, in the case of a simplicial manifold without boundary. The large-$j$ asymptotics of the EPRL model was initially investigated in [66, 74] for both Euclidean and Lorentzian cases, where the analysis concerned a single 4-simplex amplitude (EPRL vertex amplitude). It was shown that the asymptotics of the vertex amplitude is mainly a Cosine of the Regge action in a 4-simplex if the boundary data admits a nondegenerate 4-simplex geometry, and the asymptotics is non-oscillatory if the boundary data doesn't admit a nondegenerate 4-simplex geometry. There were also works found that the Regge gravity from the Euclidean/Lorentzian spin foam amplitude on a simplicial complex via a certain "double scaling limit" [68, 75].

In this chapter I present my works with Dr. Muxin Han [16, 67] that analyzes the large-$j$ asymptotic analysis of the Lorentzian EPRL spin foam amplitude to the general situation of a 4d simplicial manifold with or without boundary, with an arbitrary number of simplices. The asymptotics of the spin foam amplitude is determined by the critical configurations of the "spin foam action", and is given by a sum of the amplitudes evaluated at the critical configurations. Therefore the large-$j$ asymptotics is clarified once we find all the critical configurations and clarify their geometrical implications. Here for the Lorentzian EPRL spin foam amplitude, a critical configuration in general is given by the data $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ that solves the critical point equations, where $j_f$ is an SU(2) spin assigned to each triangle, $g_{ve}$ is an $\mathrm{SL}(2,\mathbb{C})$ group variable, and $\xi_{ef}, z_{vf}$ are two types of spinors. Here in this work we show that given a general critical configuration, there exists a partition of the simplicial complex $\mathcal{K}$ into three types of regions $\mathcal{R}_{\text{Nondeg}}, \mathcal{R}_{\text{Deg-A}}, \mathcal{R}_{\text{Deg-B}}$, where the three regions are simplicial sub-complexes with boundaries, and they may be disconnected regions. The critical configuration implies different types of geometries in different types of regions:

- The critical configuration restricted into $\mathcal{R}_{\text{Nondeg}}$ is nondegenerate in our definition of degeneracy. It implies a nondegenerate discrete Lorentzian geometry on the simplicial sub-complex $\mathcal{R}_{\text{Nondeg}}$.
- The critical configuration restricted into $\mathcal{R}_{\text{Deg-A}}$ is degenerate of type-A in our definition of degeneracy. However, it implies a nondegenerate discrete Euclidean geometry on the simplicial sub-complex $\mathcal{R}_{\text{Deg-A}}$
- The critical configuration restricted into $\mathcal{R}_{\text{Deg-B}}$ is degenerate of type-B in our definition of degeneracy. It implies a vector geometry on the simplicial sub-complex $\mathcal{R}_{\text{Deg-B}}$

With the critical configuration, we further make a subdivision of the regions $\mathcal{R}_{\text{Nondeg}}$ and $\mathcal{R}_{\text{Deg-A}}$ into sub-complexes (with boundary) $\mathcal{K}_1(\mathcal{R}_*), \cdots, \mathcal{K}_n(\mathcal{R}_*)$ ($*$=Nondeg,Deg-A) according to their Lorentzian/Euclidean oriented 4-volume $V_4(v)$ of the 4-simplices, such that $\text{sgn}(V_4(v))$ is a constant sign on each $\mathcal{K}_i(\mathcal{R}_*)$. Then in the each sub-complex $\mathcal{K}_i(\mathcal{R}_{\text{Nondeg}})$ or $\mathcal{K}_i(\mathcal{R}_{\text{Deg-A}})$, the spin foam amplitude at the critical configuration gives an exponential of Regge action in Lorentzian or Euclidean signature respectively. However we emphasize that the Regge action reproduced here contains a sign factor $\text{sgn}(V_4(v))$ related to the oriented 4-volume of the 4-simplices, i.e.

$$S = \text{sgn}(V_4) \sum_{\text{Internal } f} A_f \Theta_f + \text{sgn}(V_4) \sum_{\text{Boundary } f} A_f \Theta_f^B \tag{3.2}$$

where $A_f$ is the area of the triangle $f$ and $\Theta_f, \Theta_f^B$ are deficit angle and dihedral angle respectively. Recall that the Regge action without $\text{sgn}(V_4)$ is a discretization of Einstein-Hilbert action of GR. Therefore the Regge action reproduced here is actually a discretized Palatini action with the on-shell connection (compatible with the tetrad).

The asymptotic formula of the spin foam amplitude is given by a sum of the amplitudes evaluated at all possible critical configurations, which are the products of the amplitudes associated to different type of geometries.

Additionally, we also show that given a spin foam amplitude $A_{j_f}(\mathcal{K})$ with the spin configuration $j_f$, any pair of the non-degenerate critical configurations associated with $j_f$ are related each other by a *local* parity transformation. A similar result holds for any pair of the degenerate configuration of type-A associated with $j_f$, since it implies a nondegenerate Euclidean geometry.

## 3.2 Lorentzian spin foam amplitude

In this section I give a detail definition of the spin foam amplitude in the coherent states formulation.

Given a simplicial complex $\mathcal{K}$ (with or without boundary), the Lorentzian spin foam amplitude on $\mathcal{K}$ can be expressed in the coherent state representation:

$$A(\mathcal{K}) = \sum_{j_f} \prod_f \mu\left(j_f\right) \prod_{(v,e)} \int_{\mathrm{SL}(2,\mathbb{C})} \mathrm{d}g_{ve} \prod_{(e,f)} \int_{S^2} \mathrm{d}\hat{n}_{ef} \prod_{v\in f} \left\langle j_f, \xi_{ef} \left| Y^\dagger g_{ev} g_{ve'} Y \right| j_f, \xi_{e'f} \right\rangle \tag{3.3}$$

Here $\mu(j_f)$ is the face amplitude of the spin foam, given by $\mu(j_f) = (2j_f + 1)$. $|j_f, \xi_{e'f}\rangle$ is an SU(2) coherent state in the Spin-$j$ representation. The coherent state is labeled by the spin $j$ and a normalized 2-component spinor $|\xi_{ef}\rangle = g(\xi_{ef})|\frac{1}{2},\frac{1}{2}\rangle$ ($n_{ef} \in \mathrm{SU}(2)$), while $\hat{n}_{ef} := g(\xi_{ef}) \triangleright \hat{z}$ is a unit three-vector. $Y$ is an embedding map from the Spin-$j$ irreducible representation $\mathcal{H}^j$ of SU(2) to the unitary irreducible representation $\mathcal{H}^{(j,\gamma j)}$ of $\mathrm{SL}(2,\mathbb{C})$ with $(k,\rho) = (j, \gamma j)$. The embedding $Y$ identify $\mathcal{H}^j$ with the lowest level in the decomposition $\mathcal{H}^{(j,\gamma j)} = \oplus_{k=j}^{\infty} \mathcal{H}^k$. Therefore we define an $\mathrm{SL}(2,\mathbb{C})$ coherent state by the embedding

$$|(j_f, \gamma j_f); j_f, \xi_{ef}\rangle := Y\,|j_f, \xi_{e'f}\rangle = \Pi^{(j_f,\gamma j_f)}\Big(g(\xi_{ef})\Big)\,|(j_f, \gamma j_f); j_f, j_f\rangle\,. \tag{3.4}$$

In order to write the $\mathcal{H}^{(j_f,\gamma j_f)}$ inner product in Eq.(3.3) explicitly, we express the $\mathrm{SL}(2,\mathbb{C})$ coherent state in terms of the canonical basis [76]. The Hilbert space $\mathcal{H}^{(k,\rho)}$ can be represented as a space of homogeneous functions of two complex variables $(z^0, z^1)$ with degree $(-1 + ip + k; -1 + ip - k)$, i.e.

$$f(\lambda z^A) = \lambda^{-1+i\rho+k} \bar{\lambda}^{-1+i\rho-k} f(z^A) \tag{3.5}$$

Given a normalized two-component spinor $z^A$ ($A = 0, 1$) with $\langle z, z\rangle := \delta_{A\dot{A}} \bar{z}^{\dot{A}} z^A = 1$, we construct the SU(2) matrix

$$g(z) = \begin{pmatrix} z^0 & -\bar{z}^1 \\ z^1 & \bar{z}^0 \end{pmatrix} \equiv (z, Jz) \tag{3.6}$$

where $J(z^0, z^1)^t := (-\bar{z}^1, \bar{z}^0)^t$. The canonical basis $f^j_m(z)^{(k,\rho)} = |(k,\rho); j, m\rangle$ in the $\mathrm{SL}(2,\mathbb{C})$ unitary irreducible representation $\mathcal{H}^{(k,\rho)}$ is given by the following when restricted to the normalized spinors

$$f^j_m(z)^{(k,\rho)} = \sqrt{\frac{\dim(j)}{\pi}} D^j_{mk}\Big(g(z)\Big) \tag{3.7}$$

where $D^j_{mk}(g)$ is the SU(2) representation matrix. The canonical basis $f^j_m(z)^{(k,\rho)}$ evaluated on the non-normalized spinor $z^A$ is then given by the homogeneity

$$f^j_m(z)^{(k,\rho)} = \sqrt{\frac{\dim(j)}{\pi}} \langle z, z\rangle^{i\rho-1-j} D^j_{mk}\Big(g(z)\Big) \tag{3.8}$$

while here $D^j_{mk}\Big(g(z)\Big)$ is a analytic continuation of the SU(2) representation matrix. Thus we can write down explicitly the highest weight state in the $j$-representation and in the case of $(k,\rho) = (j, \gamma j)$

$$f^j_j(z)^{(j,\gamma j)} = \sqrt{\frac{\dim(j)}{\pi}} \langle z, z\rangle^{i\gamma j-1-j} (z^0)^{2j} \tag{3.9}$$

Therefore the coherent state is given explicitly by

$$|(j,\gamma j); j, \xi\rangle = f^j_\xi(z)^{(j,\gamma j)} = f^j_j\Big(g(\xi)^t z\Big)^{(j,\gamma j)} = \sqrt{\frac{\dim(j)}{\pi}} \langle z, z\rangle^{i\gamma j-1-j} \langle \bar{z}, \xi\rangle^{2j} \tag{3.10}$$

As a result we can write down explicitly the inner product in Eq.(3.3) in terms of a $L^2$ inner product on $\mathbb{CP}^1$ between the coherent states $f^j_\xi(z)^{(j,\gamma j)}$

$$\begin{aligned}\Big\langle j_f, \xi_{ef}\Big|Y^\dagger g_{ev} g_{ve'} Y\Big| j_f, \xi_{e'f}\Big\rangle &= \Big\langle (j_f,\gamma j_f); j_f, \xi_{ef}\Big| g_{ev} g_{ve'}\Big|(j_f,\gamma j_f); j_f, \xi_{e'f}\Big\rangle \\ &= \int_{\mathbb{CP}^1} \Omega_{z_{vf}} \overline{f^{(j_f,\gamma j_f)}_{\xi_{ef}}\Big(g^t_{ve} z_{vf}\Big)} f^{(j_f,\gamma j_f)}_{\xi_{e'f}}\Big(g^t_{ve'} z_{vf}\Big)\end{aligned} \tag{3.11}$$

where $\Omega_z = \frac{i}{2}(z_0 dz_1 - z_1 dz_0) \wedge (\bar{z}_0 d\bar{z}_1 - \bar{z}_1 d\bar{z}_0)$ is a homogeneous measure on $\mathbb{C}^2$.

We insert the result Eq.(3.11) back into Eq.(3.3) and define a new spinor variable $Z_{vef}$ and a measure on $\mathbb{CP}^1$ (a scaling invariant measure)

$$\begin{aligned} Z_{vef} &:= g^\dagger_{ve} z_{vf} \\ \Omega_{vf} &:= \frac{\Omega_{z_{vf}}}{\langle Z_{vef}, Z_{vef}\rangle \langle Z_{ve'f}, Z_{ve'f}\rangle}\end{aligned} \tag{3.12}$$

Then the spin foam amplitude $A(\mathcal{K})$ can be written as

$$A(\mathcal{K}) = \sum_{j_f} \prod_f \mu(j_f) \prod_{(v,e)} \int_{\mathrm{SL}(2,\mathbb{C})} \mathrm{d}g_{ve} \prod_{(e,f)} \int_{S^2} \mathrm{d}\hat{n}_{ef} \prod_{v\in\partial f} \int_{\mathbb{CP}^1} \left(\frac{\dim(j_f)}{\pi}\Omega_{vf}\right) e^S \tag{3.13}$$

where we have a "spin foam action" $S = \sum_f S_f$ and

$$S_f = \sum_{v\in f} S_{vf} = \sum_{v\in f}\left( j_f \ln \frac{\langle \xi_{ef}, Z_{vef}\rangle^2 \langle Z_{ve'f}, \xi_{e'f}\rangle^2}{\langle Z_{vef}, Z_{vef}\rangle \langle Z_{ve'f}, Z_{ve'f}\rangle} + i\gamma j_f \ln \frac{\langle Z_{ve'f}, Z_{ve'f}\rangle}{\langle Z_{vef}, Z_{vef}\rangle}\right). \tag{3.14}$$

In this chapter we consider the large-$j$ regime of the spin foam amplitude $A(\mathcal{K})$. Concretely, we define the partial-amplitude

$$\begin{aligned} A_{j_f}(\mathcal{K}) &:= \prod_{(v,e)} \int_{\mathrm{SL}(2,\mathbb{C})} \mathrm{d}g_{ve} \prod_{(e,f)} \int_{S^2} \mathrm{d}\hat{n}_{ef} \prod_{v\in\partial f} \int_{\mathbb{CP}^1} \left(\frac{\dim(j_f)}{\pi}\Omega_{vf}\right) e^S \\ A(\mathcal{K}) &= \sum_{j_f} \prod_f \mu(j_f)\, A_j(\mathcal{K}) \end{aligned} \tag{3.15}$$

and consider the regime in the sum $\sum_{j_f}$ where all the spins $j_f$ are large. In this regime, the spin foam amplitude is a sum over the asymptotics of partial amplitude $A_j(\mathcal{K})$ with large spins $j_f$. In the following, we study the large-$j$ asymptotics of the partial-amplitudes $A_{j_f}(\mathcal{K})$ by making the uniform scaling $j_f \mapsto \lambda j_f$ and taking the limit $\lambda \to \infty$. Each face action $S_f \mapsto \lambda S_f$ scales linearly with $\lambda$, so we can use the generalized stationary phase approximation [77] to study the asymptotical behavior of $A_{j_f}(\mathcal{K})$ in large-$j$ regime.

Before coming to the asymptotic analysis, we note that in all the following discussions, we only consider the spin configurations such that $\sum_{f\subset t_e} \epsilon_f j_f \neq 0$ with $\epsilon_f = \pm 1$ for all $e$. Therefore the geometric tetrahedron with the oriented area $j_f \hat{n}_{ef}$, $f \subset t_e$ is always assumed to be nondegenerate.

### 3.2.1 Derivation of critical point equations

We use the generalized stationary phase method to study the large-$j$ asymptotics of the above spin foam amplitude. The spin foam amplitude have been reduced to the following type of integral:

$$f(\lambda) = \int_D \mathrm{d}x\; a(x)\; e^{\lambda S(x)} \tag{3.16}$$

where $D$ is a closed manifold, $S(x)$ and $a(x)$ are smooth, complex valued functions, and Re$S \leq 0$ (this will be shown in the following for the spin foam amplitude). For large parameter $\lambda$ the dominant contributions for the above integral comes from the *critical points* $x_c$, which are the stationary point of $S(x)$ and satisfy Re$S(x_c) = 0$. The asymptotic behavior of the above integral for large $\lambda$ is given by

$$f(\lambda) = \sum_{x_c} a(x_c) \left(\frac{2\pi}{\lambda}\right)^{\frac{r(x_c)}{2}} \frac{e^{i\mathrm{Ind}H'(x_c)}}{\sqrt{|\det_r H'(x_c)|}} e^{\lambda S(x_c)} \left[1 + o(\frac{1}{\lambda})\right] \tag{3.17}$$

for isolated critical points, where $r(x_c)$ is the rank of the Hessian matrix $H_{ij}(x_c) = \partial_i \partial_j S(x_c)$ at a critical point, and $H'(x_c)$ is the invertible restriction on ker$H(x_c)^\perp$. When the critical points are not isolated, the above $\sum_{x_c}$ is replaced by a integral over a submanifold of critical points. If the $S(x)$ doesn't have any critical point $f(\lambda)$ decreases faster than any power of $\lambda^{-1}$. From the above asymptotic formula, we see that the asymptotics of the spin foam amplitude is clarified by finding all the critical points of the action and evaluating the integrand at each critical point.

In order to find the critical points of the spin foam action, first of all, we show that the spin foam action $S$ satisfies Re$S \leq 0$. For each $S_{vf}$, by using the Cauchy-Schwarz inequality

$$\begin{aligned} \mathrm{Re}S_{vf} &= j_f \ln \frac{|\langle \xi_{ef}, Z_{vef}\rangle|^2 \, |\langle Z_{ve'f}, \xi_{e'f}\rangle|^2}{\langle Z_{vef}, Z_{vef}\rangle \, \langle Z_{ve'f}, Z_{ve'f}\rangle} \\ &\leq j_f \ln \frac{\langle \xi_{ef}, \xi_{ef}\rangle \, \langle Z_{vef}, Z_{vef}\rangle \, \langle \xi_{e'f}, \xi_{e'f}\rangle \, \langle Z_{ve'f}, Z_{ve'f}\rangle}{\langle Z_{vef}, Z_{vef}\rangle \, \langle Z_{ve'f}, Z_{ve'f}\rangle} \leq 0 \end{aligned} \tag{3.18}$$

Therefore

$$\mathrm{Re}S = \sum_{f,v} \mathrm{Re}S_{vf} \leq 0 \tag{3.19}$$

From Re$S = 0$, we obtain the following equations

$$\xi_{ef} = \frac{e^{i\phi_{ev}}}{\|Z_{vef}\|} Z_{vef}, \quad \text{and} \quad \xi_{e'f} = \frac{e^{i\phi_{e'v}}}{\|Z_{ve'f}\|} Z_{ve'f} \tag{3.20}$$

where $\|Z_{vef}\| \equiv |\langle Z_{vef}, Z_{vef}\rangle|^{1/2}$. If we define $\phi_{eve'} = \phi_{ev} - \phi_{e'v}$, the above equation results in that

$$\left(g^\dagger_{ve}\right)^{-1} \xi_{ef} = \frac{\|Z_{ve'f}\|}{\|Z_{vef}\|} e^{i\phi_{eve'}} \left(g^\dagger_{ve'}\right)^{-1} \xi_{e'f} \tag{3.21}$$

Here we use the property of anti-linear map $J$

$$JgJ^{-1} = \left(g^{\dagger}\right)^{-1} \tag{3.22}$$

to Eq.(3.21), we find

$$g_{ve}\left(J\xi_{ef}\right) = \frac{\|Z_{ve'f}\|}{\|Z_{vef}\|} e^{-i\phi_{eve'}} g_{ve'}\left(J\xi_{e'f}\right) \tag{3.23}$$

Now we compute the derivative of the action $S$ on the variables $z_{vf}, \xi_{ef}, g_{ve}$ to find the stationary point of $S$. We first consider the derivative with respect to the $\mathbb{C}P^1$ variable $z_{vf}$. Given a spinor $z^A = (z_0, z_1)^t$, $z^A$ and $(Jz)^A = (-\bar{z}_1, \bar{z}_0)$ is a basis of the space $\mathbb{C}^2$ of 2-component spinors. The following variation can be written in general by

$$\delta z^A = \varepsilon (Jz)^A + \omega z^A \tag{3.24}$$

where $\varepsilon, \omega$ are complex number. Since $z \in \mathbb{CP}^1$, we can choose a partial gauge fixing that $\langle z, z\rangle = 1$, which gives $\langle \delta z, z\rangle = -\langle z, \delta z\rangle$. Thus we obtain $\omega = i\eta$ with a real number $\eta$. Moreover if we choose the variation with $\varepsilon = 0$, it leads to $\delta z^A = i\eta z^A$, which gives $\eta = 0$ for $z \in \mathbb{CP}^1$. Using the variation $\delta z^A_{vf} = \varepsilon_{vf}(Jz_{vf})^A$ and $\delta \bar{z}^A_{vf} = \bar{\varepsilon}_{vf}(J\bar{z}_{vf})^A$, and combining the equations of motion

$$\delta_{z_{vf}} S_{vf} = 0 \tag{3.25}$$

with Eq.(3.20), we obtain the following equation

$$\left\langle Jz_{vf}, g_{ve}\xi_{ef}\right\rangle = \frac{\|Z_{vef}\|}{\|Z_{ve'f}\|} e^{i\phi_{eve'}} \left\langle Jz_{vf}, g_{ve'}\xi_{e'f}\right\rangle \tag{3.26}$$

Also from Eq.(3.20), because of $\langle \xi_{ef}, \xi_{ef}\rangle = \langle \xi_{e'f}, \xi_{e'f}\rangle = 1$

$$\langle z_{vf}, g_{ve}\xi_{ef}\rangle = \frac{\|Z_{vef}\|}{\|Z_{ve'f}\|} e^{i\phi_{eve'}} \langle z_{vf}, g_{ve'}\xi_{e'f}\rangle \tag{3.27}$$

Therefore since $z^A$ and $(Jz)^A$ is a basis of the space $\mathbb{C}^2$ of 2-component spinors,

$$g_{ve}\xi_{ef} = \frac{\|Z_{vef}\|}{\|Z_{ve'f}\|} e^{i\phi_{eve'}} g_{ve'}\xi_{e'f} \tag{3.28}$$

We consider the variation with respect to $\xi_{ef}$. Since the spinor $\xi_{ef}$ is normalized, we should use $\delta\xi^A_{ef} = \omega_{ef}(J\xi_{ef})^A + i\eta_{ef}\xi^A_{ef}$ for complex infinitesimal parameter $\omega \in \mathbb{C}$ and $\eta \in \mathbb{R}$. The variation of the action vanishes automatically

$$\delta_{\xi_{ef}} S = j_f \left( 2\frac{\delta_{\xi_{ef}} \langle \xi_{ef}, Z_{vef} \rangle}{\langle \xi_{ef}, Z_{vef} \rangle} + 2\frac{\delta_{\xi_{ef}} \langle Z_{v'ef}, \xi_{ef} \rangle}{\langle Z_{v'ef}, \xi_{ef} \rangle} \right) = 0 \tag{3.29}$$

by using Eq.(3.20) and the identity $\langle J\xi_{ef}, \xi_{ef} \rangle = 0$.

Finally we consider the stationary point for the group variables $g_{ve}$. We parameterize the group with the parameter $\theta_{IJ}$ around a saddle point $g_{ve}$, i.e. $g'_{ve} = g_{ve} e^{-i\theta^{ve}_{IJ}\mathcal{J}^{IJ}}$, where $\mathcal{J}^{IJ}$ is the generator of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$. Using the equations of motion

$$\left.\frac{\partial S_{vf}}{\partial \theta^{ve}_{IJ}}\right|_{\theta^{ve}=0} = 0 \tag{3.30}$$

and applying Eq.(3.20) again, one can find

$$\sum_{f \in t_e}^{4} \varepsilon_{ef}(v) j_f \left( \left\langle \xi_{ef}, i\mathcal{J}^{IJ\dagger}\xi_{ef} \right\rangle + \left\langle \xi_{ef}, i\mathcal{J}^{IJ}\xi_{ef} \right\rangle \right) = 0 \tag{3.31}$$

where $\varepsilon_{ef}(v) = \pm 1$ is determined (up to a global sign) by the following relations

$$\varepsilon_{ef}(v) = -\varepsilon_{e'f}(v) \quad \text{and} \quad \varepsilon_{ef}(v) = -\varepsilon_{ef}(v') \tag{3.32}$$

for the triangle $f$ shared by the tetrahedra $t_e$ and $t_{e'}$ in the 4-simplex $\sigma_v$, and the dual edge $e = (v, v')$. As usual we can rewrite Lorentz Lie algebra generator $\mathcal{J}^{IJ}$ in terms of rotation part $\vec{J}$ and boost part $\vec{K}$ where where $J_i = \frac{i}{2}\epsilon_{0ijk}\mathcal{J}^{jk}$, $K_i = -i\mathcal{J}^{0i}$. In the Spin-$\frac{1}{2}$ representation, the rotation generators $\vec{J} = \frac{i}{2}\vec{\sigma}$ and the boost generators $\vec{K} = \frac{1}{2}\vec{\sigma}$. Recall that

$$\langle \xi \,|\vec{\sigma}|\, \xi \rangle = \hat{n}_\xi \quad \text{with} \quad \hat{n}_\xi = (\xi^0\bar{\xi}^1 + \xi^1\bar{\xi}^0)\hat{\mathbf{x}} - i(\xi^0\bar{\xi}^1 - \xi^1\bar{\xi}^0)\hat{\mathbf{y}} + (\xi^0\bar{\xi}^0 - \xi^1\bar{\xi}^1)\hat{\mathbf{z}} \tag{3.33}$$

we have

$$\left\langle \xi_{ef}, \vec{J}\xi_{ef} \right\rangle = -\left\langle \xi_{ef}, \vec{J}^\dagger\xi_{ef} \right\rangle = \frac{i}{2}\hat{n}_{ef} \tag{3.34}$$

$$\left\langle \xi_{ef}, \vec{K}\xi_{ef} \right\rangle = \left\langle \xi_{ef}, \vec{K}^\dagger\xi_{ef} \right\rangle = \frac{1}{2}\hat{n}_{ef} \tag{3.35}$$

Using the above relations, Eq.(3.2.1) results in the closure condition

$$\sum_{f\subset t_e}^{4}\varepsilon_{ef}(v)j_f\hat{n}_{ef}=0. \tag{3.36}$$

Thus we finish the derivation of all the critical point equations.

### 3.2.2 Analysis of critical point equations

We summarize the critical point equations for a spin foam configuration $(j_f, g_{ev}, \xi_{ef}, z_{vf})$

$$g_{ve}\left(J\xi_{ef}\right)=\frac{\|Z_{ve'f}\|}{\|Z_{vef}\|}e^{-i\phi_{eve'}}g_{ve'}\left(J\xi_{e'f}\right) \tag{3.37}$$

$$g_{ve}\xi_{ef}=\frac{\|Z_{vef}\|}{\|Z_{ve'f}\|}e^{i\phi_{eve'}}g_{ve'}\xi_{e'f} \tag{3.38}$$

$$0=\sum_{f\subset t_e}^{4}\varepsilon_{ef}(v)j_f\hat{n}_{ef} \tag{3.39}$$

where Eq.(3.39) stands for the closure condition for each tetrahedron. $\varepsilon_{ef}(v)$ is the sign factor coming from the variation with respect to $g_{ev}$. It is determined (up to a global sign) by the following relations

$$\varepsilon_{ef}(v)=-\varepsilon_{e'f}(v) \quad \text{and} \quad \varepsilon_{ef}(v)=-\varepsilon_{ef}(v') \tag{3.40}$$

for the triangle $f$ shared by the tetrahedra $t_e$ and $t_{e'}$ in the 4-simplex $\sigma_v$, and the dual edge $e=(v,v')$.

In the following, we show that Eqs.(3.37) and (3.38) give the parallel transportation condition of the bivectors. Given a spinor $\xi^A$, it naturally constructs a null vector $\xi^A\bar{\xi}^{\dot{A}}=\iota(\xi)^I\sigma_I^{A\dot{A}}$ where $\sigma_I=(1,\vec{\sigma})$. It is straight-forward to check that

$$\xi\bar{\xi}=\frac{1}{2}(1+\vec{\sigma}\cdot\hat{n}_\xi) \quad \text{with} \quad \hat{n}_\xi=(\xi^0\bar{\xi}^1+\xi^1\bar{\xi}^0)\hat{\mathbf{x}}-i(\xi^0\bar{\xi}^1-\xi^1\bar{\xi}^0)\hat{\mathbf{y}}+(\xi^0\bar{\xi}^0-\xi^1\bar{\xi}^1)\hat{\mathbf{z}} \tag{3.41}$$

$\hat{n}_\xi$ is a unit 3-vector since $\xi$ is a normalized spinor. Thus we obtain that

$$\iota(\xi)^I=\frac{1}{2}(1,\hat{n}_\xi) \tag{3.42}$$

Similarly for the spinor $J\xi$, we define the null vector $J\xi^A\overline{J\xi}^{\dot{A}}=\iota(J\xi)^I\sigma_I^{A\dot{A}}$ and obtain that

$$\iota(J\xi)^I = \frac{1}{2}(1, -\hat{n}_\xi) \tag{3.43}$$

We can write Eqs.(3.37) and (3.38) in their Spin-1 representation

$$\hat{g}_{ve}\ \iota(J\xi_{ef}) = \frac{\|Z_{ve'f}\|^2}{\|Z_{vef}\|^2}\hat{g}_{ve'}\ \iota(J\xi_{e'f}) \quad \text{and} \quad \hat{g}_{ve}\ \iota(\xi_{ef}) = \frac{\|Z_{vef}\|^2}{\|Z_{ve'f}\|^2}\hat{g}_{ve'}\ \iota(\xi_{e'f}) \tag{3.44}$$

It is obvious that if we construct a bivector[1]

$$X^{IJ}_{ef} = -4\gamma j_f \left[\iota(\xi_{ef}) \wedge \iota(J\xi_{ef})\right]^{IJ} \tag{3.45}$$

$X_{ef}$ satisfies the parallel transportation condition within a 4-simplex

$$(\hat{g}_{ve})^I{}_K(\hat{g}_{ve})^J{}_L X^{KL}_{ef} = (\hat{g}_{ve'})^I{}_K(\hat{g}_{ve'})^J{}_L X^{KL}_{e'f}. \tag{3.46}$$

We define the bivector $X^{IJ}_f$ located at each vertex $v$ of the dual face $f$ by the parallel transportation

$$X^{IJ}_f(v) := (\hat{g}_{ve})^I{}_K(\hat{g}_{ve})^J{}_L X^{KL}_{ef}. \tag{3.47}$$

which is independent of the choice of $e$ by the above parallel transportation condition. Then we have the parallel transportation relation of $X^{IJ}_f(v)$

$$X^{IJ}_f(v) = (\hat{g}_{vv'})^I{}_K(\hat{g}_{vv'})^J{}_L X^{KL}_f(v') \tag{3.48}$$

because the spinor $\xi_{ef}$ belonging to the tetrahedron $t_e$ is shared as the boundary data by two neighboring 4-simplex.

On the other hand, we can write the bivector $X^{IJ}_{ef}$ as a matrix:

$$X^{IJ}_{ef} = 2\gamma j_f \begin{pmatrix} 0 & \hat{n}^1_{ef} & \hat{n}^2_{ef} & \hat{n}^3_{ef} \\ -\hat{n}^1_{ef} & 0 & 0 & 0 \\ -\hat{n}^2_{ef} & 0 & 0 & 0 \\ -\hat{n}^3_{ef} & 0 & 0 & 0 \end{pmatrix} \tag{3.49}$$

$$\left|X^{IJ}_{ef}\right| = \sqrt{\left|\frac{1}{2}X^{IJ}_{ef}X^{ef}_{IJ}\right|} \;=\; 2\gamma j_f \tag{3.50}$$

[1]the pre-factor is a convention for simplifying the notation in the following discussion.

However the matrix $(X_{ef})^I{}_J = X_{ef}^{IK}\eta_{KJ}$ read

$$X_{ef} \equiv (X_{ef})^I{}_J = 2\gamma j_f \begin{pmatrix} 0 & \hat{n}_{ef}^1 & \hat{n}_{ef}^2 & \hat{n}_{ef}^3 \\ \hat{n}_{ef}^1 & 0 & 0 & 0 \\ \hat{n}_{ef}^2 & 0 & 0 & 0 \\ \hat{n}_{ef}^3 & 0 & 0 & 0 \end{pmatrix} = 2\gamma j_f \hat{n}_{ef} \cdot \vec{K} \tag{3.51}$$

where $\vec{K}$ denotes the boost generator of Lorentz Lie algebra $\mathfrak{sl}_2\mathbb{C}$ in the Spin-1 representation. The rotation generator in $\mathfrak{sl}_2\mathbb{C}$ is denoted by $\vec{J}$. The generators in $\mathfrak{sl}_2\mathbb{C}$ satisfies the commutation relations $[J^i, J^j] = -\epsilon^{ijk}J^k$, $[J^i, K^j] = -\epsilon^{ijk}K^k$, $[K^i, K^j] = \epsilon^{ijk}J^k$. The relation $X_{ef} = 2\gamma j_f \hat{n}_{ef} \cdot \vec{K}$ gives a representation of the bivector in terms of the $\mathfrak{sl}_2\mathbb{C}$ lie algebra generators. Moreover it is not difficult to verify that in the Spin-$\frac{1}{2}$ representation $\vec{J} = \frac{i}{2}\vec{\sigma}$ and $\vec{K} = \frac{1}{2}\vec{\sigma}$. Thus in Spin-$\frac{1}{2}$ representation

$$X_{ef} = \gamma j_f \vec{\sigma} \cdot \hat{n}_{ef} \tag{3.52}$$

For this $\mathfrak{sl}_2\mathbb{C}$ Lie algebra representation of the bivector $X_{ef}$, the parallel transportation is represented by the adjoint action of the Lie group on its Lie algebra. Therefore we have

$$g_{ve}X_{ef}g_{ev} = g_{ve'}X_{e'f}g_{e'v}, \qquad X_f(v) := g_{ve}X_{ef}g_{ev}, \qquad X_f(v) := g_{vv'}X_f(v')g_{v'v} \tag{3.53}$$

where $g_{ve} = g_{ev}^{-1}, g_{v'v} = g_{vv'}^{-1}$. We note that the above equations are valid for all the representations of $\mathrm{SL}(2, \mathbb{C})$.

There is the duality map acting on $\mathfrak{sl}_2\mathbb{C}$ by $\star\vec{J} = -\vec{K}$, $\star\vec{K} = \vec{J}$. For self-dual/anti-self-dual bivector $\vec{T}_\pm := \frac{1}{2}(\vec{J} \pm i\vec{K})$, One can verify that $\star\vec{T}_\pm = \pm i\vec{T}_\pm$. In the Spin-1 representation (bivector representation), the duality map is represented by $\star X^{IJ} = \frac{1}{2}\epsilon^{IJKL}X_{KL}$. In the Spin-$\frac{1}{2}$ representation, the duality map is represented by $\star X = iX$ since $\vec{J} = \frac{i}{2}\vec{\sigma}$ and $\vec{K} = \frac{1}{2}\vec{\sigma}$ in the Spin-$\frac{1}{2}$ representation. From Eq.(3.51), we see that

$$X_{ef} = - \star (2\gamma j_f \hat{n}_{ef} \cdot \vec{J}) \tag{3.54}$$

From its bivector representation one can see that

$$\eta_{IJ}u^I \star X_{ef}^{JK} = 0, \qquad u^I = (1, 0, 0, 0). \tag{3.55}$$

It motivates us to define a unit vector at each vertex $v$ for each tetrahedron $t_e$ by

$$N_e^I(v) := (\hat{g}_{ve})^I{}_J u^J \tag{3.56}$$

Then for all triangles $f$ in the tetrahedron $t_e$, $N_e^I(v)$ is orthogonal to all the bivectors $\star X_f(v)$ with $f$ belonging to $t_e$.

$$\eta_{IJ} N_e^I(v) \star X_f^{JK}(v) = 0. \tag{3.57}$$

In addition, from the closure constraint Eq.(3.39), we obtain for each tetrahedron $t_e$

$$\sum_{f \subset t_e} \varepsilon_{ef}(v) X_f(v) = 0. \tag{3.58}$$

We summarize the above analysis of the critical point equations Eqs.(3.37), (3.38), and (3.39) into the following proposition:

**Proposition 3.2.1.** *Given the data $(j_f, g_{ev}, \xi_{ef}, z_{vf})$ be a spin foam configuration that solves the critical point equations Eqs.(3.37), (3.38), and (3.39), we construct the bivector variables (in the $\mathfrak{sl}_2\mathbb{C}$ Lie algebra representation) for the spin foam amplitude $X_{ef} = -\star(2\gamma j_f \hat{n}_{ef} \cdot \vec{J})$ and $X_{ef}(v) := g_{ve} X_{ef} g_{ev}$, where $|X_{ef}(v)| = \sqrt{\frac{1}{2}\mathrm{Tr}\,(X_{ef}(v) X_{ef}(v))} = 2\gamma j_f$. The critical point equations implies the following equations for the bivector variables*

$$X_{ef}(v) = X_{e'f}(v) \equiv X_f(v), \qquad X_f(v) := g_{vv'} X_f(v') g_{v'v}, \tag{3.59}$$

$$\eta_{IJ} N_e^I(v) \star X_f^{JK}(v) = 0, \qquad \sum_{f \subset t_e} \varepsilon_{ef}(v) X_f(v) = 0 \tag{3.60}$$

*where $t_e$ and $t_{e'}$ are two different tetrahedra of a 4-simplex dual to $v$, $f$ is a triangle shared by the two tetrahedra $t_e$ and $t_{e'}$, and $N_e^I(v) = (\hat{g}_{ve})^I{}_J u^J$ with $u^J = (1,0,0,0)$ is a unit vector associated with the tetrahedron $t_e$. $\varepsilon_{ef}(v)$ is a sign factor determined (up to a global sign) by the following relations*

$$\varepsilon_{ef}(v) = -\varepsilon_{e'f}(v) \quad \text{and} \quad \varepsilon_{ef}(v) = -\varepsilon_{ef}(v') \tag{3.61}$$

*for the triangle $f$ shared by the tetrahedra $t_e$ and $t_{e'}$ in the 4-simplex $\sigma_v$, and the dual edge $e = (v, v')$.*

# 3.3 Nondegenerate geometry on a simplicial complex

## 3.3.1 Discrete bulk geometry

In order to relate the spin foam configurations solving the critical point equations with the a discrete Regge geometry, here we introduce the *classical* geometric variables for the discrete Lorentzian geometry on a 4-manifold [78–81].

Given a simplicial complex $\mathcal{K}$ triangulating the 4-manifold $\mathcal{M}$ with Lorentzian metric $g_{\mu\nu}$, we associate each 4-simplex $\sigma_v$ (dual to the vertex $v$) a reference frame. In this reference frame the vertices $[p_1(v), \cdots, p_5(v)]$ of the 4-simplex $\sigma_v$ have the coordinates

$$p_i(v) = \{x_i^I(v)\}_{i=1,\cdot,5} \tag{3.62}$$

Consider another 4-simplex $\sigma_{v'}$ neighboring $\sigma_v$, there is an edge $e$ connecting $v$ and $v'$, and there is a tetrahedron $t_e$ shared by $\sigma_v, \sigma_{v'}$ with vertices $[p_2(v), \cdots, p_5(v)] = [p_2(v'), \cdots, p_5(v')]$. Then it is possible to associate the edge $e = (v, v')$ uniquely an element of Poincaré group $\left\{(\Omega_e)^I{}_J, (\Omega_e)^I\right\}$, such that for the vertices $p_2, \cdots, p_5$ of $t_e$

$$(\Omega_e)^I{}_J x_i^J(v') + (\Omega_e)^I = x_i^I(v) \qquad i = 2, \cdots, 5 \tag{3.63}$$

Here the matrix $(\Omega_e)^I{}_J$ describes the change of the reference frames in $\sigma_v$ and $\sigma_{v'}$, while $(\Omega_e)^I$ describes the transportation of the frame origins from $\sigma_v$ to $\sigma_{v'}$. We assume the triangulation is orientable, and we choose the reference frames in $\sigma_v, \sigma_{v'}$ in such a way that $\Omega_e \in$ SO(1,3).

We focus on a 4-simplex $\sigma_v$ whose center is the vertex $v$. For each oriented edge $\ell = [p_i(v), p_j(v)]$ in the 4-simplex, we associate an edge vector $E_\ell^I(v) = x_i^I(v) - x_j^I(v)$. Thus under the change of reference frame from $\sigma_v$ to $\sigma_{v'}$

$$(\Omega_e)^I_J E_\ell^I(v') = E_\ell(v) \quad \forall\ \ell \subset t_e \tag{3.64}$$

In this work we assume all the edge vectors $E_\ell^I(v)$ are *spatial* in the sense of the flat metric $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$. It is straight-forward to check from the definition that the edge vectors $E_\ell^I(v)$ satisfies:

- if we reverse the orientation of $\ell$, then

$$E^I_{-\ell}(v) = -E^I_\ell(v), \tag{3.65}$$

- for all triangle $f$ in the simplex $\sigma_v$ with edge $\ell_1, \ell_2, \ell_3$, the vectors $E^I_\ell(v)$ close, i.e.

$$E^I_{\ell_1}(v) + E^I_{\ell_2}(v) + E^I_{\ell_3}(v) = 0 \tag{3.66}$$

  The set of $E^I_\ell(v)$ at $v$ satisfying Eqs.(3.65) and (3.66) is called a co-frame at the vertex $v$.

- Moreover given a tetrahedron $t$ shared by two 4-simplices $\sigma_v, \sigma_{v'}$, for all pair of edges $\ell_1, \ell_2$ of the tetrahedron, we further require that

$$\eta_{IJ} E^I_{\ell_1}(v) E^J_{\ell_2}(v) = \eta_{IJ} E^I_{\ell_1}(v') E^J_{\ell_2}(v') \tag{3.67}$$

**Definition 3.3.1.** *The collection of the vectors $E_\ell(v)$ satisfying Eqs.(3.65), (3.66), and (3.67) at all the vertices is called a co-frame on the simplicial complex $\mathcal{K}$. The discrete (spatial) metric on the each tetrahedron $t$ induced from $g_{\mu\nu}$ is given by*

$$g_{\ell_1\ell_2}(v) = \eta_{IJ} E^I_{\ell_1}(v) E^J_{\ell_2}(v) \tag{3.68}$$

*which is actually independent of $v$ because of Eq.(3.67).*

We assume the co-frame $E^I_\ell(v)$ is nondegenerate, i.e. for each 4-simplex $\sigma_v$, the set of $E^I_\ell(v)$ with $\ell \subset \partial\sigma_v$ spans a 4-dimensional vector space.

An edge $\ell$ can be denoted by its end-points, say $p_1, p_2$, i.e. $\ell = [p_1, p_2]$. There are 5 vertices $p_i, i = 1, \cdots, 5$ for a 4-simplex $\sigma_v$. Then each $p_i$ is one-to-one corresponding to a tetrahedron $t_{e_i}$ of the 4-simplex $\sigma_v$. Therefore we can denote the edge $\ell = [p_1, p_2]$ also by $\ell = (e_1, e_2)$, once a 4-simplex $\sigma_v$ is specified. Thus we can also write the co-frame $E^I_\ell(v)$ at the vertex $v$ by $E^I_{e_1e_2}(v)$. In this notation, for example Eqs.(3.65) and (3.66) become

$$E^I_{e_1e_2}(v) = -E^I_{e_2e_1}(v), \quad E^I_{e_1e_2}(v) + E^I_{e_2e_3}(v) + E^I_{e_3e_1}(v) = 0. \tag{3.69}$$

In the following we use both of the notations, according to the convenience by the context.

**Lemma 3.3.1.** *Given a co-frame $E^I_\ell(v)$ on the triangulation, it determines uniquely an SO(1,3) matrix $(\Omega_e)^I{}_J$ associated to each edge $e=(v,v')$ such that for all the edge of the tetrahedron $t_e$ shared by $\sigma_v$ and $\sigma_{v'}$*

$$(\Omega_e)^I{}_J E^J_\ell(v') = E^I_\ell(v) \qquad \forall\ \ell \subset t_e \tag{3.70}$$

*We can associate a reference frame in each 4-simplex such that SO(1,3) matrix $(\Omega_e)^I{}_J$ changing the frame from $\sigma_v$ to $\sigma_{v'}$.*

**Proof:** Given a tetrahedron $t_e$ shared by two 4-simplices $\sigma_v, \sigma_{v'}$, we consider the relation between the co-frame vectors $E^I_\ell(v)$ at the vertex $v$ and $E^I_\ell(v')$ at $v'$, for all 6 edges $\ell$ of the tetrahedron $t_e$. The spatial vectors $E^I_\ell(v)$ $\ell \subset t_e$ spans a 3-dimensional subspace, and the same holds for $E^I_\ell(v')$. We choose the time-like unit normal vectors $\hat{U}(v)$ and $\hat{U}(v')$ orthogonal to $E^I_\ell(v)$ and $E^I_\ell(v')$ respectively, and require that

$$\operatorname{sgn}\det\Big(E_{\ell_1}(v), E_{\ell_2}(v), E_{\ell_3}(v), \hat{U}(v)\Big) = \operatorname{sgn}\det\Big(E_{\ell_1}(v'), E_{\ell_2}(v'), E_{\ell_3}(v'), \hat{U}(v')\Big) \tag{3.71}$$

where $E_{\ell_1}(v), E_{\ell_2}(v), E_{\ell_3}(v)$ form a basis in the 3-dimensional subspace spanned by $E^I_\ell(v)$ $\ell \subset t_e$. From Eq.(3.71), Eq.(3.67) and $E_{\ell_i}(v)\cdot\hat{U}(v) = E_{\ell_i}(v')\cdot\hat{U}(v') = 0$, $i=1,2,3$, an SO(1,3) matrix $\Omega_e$ is determined by

$$(\Omega_e)^I{}_J E^J_{\ell_i}(v') = E^I_{\ell_i}(v) \qquad (\Omega_e)^I{}_J \hat{U}^J(v') = \hat{U}^I(v). \tag{3.72}$$

Suppose there are two SO(1,3) matrices $\Omega_e, \Omega'_e$ satisfying

$$(\Omega_e)^I{}_J E^J_{\ell_i}(v') = E^I_{\ell_i}(v) \qquad (\Omega'_e)^I{}_J E^J_{\ell_i}(v') = E^I_{\ell_i}(v) \tag{3.73}$$

we then have $\Omega_e = \Omega'_e$.

We choose a numbering $[p_1,\cdots,p_5]$ of the vertices of $\sigma_v, \sigma_{v'}$ such that $[p_2(v),\cdots,p_5(v)] = [p_2(v'),\cdots,p_5(v')]$ are the vertices of the tetrahedron $t_e$. Two reference frame in the 4-simplices $\sigma_v, \sigma_{v'}$ are specified by the coordinates $\left\{E^I_{e_2e_1}(v), E^I_{e_3e_1}(v), E^I_{e_4e_1}(v), E^I_{e_5e_1}(v)\right\}$ and $\left\{E^I_{e_2e_1}(v'), E^I_{e_3e_1}(v'), E^I_{e_4e_1}(v'), E^I_{e_5e_1}(v')\right\}$ by defining $x^I_j(v) := E^I_{e_je_1}(v)$ and similar for $x^I_j(v')$. Since

$$E_{e_2e_1} = E_{e_2e_5} - E_{e_1e_5}, \qquad E_{e_3e_1} = E_{e_3e_5} - E_{e_1e_5}, \qquad E_{e_4e_1} = E_{e_4e_5} - E_{e_1e_5} \tag{3.74}$$

and there exists a unique $(\Omega_e)^I{}_J \in$ SO(1,3) that $E^I_{e_ie_j}(v) = (\Omega_e)^I{}_J E^I_{e_ie_j}(v')$, $i,j =$

$2, \cdots, 5$, we can relate the coordinates $\left\{E^I_{e_2e_1}(v), E^I_{e_3e_1}(v), E^I_{e_4e_1}(v), E^I_{e_5e_1}(v)\right\}$ and $\left\{E^I_{e_2e_1}(v'), E^I_{e_3e_1}(v'), E^I_{e_4e_1}(v'), E^I_{e_5e_1}(v')\right\}$ in two different 4-simplices by

$$E^I_{e_ie_1}(v) = (\Omega_e)^I{}_J E^I_{e_ie_1}(v') + (\Omega_e)^I{}_J E^J_{e_1e_5}(v') - E^I_{e_1e_5}(v), \quad i = 2,3,4,5 \tag{3.75}$$

The coordinates of $p_2, \cdots, p_5$ are given by $x^I_j(v) := E^I_{e_je_1}(v)$ with respective the reference frame in $\sigma_v$, thus the Poincaré transformation relating two reference frames are given by an SO(1,3) matrix and a translation $\left\{(\Omega_e)^I{}_J, (\Omega_e)^I\right\}$, where the translation vector $(\Omega_e)^I$ is given by

$$(\Omega_e)^I := (\Omega_e)^I{}_J E^J_{e_1e_5}(v') - E^I_{e_1e_5}(v) \tag{3.76}$$

□

The orientation of a 4-simplex $\sigma_v$ is represented by an ordering of its 5 vertices, i.e. a tuple $[p_1, \cdots, p_5]$. Two orientations are opposite to each other if the two orderings are related by an odd permutation, e.g. $[p_1, p_2, \cdots, p_5] = -[p_2, p_1 \cdots, p_5]$. We say that two neighboring 4-simplices $\sigma, \sigma'$ are consistently oriented, if the orientation of their shared tetrahedron $t$ induced from $\sigma$ is opposite to the orientation induced from $\sigma'$. For example, $\sigma = [p_1, p_2, \cdots, p_5]$ and $\sigma' = -[p'_1, p_2, \cdots, p_5]$ are consistently oriented since the opposite orientations $t = \pm[p_2, \cdots, p_5]$ are induced respectively from $\sigma$ and $\sigma'$. The simplicial complex $\mathcal{K}$ is said to be orientable if it is possible to orient consistently all pair of neighboring 4-simplices. Such a choice of consistent 4-simplex orientations is called a global orientation. We assume we define a global orientation of the triangulation $\mathcal{K}$. Then for each 4-simplex $\sigma_v = [p_1, p_2, \cdots, p_5]$, we define an oriented volume (assumed to be nonvanishing as the nondegeneracy)

$$V_4(v) := \det\Big(E_{e_2e_1}(v), E_{e_3e_1}(v), E_{e_4e_1}(v), E_{e_5e_1}(v)\Big) \tag{3.77}$$

In general the oriented 4-volume $V_4(v)$ can be positive or negative for different 4-simplices.

**Definition 3.3.2.** *Given two neighboring 4-simplices $\sigma_v$ and $\sigma_{v'}$, if their oriented volumes are both positive or both negative, i.e.* $\mathrm{sgn}(V_4(v)) = \mathrm{sgn}(V_4(v'))$. *The SO(1,3) matrix $(\Omega_e)^I{}_J$, $e = (v, v')$ is the discrete spin connection compatible with $E_\ell(v)^I$.*

For each vertex $v$ and a dual edge $e$ connecting $v$, we define a time-like vector $U_e(v)$

at the vertex $v$ by (choosing any $j \neq k$, the definition is independent of the choice of $j$ by Eqs.(3.65) and (3.66))

$$U^{e_k}_I(v) := \frac{1}{3!V_4(v)} \sum_{l,m,n} \epsilon^{jklmn} \epsilon_{IJKL} E^J_{e_l e_j}(v) E^K_{e_m e_j}(v) E^L_{e_n e_j}(v) \tag{3.78}$$

In total there are 5 vectors $U_e(v)$ at each vertex $v$. Using Eq.(3.65) and (3.66), one can show that

$$U^{e_j}_J(v) E^J_{e_k e_l}(v) = \delta_{jk} - \delta_{jl} \tag{3.79}$$

Thus we call the collection of $U_e(v)$ a discrete frame since $E_{e_1 e_2}(v)$ is called a discrete co-frame. Moreover from this equation we see that $U^J_e(v)$ is a vector at $v$ normal to the tetrahedron $t_e$. If we sum over all 5 frame vectors $U_e(v)$ at $v$ in Eq.(3.79)

$$\sum_{j=1}^{5} U^{e_j}_J(v) E^J_{e_k e_l}(v) = \sum_{j=1}^{5} \delta_{jk} - \sum_{j=1}^{5} \delta_{jl} = 0 \quad \forall\ e_k, e_l \tag{3.80}$$

which shows the closure of $U_e(v)$ at each vertex $v$, i.e.

$$\sum_{e=1}^{5} U_e(v) = 0 \tag{3.81}$$

by the nondegenercy of $E_{ee'}(v)$. Eq.(3.81) shows that the 5 vectors $U_e(v)$ are all out-pointing or all in-pointing normal vectors to the tetrahedra. Also following from Eq.(3.79) (fix $l = 1$ and let $j = 2, 3, 4, 5$), we have that the $4 \times 4$ matrix $(U^{e_2}(v), U^{e_3}(v), U^{e_4}(v), U^{e_5}(v))^t$ is the inverse of the matrix $(E_{e_2 e_1}(v), E_{e_3 e_1}(v), E_{e_4 e_1}(v), E_{e_5 e_1}(v))$. Therefore

$$\frac{1}{V_4(v)} = \det\Big(U^{e_2}(v), U^{e_3}(v), U^{e_4}(v), U^{e_5}(v)\Big). \tag{3.82}$$

It implies $(i, j, k, l = 2, 3, 4, 5)$

$$V_4(v) \epsilon^{IJKL} U^{e_i}_I(v) U^{e_j}_J(v) U^{e_k}_K(v) U^{e_l}_L(v) = \epsilon^{ijkl} \tag{3.83}$$

$$V_4(v) \epsilon_{ijkl} U^{e_i}_I(v) U^{e_j}_J(v) U^{e_k}_K(v) U^{e_l}_L(v) = \epsilon_{IJKL} \tag{3.84}$$

where the above $\epsilon_{ijkl} = \epsilon^{ijkl}$, $\epsilon_{IJKL} = \epsilon^{IJKL}$ are all Levi-Civita symbols. Then using the fact that the matrix $U_I^{e_i}(v)$ is the inverse of $E^I_{e_ie_1}(v)$, we can verify that

$$E^I_{e_ke_j}(v) = \frac{V_4(v)}{3!}\sum_{l,m,n}\epsilon_{jklmn}\epsilon^{IJKL}U_J^{e_l}(v)U_K^{e_m}(v)U_L^{e_n}(v) \tag{3.85}$$

$$V_4(v)U_{[I}^{e_i}(v)U_{J]}^{e_j}(v) = \frac{1}{2}\sum_{m,n}\epsilon^{kijmn}\epsilon_{IJKL}E^K_{e_me_k}(v)E^L_{e_ne_k}(v) \tag{3.86}$$

where the last equation is a relation for the area bivector $E_\ell(v)\wedge E_{\ell'}(v)$ of each triangle $f$. For example, given a triangle $f$ shared by $t_{e_4}$ and $t_{e_5}$ in a 4-simplex $\sigma_v$. one has

$$\star\left[E_{e_1e_2}(v)\wedge E_{e_2e_3}(v)\right] = V_4(v)\left[U^{e_4}(v)\wedge U^{e_5}(v)\right] \tag{3.87}$$

where $\star[E_1\wedge E_2]\equiv\epsilon_{IJKL}E_1^KE_2^L$.

### 3.3.2 Discrete boundary geometry

All the above discussions are considering the discrete geometry in the bulk of the triangulation, where all the co-frame vectors $E_\ell(v)$ and frame vectors $U_e(v)$ are located at internal vertices $v$. Now we consider a triangulation with boundary, where the boundary is a simplical complex $\partial\mathcal{K}$ built by tetrahedra triangulating a boundary 3-manifold. On the boundary $\partial\mathcal{K}$, each triangle is shared by precisely two boundary tetrahedra. This triangle is dual to a unique boundary link $l$, connecting the centers of the two boundary tetrahedra sharing the triangle. We denote this triangle $f_l$. On the other hand, from the viewpoint of the whole triangulation $\mathcal{K}$, there is a unique face dual to the triangle $f_l$, where two edges $e_0,e_1$ of this dual face are dual to the two boundary tetrahedra $t_{e_0},t_{e_1}$ sharing $f_l$. This dual face intersects the boundary uniquely by the link $l$[2]. Thus we denote this dual face also by $f_l$ because of the one-to-one correspondence of the duality for $\mathcal{K}$. See FIG.3.1 for an example of a face dual to a boundary triangle.

The end-points $s(l),t(l)$ of the boundary link $l$ are centers of the tetrahedra $t_{e_0},t_{e_1}$ respectively. For each edge $\ell$ of the tetrahedron $t_{e_i}$ $(i=0,1)$, we associate a spatial vector $E_\ell(e_i)$ at the center of $t_{e_i}$, satisfying the following requirement:

- Given the time-like unit vector $u^I=(1,0,0,0)$, all the vectors $E_\ell(e_i)$ $(i=0,1)$

[2]If the dual face intersects the boundary by more than one link, then it means that the triangle $f_l$ is shared by more than two tetrahedra, which is impossible for a 3-dimensional triangulation.

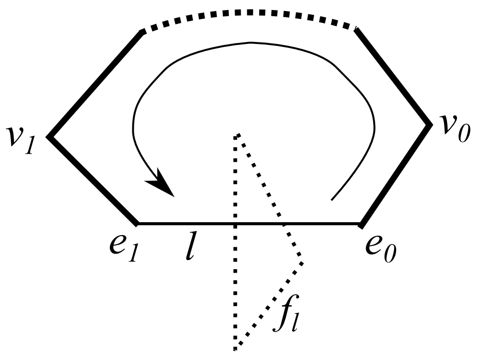


Fig. 3.1 The face dual to a boundary triangle $f_l$ shared by two tetrahedra $t_{e_0}, t_{e_1}$.

are orthogonal to $u^I$, i.e.

$$u_I E^I_\ell(e_i) = 0 \quad \forall\, \ell \in t_{e_i}. \tag{3.88}$$

- If we reverse the orientation of $\ell$, then

$$E_{-\ell}(e_i) = -E_\ell(e_i) \quad \forall\, \ell \in t_{e_i}. \tag{3.89}$$

- For all triangle $f$ of the boundary tetrahedron $t_{e_i}$ with edge $\ell_1, \ell_2, \ell_3$, the vectors $E_\ell(e_i)$ close, i.e.

$$E_{\ell_1}(e_i) + E_{\ell_2}(e_i) + E_{\ell_3}(e_i) = 0. \tag{3.90}$$

- There is a internal vertex $v_i$ as one of the end-points of the dual edge $e_i$ $(i = 0, 1)$, i.e. the boundary tetrahedron $t_{e_i}$ belongs to the boundary of the 4-simplex $\sigma_{v_i}$. Then we require that

$$\eta_{IJ} E^I_{\ell_1}(e_i) E^J_{\ell_2}(e_i) = \eta_{IJ} E^I_{\ell_1}(v_i) E^J_{\ell_2}(v_i) \quad \forall\, \ell_1, \ell_2 \in t_{e_i}. \tag{3.91}$$

The set of $E^I_\ell(e_i)$ $(i = 0, 1)$ at the center of $t_{e_i}$ satisfying the above requirements is called a boundary (3-dimensional) co-frame at the center of $t_{e_i}$ (at the node $s(l)$). The discrete metric

$$g_{\ell_1 \ell_2}(e_i) := \eta_{IJ} E^I_{\ell_1}(e_i) E^J_{\ell_2}(e_i) \tag{3.92}$$

is the induced metric on the boundary $\partial\mathcal{K}$.

Consider a boundary tetrahedron $t_{e_i}$ belonging to a 4-simplex $\sigma_{v_i}$, then the edge $e_i$ dual to $t_{e_i}$ connects to a boundary node (the center of $t_{e_i}$). We choose 3 linearly independent co-frame vectors $E_{\ell_1}(e_i), E_{\ell_2}(e_i), E_{\ell_3}(e_i)$ at the center of $t_{e_i}$ associated with 3 edges $\ell_1, \ell_2, \ell_3$, and also choose 3 linearly independent co-frame vectors $E_{\ell_1}(v_i), E_{\ell_2}(v_i), E_{\ell_3}(v_i)$ at the vertex $v_i$ associated with the same set of edges. Given a unit vector $\hat{U}(v_i)$ orthogonal to $E_{\ell_1}(v_i), E_{\ell_2}(v_i), E_{\ell_3}(v_i)$ such that

$$\operatorname{sgn}\det\Big(E_{\ell_1}(v_i), E_{\ell_2}(v_i), E_{\ell_3}(v_i), \hat{U}(v_i)\Big) = \operatorname{sgn}\det\Big(E_{\ell_1}(e_i), E_{\ell_2}(e_i), E_{\ell_3}(e_i), u\Big) \quad (3.93)$$

by the requirement Eq.(3.91), there exist a unique SO(1,3) matrix $\Omega_{e_i}$ such that

$$(\Omega_{e_i})^I{}_J E^J_{\ell_j}(e_i) = E^I_{\ell_j}(v_i) \qquad (\Omega_{e_i})^I{}_J u^J = \hat{U}^I(v_i). \quad (3.94)$$

Thus $\Omega_{e_i}$ is identify as the spin connection compatible with $E_\ell(v_i), E_\ell(e_i)$.

Consider a dual face bounded by a boundary link $l$ (see, e.g. FIG.3.1), by using the defining requirement of the co-frames in the bulk and on the boundary, i.e. Eqs.(3.67) and (3.91), we have

$$\eta_{IJ} E^I_{\ell_j}(e_0) E^J_{\ell_k}(e_0) = \eta_{IJ} E^I_{\ell_j}(e_1) E^J_{\ell_k}(e_1) \quad (3.95)$$

where $\ell_j, \ell_k$ are two of the three edges of the triangle $f_l$ dual to the face. Therefore we obtain the *shape-matching* condition between the triangle geometries of $f_l$ viewed in the frame of $t_{e_0}$ and $t_{e_1}$. More precisely, there exists an SO(3) matrix $\hat{g}_l$ such that for all the three $\ell$'s forming the boundary of the triangle $f_l$

$$(\hat{g}_l)^I{}_J E^J_\ell(e_0) = E^I_\ell(e_1) \quad (3.96)$$

by the fact that both $E_\ell(e_0)$ and $E_\ell(e_1)$ are orthogonal to $u^I = (1,0,0,0)$.

Now we consider a single boundary tetrahedron $t_e$ dual to an edge $e$ connecting to the boundary. Since all the boundary co-frame vectors $E_\ell(e)$ at the center of $t_e$ are orthogonal to the time-like unit vector $u^I = (1,0,0,0)$, we now only consider the 3-dimensional spatial subspace orthogonal to $u^I = (1,0,0,0)$. We further assume the boundary tetrahedral geometry is nondegenerate, i.e. the (oriented) 3-volume of the tetrahedron

$$V_3(e) = \det\Big(E_{\ell_1}(e), E_{\ell_2}(e), E_{\ell_3}(e)\Big) \quad (3.97)$$

is nonvanishing, where $\ell_1, \ell_2, \ell_3$ are the three edges of $t_e$ connecting to a vertex $p$ of $t_e$. Since there are 4 vertices of $t_e$ and an edge $\ell$ is determined by its end-points $p_i, p_j$, we denote $E_\ell(e)$ by $E_{p_ip_j}(e)$. Choose a vertex $p_1$ and construct the nondegenerate $3 \times 3$

matrix

$$\Big(E_{p_2p_1}(e), E_{p_3p_1}(e), E_{p_4p_1}(e)\Big) \tag{3.98}$$

we construct is inverse

$$\Big(n_{p_2}(e), n_{p_3}(e), n_{p_4}(e)\Big)^t \tag{3.99}$$

with $n_{p_i}(e) \cdot E_{p_jp_1}(e) = \delta_{ij}$. Repeat the same construction for all the other 3 vertices $p_2, p_3, p_4$, we obtain four 3-vector $n_{p_i}(e)$ such that

$$n_{p_i}(e) \cdot E_{p_jp_k}(e) = \delta_{ij} - \delta_{ik}. \tag{3.100}$$

From this relation, one can verify that: (i) The 3-vector $n_{p_i}(e)$ is orthogonal to the triangle $(p_j, p_k, p_l)$ spanned by $E_{p_jp_k}(e), E_{p_jp_l}(e), E_{p_lp_k}(e)$ with $i \neq j, k, l$. Therefore we denote $n_p(e)$ by $n_{ef}$ where $f$ is the triangle determined by the 3 vertices other than $p$. (ii) the four $n_{ef}$ satisfy the closure condition

$$\sum_{f=1}^{4} n_{ef} = 0. \tag{3.101}$$

We call the set of $n_{ef}$ a 3-dimensional frame at the center of $t_e$. Explicitly, the vector $n_{ef}$ is given by

$$n_{ef} = V_3(e)^{-1} E_{\ell_1}(e) \times E_{\ell_2}(e) \quad \text{or} \quad n_{p_1}(e) = V_3(e)^{-1} E_{p_2p_3}(e) \times E_{p_3p_4}(e) \tag{3.102}$$

The norm $|n_{ef}| = 2A_f/|V_3(e)|$ is proportional to the area of the triangle $A_f = \frac{1}{2}\left|E_{\ell_1}(e) \times E_{\ell_2}(e)\right|$.

# 3.4 Geometric interpretation of nondegenerate critical configuration

## 3.4.1 Classical geometry from spin foam critical configuration

Now we come back to the discussion of the critical point of spin foam amplitude. The purpose of this section is to make a relation between the solution of the critical point equations Eqs.(3.37), (3.38), and (3.39) and a (Lorentzian) discrete geometry described in Section 3.3.

Given a spin foam configuration $(j_f, g_{ev}, \xi_{ef}, z_{vf})$ that solves the critical point equations, let's recall Proposition 3.2.1 and consider a triangle $f$ shared by two tetrahedra $t_e$ and $t_{e'}$ of a 4-simplex $\sigma_v$. In Eq.(3.59), there are the simplicity conditions $N_I^e(v) \star X_{ef}^{IJ}(v) = 0$ and $N_I^{e'}(v) \star X_{e'f}^{IJ}(v) = 0$ from the viewpoint of the two tetrahedra $t_e$ and $t_{e'}$. The two simplicity conditions implie that there exists two 4-vectors $M_{ef}^I(v)$ and $M_{ef}^I(v)$ such that $X_{ef}(v) = N_e(v) \wedge M_{ef}(v)$ and $X_{e'f}(v) = N_{e'}(v) \wedge M_{e'f}(v)$. However we have in Eq.(3.59) the gluing condition $X_{ef}(v) = X_{e'f}(v) = X_f(v)$, which implies that $N_{e'}(v)$ belongs to the plane spanned by $N_e(v), M_{ef}(v)$, i.e. $N_{e'}(v) = a_{ef} M_{ef}(v) + b_{ef} N_e(v)$. If we assume the following nondegeneracy condition[3]:

$$\prod_{e_1,e_2,e_3,e_4=1}^{5} \det\Big(N_{e_1}(v), N_{e_2}(v), N_{e_3}(v), N_{e_4}(v)\Big) \neq 0 \tag{3.103}$$

then $N_e(v), N_{e'}(v)$ cannot be parallel with each other, for all pairs of $e, e'$, which excludes the case of vanishing $a_{ef}$ in the above. Denoting $\alpha_{ee'} = a_{ef}^{-1}$, we obtain that $M_{ef}(v) = \alpha_{ee'} N_{e'}(v) - \alpha_{ee'} b_{ef} N_e(v)$. Therefore

$$X_f(v) = \alpha_{ee'}(v)\,[N_e(v) \wedge N_{e'}(v)] \tag{3.104}$$

for all $f$ shared by $t_e$ and $t_{e'}$. Note that within a simplex $\sigma_v$ there is a one-to-one correspondence between a pair of tetrahedra $t_e$ and $t_{e'}$ and a triangle $f$ shared by them. Thus we can write the bivector $X_f(v) \equiv X_{ee'}(v) = \alpha_{ee'}(v)\,[N_e(v) \wedge N_{e'}(v)]$.

We label the 5 tetrahedra of $\sigma_v$ by $t_{e_i}$, $i = 1, \cdots, 5$. Then Eq.(3.104) reads

$$X_{e_i e_j}(v) = \alpha_{ij}(v)\left[N_{e_i}(v) \wedge N_{e_j}(v)\right] \tag{3.105}$$

Then the closure condition $\sum_{j=1}^{4} \varepsilon_{e_i e_j}(v) X_{e_i e_j}(v) = 0$[4] gives that $\forall\ i = 1, \cdots, 5$

$$0 = \sum_{j=1}^{4} \varepsilon_{e_i e_j}(v)\alpha_{ij}(v)\left[N_{e_i}(v) \wedge N_{e_j}(v)\right] = N_{e_i}(v) \wedge \sum_{j=1}^{4} \varepsilon_{e_i e_j}(v)\alpha_{ij}(v) N_{e_j}(v) \tag{3.106}$$

which implies that for a choice of diagonal element $\beta_{ii}(v)$,

$$\sum_{j=1}^{5} \beta_{ij}(v) N_{e_j}(v) = 0 \tag{3.107}$$

[3]Note that the nondegenerate here is purely a condition for the group variables $g_{ve}$ since $N_e(v) = g_{ve}(1, 0, 0, 0)^t$.

[4]Here $\varepsilon_{e_i e_j}(v) = -\varepsilon_{e_j e_i}(v)$ and $X_{e_i e_j}(v) = X_{e_j e_i}(v)$.

where we denote $\beta_{ij}(v) := \varepsilon_{e_ie_j}(v)\alpha_{ij}(v)$. Here $\beta_{ii}(v)$ must be chosen as nonzero, because if $\beta_{ii}(v) = 0$, Eq.(3.107) would reduce to $\sum_{j\neq i}\beta_{ij}(v)N_{e_j}(v) = 0$, which gives all the coefficients $\beta_{ij}(v) = 0$ by linearly independence of any four $N_e(v)$ (from the nondegeneracy Eq.(3.103)).

We consider

$$\begin{aligned} 0 &= \beta_{km}(v)\sum_{j=1}^{5}\beta_{lj}(v)N_{e_j}(v) - \beta_{lm}(v)\sum_{j=1}^{5}\beta_{kj}(v)N_{e_j}(v) \\ &= \sum_{j\neq m}\Big[\beta_{km}(v)\beta_{lj}(v) - \beta_{lm}(v)\beta_{kj}(v)\Big]N_{e_j}(v) \end{aligned} \tag{3.108}$$

Since we assume the nondegeneray condition Eq.(3.103), any four of the five $N_e(v)$ are linearly independent. Thus

$$\beta_{km}(v)\beta_{lj}(v) = \beta_{lm}(v)\beta_{kj}(v) \tag{3.109}$$

Let us pick one $j_0$ for each 4-simplex, and ask $l = j = j_0$ we obtain

$$\beta_{km}(v) = \frac{\beta_{kj_0}(v)\beta_{mj_0}(v)}{\beta_{j_0j_0}(v)}. \tag{3.110}$$

Therefore we have the factorization of $\beta_{ij}(v)$

$$\beta_{ij}(v) = \mathrm{sgn}(\beta_{j_0j_0}(v))\beta_i(v)\beta_j(v) \tag{3.111}$$

where $\beta_j(v) = \beta_{jj_0}(v)\Big/\sqrt{|\beta_{j_0j_0}(v)|}$. We denote $\mathrm{sgn}(\beta_{j_0j_0}(v)) = \tilde{\varepsilon}(v)$ which is a constant within a 4-simplex $\sigma_v$. Thus we have the following expression of the bivector $\varepsilon_{e_ie_j}(v)X_{e_ie_j}(v)$

$$\varepsilon_{e_ie_j}(v)X_{e_ie_j}(v) = \tilde{\varepsilon}(v)\Big(\beta_i(v)N_{e_i}(v)\Big)\wedge\Big(\beta_j(v)N_{e_j}(v)\Big) \tag{3.112}$$

The Eq.(3.107) takes the form

$$\sum_{j=1}^{5}\beta_j(v)N_{e_j}(v) = 0. \tag{3.113}$$

Now we construct the frame vectors $U_{e_i}(v)$ for a classical discrete geometry at each vertex $v$[5]:

$$U_I^{e_i}(v) := \pm\frac{\beta_i(v)N_I^{e_i}(v)}{\sqrt{|V_4(v)|}} \tag{3.114}$$

with

$$V_4(v) := \det\Big(\beta_2(v)N^{e_2}(v), \beta_3(v)N^{e_2}(v), \beta_4(v)N^{e_2}(v), \beta_5(v)N^{e_2}(v)\Big) \tag{3.115}$$

where $U^I_{e_i}(v)$ are time-like 4-vectors by Eq.(3.56), and any four of the five frame vectors $U_{e_i}(v)$ span a 4-dimensional vector space by the assumption of nondegeneracy. Moreover the frame vectors satisfy the closure condition

$$\sum_{j=1}^{5} U_{e_j}(v) = 0 \tag{3.116}$$

and

$$\frac{1}{V_4(v)} = \det\Big(U^{e_2}(v), U^{e_3}(v), U^{e_4}(v), U^{e_5}(v)\Big) \tag{3.117}$$

and

$$\varepsilon_{e_ie_j}(v)X^{e_ie_j}_{IJ}(v) = \tilde{\varepsilon}(v)|V_4(v)|\left[U_{e_i}(v)\wedge U_{e_j}(v)\right]_{IJ} = \varepsilon(v)V_4(v)\left[U_{e_i}(v)\wedge U_{e_j}(v)\right]_{IJ}. \tag{3.118}$$

where $\varepsilon(v) = \tilde{\varepsilon}(v)\mathrm{sgn}(V_4(v))$. We emphasize that these frame vectors $U_e(v)$ are constructed from spin foam configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ that solves the critical point equations. Note that the oriented 4-volume $V_4(v)$ in general can be either positive or negative for different 4-simplices. However for a nondegenerate critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, we can always make a subdivision of the triangulation, such that $\mathrm{sgn}(V_4(v))$ is a constant within each sub-triangulation.

Fix an edge $e_1$ at the vertex $v$, we construct the inverse of the nondegenerate matrix $\Big(U^{e_2}(v), U^{e_3}(v), U^{e_4}(v), U^{e_5}(v)\Big)^t$, denoted by $E^I_{e_ie_1}(v)$ such that

$$U_I^{e_i}(v)E^I_{e_je_1}(v) = \delta^i_j \qquad i,j = 2,3,4,5 \tag{3.119}$$

Explicitly, for example

$$E^I_{e_2e_1}(v) = V_4(v)\epsilon^{IJKL}U^{e_3}_J(v)U^{e_4}_K(v)U^{e_5}_L(v) \tag{3.120}$$

[5]We denote the dual vector $N^e_I$ by $N^e$ and the vector $N^I_e$ by $N_e$, and the same convention holds for $U_e$ and $U^e$.

Note that $E^I_{e_ie_1}(v)$ is determined only up to a sign from the data $N_e(v)$ since Eq.(3.114). However if we fix $e_2$ instead of $e_1$, and find the inverse of $\Big(U^{e_1}(v), U^{e_3}(v), U^{e_4}(v), U^{e_5}(v)\Big)^t$, denoted by $E^I_{e_ie_2}(v)$, then

$$U^{e_i}_I(v)E^I_{e_je_2}(v) = \delta^i_j \qquad i,j = 1,3,4,5 \tag{3.121}$$

and

$$E^I_{e_1e_2}(v) = -V_4(v)\epsilon^{IJKL}U^{e_3}_J(v)U^{e_4}_K(v)U^{e_5}_L(v) \tag{3.122}$$

where the minus sign comes from $V_4(v)$, because from the closure condition $\sum_{j=1}^5 U_{e_j}(v) = 0$

$$\det\Big(U^{e_2}(v), U^{e_3}(v), U^{e_4}(v), U^{e_5}(v)\Big) = -\det\Big(U^{e_1}(v), U^{e_3}(v), U^{e_4}(v), U^{e_5}(v)\Big). \tag{3.123}$$

Therefore we find

$$E^I_{e_1e_2}(v) = -E^I_{e_2e_1}(v). \tag{3.124}$$

Then we can fix $e_3$, $e_4$, $e_5$, and do the same manipulation as above, to obtain $E_{e_ie_j}(v)$ $i,j = 1,\cdots,5$ such that

$$U^{e_i}_I(v)E^I_{e_je_k}(v) = \delta^i_j - \delta^i_k \quad \text{and} \quad E^I_{e_ie_j}(v) = -E^I_{e_je_i}(v) \tag{3.125}$$

from which we can see that all $E^I_{e_je_k}(v)$ are spatial vectors. One can also verify immediately that

$$U^{e_i}_I(v)\Big(E^I_{e_je_k}(v) + E^I_{e_ke_l}(v) + E^I_{e_le_j}(v)\Big) = 0 \qquad \forall i = 1,\cdots,5 \tag{3.126}$$

By the nondegeneracy of $U^{e_i}_I(v)$, one has

$$E^I_{e_je_k}(v) + E^I_{e_ke_l}(v) + E^I_{e_le_j}(v) = 0 \tag{3.127}$$

Comparing Eqs.(3.125) and (3.127) with Eq.(3.69), we see that the collection of $E_{ee'}(v)$ at $v$ is a co-frame at the vertex $v$. The bivector $X_{ee'}(v)$ can also be expressed by $E_{ee'}(v)$

$$\varepsilon_{e_4e_5}(v)X^{IJ}_{e_4e_5}(v) = \varepsilon(v)\star\Big[E_{e_1e_2}(v)\wedge E_{e_2e_3}(v)\Big]^{IJ} \tag{3.128}$$

which will also be denoted by $\varepsilon_{ef}(v)X^{IJ}_f(v) = \varepsilon(v)\star\Big[E_{\ell_1}(v)\wedge E_{\ell_2}(v)\Big]^{IJ}$.

The above work are done essentially with in a single 4-simplex $\sigma_v$. Now we consider two neighboring 4-simplices $\sigma_v, \sigma_{v'}$ while their center $v, v'$ are connected by the dual edge $e$. Since we only consider two simplices, we introduce a short-hand notation:

$$U_0 := U_e(v) \quad U'_0 := g_{vv'} U_e(v') \tag{3.129}$$

$$U_i := U_{e_i}(v) \quad U'_i := g_{vv'} U_{e'_i}(v') \tag{3.130}$$

$$E_{ij} := E_{e_i e_j}(v) \quad E'_{ij} := g_{vv'} E_{e'_i e'_j}(v') \tag{3.131}$$

where $i, j = 1, \cdots, 4$ labels the edges connecting to $v$ or $v'$ other than $e$, $E_{ij}$ and $E'_{ij}$ are orthogonal to $U_0$ and $U'_0$ respectively from Eq.(3.125). Here $g_{vv'} = g_{ve} g_{ev'}$ comes from the spin foam configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ that solves the critical point equations. From the closure condition of $U_e(v)$ we have

$$U_0 = -\sum_i U_i \quad \text{and} \quad U'_0 = -\sum_i U'_i \tag{3.132}$$

By definition $N_e(v) = g_{ve} u$ and $N_e(v') = g_{v'e} u$ where $u = (1, 0, 0, 0)^t$, thus $N_e(v) = g_{vv'} N_e(v')$ with $e = (v, v')$. Thus from the definition of $U_e(v)$ in Eq.(3.114), we find

$$\frac{U'_0}{|U'_0|} = \tilde{\varepsilon} \frac{U_0}{|U_0|} \tag{3.133}$$

where $\tilde{\varepsilon} = \pm$. On the other hand, from the parallel transportation relation $X_f(v) = g_{vv'} X_f(v') g_{v'v}$ and $\varepsilon_{ef}(v) = -\varepsilon_{ef}(v')$ for $e = (v, v')$, we have

$$\varepsilon_{0i} X^{0i}_{IJ} = \varepsilon V (U^0 \wedge U^i)_{IJ} = -\varepsilon' V' (U'^0 \wedge U'^i)_{IJ} \tag{3.134}$$

where $X_{0i}$ is the bivector corresponds to the dual face $f$ determined by $e, e_i, e'_i$, the sign factor $\varepsilon_{0i} = \varepsilon_{ef}(v)$, the sign factors $\varepsilon$ and $\varepsilon'$ are short-hand notations of $\varepsilon(v)$ and $\varepsilon(v')$ respectively, and

$$\frac{1}{V} = \det\left(U^1, U^2, U^3, U^4\right) \qquad -\frac{1}{V'} = \det\left(U'^1, U'^2, U'^3, U'^4\right) \tag{3.135}$$

Here the minus sign for $1/V'$ is because the compatible orientations of $\sigma_v$ and $\sigma_{v'}$ are $[p_0, p_1, p_2, p_3, p_4]$ and $-[p_0, p_1, p_2, p_3, p_4]$. Thus we should set $\epsilon_{01234}(v) = -\epsilon_{01234}(v') = 1$.

Eqs.(3.133) and (3.134) tell us that $U_I^0$ is proportional to $U_I'^0$ and $U_I'^i$ is a linear combination of $U_I^i$ and $U_I^0$. Explicitly

$$U_I'^i = -\varepsilon\varepsilon'\tilde{\varepsilon}\frac{|U_0|V}{|U_0'|V'}U_I^i + a_i U_I^0 \tag{3.136}$$

where $a_i$ are the coefficients such that $\sum_i U_i' = -U_0'$. Using this expression of $U'^i$, we have

$$\begin{aligned} -\frac{1}{V'} &= \det\left(U'^1, U'^2, U'^3, U'^4\right) = \det\left(U'^0, U'^1, U'^2, U'^3\right) \\ &= \tilde{\varepsilon}\frac{|U_0'|}{|U_0|}\left(-\varepsilon\varepsilon'\tilde{\varepsilon}\frac{|U_0|V}{|U_0'|V'}\right)^3 \det\left(U^0, U^1, U^2, U^3\right) \\ &= -\varepsilon\varepsilon'\left(\frac{|U_0|V}{|U_0'|V'}\right)^2\frac{1}{V'} \end{aligned} \tag{3.137}$$

which results in $\varepsilon = \varepsilon'$. Therefore $\varepsilon(v) = \varepsilon(v') = \varepsilon$ is a global sign on the entire triangulation. Now for the bivectors $X_{0i}(v)$ and $X_{0i}(v')$ ($X^{ji}(v) = X^{ij}(v)$ and $\varepsilon_{ij}(v) = -\varepsilon_{ji}(v)$)

$$\varepsilon_{0i}(v)X_{IJ}^{0i}(v) = \varepsilon\frac{1}{2}\sum_{m,n}\varepsilon^{k0imn}(v)\epsilon_{IJKL}E_{mk}^K(v)E_{nk}^L(v) \tag{3.138}$$

$$\varepsilon_{0i}(v')X_{IJ}^{0i}(v') = \varepsilon\frac{1}{2}\sum_{m,n}\varepsilon^{k0imn}(v')\epsilon_{IJKL}E_{mk}^K(v')E_{nk}^L(v') \tag{3.139}$$

Since $\varepsilon_{0i}(v) = -\varepsilon_{0i}(v')$ and $\varepsilon^{kijmn}(v) = -\varepsilon^{kijmn}(v')$, we can set $\varepsilon_{0i}(v)\varepsilon^{k0imn}(v) = \varepsilon_{0i}(v')\varepsilon^{k0imn}(v') = \varepsilon_{0i}\varepsilon^{k0imn}$. Therefore

$$\varepsilon_{0i}X_{IJ}^{0i}(v) = \varepsilon\frac{1}{2}\sum_{m,n}\varepsilon^{k0imn}\epsilon_{IJKL}E_{mk}^K(v)E_{nk}^L(v) \tag{3.140}$$

$$\varepsilon_{0i}X_{IJ}^{0i}(v') = \varepsilon\frac{1}{2}\sum_{m,n}\varepsilon^{k0imn}\epsilon_{IJKL}E_{mk}^K(v')E_{nk}^L(v') \tag{3.141}$$

Given a triangle $f$, we can choose $E_{\ell_1}(v), E_{\ell_2}(v)$ (e.g. $\ell_1 = (p_m, p_k)$ and $\ell_2 = (p_n, p_k)$ with $\varepsilon_{0i} = 1$ and $\varepsilon^{k0imn} = 1$) such that

$$X_f^{IJ}(v) = \varepsilon \star\Big[E_{\ell_1}(v)\wedge E_{\ell_2}(v)\Big]^{IJ} \quad \text{and} \quad X_f^{IJ}(v') = \varepsilon \star\Big[E_{\ell_1}(v')\wedge E_{\ell_2}(v')\Big]^{IJ} \tag{3.142}$$

On the other hand, Eq.(3.137) also implies that $|U_0|V = \pm|U_0'|V'$. Thus we define a sign factor $\mu := -\tilde{\varepsilon}|U_0|V\Big/|U_0'|V' = \pm 1$ such that from Eq.(3.136)

$$U'^i_I = \mu U^i_I + a_i U^0_I \qquad \mu = -\tilde{\varepsilon}\ \mathrm{sgn}(VV') \tag{3.143}$$

Therefore we obtain the relation between $E_{ij}$ and $E'_{ij}$ (using $\varepsilon_{jklm0}(v') = -\varepsilon_{jklm0}(v)$)

$$\begin{aligned} E'^I_{jk} &= V'\varepsilon_{jklm}(v')\epsilon^{IJKL}\ U'^l_J\ U'^m_K\ U'^0_L \\ &= -\tilde{\varepsilon}\frac{|U_0'|}{|U_0|}\mu^2 V'\varepsilon_{jklm}(v)\epsilon^{IJKL}\ U^l_J\ U^m_K\ U^0_L \\ &= \mu^3 V\varepsilon_{jklm}(v)\epsilon^{IJKL}\ U^l_J\ U^m_K\ U^0_L \\ &= \mu E^I_{jk} \end{aligned} \tag{3.144}$$

which means that for all tetrahedron edge $\ell$ of the tetrahedron $t_e$ dual to $e = (v, v')$, the co-frame vectors $E_\ell(v)$ and $E_\ell(v')$ at neighboring vertices $v$ and $v'$ are related by parallel transportation up to a sign $\mu_e$, i.e.

$$\mu_e E_\ell(v) = g_{vv'} E_\ell(v') \quad \forall\ \ell \subset t_e \tag{3.145}$$

This relation shows that the vectors $E_\ell(v)$ (constructed from spin foam critical point configuration) satisfy the metricity condition Eq.(3.67). Therefore the collection of co-frame vectors $E_\ell(v)$ at different vertices consistently forms a discrete co-frame of the whole triangulation. At the critical configuration, we define an SO(1,3) matrix $\Omega_{vv'}$ relating $g_{vv'}$ (in the Spin-1 representation) by the sign $\mu_e$, i.e.

$$g_{vv'} = \mu_e \Omega_{vv'} \tag{3.146}$$

By Lemma 3.3.1 and Definition 3.3.2, the SO(1,3) matrix $\Omega_{vv'}$ is a discrete spin connection compatible with the co-frame if $\mathrm{sgn}(V_4(v)) = \mathrm{sgn}(V_4(v'))$.

If $\mathrm{sgn}(V_4(v)) = \mathrm{sgn}(V_4(v'))$, $\mu_e = -\tilde{\varepsilon}\,\mathrm{sgn}(V_4(v)V_4(v')) = -\tilde{\varepsilon}$. Thus from Eq.(3.133),

$$\frac{U_0'}{|U_0'|} = -\mu_e \frac{U_0}{|U_0|} \tag{3.147}$$

the tetrahedron normal $U_e(v)/|U_e(v)|$ is always opposite to $\Omega_e U_e(v')/|U_e(v')|$ when $\mathrm{sgn}(V_4(v)) = \mathrm{sgn}(V_4(v'))$.

Since in Spin-1 representation $g_{vv} \in \mathrm{SO}^+(1,3)$ and $\Omega \in \mathrm{SO}(1,3)$, $\mu_e = -1$ corresponds the case that $\Omega_{vv'} \in \mathrm{SO}^-(1,3)$. It means that in the case of $\mu_e = -1$ if we choose the unit vectors $\hat{U}(v), \hat{U}(v')$ orthogonal to $E_\ell(v), E_\ell(v)$ ($\ell \subset t_e$) such that

$$\operatorname{sgn}\det\Big(E_{\ell_1}(v), E_{\ell_2}(v), E_{\ell_3}(v), \hat{U}(v)\Big) = \operatorname{sgn}\det\Big(E_{\ell_1}(v'), E_{\ell_2}(v'), E_{\ell_3}(v'), \hat{U}(v')\Big) \tag{3.148}$$

then one of $\hat{U}(v), \hat{U}(v')$ is future-pointing and the other is past-pointing.

### 3.4.2 Boundary data for spin foam critical configuration

Given a spin foam configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ that solves critical point equations. The boundary data of the spin foam amplitude is given by the boundary spins and the normalized spinors $(j_f, \xi_{ef})$ for the boundary triangles $f$. Eq.(3.45) naturally associates a bivector $X_{ef}$ to each pair $(j_f, \xi_{ef})$ for each $(e, f)$. From Eq.(3.2.2),

$$X_{ef}^{IJ} = 2\gamma j_f \left[\hat{n}_{ef} \wedge u\right] \tag{3.149}$$

The spatial 3-vectors $j_f \hat{n}_{ef}$ satisfy the critical point equation Eq.(3.39)

$$\sum_f \varepsilon_{ef} j_f \hat{n}_{ef} = 0 \tag{3.150}$$

where $v$ is the vertex connecting to the edge $e$. We define $V_3(e)$ such that

$$\det\Big(\varepsilon_{ef_2} j_{f_2} \hat{n}_{ef_2}, \varepsilon_{ef_3} j_{f_3} \hat{n}_{ef_3}, \varepsilon_{ef_4} j_{f_4} \hat{n}_{ef_4}\Big) = \operatorname{sgn}(V_3(e))\, |V_3(e)|^2 \tag{3.151}$$

We rescale each vector $\varepsilon_{ef} j_f \hat{n}_{ef}$ by

$$n_{ef} := \frac{\varepsilon_{ef}\gamma j_f \hat{n}_{ef}}{|V_3(e)|} \quad \text{then} \quad \sum_f n_{ef} = 0 \quad \text{and} \quad \det\Big(n_{ef_2}, n_{ef_3}, n_{ef_4}\Big) = \frac{1}{V_3(e)}. \tag{3.152}$$

We assume the nondegeneracy of the boundary data, i.e. any three of the four vectors $n_{ef}$ span the 3-dimensional spatial subspace, in another word, the following product of determinants is nonvanishing

$$\prod_{f_1,f_2,f_3=1}^{4} \det\left(n_{ef_1}, n_{ef_2}, n_{ef_3}\right) \neq 0. \tag{3.153}$$

The nondegeneracy of the tetrahedron Eq.(3.153) is implied by the nondegeneracy condition in the bulk Eq.(3.103). The reason is the following: By the parallel transportation relation $X_f(v) = g_{ve} X_{ef} g_{ev}$ and $X_{ef} = 2\gamma j_f \hat{n}_{ef} \wedge u$, the bivector $X_f(v)$ is then given by $X_f(v) = V_{ef}(v) \wedge N_e(v)$, where $N_e(v) = g_{ve}u$ and $V_{ef}(v) := 2\gamma j_f g_{ve}\hat{n}_{ef}$ is orthogonal to $N_e(v)$. For $f$ the triangle shared by $t_e$ and $t_{e_i}$ $(i = 1, \cdots, 4)$, we know that $X_f(v) = \alpha_{e_ie}(v)N_{e_i}(v) \wedge N_e(v)$. Therefore the vector $V_{ef}(v)$ is a linear combination of $N_{e_i}$ and $N_e$. The nondegeneracy condition Eq.(3.103) in 4-dimensions implies the 4 unit vectors, say $N_e$ and any 3 out of 4 vectors $N_{e_i}$, are linear independent and span a 4-dimensional vector space. Thus any 3 out of the 4 vectors $V_{ef}(v)$ must be linear independent and span a 3-dimensional subspace orthogonal to $N_e(v)$. Then Eq.(3.153) is a result from parallel transporting $V_{ef}(v)$ back to the center of $t_e$.

We now denote $n_{ef} \equiv n_{p_1}(e)$, where the triangle $f$ is determined by $(p_2, p_3, p_4)$. Now we construct the spatial 3-vectors $E_{p_1p_2}(e)$, such that the matrix $\Big(E_{p_2p_1}(e), E_{p_3p_1}(e), E_{p_4p_1}(e)\Big)$ is the inverse of $\Big(n_{p_2}(e), n_{p_3}(e), n_{p_4}(e)\Big)^t$. Therefore we have

$$n_{p_i}(e) \cdot E_{p_jp_k}(e) = \delta_{ij} - \delta_{ik} \tag{3.154}$$

The 3-vectors $E_{p_ip_j}(e)$ are associated to the edges $\ell = (p_i, p_j)$ of the tetrahedron $t_e$, so it can be denoted by $E_\ell(e)$. Note that $E_\ell(e)$ is determined up to an overall rescaling, since the set of $n_{ef}$ is defined up to an overall scaling $\alpha \in \mathbb{R}$. In the following we are going to show that the vectors $E_\ell(e)$ are co-frame vectors on the boundary.

First of all, Eqs.(3.88), (3.89) and (3.91) can be verified immediately from Eq.(3.154). Since $\Big(E_{p_2p_1}(e), E_{p_3p_1}(e), E_{p_4p_1}(e)\Big)$ is the inverse of $\Big(n_{p_2}(e), n_{p_3}(e), n_{p_4}(e)\Big)^t$, we have

$$\det\Big(E_{p_2p_1}(e), E_{p_3p_1}(e), E_{p_4p_1}(e)\Big) = V_3(e) \tag{3.155}$$

we also have

$$\varepsilon_{ef}\gamma j_f \hat{n}_{p_j}(e) = |V_3(e)| n_{p_j}(e) = \varepsilon(e)V_3(e)n_{p_j}(e) = \varepsilon(e)\frac{1}{2}\sum_{k,l}\varepsilon_{ijkl}E_{p_kp_i}(e) \times E_{p_lp_i}(e) \tag{3.156}$$

where we have define a sign factor $\varepsilon(e) = \mathrm{sgn}(V_3(e))$. Equivalently for the bivector $X_{ef}$, there exists $E_{\ell_1}(e), E_{\ell_2}(e)$ such that

$$X_{ef}^{IJ} = 2\gamma j_f \left[\hat{n}_{ef} \wedge u\right]^{IJ} = \varepsilon(e) \star \Big[E_{\ell_1}(e) \wedge E_{\ell_2}(e)\Big]^{IJ}. \tag{3.157}$$

Consider a internal vertex $v$ which connected by the edge $e$, we introduce the short-hand notation:

$$E_{ij} := E_{p_ip_j}(e) \qquad E'_{ij} := g_{ev}E_{p_ip_j}(v) \qquad \varepsilon(e) := \varepsilon' \qquad \hat{n}_j := \hat{n}_{p_j}(e) \tag{3.158}$$

Since $X_f(v) = g_{ve}X_{ef}g_{ev}$, for each triangle determined by $(p_i, p_j, p_k)$

$$\varepsilon' \frac{1}{2}\sum_{k,l} \varepsilon_{ijkl} \star E_{ki} \wedge E_{li} = \varepsilon \frac{1}{2}\sum_{k,l} \varepsilon_{ijkl} \star E'_{ki} \wedge E'_{li} = 2\varepsilon_{ef}\gamma j_f \left[\hat{n}_j \wedge u\right] \tag{3.159}$$

We also have $N_e(v) = g_{ve}u$. So $E'_{ij}$ is orthogonal to $u^I = (1, 0, 0, 0)$ since $E_\ell(v)$ $(\ell \subset t_e)$ orthogonal to $N_e(v)$. Thus

$$\varepsilon' V n_j = 2\varepsilon_{ef}\gamma j_f \hat{n}_j = \varepsilon \frac{1}{2}\sum_{k,l} \varepsilon_{ijkl} E'_{ki} \times E'_{li} \tag{3.160}$$

which implies that the $3 \times 3$ matrix given by $E'_{ki}$ (with $i$ fixed) is the inverse of the matrix given by $n_j$, $j \neq i$, up to an overall constant, i.e.

$$n_i \cdot E'_{jk} = \varepsilon\varepsilon' \frac{V'_3}{V_3}(\delta_{ij} - \delta_{ik}) \tag{3.161}$$

we have used the short-hand notation

$$V_3 = V_3(e) = \det\Big(E_{21}(e), E_{31}(e), E_{41}(e)\Big) \tag{3.162}$$

$$V'_3 = V'_3(e) = \det\Big(E'_{21}(e), E'_{31}(e), E'_{41}(e)\Big) \tag{3.163}$$

Comparing Eq.(3.161) and Eq.(3.154) we determine that $E_{jk}$ is proportional to $E'_{jk}$:

$$E'_{jk} = \varepsilon\varepsilon' \frac{V'_3}{V_3} E_{jk}. \tag{3.164}$$

since the matrix given by $n_i$ has unique inverse. Insert this relation back into Eq.(3.159), we obtain that

$$\varepsilon \left(\frac{V'_3}{V_3}\right)^2 = \varepsilon' \tag{3.165}$$

which tell us that

$$\varepsilon' = \varepsilon \quad \text{and} \quad \left|\frac{V'_3}{V_3}\right| = 1 \tag{3.166}$$

As a result we find the relations

$$X_{ef}^{IJ} = \varepsilon \star \Big[E_{\ell_1}(e) \wedge E_{\ell_2}(e)\Big]^{IJ} \quad \text{and} \quad \mu_e E_\ell(e) = g_{ev} E_\ell(v) \quad \forall\, \ell \subset t_e \tag{3.167}$$

where $\varepsilon = \pm 1$ is the global sign factor of the whole triangulation, and $\mu_e = \mathrm{sgn}(V_3)\mathrm{sgn}(V_3') = \pm 1$. From the second relation above, we obtain the metricity condition Eq.(3.91). Therefore we confirm that $E_\ell(e)$ is a boundary co-frame constructed from spin foam critical configuration. The group element $g_{ev}$ equals to the spin connection $\Omega_{ev}$ up to a sign, i.e.

$$g_{ev} = \mu_e \Omega_{ev}. \tag{3.168}$$

Since $\varepsilon$ is a global sign of the entire triangulation and $\varepsilon = \mathrm{sgn}(V_3(e))$ on the boundary, then prior to the construction, one has to choose a consistent orientation of the boundary triangulation such that $\mathrm{sgn}(V_3(e)) = \mathrm{sgn}(V_3(e'))$ for each pair of tetrahedra $t_e, t_{e'}$.

By the following relations (we choose the orientation of the 4-simplex $\sigma_v = [p_0, p_1, p_2, p_3, p_4]$):

$$V_3 = \epsilon_{IJK} E_{21}^I E_{31}^J E_{41}^K \quad V_3' = \epsilon_{IJK} E_{21}'^I E_{31}'^J E_{41}'^K \quad (g_{ev} U^0)_I = \frac{-1}{V_4} \epsilon_{IJKL} E_{21}'^I E_{31}'^J E_{41}'^K \tag{3.169}$$

we obtain that

$$V_3' = -V_4 U_I^0 (g_{ve} u)^I = -V_4 U_I^0 N_0^I, \quad \text{where} \quad u = (1,0,0,0)^t \tag{3.170}$$

Then for an edge $e$ connecting to the boundary

$$\mu_e = -\varepsilon\ \mathrm{sgn}(V_4(v))\mathrm{sgn}(U_I^0(v) N_0^I(v)) \tag{3.171}$$

which implies that if we choose $\varepsilon = \mathrm{sgn}(V_3(e)) = +1$ globally on the boundary, and if $V_4(v) > 0$, $\mu_e = +1$ when $U_0(v)$ is future-pointing and $\mu_e = -1$ when $U_0(v)$ is past-pointing, while $N_0(v) = g_{ve} u$ is always future-pointing.

**Lemma 3.4.1.** *Given $f$ either an internal face or a boundary face, the product $\prod_{e\subset\partial f} \mu_e$ doesn't change when $U_e(v)$ flips sign for any 4-simplex $\sigma_v$, recall that the five normals $U_e(v)$ at $\sigma_v$ are defined up to a overall sign. Therefore the product $\prod_{e\subset\partial f} \mu_e$ is determined by the spin foam critical configuration.*

**Proof:** For a internal edge $e = (v, v')$, we have

$$\mu_e = -\tilde{\varepsilon}_e \mathrm{sgn}\Big(V_4(v)V_4(v')\Big) = \mathrm{sgn}\Big(U_e^I(v)(g_{vv'}U_e)^I(v')\Big)\mathrm{sgn}\Big(V_4(v)V_4(v')\Big) \quad (3.172)$$

where we recall that $\tilde{\varepsilon}_e U_e(v)/|U_e(v)| = g_{vv'}U_e(v')/|U_e(v')|$. Combine with Eq.(3.171), it is easy to see that if we flip simultaneously the sign of all the five $U_e(v)$ at any $\sigma_v$ ($v \in \partial f$), the product $\prod_{e\subset\partial f}\mu_e$ doesn't change, for $f$ either an internal face or a boundary face. □

We recall FIG.3.1, where the triangle $f_l$ is shared by two boundary tetrahedra $t_{e_0}, t_{e_1}$. Because of Eq.(3.167), we parallel transport three co-frame vectors $E_\ell(e_0)$ corresponding to the three edges of the triangle $f_l$,

$$(\prod_e \mu_e)E_\ell(e_1) = G_{f_l}(e_1, e_0)E_\ell(e_0) \quad \forall\ \ell \subset f_l \quad (3.173)$$

where $G_{f_l}(e_1, e_0) := \overleftarrow{\prod}_e g_e$ is a product of the edge holonomy $g_e$ over all the internal edges $e$ of the dual face $f_l$. Therefore the triangle formed by the three $E_\ell(e_0)$ ($\ell \subset f_l$) matches in shape with the triangle formed by $E_\ell(e_1)$ ($\ell \subset f_l$). Since both $E_\ell(e_0)$ and $E_\ell(e_1)$ are orthogonal to the unit time-like vector $u = (1, 0, 0, 0)$. There exists an O(3) matrix $\hat{g}_l$ such that

$$\hat{g}_l E_\ell(e_0) = E_\ell(e_1) \quad \text{and} \quad \hat{g}_l \hat{n}_{e_0 f_l} = \hat{n}_{e_1 f_l} \quad (3.174)$$

These relations give the restrictions of the boundary data for the spin foam amplitude. We call the boundary condition given by Eq.(3.174) the (nondegenerate) *Regge boundary condition*. The above analysis shows that the spin foam boundary data must satisfy the Regge boundary condition in order to have nondegenerate solutions of the critical point equations Eqs.(3.37), (3.38), (3.39).

### 3.4.3 Reconstruction theorem

Now we summarize the reconstruction results in this section as a theorem:

**Theorem 3.4.2.** *(Construction of Classical Geometry from Spin Foam Critical Configuration)*

- *Given the data $(j_f, g_{ev}, \xi_{ef}, z_{vf})$ be a nondegenerate spin foam configuration that solves the critical point equations Eqs.(3.37), (3.38), and (3.39), there exists* *a discrete classical Lorentzian geometry on $\mathcal{M}$, represented by a set of spatial co-frame vectors $E_\ell(v)$ satisfying Eqs.(3.65), (3.66) and (3.67) in the bulk, and $E_\ell(e)$ satisfying Eqs.(3.88), (3.89), (3.90) and (3.91) on the boundary, such that the bivectors $X_f(v)$ and $X_{ef}$ in Proposition 3.2.1 is written by*

$$X_f^{IJ}(v) = \varepsilon \star \Big[E_{\ell_1}(v) \wedge E_{\ell_2}(v)\Big]^{IJ}, \qquad X_{ef}^{IJ} = \varepsilon \ \star \Big[E_{\ell_1}(e) \wedge E_{\ell_2}(e)\Big]^{IJ} \tag{3.175}$$

*where $\ell_1, \ell_2$ are edges of the triangle $f$. The above equation is a relation between the spin foam data $X_f(v), X_{ef}$ and a classical geometric data $E_\ell(v)$. Such a relation is determined up to a global sign $\varepsilon$ on the whole triangulation. Moreover the above co-frame is unique up to inversion $E_\ell \mapsto -E_\ell$ at each $v$ or $t_e$. With the co-frame vectors $E_\ell(v), E_\ell(e)$, we can construct a discrete metric $g_{\ell_1\ell_2}(v), g_{\ell_1\ell_2}(e)$ in the bulk and on the boundary*

$$g_{\ell_1\ell_2}(v) = \eta_{IJ} E^I_{\ell_1}(v) E^J_{\ell_2}(v) \qquad g_{\ell_1\ell_2}(e) = \eta_{IJ} E^I_{\ell_1}(e) E^J_{\ell_2}(e). \tag{3.176}$$

- *The norm of the bivector $|X_f(v)| = |E_{\ell_1}(v) \wedge E_{\ell_2}(v)| = 2\gamma j_f$. Thus $\gamma j_f$ is understood as the area of the triangle $f$[6].*

- *If the triangulation has boundary, one has to choose a consistent orientation of the boundary triangulation such that $\mathrm{sgn}(V_3(e)) = \mathrm{sgn}(V_3(e'))$ for each pair of tetrahedra $t_e, t_{e'}$ (recall Eq.(3.151)). Then the global sign $\varepsilon$ is specified by the orientation of the boundary, i.e. $\varepsilon = \mathrm{sgn}(V_3(e))$.*

- *Equivalently the bivectors in the bulk can be expressed by the frame $U_e(v)$ associated with $E_\ell(v)$*

$$X_f^{IJ}(v) = \varepsilon \ V_4(v) \Big[U_e(v) \wedge U_{e'}(v)\Big]^{IJ} \tag{3.177}$$

*where $e, e'$ are the dual edges of the dual face $f$, and $V_4(v)^{-1}$ is the determinant of the matrix defined by the frame co-vectors $U_I^{e_i}(v)$, $i = 2, 3, 4, 5$, i.e.*

$$\frac{1}{V_4(v)} = \det\Big(U^{e_2}(v), U^{e_3}(v), U^{e_4}(v), U^{e_5}(v)\Big). \tag{3.178}$$

[6] $|E_1 \wedge E_2|^2 = \frac{1}{2}(E_1^I E_2^J - E_1^J E_2^I)(E^1_I E^2_J - E^1_J E^2_I) = |E_1|^2|E_2|^2(1-\cos^2\theta) = (2A_f)^2$ where $E_1 \cdot E_2 = |E_1||E_2|\cos\theta$. $|E_1 \wedge E_2|$ corresponds to the area of a parallelogram (two times the area of the triangle) determined by $E_1$ and $E_2$.

*For the bivector on the boundary, from Eq.(3.2.2)*

$$X_{ef}^{IJ} = 2\gamma j\, [\hat{n}_{ef} \wedge u]^{IJ} \tag{3.179}$$

*where* $u = (1, 0, 0, 0)$ *and* $j\hat{n}_{ef}$ *is the oriented area of the boundary triangle.*

- *Given a dual edge* $e$*, for all tetrahedron edge* $\ell$ *of the tetrahedron* $t_e$ *dual to* $e = (v, v')$*, the associated co-frame vectors* $E_\ell(v)$ *and* $E_\ell(v')$ *at neighboring vertices* $v$ *and* $v'$ *are related by parallel transportation up to a sign* $\mu_e$*, i.e.*

$$\mu_e E_\ell(v) = g_{vv'} E_\ell(v') \quad \forall\ \ell \subset t_e \tag{3.180}$$

*If the dual edge* $e$ *connects the boundary, we have similarly*

$$\mu_e E_\ell(v) = g_{ve} E_\ell(e) \quad \forall\ \ell \subset t_e. \tag{3.181}$$

*We define the SO(1,3) matrices* $\Omega_{vv'}, \Omega_{ve}$ *by*

$$\Omega_{vv'} = \mu_e g_{vv'} \quad \Omega_{ve} = \mu_e g_{ve}. \tag{3.182}$$

*The simplicial complex* $\mathcal{K}$ *can be subdivided into sub-complexes* $\mathcal{K}_1, \cdots, \mathcal{K}_n$ *such that (1) each* $\mathcal{K}_i$ *is a simplicial complex with boundary, (2) within each sub-complex* $\mathcal{K}_i$*,* $\mathrm{sgn}(V_4(v))$ *is a constant. Then within each sub-complex* $\mathcal{K}_i$*, the SO(1,3) matrices* $\Omega_{vv'}, \Omega_{ve}$ *are the discrete spin connection compatible with the co-frame* $E_\ell(v)$ *and* $E_\ell(v')$*.*

- *Given the boundary triangles* $f$ *and boundary tetrahedra* $t_e$*, in order to have nondegenerate solutions of the critical point equations Eqs.(3.37), (3.38), (3.39), the spin foam boundary data* $(j_f, \xi_{ef})$ *must satisfy the (nondegenerate) Regge boundary condition: (1) For each boundary tetrahedron* $t_e$ *and its triangles* $f$*,* $(j_f, \xi_{ef})$ *determines 4 triangle normals* $\hat{n}_{ef}$ *that spans a 3-dimensional spatial subspace. (2) Given the tetrahedra* $t_{e_0}, t_{e_1}$ *sharing the triangle* $f$*, the triangle normals* $\hat{n}_{e_0 f}$ *and* $\hat{n}_{e_1 f}$ *are related by an O(3) matrix* $g_l$ *(l the link dual to* $f$ *on the boundary)*

$$\hat{g}_l \hat{n}_{e_0 f} = \hat{n}_{e_1 f}. \tag{3.183}$$

*(3) The boundary triangulation is consistently oriented such that the orientation* $\mathrm{sgn}(V_3(e))$ *(recall Eq.(3.151)) is a constant on the boundary. If the Regge bound-*

*ary condition is satisfied, there are nondegenerate solutions of the critical point equations, and the solutions implies the shape-matching of the triangle $f$ shared by the tetrahedra $t_{e_0}$ and $t_{e_1}$. If the Regge boundary condition is not satisfied, there is no nondegenerate critical configuration.*

## 3.5 Spin foam amplitude at nondegenerate critical configuration

Given a nondegenerate critical configuration $(j_f, g_{ev}, \xi_{ef}, z_{vf})$, the previous discussions show us that we can construct a discrete classical geometry from the critical configuration. Moreover we can make a subdivision of the triangulation into sub-triangulations $\mathcal{K}_1, \cdots, \mathcal{K}_n$, such that (1) each $\mathcal{K}_i$ is a simplicial complex with boundary, (2) within each sub-complex $\mathcal{K}_i$, $\mathrm{sgn}(V_4(v))$ is a constant. To study the spin foam (partial-)amplitude $A_j(\mathcal{K})$ at a nondegenerate critical configuration, we only need to study the amplitude $A_j(\mathcal{K}_i)$ on the sub-triangulation $\mathcal{K}_i$ where $\mathrm{sgn}(V_4(v))$ is a constant. Then the behavior of $A_j(\mathcal{K})$ can be expressed as a product

$$A_j(\mathcal{K})\Big|_{\text{critical}} = \prod_i A_j(\mathcal{K}_i)\Big|_{\text{critical}} \tag{3.184}$$

Therefore in the following analysis of this section we always assume the triangulation has a boundary and $\mathrm{sgn}(V_4)$ is a constant on the triangulation.

### 3.5.1 Internal faces

We have shown previously that the action $S$ of the spin foam amplitude can be written as a sum $S = \sum_f S_f$. We first consider the internal faces whose edges are not contained in the boundary of the triangulation. Each internal "face action" $S_f$ evaluated at the critical point defined by Eqs.(3.37), (3.38), and (3.39) takes the form

$$S_f = 2i\gamma j_f \sum_{v\in\partial f} \ln \frac{||Z_{ve'f}||}{||Z_{vef}||} - 2ij_f \sum_{v\in\partial f} \phi_{eve'} = -2ij_f \left(\gamma \sum_{v\in\partial f} \theta_{eve'} + \sum_{v\in\partial f} \phi_{eve'}\right) \tag{3.185}$$

where we have denoted

$$\frac{||Z_{vef}||}{||Z_{ve'f}||} := e^{\theta_{eve'}} \tag{3.186}$$

Recall Eqs.(3.37) and (3.38), and consider the following successive actions on $\xi_{ef}$ of $g_{e'v}g_{ve}$ around the entire boundary of the face $f$

$$\overleftarrow{\prod_{v\in\partial f}} g_{e'v}g_{ve}J\xi_{ef} = e^{-\sum_v \theta_{eve'} - i\sum_v \phi_{eve'}} J\xi_{ef} \tag{3.187}$$

$$\overleftarrow{\prod_{v\in\partial f}} g_{e'v}g_{ve}\xi_{ef} = e^{\sum_v \theta_{eve'} + i\sum_v \phi_{eve'}} \xi_{ef} \tag{3.188}$$

Thus $\xi_{ef}$ is a eign-vector of the loop holonomy $\overleftarrow{\prod}_{v\in\partial f} g_{e'v}g_{ve}$. Since $\xi_{ef}, J\xi_{ef}$ are normalized spinors and $\langle J\xi_{ef}, \xi_{ef}\rangle = 0$, thus we represent them by

$$\xi_{ef} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad J\xi_{ef} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.189}$$

We express this loop holonomy by an arbitrary $\mathrm{SL}(2,\mathbb{C})$ matrix

$$G_f(e) := \overleftarrow{\prod_{v\in\partial f}} g_{e'v}g_{ve} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.190}$$

Thus the eigenvalue equations for arbitrary complex number $\alpha$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{\alpha}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{-\alpha}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.191}$$

implies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix} = e^{\alpha\vec{\sigma}\cdot\hat{z}} \tag{3.192}$$

By rotating $\hat{z}$ to the unit 3-vector $\hat{n}_{ef}$, we obtain a representation-independent expression of the loop holonomy $G_f(e)$

$$G_f(e) = \exp\left[\sum_{v\in\partial f} (\theta_{eve'} + i\phi_{eve'})\,\vec{\sigma}\cdot\hat{n}_{ef}\right]. \tag{3.193}$$

which is an exponential map from Lie algebra variable[7].

[7]Note that not all the elements in $\mathrm{SL}(2,\mathbb{C})$ can be written in an exponential form, because of the noncompactness.

Consider the following identity: for any complex number $\alpha$ and unit vector $\hat{n}$,

$$\mathrm{Tr}\left[\frac{1}{2}\left(1+\vec{\sigma}\cdot\hat{n}\right)e^{\alpha\vec{\sigma}\cdot\hat{n}}\right]=e^{\alpha} \tag{3.194}$$

which can be proved by the identities of Pauli matrices: $(\vec{\sigma}\cdot\hat{n})^{2k}=1_{2\times 2}$ and $(\vec{\sigma}\cdot\hat{n})^{2k+1}=\vec{\sigma}\cdot\hat{n}$. Using this identity, we have

$$\ln\mathrm{Tr}\left[\frac{1}{2}\left(1+\vec{\sigma}\cdot\hat{n}_{ef}\right)G_f(e)\right]=\sum_{v\in\partial f}\theta_{eve'}+i\sum_{v\in\partial f}\phi_{eve'} \tag{3.195}$$

$$\ln\mathrm{Tr}\left[\frac{1}{2}\left(1+\vec{\sigma}\cdot\hat{n}_{ef}\right)G^\dagger_f(e)\right]=\sum_{v\in\partial f}\theta_{eve'}-i\sum_{v\in\partial f}\phi_{eve'} \tag{3.196}$$

where we use the fact that $\vec{\sigma}$ are Hermitian matrices. Insert these into the expression of the face action $S_f$

$$\begin{aligned} S_f &= -(i\gamma+1)j_f\ln\mathrm{Tr}\left[\frac{1}{2}\left(1+\vec{\sigma}\cdot\hat{n}_{ef}\right)G_f(e)\right] \\ &\quad -(i\gamma-1)j_f\ln\mathrm{Tr}\left[\frac{1}{2}\left(1+\vec{\sigma}\cdot\hat{n}_{ef}\right)G^\dagger_f(e)\right] \end{aligned} \tag{3.197}$$

We define the following variables by making a parallel transport to a vertex $v$

$$\hat{X}_f(v):=g_{ve}\vec{\sigma}\cdot\hat{n}_{ef}g_{ev},\quad \hat{X}^\dagger_f(v):=g^\dagger_{ev}\vec{\sigma}\cdot\hat{n}_{ef}g^\dagger_{ve} \tag{3.198}$$

$$G_f(v):=g_{ve}G_f(e)g_{ev},\quad G^\dagger_f(v):=g^\dagger_{ev}G_f(e)g^\dagger_{ve} \tag{3.199}$$

where one can see that $\hat{X}_f(v)$ is related to the bivector in Proposition 3.2.1 by $\hat{X}_f(v)=X_f(v)/\gamma j_f$. In terms of these new variables at the vertex $v$, the face action is written as

$$\begin{aligned} S_f &= -(i\gamma+1)j_f\ln\mathrm{Tr}\left[\frac{1}{2}\left(1+\hat{X}_f(v)\right)G_f(e)\right] \\ &\quad -(i\gamma-1)j_f\ln\mathrm{Tr}\left[\frac{1}{2}\left(1+\hat{X}^\dagger_f(v)\right)G^\dagger_f(e)\right] \end{aligned} \tag{3.200}$$

According to Theorem 3.4.2, at the critical point, the bivector $\hat{X}_f(v)$ is written as

$$\hat{X}_f(v)=2\varepsilon\frac{\star E_{\ell_1}(v)\wedge E_{\ell_2}(v)}{|\star E_{\ell_1}(v)\wedge E_{\ell_2}(v)|} \tag{3.201}$$

and the spin foam edge holonomy $g_{vv'}$ equals to the spin connection $\Omega_{vv'}$ up to a sign $\mu_e = e^{i\pi n_e}$, i.e.

$$g_{vv'} = e^{i\pi n_e}\Omega_{vv'}. \tag{3.202}$$

The spin foam loop holonomy (in its Spin-1 representation) at the critical point satisfies

$$G_f(v)E_\ell(v) = e^{i\pi\sum_{e\subset f} n_e}E_\ell(v) = \cos\Big(\pi\sum_{e\subset f} n_e\Big)E_\ell(v) \tag{3.203}$$

We pick out a $E_\ell(v)$ as one of the edge of the triangle dual to $f$ and construct $E_{\ell'}(v)$ as a linear combination of the edge vectors $E_{\ell_1}(v), E_{\ell_2}(v)$ and orthogonal to $E_\ell(v)$. We normalize $E_\ell(v), E_{\ell'}(v)$ and represented them by

$$\hat{E}_\ell(v) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{E}_{\ell'}(v) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.204}$$

We have shown that the loop holonomy $G_f(v)$ can be written as an exponential form, i.e. $G_f(v) = e^{Y_f(v)}$. If we represent $Y_f(v)$ by a $4\times 4$ matrix, from Eq.(3.203), $Y_f(v)$ must be given by

$$Y_f(v) = \begin{pmatrix} D_{11} & D_{12} & 0 & 0 \\ D_{21} & D_{22} & 0 & 0 \\ 0 & 0 & 0 & -\pi\sum_e n_e \\ 0 & 0 & \pi\sum_e n_e & 0 \end{pmatrix} \tag{3.205}$$

where $D_{ij}$ is a pure boost leaving the 2-plane spaned by $E_\ell(v), E_{\ell'}(v)$ invariant. Then the spin-1 representation of the loop holonomy $G_f(v)$ can be expressed as

$$G_f(v) = e^{\varepsilon\frac{1}{2}\vartheta_f\hat{X}_f(v)+\frac{1}{2}\pi\sum_{e\subset f} n_e\star\hat{X}_f(v)} \tag{3.206}$$

where $\vartheta_f$ is an arbitrary number. Since the duality map $\star = i$ in the spin-$\frac{1}{2}$ representation, thus

$$G_f(v) = e^{\varepsilon\frac{1}{2}\vartheta_f\hat{X}_f(v)+i\frac{1}{2}\pi\sum_{e\subset f} n_e\hat{X}_f(v)} \tag{3.207}$$

in the spin-$\frac{1}{2}$ representation, where $G_f(v)\in \mathrm{SL}(2,\mathbb{C})$.

We now determine the physical meaning of the parameter $\vartheta_f$. $\mathrm{sgn}(V_4(v))$ is a constant on the triangulation for the oriented 4-volumes of the 4-simplices. By the relation between spin foam variable $g_{vv'}$ and the spin connection: $g_{vv'} = \mu_e \Omega_{vv'}$, we have for the spin connection

$$\begin{aligned}\Omega_f(v) &= e^{i\pi \sum_e n_e} G_f(v) \\ &= e^{i\pi \sum_e n_e} e^{\frac{\star E_{\ell_1}(v)\wedge E_{\ell_2}(v)}{|\star E_{\ell_1}(v)\wedge E_{\ell_2}(v)|}\vartheta_f + \frac{E_{\ell_1}(v)\wedge E_{\ell_2}(v)}{|E_{\ell_1}(v)\wedge E_{\ell_2}(v)|}\pi \sum_e n_e} \in \mathrm{SO}(1,3)\end{aligned} \tag{3.208}$$

We consider a discretization of classical Einstein-Hilbert action $\int R\sqrt{-g}\mathrm{d}^4x$: For each dual face $f$

$$\begin{aligned}\mathrm{Tr}\left[\int_{\Delta_f} \mathrm{sgn}\det(e^I_\mu) \star [e\wedge e] \int_f R\right] &\simeq \mathrm{sgn}(V_4)\frac{1}{2}\mathrm{Tr}\left[\star\Big(E_{\ell_1}(v)\wedge E_{\ell_2}(v)\Big)\ln \Omega_f^{\mathrm{boost}}(v)\right] \\ &= \mathrm{sgn}(V_4) A_f \vartheta_f\end{aligned} \tag{3.209}$$

This formula should be understood by ignoring the higher order correction in the continuum limit. Here we use $\Delta_f$ to denote the triangle dual to $f$. $e^I_\mu$ is a co-tetrad in the continuum. $R$ is the local curvature from the $\mathfrak{sl}_2\mathbb{C}$-valued local spin connection compatible with $e^I_\mu$. Only the pure boost part $\Omega_f^{\mathrm{boost}}(v) = e^{\frac{\star E_{\ell_1}(v)\wedge E_{\ell_2}(v)}{|\star E_{\ell_1}(v)\wedge E_{\ell_2}(v)|}\vartheta_f}$ of the spin connection $\Omega_f(v)$ contributes the curvature $R$ in the discrete context. When $e^{i\pi\sum_e n_e} = -1$, the factor $e^{i\pi \sum_e n_e} e^{\frac{E_{\ell_1}(v)\wedge E_{\ell_2}(v)}{|E_{\ell_1}(v)\wedge E_{\ell_2}(v)|}\pi \sum_e n_e}$ flips the overall sign of the reference frame at $v$ and rotates $\pi$ on the 2-plane spanned by $E_{\ell_1}(v), E_{\ell_2}(v)$. It serves for the case that the time-orientation of the reference frame is flipped by $\Omega_f$, while the triangle spanned by $E_{\ell_1}(v), E_{\ell_2}(v)$ is kept unchange. Such an operation doesn't change the quantity[8]

$$\mathrm{Tr}\left[\int_{\Delta_f} \mathrm{sgn}\det(e^I_\mu) \star [e\wedge e] \int_f R\right] \tag{3.210}$$

$A_f = \frac{1}{2}|\star E_{\ell_1}(v)\wedge E_{\ell_2}(v)|$ is the area of the triangle dual to $f$. Compare Eq.(3.209) with the Regge action of discrete GR, we identify that $\mathrm{sgn}(V_4)\vartheta_f$ is the deficit angle $\Theta_f$ of $f$ responsible to the curvature $R$ from the spin connection.

$$\Theta_f = \mathrm{sgn}(V_4)\vartheta_f \tag{3.211}$$

where we keep in mind that $\mathrm{sgn}(V_4)$ is a constant sign on the (sub-)triangulation.

[8] $\Omega_f(v) \in \mathrm{SO}^-(1,3)$ comes from an oriented but time-unoriented orthonormal frame boundle, where the co-tetrad $e^I_\mu$ can flip sign. However, the local spin connection $\Gamma^{IJ}_\alpha = e^I_\mu \nabla_\alpha e^{\mu J}$ doesn't change as $e^I_\mu \mapsto -e^I_\mu$ and coincides with the spin connection on the oriented and time-oriented orthonormal frame bundle. The same holds also for the curvature $R$ from the spin connection.

Insert the expression of $G_f(v)$ into Eq.(3.200), we obtain for a internal face $f$

$$S_f = -i\ \varepsilon\ \mathrm{sgn}(V_4)\ \gamma j_f \Theta_f - i\pi j_f \sum_{e\subset f} n_e \tag{3.212}$$

where we have used again the relations of Pauli matrices $(\vec{\sigma}\cdot\hat{n})^{2k} = 1_{2\times 2}$ and $(\vec{\sigma}\cdot\hat{n})^{2k+1} = \vec{\sigma}\cdot\hat{n}$, as well as the following relation

$$\mathrm{Tr}\left(\hat{X}_f(v)\cdots\hat{X}_f(v)\right) = \mathrm{Tr}\Big(g_{ve}\vec{\sigma}\cdot\hat{n}_{ef}g_{ev}\cdots g_{ve}\vec{\sigma}\cdot\hat{n}_{ef}g_{ev}\Big) = \mathrm{Tr}\Big(\vec{\sigma}\cdot\hat{n}_{ef}\cdots\vec{\sigma}\cdot\hat{n}_{ef}\Big). \tag{3.213}$$

Finally we sum over all the internal faces and construct the total internal action $S_{\text{int}} = \sum_{f\ \text{internal}} S_f$

$$S_{\text{internal}} = -i\ \varepsilon\ \mathrm{sgn}(V_4) \sum_{f\ \text{internal}} \gamma j_f \Theta_f - i\pi \sum_{f\ \text{internal}} j_f \sum_{e\in\partial f} n_e. \tag{3.214}$$

where $\gamma j_f$ is understood as the area of the triangle $f$, and $\sum_f \gamma j_f \Theta_f$ is the Regge action for discrete GR.

### 3.5.2 Boundary faces

Let's consider a face $f$ dual to a boundary triangle (see FIG.3.1). The corresponding face action $S_f$ reads

$$S_f = 2i\gamma j_f \sum_v \ln\frac{||Z_{ve'f}||}{||Z_{vef}||} - 2ij_f\sum_v \phi_{eve'} = -2ij_f\left(\gamma\sum_v \theta_{eve'} + \sum_v \phi_{eve'}\right) \tag{3.215}$$

where the sum is over all the internal verices $v$ around the face $f$, and we have also used the notation $||Z_{vef}||/||Z_{ve'f}|| := e^{\theta_{eve'}}$.

On the boundary of the face $f$, there are at least two edges connecting to the nodes on the boundary of the triangulation. We suppose there is an edge $e_0$ of the face $f$ connecting a boundary node, associated with a boundary spinor $\xi_{e_0f}$. Recall Eqs.(3.37) and (3.38), and consider the following successive action on $\xi_{e_0f}$ of $g_{e'v}g_{ve}$ along the boundary of the face $f$, until reaching another edge $e_1$ connecting to another boundary node. We denote by $p_{e_1e_0}$ the path from $e_0$ to $e_1$

$$g_{e_1v'}g_{v'e'}\cdots g_{ev}g_{ve_0}J\xi_{e_0f} = J\xi_{e_1f}\exp\left[-\sum_{v\in p_{e_1e_0}}\theta_{eve'} - i\sum_{v\in p_{e_1e_0}}\phi_{eve'}\right]$$

$$g_{e_1v'}g_{v'e'}\cdots g_{ev}g_{ve_0}\xi_{e_0f} = \xi_{e_1f}\exp\left[\sum_{v\in p_{e_1e_0}}\theta_{eve'} + i\sum_{v\in p_{e_1e_0}}\phi_{eve'}\right] \tag{3.216}$$

We denote the holonomy along the path $p_{e_1e_0}$ by

$$G_f(e_1,e_0) := g_{e_1v'}g_{v'e'}\cdots g_{ev}g_{ve_0} \tag{3.217}$$

and construct a SU(2) matrix from the normalized spinor $\xi$ by

$$g(\xi) = (\xi, J\xi) \in \mathrm{SU}(2) \tag{3.218}$$

If we denote by

$$\alpha = \sum_{v\in p_{e_1e_0}}\theta_{eve'} + i\sum_{v\in p_{e_1e_0}}\phi_{eve'} \tag{3.219}$$

Eq.(3.216) can be expressed as a matrix equation

$$G_f(e_1,e_0)\ g(\xi_{e_0f}) = g(\xi_{e_1f})\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha}\end{pmatrix} \tag{3.220}$$

Therefore $G_f(e_1,e_0)$ can be solved immediately

$$G_f(e_1,e_0) = g(\xi_{e_1f})\ e^{\sum_v(\theta_{eve'}+i\phi_{eve'})\vec{\sigma}\cdot\hat{z}}\ g(\xi_{e_0f})^{-1} \tag{3.221}$$

We again employ the identity Eq.(3.194) to obtain

$$\ln\mathrm{Tr}\left[\frac{1}{2}\left(1+\vec{\sigma}\cdot\hat{z}\right)g(\xi_{e_1f})^{-1}G_f(e_1,e_0)g(\xi_{e_0f})\right] = \sum_v\left(\theta_{eve'}+i\phi_{eve'}\right)$$

$$\ln\mathrm{Tr}\left[\frac{1}{2}\left(1+\vec{\sigma}\cdot\hat{z}\right)g(\xi_{e_0f})^{-1}G^\dagger_f(e_1,e_0)g(\xi_{e_1f})\right] = \sum_v\left(\theta_{eve'}-i\phi_{eve'}\right) \tag{3.222}$$

Insert these relations into the face action $S_f$

$$\begin{aligned} S_f &= -(i\gamma+1)j_f\ln\mathrm{Tr}\left[\frac{1}{2}\left(1+\vec{\sigma}\cdot\hat{z}\right)g(\xi_{e_1f})^{-1}G_f(e_1,e_0)g(\xi_{e_0f})\right] \\ &\quad -(i\gamma-1)j_f\ln\mathrm{Tr}\left[\frac{1}{2}\left(1+\vec{\sigma}\cdot\hat{z}\right)g(\xi_{e_0f})^{-1}G^\dagger_f(e_1,e_0)g(\xi_{e_1f})\right] \end{aligned} \tag{3.223}$$

Recall that at the critical configuration $G_f(e_1,e_0)$ coincides with the spin connection $\Omega_f(e_1,e_0)$ up to a sign. Given the co-frame vectors $E_\ell(e_0)$ and $E_\ell(e_1)$ with $\ell$ the edges of the triangle $f$.

$$(\prod_e \mu_e)E_\ell(e_1) = G_f(e_1,e_0)E_\ell(e_0) \quad \forall\ \ell\subset f \tag{3.224}$$

$$G_f(e_1,e_0) = (\prod_e \mu_e)\Omega_f(e_1,e_0) \tag{3.225}$$

where the product $\prod_e$ is over all the edges along the path $p_{e_1e_0}$.

Here we are going to give an explicit expression for $G_f(e_1,e_0)$ from Eq.(3.225). We first define three new vectors $\tilde{E}_\ell(e_i)$ for the three $\ell$'s of the triangle $f$

$$\tilde{E}_\ell(e_i) = \hat{g}(\xi_{e_if})^{-1}E_\ell(e_i) \quad i=0,1 \tag{3.226}$$

where $\hat{g}(\xi_{e_if})$ is the spin-1 representation of $g(\xi_{e_if})\in \mathrm{SU}(2)$. Thus

$$\hat{g}(\xi_{e_1f})^{-1}G_f(e_1,e_0)\hat{g}(\xi_{e_0f})\tilde{E}_\ell(e_0) = (\prod_e \mu_e)\tilde{E}_\ell(e_1) \tag{3.227}$$

The co-frame vectors $E_\ell(e)$ of a triangle $f$ is orthogonal to $\hat{n}_{ef}$, which is given by $\hat{n}_{ef}=\hat{g}(\xi_{ef})\hat{z}$. Thus the triangles formed by $\tilde{E}_\ell(e_i)$ $(i=0,1)$ are both on the 2-plane (the $xy$-plane) orthogonal to $u=(1,0,0,0)$ and $\hat{z}=(0,0,0,1)$, then they are related by a rotation $e^{\zeta_f J_3}$ on the $xy$-plane

$$\tilde{E}_\ell(e_1) = e^{\zeta_f J_3}\tilde{E}_\ell(e_0) \quad \forall\ \ell\subset f. \tag{3.228}$$

Therefore $\hat{g}(\xi_{e_1f})^{-1}G_f(e_1,e_0)\hat{g}(\xi_{e_0f})$ is the above rotation plus a pure boost along the $z$-direction and a rotation taking care the sign factor $\prod_e \mu_e$, both of which leaves the vector on $xy$-plane invariant. Hence

$$G_f(e_1,e_0) = \hat{g}(\xi_{e_1f})e^{\vartheta_f^B K_3}e^{\pi\sum_e n_e J_3}e^{\zeta_f J_3}\hat{g}(\xi_{e_0f})^{-1} \tag{3.229}$$

where $\vartheta_f^B$ is an arbitrary number. The rotation $e^{\zeta_f J_3}$ corresponds to a gauge transformation in the context of twisted geometry [12, 13]. Here we can always absorb $e^{\zeta_f J_3}$ into one of $\hat{g}(\xi_{e_if})$, which leads to a redefinition of the boundary data $\xi_{e_if}$. Such a redefinition doesn't change the triangle normal $\hat{n}_{ef}$ thus doesn't change the bivector $X_{ef}$. Then all the above analysis about constructing discrete geometry is unaffected. The boundary data after this redefinition is the Regge boundary data employed in

[66, 74]. With this setting, we obtain

$$G_f(e_1,e_0) = \hat{g}(\xi_{e_1 f}) e^{\vartheta_f^B K_3} e^{\pi \sum_e n_e J_3} \hat{g}(\xi_{e_0 f})^{-1}. \tag{3.230}$$

for an explicit expression of $G_f(e_1,e_0)$, and

$$\tilde{E}_\ell(e_0) = \tilde{E}_\ell(e_1) = \tilde{E}_\ell \tag{3.231}$$

for the edges of triangle $\ell$. The three vectors $\tilde{E}_\ell$ determines the triangle geometry of $f$ in the frame at $f$. From Eq.(3.225), we obtain the spin connection compatible with the co-frame

$$\Omega_f(e_1,e_0) = e^{i\pi \sum_e n_e} \hat{g}(\xi_{e_1 f}) e^{\vartheta_f^B K_3} e^{\pi \sum_e n_e J_3} \hat{g}(\xi_{e_0 f})^{-1}. \tag{3.232}$$

When $e^{i\pi\sum_e n_e} = 1$, the spin connection $\Omega_f(e_1,e_0) \in \mathrm{SO}^+(1,3)$, and when $e^{i\pi\sum_e n_e} = -1$, $\Omega_f(e_1,e_0) \in \mathrm{SO}^-(1,3)$.

We now determine the physical meaning of the parameter $\vartheta_f^B$ in the expression of $G_f(e_1,e_0)$. It is related to the dihedral angle $\Theta_f^B$ of the two boundary tetrahedra $t_{e_0}, t_{e_1}$ at the triangle $f$ sheared by them. The two tetrahedra $t_{e_0}, t_{e_1}$ belongs to different 4-simplicies $\sigma_{v_0}, \sigma_{v_1}$, while the curvature from spin connection between $\sigma_{v_0}, \sigma_{v_1}$ are given by the pure boost part of $\Omega_f(v_1,v_0)$ along the internal edges of the face $f$. This curvature is responsible to the dihedral angle between $t_{e_0}, t_{e_1}$. The dihedral boost between the normals of $t_{e_0}, t_{e_1}$ at the triangle $f$ is given by the pure boost part of

$$\hat{g}(\xi_{e_1 f})^{-1} \Omega_f(e_1,e_0) \hat{g}(\xi_{e_0 f}) = e^{i\pi\sum_e n_e} e^{\vartheta_f^B K_3} e^{\pi\sum_e n_e J_3} \tag{3.233}$$

The above transformation leaves the triangle geometry $\tilde{E}_\ell$ invariant in both case of $e^{i\pi\sum_e n_e} = \pm 1$. We consider the unit normal of the tetrahedron $t_{e_0}$ (viewed in its own frame) $u^I = (1,0,0,0)^t$, parallel transported by $G_f(e_1,e_0)$ (from the frame of $t_{e_0}$ to the frame of $t_{e_1}$)

$$G_f(e_1,e_0)^I{}_J u^J = e^{\vartheta_f^B K_3} u = (\cosh\vartheta_f^B, 0, 0, \sinh\vartheta_f^B)^t \tag{3.234}$$

Contract this equation with the unit normal $u^I = (1,0,0,0)^t$ viewed in the frame of $t_{e_1}$, we obtain that for the dihedral angle $\Theta_f^B$

$$\cosh\Theta_f^B = -u_I G_f(e_1,e_0)^I{}_J u^J = \cosh\vartheta_f^B \tag{3.235}$$

which implies that $\Theta_f^B = \pm\vartheta_f^B$. By a generalization of the analysis in [66, 74], we can conclude that

**Lemma 3.5.1.** *The dihedral angle $\Theta_f^B$ at the triangle $f$ relates to the parameter $\vartheta_f^B$ by*

$$\Theta_f^B = \varepsilon \ \mathrm{sgn}(V_4)\vartheta_f^B \tag{3.236}$$

**Proof:** In the tetrahedra $t_{e_0}$ and $t_{e_1}$, both pairs of the vectors $E_{\ell_1}(e_0), E_{\ell_2}(e_0)$ and $E_{\ell_1}(e_1), E_{\ell_2}(e_1)$ are orthogonal to $u = (1,0,0,0)^t$. Thus at the vertex $v$, both $E_{\ell_1}(v)$ and $E_{\ell_2}(v)$ are orthogonal to

$$F_{e_0}(v) = G_f(v,e_0) \rhd u \qquad F_{e_1}(v) = G_f(v,e_1) \rhd u \tag{3.237}$$

Thus both $F_{e_0}(v)$ and $F_{e_1}(v)$ are future-pointing since $G_f(v,e) \in \mathrm{SL}(2,\mathbb{C})$. Eq.(3.235) implies that

$$\left|\eta_{IJ}F^I_{e_0}(v)F^J_{e_1}(v)\right| = \cosh\Theta_f^B. \tag{3.238}$$

We define a dihedral boost from the dihedral angle $\Theta_f^B$ by

$$\begin{aligned} D(e_1,e_0) &= \exp\left[|\Theta_f^B|\frac{F_{e_0}(v)\wedge F_{e_1}(v)}{|F_{e_0}(v)\wedge F_{e_1}(v)|}\right] \\ &= \exp\left[\Theta_f^B\frac{U_e(v)\wedge U_{e'}(v)}{|U_e(v)\wedge U_{e'}(v)|}\right] \end{aligned} \tag{3.239}$$

where we have chosen the sign of the dihedral angle such that

$$\begin{aligned} &\text{If} \quad \frac{F_{e_1}(v)\wedge F_{e_0}(v)}{|F_{e_1}(v)\wedge F_{e_0}(v)|} = \frac{U_e(v)\wedge U_{e'}(v)}{|U_e(v)\wedge U_{e'}(v)|} : \quad |\Theta_f^B| = -\Theta_f^B \\ &\text{If} \quad \frac{F_{e_1}(v)\wedge F_{e_0}(v)}{|F_{e_1}(v)\wedge F_{e_0}(v)|} = -\frac{U_e(v)\wedge U_{e'}(v)}{|U_e(v)\wedge U_{e'}(v)|} : \quad |\Theta_f^B| = \Theta_f^B \end{aligned} \tag{3.240}$$

with $V_4(v)U_e(v)\wedge U_{e'}(v) = G_f(v,e_0)\rhd \star E_{\ell_1}(e_0)\wedge E_{\ell_2}(e_0)$.

On the other hand, the boost generator $K_3$ can be related to the bivector $X_{ef}^{IJ} = 2\gamma j_f(\hat{n}_{ef}\wedge u)^{IJ}$

$$K_3 = -\hat{z}\wedge u = -g(\xi_{ef})^{-1}\otimes g(\xi_{ef})^{-1}(\hat{n}_{ef}\wedge u) = -g(\xi_{ef})^{-1}\otimes g(\xi_{ef})^{-1}\,\frac{1}{2\gamma j}X_{ef} \tag{3.241}$$

At the critical configuration the bivector $X_{ef}$ is given by Eq.(3.175), which results in that

$$K_3 = -\varepsilon g(\xi_{ef})^{-1} \otimes g(\xi_{ef})^{-1} \frac{\star E_{\ell_1}(e) \wedge E_{\ell_2}(e)}{|E_{\ell_1}(e) \wedge E_{\ell_2}(e)|} = -\varepsilon \frac{\star \tilde{E}_{\ell_1} \wedge \tilde{E}_{\ell_2}}{|\tilde{E}_{\ell_1} \wedge \tilde{E}_{\ell_2}|} \tag{3.242}$$

where $\frac{\star \tilde{E}_{\ell_1} \wedge \tilde{E}_{\ell_2}}{|\tilde{E}_{\ell_1} \wedge \tilde{E}_{\ell_2}|}$ is the (unit) bivector corresponding to the triangule $f$. Therefore for the bivector at the vertex $v$

$$\begin{aligned} \frac{U_e(v) \wedge U_{e'}(v)}{|U_e(v) \wedge U_{e'}(v)|} &= \operatorname{sgn}(V_4) G_f(v, e_0) \rhd \frac{\star E_{\ell_1}(e_0) \wedge E_{\ell_2}(e_0)}{|E_{\ell_1}(e_0) \wedge E_{\ell_2}(e_0)|} \\ &= -\operatorname{sgn}(V_4)\ \varepsilon\ G_f(v, e_0) g(\xi_{e_0 f}) K_3 g(\xi_{e_0 f})^{-1} G_f(v, e_0)^{-1} \end{aligned} \tag{3.243}$$

Then we obtain the following expression of $D(e_1, e_0)$:

$$D(e_1, e_0) = G_f(v, e_0) g(\xi_{e_0 f}) e^{-\varepsilon\ \operatorname{sgn}(V_4) \Theta_f^B K_3} g(\xi_{e_0 f})^{-1} G_f(v, e_0)^{-1}. \tag{3.244}$$

One can check that $D(e_1, e_0)$ gives a dihedral boost from $F_{e_0}(v)$ to $F_{e_1}(v)$, i.e.

$$D(e_1, e_0) F_{e_0}(v) = F_{e_1}(v) \tag{3.245}$$

If we represent the vector $F_e(v)$ by the $2 \times 2$ matrix $F_e = F_e^I \sigma_I$, we have $F_e(v) = G_f(v, e) G_f(v, e)^\dagger$, Eq.(3.245) can be expressed as

$$D(e_1, e_0) G_f(v, e_0) G_f(v, e_0)^\dagger D(e_1, e_0)^\dagger = G_f(v, e_1) G_f(v, e_1)^\dagger \tag{3.246}$$

By using Eq.(3.244), we obtain that ($J_3^\dagger = -J_3$)

$$G_f(v, e_0) g(\xi_{e_0 f}) e^{-2\varepsilon\ \operatorname{sgn}(V_4) \Theta_f^B K_3} g(\xi_{e_0 f})^\dagger G_f(v, e_0)^\dagger = G_f(v, e_1) G_f(v, e_1)^\dagger \tag{3.247}$$

From the expression Eq.(3.230) of $G_f(e_1, e_0) = G_f(v, e_1)^{-1} G_f(v, e_0)$ in terms of $\vartheta_f^B$, we obtain

$$G_f(e_0, e_1) G_f(e_0, e_1)^\dagger = \hat{g}(\xi_{e_0 f}) e^{-2\vartheta_f^B K_3} \hat{g}(\xi_{e_0 f})^{-1} \tag{3.248}$$

Combining Eqs.(3.247) and (3.248), we obtain

$$e^{-2\varepsilon\ \operatorname{sgn}(V_4) \Theta_f^B K_3} = e^{-2\vartheta_f^B K_3} \tag{3.249}$$

which results in

$$\vartheta_f^B = \varepsilon\ \operatorname{sgn}(V_4) \Theta_f^B. \tag{3.250}$$

□

The Eq.(3.230) is now related to the dihedral angle $\Theta_f^B$

$$\hat{g}(\xi_{e_1 f})^{-1} G_f(e_1, e_0) \hat{g}(\xi_{e_0 f}) = e^{\varepsilon\ \mathrm{sgn}(V_4)\Theta_f^B K_3} e^{\pi \sum_e n_e J_3}. \tag{3.251}$$

Recall that in Spin-$\frac{1}{2}$ representation $\vec{J} = \frac{i}{2}\vec{\sigma}$ and $\vec{K} = \frac{1}{2}\vec{\sigma}$, thus in Spin-$\frac{1}{2}$ representation:

$$g(\xi_{e_1 f})^{-1} G_f(e_1, e_0) g(\xi_{e_0 f}) = e^{\frac{1}{2}\varepsilon\ \mathrm{sgn}(V_4)\Theta_f^B \sigma_3} e^{\frac{i}{2}\pi \sum_e n_e \sigma_3} \tag{3.252}$$

Insert this relation back into Eq.(3.223),

$$S_f = -i\varepsilon\ \mathrm{sgn}(V_4)\gamma j_f \Theta_f^B - i j_f \pi \sum_{e \subset p_{e_1 e_0}} n_e \tag{3.253}$$

Then the total boundary action $S_{\text{boundary}} = \sum_{\text{boundary } f} S_f$:

$$S_{\text{boundary}} = -i\ \varepsilon\ \mathrm{sgn}(V_4) \sum_{\text{boundary } f} \gamma j_f \Theta_f^B - i\pi \sum_{\text{boundary } f} j_f \sum_{e \subset p_{e_1 e_0}} n_e. \tag{3.254}$$

### 3.5.3 Spin foam amplitude at nondegenerate critical configuration

In this subsection we summarize our result and give spin foam amplitude at a general nondegenerate critical configuration. First of all, we say a spin configuration $j_f$ is Regge-like, if with $j_f$ on each face the critical point equations Eqs.(3.37), (3.38), and (3.39) have nondegenerate solution $(j_f, g_{ve}, \xi_{ef}, z_{vf})$. For a non-Regge-like spin configuration $j_f$, the critical point equations have no nondegenerate solutions.

Given a Regge-like spin configuration $j_f$ and find a solution $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ of the critical point equations, we construct the following variables as in Section 3.4:

- A co-frame $E_\ell(v), E_\ell(e)$ of the triangulation (bulk and boundary) can be constructed from the solution $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, unique up to a simultaneously sign flipping $E_\ell \to -E_\ell$ within a 4-simplex, such that the Regge-like spin configuration $j_f$ satisfies

$$2\gamma j_f = \left|E_{\ell_1}(v) \wedge E_{\ell_2}(v)\right|. \tag{3.255}$$

From the co-frame we can construct a unique discrete metric on the whole triangulation (bulk and boundary)

$$g_{\ell_1\ell_2}(v) = \eta_{IJ} E^I_{\ell_1}(v) E^J_{\ell_2}(v) \qquad g_{\ell_1\ell_2}(e) = \eta_{IJ} E^I_{\ell_1}(e) E^J_{\ell_2}(e). \tag{3.256}$$

So $\gamma j_f$ is the triangle area from the discrete metric $g_{\ell_1\ell_2}$

- For each dual edge $e$ we specify a sign factor $\mu_e = e^{i\pi n_e}$ that equals 1 or $-1$ with $n_e$ equals 0 or 1, such that the spin foam group element $g_{vv'}$ (in the Spin-1 representation) is related to an SO(1,3) matrix $\Omega_{vv'}$ by this sign factor, i.e.

$$g_{vv'} = e^{i\pi n_e} \Omega_{vv'} \tag{3.257}$$

where $\Omega_{vv'}$ is compatible with the co-frame $E_\ell(v)$, i.e.

$$(\Omega_{vv'})^I{}_J E^J_\ell(v') = E^I_\ell(v) \tag{3.258}$$

If $\mathrm{sgn}(V_4(v)) = \mathrm{sgn}(V_4(v'))$, $\Omega_{vv'}$ is the unique discrete spin connection compatible with the co-frame. In addition, we note that each $\mu_e$ is not invariant under the sign flipping $E_\ell \to -E_\ell$, but the product $\prod_{e\subset\partial f} \mu_e$ is invariant for any (internal or boundary) face $f$ (see Lemma.3.4.1).

- There is a global sign factor $\varepsilon$ that equals 1 or $-1$, to relate the bivectors $X_f(v)$ in the bulk and $X_{ef}$ on the boundary to the co-frame:

$$X^{IJ}_f(v) = \varepsilon \ \star \Big[ E_{\ell_1}(v) \wedge E_{\ell_2}(v) \Big]^{IJ}, \qquad X^{IJ}_{ef} = \varepsilon \ \star \Big[ E_{\ell_1}(e) \wedge E_{\ell_2}(e) \Big]^{IJ}. \tag{3.259}$$

If the triangulation $\mathcal{K}$ has boundary, the global sign factor $\varepsilon = \pm 1$ is specified by the orientation of the boundary triangulation, i.e. $\varepsilon = \mathrm{sgn}(V_3)$ for the boundary tetrahedra.

Therefore a nondegenerate solution $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ of the spin foam critical point equations specifies uniquely a set of variables $(g_{\ell_1\ell_2}, n_e, \varepsilon)$, which include a discrete metric and two types of sign factors.

The previous analysis shows that, given a general critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, we can divide the triangulation $\mathcal{K}$ into sub-triangulations $\mathcal{K}_1, \cdots, \mathcal{K}_n$, where each of the sub-triangulations is a triangulation with boundary, with a constant $\mathrm{sgn}(V_4(v))$. On each of the sub-triangulation $\mathcal{K}_i$, the spin foam action $S$ evaluated at $(j_f, g_{ve}, \xi_{ef}, z_{vf})_{\mathcal{K}_i}$

is a function of the variables $(g_{\ell_1\ell_2}, n_e, \varepsilon)$ and behaves mainly as a Regge action:

$$\begin{aligned}
S(g_{\ell_1\ell_2}, n_e, \varepsilon)\Big|_{\mathcal{K}_i} &= S_{\text{internal}}(g_{\ell_1\ell_2}, n_e, \varepsilon) + S_{\text{boundary}}(g_{\ell_1\ell_2}, n_e, \varepsilon) \\
&= -i\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{internal } f} \gamma j_f \Theta_f - i\pi \sum_{\text{internal } f} j_f \sum_{e\subset\partial f} n_e \\
&\quad -i\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{boundary } f} \gamma j_f \Theta^B_f - i\pi \sum_{\text{boundary } f} j_f \sum_{e\subset\partial f} n_e \\
&= -i\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{internal } f} \gamma j_f \Theta_f - i\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{boundary } f} \gamma j_f \Theta^B_f \\
&\quad -i\pi \sum_e n_e \sum_{f\subset t_e} j_f \qquad (3.260)
\end{aligned}$$

where we note that the areas $\gamma j_f$, deficit angles $\Theta_f$, and dihedral angles $\Theta^B_f$ are uniquely determined by the discrete metric $g_{\ell_1\ell_2}$. Moreover for each tetrahedron $t$, the sum of face spins $\sum_{f\subset t} j_f$ is an integer. If the spins $j_f$ are integers, $\sum_{f\subset t} j_f$ then is an even integer, so $e^{-i\pi\sum_e n_e \sum_{f\subset t_e} j_f} = 1$ so the second term in the above formula doesn't contribute the exponential $e^{\lambda S_{\text{int}}}$. For half-integer spins, $e^{-i\pi\sum_e n_e \sum_{f\subset t_e} j_f} = \pm 1$ gives an overall sign factor. Therefore in general at a nondegenerate spin foam configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ that solves the critical point equations,

$$e^{\lambda S}\Big|_{\mathcal{K}_i} = \pm \exp\lambda \left[-i\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{internal } f} \gamma j_f \Theta_f - i\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{boundary } f} \gamma j_f \Theta^B_f\right]. \qquad (3.261)$$

There exists two ways to make the overall sign factor disappear: (1) only consider integer spins $j_f$, or (2) modify the embedding from SU(2) unitary irreducible representations to $\text{SL}(2,\mathbb{C})$ unitary irreducible representations by $j_f \mapsto (p_f, k_f) := (2\gamma j_f, 2j_f)$, then the spin foam action $S$ is replaced by $2S$. In these two cases the exponential $e^{\lambda S}$ at the critical configuration is independent of the variable $n_e$.

On the triangulation $\mathcal{K} = \cup_{i=1}^n \mathcal{K}_i$, $e^{\lambda S}$ is given by a product over all the sub-triangulations:

$$\begin{aligned}
e^{\lambda S} &= \prod_{i=1}^n e^{\lambda S}\Big|_{\mathcal{K}_i} \\
&= \prod_{i=1}^n \exp\lambda\Big[-i\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{internal } f} \gamma j_f \Theta_f - i\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{boundary } f} \gamma j_f \Theta^B_f \\
&\qquad -i\pi \sum_e n_e \sum_{f\subset t_e} j_f\Big]_{\mathcal{K}_i} \qquad (3.262)
\end{aligned}$$

Suppose the oriented 4-volumes are different between two sub-triangulation $\mathcal{K}_i$ and $\mathcal{K}_j$

sharing a boundary, the spin foam amplitude at this critical configuration exhibits a transition between two different spacetime regions with different spacetime orientation. The spacetime orientation is not continuous on the boundary between $\mathcal{K}_i$ and $\mathcal{K}_j$.

We recall the difference between Einstein-Hilbert action and Palatini action

$$\mathcal{L}_{EH} = R\ \underline{\varepsilon} = \operatorname{sgn}\det(e^I_\mu) \star [e \wedge e]_{IJ} \wedge R^{IJ} = \operatorname{sgn}\det(e^I_\mu)\mathcal{L}_{Pl} \tag{3.263}$$

where $\mathcal{L}_{EH}$ and $\mathcal{L}_{Pl}$ denote the Lagrangian densities of Einstein-Hilbert action and Palatini action respectively, and $\underline{\varepsilon}$ is a chosen volume form compatible with the metric $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$. Since the Regge action is a discretization of the Einstein-Hilbert action, we may consider the resulting action

$$-i\ \varepsilon \sum_{i=1}^{n} \left[ \operatorname{sgn}(V_4) \sum_{\text{internal } f} \gamma j_f \Theta_f + \operatorname{sgn}(V_4) \sum_{\text{boundary } f} \gamma j_f \Theta^B_f \right]_{\mathcal{K}_i} \tag{3.264}$$

as a discretized Palatini action with on-shell connection, where the on-shell connection means that the discrete connection is the spin connection compatible with the co-frame.

According to the properties of Regge geometry, given a collection of Regge-like areas $\gamma j_f$, the discrete metric $g_{\ell_1\ell_2}(v)$ is uniquely determined at each vertex $v$. Furthermore since the areas $\gamma j_f$ are Regge-like, There exists a discrete metric $g_{\ell_1\ell_2}$ in the entire bulk of the triangulation, such that the neighboring 4-simplicies are consistently glued together, as we constructed previously. This discrete metric $g_{\ell_1\ell_2}$ is obviously unique by the uniqueness of $g_{\ell_1\ell_2}(v)$ at each vertex. Therefore given the partial-amplitude $A_{j_f}(\mathcal{K})$ in Eq.(3.15) with a specified Regge-like $j_f$, all the critical configurations $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ of $A_{j_f}(\mathcal{K})$ corresponds to the same discrete metric $g_{\ell_1\ell_2}$, provided a Regge boundary data. The critical configurations from the same Regge-like $j_f$ is classified in the next section.

As a result, given a Regge-like spin configurations $j_f$ and a Regge boundary data, the partial amplitude $A_{j_f}(\mathcal{K})$ has the following asymptotics

$$\begin{aligned} A_{j_f}(\mathcal{K})\Big|_{\text{Nondeg}} \sim & \sum_{x_c} a(x_c)\left(\frac{2\pi}{\lambda}\right)^{\frac{r(x_c)}{2}-N(v,f)} \frac{e^{i\text{Ind}H'(x_c)}}{\sqrt{|\det_r H'(x_c)|}}\left[1+o\left(\frac{1}{\lambda}\right)\right]\times \\ & \times\ \exp -i\lambda\sum_{i=1}^{n(x_c)}\Big[\varepsilon\ \text{sgn}(V_4)\sum_{\text{internal } f}\gamma j_f\Theta_f + \\ & +\varepsilon\ \text{sgn}(V_4)\sum_{\text{boundary } f}\gamma j_f\Theta_f^B+\pi\sum_e n_e\sum_{f\subset t_e} j_f\Big]_{\mathcal{K}_i(x_c)} \end{aligned} \tag{3.265}$$

where $x_c \equiv (j_f, g_{ve}, \xi_{ef}, z_{vf})$ labels the nondegenerate critical configurations, $r(x_c)$ is the rank of the Hessian matrix at $x_c$, and $N(v,f)$ is the number of the pair $(v,f)$ with $v \in \partial f$ (recall Eq.(3.15), there is a factor of $\dim(j_f)$ for each pair of $(v,f)$). $a(x_c)$ is the evaluation of the integration measures at $x_c$, which doesn't scale with $\lambda$. Here $\Theta_f$ and $\Theta_f^B$ only depend on the metric $g_{\ell_1\ell_2}$, which is uniquely determined by the Regge-like spin configuration $j_f$ and the Regge boundary data. Note that different critical configurations $x_c$ may have different subdivisions of the triangulation into sub-triangulations $\mathcal{K}_1(x_c), \cdots, \mathcal{K}_{n(x_c)}(x_c)$.

## 3.6 Parity inversion

We consider a tetrahedron $t_e$ associated with spins $j_{f_1}, \cdots, j_{f_4}$, we know that the set of four spinors $\xi_{ef_1}, \cdots, \xi_{ef_4}$, modulo diagonal SU(2) gauge transformation, is equivalent to the shape of the tetrahedron, if the closure condition is satisfied [34, 82]. Given a nondegenerate critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, as we discussed previously, the Regge-like spin configuration $j_f$ determines a discrete metric $g_{\ell_1\ell_2}$, which determines the shape of all the tetrahedra in the triangulation. At the critical configuration the closure condition of tetrahedron is always satisfied, so the spinors $\xi_{ef_1}, \cdots, \xi_{ef_4}$ for each tetrahedron are determined by the Regge-like spins $j_f$, up to a diagonal SU(2) action on the spinors $\xi_{ef_1}, \cdots, \xi_{ef_4}$, which is a gauge transformation of the spin foam action[9]. Therefore the gauge equivalence class of the critical configurations $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ with the same Regge-like spins $j_f$ must have the same set of spinors $\xi_{ef}$. Thus with a given Regge-like spin configuration $j_f$, the degrees of freedom of the nondegenerate critical configurations are the variables $g_{ve}$ and $z_{vf}$. The degrees of freedom of $g_{ve}$ and $z_{vf}$ are factorized into the 4-simplices. Given the Regge-like spins $j_f$ and spinors

[9]The SU(2) transformation $\xi_{ef} \mapsto h_e\xi_{ef}$ and $g_{ve} \mapsto g_{ve}h_e^{-1}$ ($h_e \in$ SU(2)) is a gauge transformation of the spin foam action $S$.

$\xi_{ef}$, within each 4-simplex, the solutions of $g_{ve}$ and $z_{vf}$ from critical point equations are completely classified in [66, 74], which are the two solutions related by a *parity* transformation.

Given a nondegenerate critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, it generates many other critical configurations $(j_f, \tilde{g}_{ve}, \xi_{ef}, \tilde{z}_{vf})$ which are the solutions of the critical point equations Eqs.(3.37), (3.38), and (3.39). In at least one simplex or some 4-simplices $\tilde{\sigma}_v$

$$\tilde{g}_{ve} = J g_{ve} J^{-1} = (g_{ve}^{\dagger})^{-1} \quad \text{and} \quad \frac{||\tilde{Z}_{ve'f}||}{||\tilde{Z}_{vef}||} = \frac{||Z_{vef}||}{||Z_{ve'f}||} \tag{3.266}$$

while in the other 4-simplices $\tilde{g}_{ve} = g_{ve}$ and $\tilde{z}_{vf} = z_{vf}$. In [66, 74], such a solution-generating map $g_{ve} \mapsto \tilde{g}_{ve}$ and $z_{vf} \mapsto \tilde{z}_{vf}$ is called a *parity*, because $N_e(v) = g_{ve} \rhd (1,0,0,0)^t$ and $\tilde{N}_e(v) = \tilde{g}_{ve} \rhd (1,0,0,0)^t$ are different by a parity inversion. The parity inversion between $N_e(v)$ and $\tilde{N}_e(v)$ can be shown by using the Hermitian matrix representation of the vectors $V = V^0 \mathbf{1} + V^j \sigma_j$, thus

$$\tilde{N}_e(v) = \tilde{g}_{ve} \tilde{g}_{ve}^{\dagger} = J g_{vf} g_{vf}^{\dagger} J^{-1} = J N_e(v) J^{-1} = N_e^0(v) \mathbf{1} - N_e^j(v) \sigma_j \tag{3.267}$$

since $J \vec{\sigma} J^{-1} = -\vec{\sigma}$. We denote the parity inversion in $(\mathbb{R}^4, \eta_{IJ})$ by $\mathbf{P} = \mathrm{diag}(1, -1, -1, -1)$ then we have $\tilde{N}_e(v) = \mathbf{P} N_e(v)$ in the simplices $\tilde{\sigma}_v$ where $\tilde{g}_{ve} \neq g_{ve}$.

Within a single 4-simplex there are in total 2 parity-related solutions of $(g_{ve}, z_{vf})$ in the nondegenereate case [66, 74]. Therefore in a general simplicial complex with $N$ simplices, given a Regge-like spin configuration $j_f$, there are in total $2^N$ nondegenerate critical configurations $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ that solve the critical point equations. Any two critical configurations are related by the parity transformation in one 4-simplex or many 4-simplices.

We define the bivectors $\tilde{X}_f(v) = \tilde{g}_{ve} \otimes \tilde{g}_{ve} \rhd X_{ef}$ within the 4-simplices $\tilde{\sigma}_v$, where

$$X_{ef}^{IJ} = 2\gamma j_f \left[\hat{n}_{ef} \wedge u\right] \qquad u = (1,0,0,0)^t \tag{3.268}$$

Consider the Hermitian matrix representation of $\hat{n}_{ef}$, the action $\tilde{g}_{ve} \rhd \hat{n}_{ef}$ is given by (note that $J^2 = -1$)

$$\tilde{g}_{ve}(\hat{n}_{ef} \cdot \vec{\sigma}) \tilde{g}_{ve}^{\dagger} = J g_{ve} J^{-1} (\hat{n}_{ef} \cdot \vec{\sigma}) J g_{ve}^{\dagger} J^{-1} = -J g_{ve} (\hat{n}_{ef} \cdot \vec{\sigma}) g_{ve}^{\dagger} J^{-1} = -\mathbf{P} g_{ve} (\hat{n}_{ef} \cdot \vec{\sigma}) g_{ve}^{\dagger} \tag{3.269}$$

while we have shown $\tilde{g}_{ve} \rhd u = \mathbf{P}\left(g_{ve} \rhd u\right)$, thus we obtain that

$$\tilde{X}_f(v) = -(\mathbf{P}\otimes\mathbf{P})X_f(v) \tag{3.270}$$

Recall the construction in Section 3.4 and Eq.(3.104)

$$X_f(v) = \alpha_{ee'}(v)N_e(v)\wedge N_{e'}(v) \tag{3.271}$$

Following the same argument towards Eq.(3.104), we obtain that for the bivectors and normals constructed from $\tilde{g}_{ve}$

$$\tilde{X}_f(v) = \tilde{\alpha}_{ee'}(v)\tilde{N}_e(v)\wedge\tilde{N}_{e'}(v) \quad \Rightarrow \quad -(\mathbf{P}\otimes\mathbf{P})X_f(v) = \tilde{\alpha}_{ee'}(v)\mathbf{P}N_e(v)\wedge\mathbf{P}N_{e'}(v) \tag{3.272}$$

Then we have the relation

$$\tilde{\alpha}_{ee'}(v) = -\alpha_{ee'}(v) \quad \text{ and } \quad \tilde{\beta}_{ee'}(v) = -\beta_{ee'}(v) \tag{3.273}$$

where $\beta_{ee'}(v) = \alpha_{ee'}(v)\varepsilon_{ee'}(v)$. Following the same procedure as in Section 3.4, we denote $\tilde{\beta}_{e_ie_j}$ by $\tilde{\beta}_{ij}$ and construct the closure condition for the 4-simplex $\tilde{\sigma}_v$

$$\sum_{j=1}^{5}\tilde{\beta}_{ij}(v)\tilde{N}_{e_j}(v) = 0 \tag{3.274}$$

by choosing the nonvanishing diagonal elements $\tilde{\beta}_{ii}$. Since we have the closure condition $\sum_{j=1}^{5}\beta_{ij}N_{e_j}(v) = 0$, the parity inversion $\tilde{N}_e(v) = \mathbf{P}N_e(v)$, and $\tilde{\beta}_{ij}(v) = -\beta_{ij}(v)$ for $i \neq j$, we obtain that the diagonal elements $\tilde{\beta}_{ii}(v) = -\beta_{ii}(v)$. Furthermore we can show that $\tilde{\beta}_{ij}$ can be factorized in the same way as in Section 3.4

$$\tilde{\beta}_{ij}(v) = \mathrm{sgn}(\tilde{\beta}_{j_0j_0}(v))\tilde{\beta}_i(v)\tilde{\beta}_j(v) \qquad \tilde{\beta}_j(v) = \tilde{\beta}_{jj_0}(v)\Big/\sqrt{|\tilde{\beta}_{j_0j_0}(v)|} \tag{3.275}$$

which results in that

$$\mathrm{sgn}(\tilde{\beta}_{j_0j_0}(v)) = -\mathrm{sgn}(\beta_{j_0j_0}(v)) \quad \text{ and } \quad \tilde{\beta}_j(v) = -\beta_j(v) \tag{3.276}$$

We construct the 4-volume for $\tilde{\beta}_j(v)\tilde{N}_{e_j}(v)$

$$\tilde{V}_4(v) := \det\left(\tilde{\beta}_2(v)\tilde{N}^{e_2}(v), \tilde{\beta}_3(v)\tilde{N}^{e_3}(v), \tilde{\beta}_4(v)\tilde{N}^{e_4}(v), \tilde{\beta}_5(v)\tilde{N}^{e_5}(v)\right) = -V_4(v) \tag{3.277}$$

by the parity inversion. Since in Section.3.4 we define the sign factor $\varepsilon(v) = \mathrm{sgn}(\beta_{j_0 j_0}(v))\mathrm{sgn}(V_4(v))$, then we have for the parity inversion

$$\tilde{\varepsilon}(v) = \mathrm{sgn}(\tilde{\beta}_{j_0 j_0}(v))\mathrm{sgn}(\tilde{V}_4(v)) = \varepsilon(v) \tag{3.278}$$

Note that one should not confuse the $\tilde{\varepsilon}$ here with the $\tilde{\varepsilon}$ appeared in section 3.4. This result shows that the parity configuration $(j_f, \tilde{g}_{ve}, \xi_{ef}, \tilde{z}_{vf})$ results in an identical global sign factor $\varepsilon$ for the bivector (recall the proof of Theorem 3.4.2).

The fact that the parity flips the sign of the oriented 4-volume, $\tilde{V}_4(v) = -V_4(v)$, has some interesting consequences: First of all, we mentioned that given a set of Regge-like spins, different nondegenerate critical configurations $x_c = (j_f, g_{ve}, \xi_{ef}, z_{vf})$ may lead to different subdivisions of the triangulation $\mathcal{K}$ into sub-triangulation $\mathcal{K}_1(x_c), \cdots, \mathcal{K}_{n(x_c)}(x_c)$, where on each sub-triangulation $\mathrm{sgn}(V_4(v))$ is a constant. Now we understand that the difference of the subdivisions comes from a local parity transformation, which flips the sign of the oriented 4-volume. On the other hand, given a nondegenerate critical configuration $x_c = (j_f, g_{ve}, \xi_{ef}, z_{vf})$, there exists another nondegenerate critical configuration $\tilde{x}_c = (j_f, \tilde{g}_{ve}, \xi_{ef}, \tilde{z}_{vf})$, naturally associated with $x_c$, obtained by a global parity (parity transformation in all simplices) on the triangulation. The global parity flips the sign of the oriented volume $V_4(v)$ everywhere, thus flip the sign of the spin foam action at the nondegenerate critical configuration (the deficit angle, dihedral angle, and $\sum_{e\subset\partial f} n_e$ are unchanged under the global parity, which is shown in the following), i.e.[10]

$$S(\tilde{x}_c) = -S(x_c) \tag{3.279}$$

if $\tilde{x}_c$ and $x_c$ are related by a global parity transformation.

Since the frame vectors $U_e(v) = \pm\frac{\beta_e(v)N_e(v)}{\sqrt{|V_4(v)|}}$ are defined up to a sign, the frame $\tilde{U}_e(v)$ constructed from parity configuration relates $U_e(v)$ only by a parity inversion

$$\tilde{U}_e(v) = \mathbf{P}U_e(v) \tag{3.280}$$

[10] The sign in front of the term $i\pi\sum_e n_e \sum_{f\subset t_e} j_f$ is unimportant.

The same relation holds for the co-frame $\tilde{E}_\ell(v)$

$$\tilde{E}_\ell(v) = \mathbf{P}E_\ell(v) \tag{3.281}$$

from the relation

$$\tilde{U}^{e_j}_I(v)\tilde{E}^I_{e_k e_l}(v) = \delta^j_k - \delta^j_l \tag{3.282}$$

We then obtain the same relation relating the bivector and co-frame/frame as in Theorem 3.4.2

$$\tilde{X}_f(v) = \varepsilon\ \tilde{V}_4\left[\tilde{U}_e(v)\wedge\tilde{U}_{e'}(v)\right] \quad \text{and} \quad \tilde{X}_f(v) = \varepsilon\ \star\left[\tilde{E}_{\ell_1}(v)\wedge\tilde{E}_{\ell_2}(v)\right] \tag{3.283}$$

which is consistent because of the relations $\tilde{X}_f(v) = -(\mathbf{P}\otimes\mathbf{P})X_f(v)$, $\tilde{U}_e(v) = \mathbf{P}U_e(v)$, $\tilde{E}_\ell(v) = \mathbf{P}E_\ell(v)$, $\tilde{V}_4(v) = -V_4(v)$, and $\epsilon_{IJKL}\mathbf{P}^I{}_M\mathbf{P}^J{}_N\mathbf{P}^K{}_P\mathbf{P}^L{}_Q = -\epsilon_{MNPQ}$. Here we emphasize that the sign factor $\varepsilon$ for the parity configuration $(j_f, \tilde{g}_{ve}, \xi_{ef}, \tilde{z}_{vf})$ is the same as the original configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, thus is consistent with the fact that $\varepsilon$ is a global sign factor on the entire triangulation, i.e. the local/global parity inversion of the critical configuration doesn't change the global sign $\varepsilon$.

The local/global parity inversion $\tilde{E}_\ell(v) = \mathbf{P}E_\ell(v)$ doesn't change the discrete metric $g_{\ell_1\ell_2}(v) = \eta_{IJ}E^I_{\ell_1}(v)E^J_{\ell_2}(v)$, so the parity configuration $(j_f, \tilde{g}_{ve}, \xi_{ef}, \tilde{z}_{vf})$ leads to the same discrete metric as $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, but gives an O(1,3) gauge transformation (parity inversion) for the co-frame $E_\ell(v)$. The SO(1,3) matrix $\Omega_{vv'} \in$ SO(1,3) is uniquely compatible with the co-frame $E_\ell(v)$ and is a discrete spin connection when $\mathrm{sgn}(V_4(v)) = \mathrm{sgn}(V_4(v'))$, as it was shown in Section 3.3. Given a nondegenerate critical configuration with a subdivision of the triangulation into sub-triangulations, in each of which $\mathrm{sgn}(V_4(v))$ is a constant, we consider a global parity transformation which doesn't change the subdivision but flip the signs of $\mathrm{sgn}(V_4(v))$ in all sub-triangulations. Given a spin connection $\Omega_{vv'}$ with $\sigma_v, \sigma_{v'}$ are both in the same sub-triangulation, i.e. $\mathrm{sgn}(V_4(v)) = \mathrm{sgn}(V_4(v'))$, the spin connection $\tilde{\Omega}_{vv'} \in$ SO(1,3) after a parity transformation in both $\sigma_v, \sigma_{v'}$ is given by

$$\tilde{\Omega}_{vv'} = \mathbf{P}\Omega_{vv'}\mathbf{P} \tag{3.284}$$

since $\tilde{\Omega}_{vv'}$ is uniquely determined by

$$\tilde{\Omega}_{vv'}\tilde{E}_\ell(v') = \tilde{E}_\ell(v) \qquad \ell \subset t_e,\ e = (v, v') \tag{3.285}$$

On the other hand we can check from

$$\tilde{g} = JgJ^{-1} \qquad \tilde{g}(-\vec{\sigma})\tilde{g}^\dagger = \mathbf{P} \rhd g\vec{\sigma}g^\dagger \tag{3.286}$$

that given a 4-vector $V^I$

$$\tilde{g}\mathbf{P}(V^I\sigma_I)\tilde{g}^\dagger = \mathbf{P}(gV^I\sigma_I g^\dagger) \quad \text{i.e.} \quad \tilde{g}\mathbf{P}V = \mathbf{P}gV \quad \text{in Spin-1 representation} \tag{3.287}$$

Let $V = E_\ell(v')$, using $g_{vv'} = \mu_e \Omega_{vv'}$,

$$\tilde{g}_{vv'}\tilde{E}_\ell(v') = \tilde{g}_{vv'}\mathbf{P}E_\ell(v') = \mathbf{P}g_{vv'}E_\ell(v') = \mu_e\mathbf{P}E_\ell(v) = \mu_e\tilde{E}_\ell(v) \tag{3.288}$$

Therefore we obtain from $\tilde{g}_{vv'} = \tilde{\mu}_e\tilde{\Omega}_{vv'}$ that the sign $\mu_e$ is invariant under the parity transformation:

$$\mu_e = \tilde{\mu}_e \tag{3.289}$$

when $e$ is a internal edge. In case $t_e$ is a boundary tetrahedron, the parity transformation changes the co-frame $E_\ell(v) \mapsto \tilde{E}_\ell(v) = \mathbf{P}E_\ell(v)$ at the vertex $v$, while leaves the boundary co-frame $E_\ell(e)$ invariant. Therefore the spin connection $\tilde{\Omega}_{ve} \in$ SO(1,3) is uniquely determined by

$$\tilde{\Omega}_{ve}E_\ell(e) = \tilde{E}_\ell(v) \qquad \ell \subset t_e, \tag{3.290}$$

Before the parity transformation, $\Omega_{ve}E_\ell(e) = E_\ell(v)$ determines uniquely the spin connection $\Omega_{ve}$. Then the relation between $\tilde{\Omega}_{ve}$ and $\Omega_{ve}$ is given by

$$\tilde{\Omega}_{ve} = \mathbf{P}\Omega_e\mathbf{T} \quad \text{where} \quad \mathbf{T} = \mathrm{diag}(-1,1,1,1) \tag{3.291}$$

by the fact that the co-frame vectors $E_\ell(e)$ are orthogonal to $(1,0,0,0)^t$ and both $\tilde{\Omega}_e$ and $\Omega_e$ belong to SO(1,3). Here the matrix $\mathbf{T}$ is a time-reversal in the Minkowski space, which leaves $E_\ell(e)$ invariant. Given a spatial vector $V^I$ orthogonal to $(1,0,0,0)^t$

$$\tilde{g}(V^i\sigma_i)\tilde{g}^\dagger = -\mathbf{P}g(V^i\sigma_i)g^\dagger \quad \text{i.e.} \quad \tilde{g}V = -\mathbf{P}gV \quad \text{in Spin-1 representation} \tag{3.292}$$

Let $V = E_\ell(e)$, using $g_{ve} = \mu_e\Omega_{ve}$ in Spin-1 representation

$$\tilde{g}_{ve}E_\ell(e) = -\mathbf{P}g_{ve}E_\ell(e) = -\mu_e\mathbf{P}E_\ell(v) = -\mu_e\tilde{E}_\ell(v) \tag{3.293}$$

Therefore we obtain from $\tilde{g}_{ve}E_\ell(e) = \tilde{\mu}_e\tilde{E}_\ell(v)$ that

$$\mu_e = -\tilde{\mu}_e \tag{3.294}$$

for an edge connecting to the boundary. A boundary triangle is shared by exactly two boundary tetrahedra, in the dual language, a boundary face has exactly two edges connecting to the boundary. Thus the product $\prod_{e\subset\partial f}\mu_e$ is invariant under the parity transformaiton, i.e.

$$\prod_{e\subset\partial f}\mu_e = \prod_{e\subset\partial f}\tilde{\mu}_e \tag{3.295}$$

for either a boundary face or an internal face. If we write $\mu_e = e^{i\pi n_e}$ and $\tilde{\mu}_e = e^{i\pi\tilde{n}_e}$, then we have

$$\sum_{e\subset\partial f} n_e = \sum_{e\subset\partial f}\tilde{n}_e \tag{3.296}$$

We consider $\tilde{\Omega}_f(v)$ a loop holonomy of the spin connection along the boundary of an internal face $f$, based at the vertex $v$, which is constructed from a global parity configuration $(j_f, \tilde{g}_{ve}, \xi_{ef}, \tilde{z}_{vf})$ with $\tilde{g}_{ve} \neq g_{ve}$ at all the vertices. It is different from the original $\Omega_f(v)$ by

$$\tilde{\Omega}_f(v) = \mathbf{P}\Omega_f(v)\mathbf{P} \tag{3.297}$$

From Eq.(3.208), $\Omega_f(v)$ can be expressed in terms of the co-frame vectors $E_{\ell_1}(v), E_{\ell_2}(v)$ for the edges $\ell_1, \ell_2$ of the triangle $f$

$$\Omega_f(v) = e^{i\pi\sum_e n_e} e^{\frac{\star E_{\ell_1}(v)\wedge E_{\ell_2}(v)}{|\star E_{\ell_1}(v)\wedge E_{\ell_2}(v)|}\mathrm{sgn}(V_4)\Theta_f + \frac{E_{\ell_1}(v)\wedge E_{\ell_2}(v)}{|E_{\ell_1}(v)\wedge E_{\ell_2}(v)|}\pi\sum_e n_e}$$

$$\tilde{\Omega}_f(v) = e^{i\pi\sum_e \tilde{n}_e} e^{\frac{\star \tilde{E}_{\ell_1}(v)\wedge \tilde{E}_{\ell_2}(v)}{|\star \tilde{E}_{\ell_1}(v)\wedge \tilde{E}_{\ell_2}(v)|}\mathrm{sgn}(\tilde{V}_4)\tilde{\Theta}_f + \frac{\tilde{E}_{\ell_1}(v)\wedge \tilde{E}_{\ell_2}(v)}{|\tilde{E}_{\ell_1}(v)\wedge \tilde{E}_{\ell_2}(v)|}\pi\sum_e \tilde{n}_e} \tag{3.298}$$

From the previous results $\mathrm{sgn}(\tilde{V}_4) = -\mathrm{sgn}(V_4)$, $\sum_e n_e = \sum_e \tilde{n}_e$ and the relation $\mathbf{P}\otimes\mathbf{P}(\star E_1\wedge E_2) = -\star\mathbf{P}E_1\wedge\mathbf{P}E_2$, we obtain that

$$\Theta_f = \tilde{\Theta}_f \tag{3.299}$$

which is consistent with the fact that the deficit angle $\Theta_f$ is determined by the metric $g_{\ell_1\ell_2}$ which is invariant under the parity transformation.

For the holonomy $\Omega_f(e_1, e_0)$ for a boundary face $f$, under a global parity

$$\tilde{\Omega}_f(e_1, e_0) = \mathbf{T}\Omega_f(e_1, e_0)\mathbf{T} \tag{3.300}$$

Recall Eq.(3.232), we have for both $\tilde{\Omega}_f(e_1, e_0)$ and $\Omega_f(e_1, e_0)$

$$\hat{g}(\xi_{e_1 f})^{-1}\Omega_f(e_1, e_0)\hat{g}(\xi_{e_0 f}) = e^{i\pi\sum_e n_e} e^{\varepsilon\ \mathrm{sgn}(V_4)\Theta_f^B K_3} e^{\pi\sum_e n_e J_3}$$

$$\hat{g}(\xi_{e_1 f})^{-1}\tilde{\Omega}_f(e_1, e_0)\hat{g}(\xi_{e_0 f}) = e^{i\pi\sum_e \tilde{n}_e} e^{\varepsilon\ \mathrm{sgn}(\tilde{V}_4)\tilde{\Theta}_f^B K_3} e^{\pi\sum_e \tilde{n}_e J_3} \tag{3.301}$$

Since $\mathbf{T}$ commutes with $\hat{g}(\xi_{ef}) \in \mathrm{SU}(2)$ and $\mathbf{T}K_3\mathbf{T} = -K_3, \mathbf{T}J_3\mathbf{T} = J_3$, we obtain that

$$\Theta_f^B = \tilde{\Theta}_f^B \tag{3.302}$$

and consistent with the fact that the dihedral angle $\Theta_f^B$ is determined by the metric $g_{\ell_1\ell_2}$ which is invariant under the parity transformation.

Before we come to the next section, we emphasize that given a Regge-like spin configuration $j_f$, there exists only two nondegenerate critical configurations $(j_f, g^c_{ve}, \xi_{ef}, z^c_{vf})$ such that the oriented 4-volume has a constant sign on the triangulation, i.e. $\mathrm{sgn}(V_4(v))$ is a constant for all $\sigma_v$. The existence can be shown in the following way: given a nondegenerate critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, it determines a subdivision of the triangulation into sub-triangulations $\mathcal{K}_1, \cdots, \mathcal{K}_n$, where on each $\mathcal{K}_i$, $\mathrm{sgn}(V_4(v))$ is a constant, but $\mathrm{sgn}(V_4(v))$ is not a constant for neighboring $\mathcal{K}_i$ and $\mathcal{K}_j$. However we can always make a parity transformation for all the simplices within some sub-triangulations, to flip the sign of the oriented 4-volume, such that $\mathrm{sgn}(V_4(v))$ is a constant on the entire triangulation. Any two nondegenerate solutions $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ are related by a (local) parity transformation, which flips the sign of $V_4(v)$ at least locally. There exists two nondegenerate critical configurations $(j_f, g^c_{ve}, \xi_{ef}, z^c_{vf})$ such that the oriented 4-volume has a constant sign on the entire triangulation, while the two configurations are related by a global parity transformation. If there was another nondegenerate critical configurations such that the oriented 4-volume has a constant sign on the entire triangulation, it must relate the existed two configurations by a local parity transformation, which flips $\mathrm{sgn}(V_4(v))$ only locally thus breaks the constancy of $\mathrm{sgn}(V_4(v))$.

# 3.7 Asymptotics of degenerate amplitudes

## 3.7.1 Degenerate critical configurations

The previous discussions of the critical configuration and asymptotic formula are under the nondegenerate assumption:

$$\prod_{e_1,e_2,e_3,e_4=1}^{5} \det\Big(N_{e_1}(v), N_{e_2}(v), N_{e_3}(v), N_{e_4}(v)\Big) \neq 0 \tag{3.303}$$

where $N_e(v) = g_{ve}(1,0,0,0)^t$, i.e. any four of the five normal vectors $N_e(v)$ form a linearly independent set and span the 4-dimensional Minkowski space.

Now we consider a degenerate critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ that solves the critical equations Eqs.(3.37), (3.38), and (3.39), but violates the above nondegenerate assumption at *all* vertices on a triangulation (with boundary). if we assume the nondegeneracy of the tetrahedra, i.e. given a tetrahedron $t_e$, the 4 vectors $\hat{n}_{ef}$ obtained from the spinors $\xi_{ef}$ span a 3-dimensional subspace, then the Lemma 3 in the first reference of [66, 74] shows that within each 4-simplex, all five normals $N_e(v)$ from the degenerate critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ are parallel and more precisely $N_e(v) = u = (1,0,0,0)$[11]. By definition $N_e(v) = g_{ve}(1,0,0,0)^t$, we find that all the group variables $g_{ve} \in \mathrm{SU}(2)$ for a degenerate critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$. For the bivectors $\star X_f(v)$, they are all orthogonal to the same unit vector $u = (1,0,0,0)$.

From $\star X_f(v)\cdot u = 0$, we can write the bivector $X_f(v) = V_f(v)\wedge u$ for a vector $V_f(v)$ orthogonal to $u$. The vector $V_f(v)$ can be determined by the parallel transportation $X_f(v) = g_{ve}X_{ef}g_{ev}$ and $X_{ef} = 2\gamma j_f \hat{n}_{ef}\wedge u$, thus

$$V_f(e) = 2\gamma j_f \hat{n}_{ef} \qquad V_f(v) = 2\gamma j_f \; g_{ve}\hat{n}_{ef} \tag{3.304}$$

The above relation doesn't depend on the choice of $e$ (recall Proposition 3.2.1). From the closure condition Eq.(3.39), we have

$$\sum_{f\subset t_e} \varepsilon_{ef}(v) V_f(v) = 0 \tag{3.305}$$

Therefore a degenerate critical configuration $(j_f, g_{ve}, \xi_{ef})$ assign uniquely a spatial vector $V_f(v)\perp u$ at the vertex $v$ for each triangle $f$, satisfying the closure condition Eq.(3.305). The collection of the vectors $V_f(v)$ is referred as a *vector geometry* in

[11] Recall that we have fixed $g_{ve_5} = 1$ to make the vertex amplitude finite.

[66, 74].

Since $g_{ev} \in \mathrm{SU}(2)$ in the degenerate critical configuration $(j_f, g_{ve}, \xi_{ef})$, we have immediately $\frac{||Z_{ve'f}||}{||Z_{vef}||} = 1$. Then for each face action $S_f$ (internal face or boundary face)

$$S_f = 2i\gamma j_f \sum_v \ln \frac{||Z_{ve'f}||}{||Z_{vef}||} - 2ij_f \sum_v \phi_{eve'} = -2i\ j_f \sum_v \phi_{eve'} \tag{3.306}$$

In the same way as we did for the nondegenerate amplitude, we make use of Eqs.(3.37) and (3.38), which now take the following forms

$$g_{ve}\left(J\xi_{ef}\right) = e^{-i\phi_{eve'}} g_{ve'}\left(J\xi_{e'f}\right)$$

$$g_{ve}\xi_{ef} = e^{i\phi_{eve'}} g_{ve'}\xi_{e'f} \tag{3.307}$$

First of all, for a internal face $f$, we again consider the successive actions on $\xi_{ef}$ of $g_{e'v}g_{ve}$ around the entire boundary of the face $f$,

$$\overleftarrow{\prod_{v\in\partial f}} g_{e'v}g_{ve}J\xi_{ef} = e^{-i\sum_v \phi_{eve'}} J\xi_{ef}$$

$$\overleftarrow{\prod_{v\in\partial f}} g_{e'v}g_{ve}\xi_{ef} = e^{+i\sum_v \phi_{eve'}} \xi_{ef} \tag{3.308}$$

where $g_{ve} \in \mathrm{SU}(2)$. In the same way as we did for the nondegenerate case, the above equations imply that for the loop holonomy $G_f(e) = \overleftarrow{\prod}_{v\in\partial f} g_{e'v}g_{ve}$,

$$G_f(e) = \exp\left[i \sum_{v\in\partial f} \phi_{eve'}\vec{\sigma}\cdot\hat{n}_{ef}\right]. \tag{3.309}$$

For a boundary face $f$, again in the same way as we did for the nondegenerate case, we obtain

$$G_f(e_1, e_0) = g(\xi_{e_1 f})\ e^{i\sum_v \phi_{eve'}\vec{\sigma}\cdot\hat{z}}\ g(\xi_{e_0 f})^{-1}. \tag{3.310}$$

We then need to determine the physical interpretation of the angle $\sum_{v\in\partial f}\phi_{eve'}$ in different cases.

Recall the degenerate critical equations Eq.(3.307) together with the closure condition Eq.(3.39), we find they are essentially the same as the critical equations in [67] for a Euclidean spin foam amplitude:

$$g^{\pm}_{ve}\left(J\xi_{ef}\right) = e^{-i\phi^{\pm}_{eve'}} g^{\pm}_{ve'}\left(J\xi_{e'f}\right)$$

$$g^{\pm}_{ve}\xi_{ef} = e^{i\phi^{\pm}_{eve'}} g^{\pm}_{ve'}\xi_{e'f}$$

$$0 = \sum_{f\subset t_e}^{4} \varepsilon_{ef}(v) j_f \hat{n}_{ef} \tag{3.311}$$

where the equations for self-dual or anti-self-dual sector are essentially the same, and both of them are the same as the above degenerate critical equation for Lorentzian amplitude. Therefore given a degenerate critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ for the Lorentzian amplitude, there exists a critical configuration $(j_f, g^{\pm}_{ve}, \xi_{ef})$ for the Euclidean amplitude in [67], such that $g_{ve} = g^{+}_{ve}$. In the following, we classify the degenerate Lorentzian critical configurations into two type (type A and type B) and discuss the uniqueness of the corresponding Euclidean critical configurations:

**Type-A configuration:** A degenerate Lorentzian critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ corresponds to an Euclidean critical configuration $(j_f, g^{\pm}_{ve}, \xi_{ef})$, which is nondegenerate at each 4-simplex $\sigma_v$ of the triangulation, i.e. any four of the five normals $N_e(v) = (g^{+}_{ve}, g^{-}_{ve}) \rhd (1,0,0,0)^t$ span a 4-dimensional vector space. Since the Euclidean spins $j_f$ and spinors $\xi_{ef}$ are uniquely specified by the Lorentzian configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, we only need to consider how many solutions $(g^{+}_{ve}, g^{-}_{ve})$ in Eq.(3.311) if the variables $j_f$ and $\xi_{ef}$ are fixed. It is shown in [66, 74] that for a 4-simplex $\sigma_v$, there are only two solutions in the nondegenerate case[12]

$$(g^{+}_{ve}, g^{-}_{ve}) = (g^{1}_{ve}, g^{2}_{ve}) \quad \text{and} \quad (g^{+}_{ve}, g^{-}_{ve}) = (g^{2}_{ve}, g^{1}_{ve}) \tag{3.312}$$

Then the correspondence $g_{ve} = g^{+}_{ve}$ fix uniquely a solution $(g^{+}_{ve}, g^{-}_{ve})$ for the Euclidean critical configuration $(j_f, g^{\pm}_{ve}, \xi_{ef})$.

**Type-A configuration:** The degenerate Lorentzian critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ could always correspond to a degenerate Euclidean critical configuration $(j_f, g^{\pm}_{ve}, \xi_{ef})$

[12] The notion of nondegenercy here is different from the notion in [66, 74]. In the Lemma 4 of the first reference of [66, 74], there are 4 solutions in a 4-simplex $(g^{1}_{ve}, g^{2}_{ve})$, $(g^{2}_{ve}, g^{1}_{ve})$, $(g^{1}_{ve}, g^{1}_{ve})$, $(g^{2}_{ve}, g^{2}_{ve})$ for the nondegenerate case (in the sense of [66, 74]). However the two solutions $(g^{1}_{ve}, g^{1}_{ve})$, $(g^{2}_{ve}, g^{2}_{ve})$ are degenerate in our notion of degeneracy.

with $g^+_{ve} = g^-_{ve}$ by $(g^+_{ve}, g^-_{ve}) = (g_{ve}, g_{ve})$, even the data $j_f$ and $\xi_{ef}$ can have two nondegenerate solutions as above. Then in this case, we alway make the above nondegenerate choice as the canonical choice.

**Type-B configuration:** The data $j_f$ and $\xi_{ef}$ in a degenerate Lorentzian critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ lead to only one Euclidean solutions $(g_{ve}, g_{ve}) \in$ SO(4) for Eq.(3.311) in each 4-simplex $\sigma_v$. Then the Euclidean configuration $(j_f, g^\pm_{ve}, \xi_{ef})$ is degenerate in $\sigma_v$ in the sense of [66, 74]. Then obviously the correspondence is unique by $g_{ve} \mapsto (g_{ve}, g_{ve})$.

### 3.7.2 Type-A degenerate critical configuration: Euclidean geometry

First of all, we consider a type A degenerate Lorentzian critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ on the triangulation (with boundary). The corresponding Euclidean critical configuration $(j_f, g^\pm_{ve}, \xi_{ef})$ is nondegenerate everywhere. We can construct a nondegenerate discrete Euclidean geometry on the triangulation such that (see [67], see also [64])

- An Euclidean co-tetrad $E_\ell(v), E_\ell(e)$ of the triangulation (bulk and boundary) can be constructed from $(j_f, g^\pm_{ve}, \xi_{ef})$, unique up to a sign fliping $E_\ell \to -E_\ell$, such that the spins $j_f$ satisfies

$$2\gamma j_f = |E_{\ell_1}(v) \wedge E_{\ell_2}(v)| \,. \tag{3.313}$$

  From the co-tetrad we can construct a unique discrete metric with Euclidean signature on the whole triangulation (bulk and boundary)

$${}^E g_{\ell_1\ell_2}(v) = \delta_{IJ} E^I_{\ell_1}(v) E^J_{\ell_2}(v) \qquad {}^E g_{\ell_1\ell_2}(e) = \delta_{IJ} E^I_{\ell_1}(e) E^J_{\ell_2}(e). \tag{3.314}$$

  So $\gamma j_f$ is the triangle area from the discrete metric ${}^E g_{\ell_1\ell_2}$.

- For the bivectors in the bulk,

$$j_f (g^+_{ve}, g^-_{ve})(\hat{n}_{ef}, \hat{n}_{ef}) = \varepsilon \star E_{\ell_1}(v) \wedge E_{\ell_2}(v) \tag{3.315}$$

  For the bivector on the boundary

$$j_f (\hat{n}_{ef}, \hat{n}_{ef}) = \varepsilon \star E_{\ell_1}(e) \wedge E_{\ell_2}(e) \tag{3.316}$$

where $\varepsilon$ is a global sign on the entire triangulation. If the triangulation has boundary, the sign factor $\varepsilon$ is specified by the orientation of the boundary triangulation, i.e. $\varepsilon = \mathrm{sgn}(V_3)$ for the boundary tetrahedra.

- The SO(4) group variable $(g_e^+, g_e^-)$ equals to the Euclidean spin connection ${}^E\Omega_e$ compatible with $E_\ell(v)$, up to a sign $\mu_e = e^{i\pi n_e}$ ($n_e = 0, 1$), i.e.

$$(g_e^+, g_e^-) = \mu_e {}^E\Omega_e \tag{3.317}$$

in the Spin-1 representation. Here ${}^E\Omega_e \in \mathrm{SO}(4)$ is compatible with the co-frame $E_\ell(v), E_\ell(e)$

$$({}^E\Omega_{vv'})^I{}_J E_\ell^J(v') = E_\ell^I(v) \quad \text{and} \quad ({}^E\Omega_{ve})^I{}_J E_\ell^J(e) = E_\ell^I(v) \tag{3.318}$$

If $\mathrm{sgn}(V_4(v)) = \mathrm{sgn}(V_4(v'))$, $\Omega_{vv'}$ is the unique discrete spin connection compatible with the co-frame. In addition, we note that each $\mu_e$ is not invariant under the sign flipping $E_\ell \to -E_\ell$, but the product $\prod_{e\subset\partial f} \mu_e$ is invariant for any (internal or boundary) face $f$ (see Lemma.3.4.1).

Therefore in this way, a type-A degnerate *Lorentzian* critical configuration determines uniquely a triple of (Euclidean) variables $({}^E g_{\ell_1\ell_2}, n_e, \varepsilon)$ corresponding to a *Euclidean* Geometry and two types of sign factors.

Given a nondegenerate Euclidean critical configuration $(j_f, g_{ve}^\pm, \xi_{ef})$, in the same way as the nondegenerate Lorentzian critical configuration, it determines a subdivision of the triangulation into sub-triangulations (with boundaries) $\mathcal{K}_1, \cdots, \mathcal{K}_n$, on each of the sub-triangulation, the sign of the oriented 4-volume $\mathrm{sgn}(V_4(v))$ is a constant.

Now we discuss the spin foam amplitude at a Type-A degenerate configuration, while we restrict our attention into a sub-triangulation $\mathcal{K}_i$ where $\mathrm{sgn}(V_4(v))$ is a constant. For a internal face $f$, it is shown in [67] that the loop holonomy along the boundary of $f$ is given by

$$\Big(G_f^+(e), G_f^-(e)\Big) = \left(e^{\frac{i}{2}\left[\varepsilon\ \mathrm{sgn}(V_4)^E\Theta_f + \pi\sum_e n_e\right]\vec{\sigma}\cdot\hat{n}_{ef}}, e^{-\frac{i}{2}\left[\varepsilon\ \mathrm{sgn}(V_4)^E\Theta_f - \pi\sum_e n_e\right]\vec{\sigma}\cdot\hat{n}_{ef}}\right) \tag{3.319}$$

where ${}^E\Theta_f$ is the deficit angle from the Euclidean spin connection compatible with the metric ${}^E g_{\ell_1\ell_2}$. By the above identification $g_{ve} = g_{ve}^+$ between the degenerate Lorentzian critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ and a nondegenerate Euclidean critical configuration $(j_f, g_{ve}^\pm, \xi_{ef})$. We obtain that for the degenerate Lorentzian critical configuration, the loop holonomy $G_f(e) = G_f^+(e)$. Comparing with Eq.(3.309),

$$\sum_{v\in\partial f} \phi_{eve'} = \frac{1}{2}\left[\varepsilon \ \mathrm{sgn}(V_4)^E\Theta_f + \pi\sum_e n_e\right] \tag{3.320}$$

Therefore the angle $\sum_{v\in\partial f}\phi_{eve'}$ has the physical meaning as a deficit angle in a corresponding Euclidean geometry. Then the face action (as a function of $({}^Eg_{\ell_1\ell_2}, n_e, \varepsilon)$) reads

$$S_f({}^Eg_{\ell_1\ell_2}, n_e, \varepsilon) = -i\varepsilon \ \mathrm{sgn}(V_4) \ j_f{}^E\Theta_f - i\pi\sum_e n_e j_f \tag{3.321}$$

for a internal face $f$.

For a boundary face $f$, we have the path holonomy along its internal boundary $p_{e_1e_0}$ is given by

$$\begin{aligned}&\Big(G_f^+(e_1,e_0), G_f^-(e_1,e_0)\Big)\\ =\ & g(\xi_{e_1f})\left( e^{\frac{i}{2}\left[\varepsilon \ \mathrm{sgn}(V_4)^E\Theta_f^B+\pi\sum_e n_e\right]\sigma_3} \ , \ e^{-\frac{i}{2}\left[\varepsilon \ \mathrm{sgn}(V_4)^E\Theta_f^B-\pi\sum_e n_e\right]\sigma_3} \right) g(\xi_{e_0f})^{-1}\end{aligned} \tag{3.322}$$

where ${}^E\Theta_f^B$ is the dihedral angle (determined by the metric ${}^Eg_{\ell_1\ell_2}$) between two boundary tetrahedra $t_{e_0}, t_{e_1}$ at the triangle $f$ shared by them. The degenerate Lorentzian critical configuration $G_f(e_1,e_0)$ is identify with $G_f^+(e_1,e_0)$ here. Comparing to Eq.(3.310) we obtain that

$$\sum_{v\in p_{e_1e_0}} \phi_{eve'} = \frac{1}{2}\left[\varepsilon \ \mathrm{sgn}(V_4)^E\Theta_f^B + \pi\sum_e n_e\right] \tag{3.323}$$

Therefore the face action $S_f$ for a boundary face $f$ is given by

$$S_f({}^Eg_{\ell_1\ell_2}, n_e, \varepsilon) = -i\varepsilon \ \mathrm{sgn}(V_4) \ j_f{}^E\Theta_f^B - i\pi\sum_e n_e j_f. \tag{3.324}$$

As a result, at a type-A degenerate critical configuration (restricted to a subtriangulation $\mathcal{K}_i$), the Lorentzian spin foam action $S$ is a function of the variables $({}^Eg_{\ell_1\ell_2}, n_e, \varepsilon)$ and behaves mainly as an Euclidean Regge action:

$$\begin{aligned} S({}^Eg_{\ell_1\ell_2}, n_e, \varepsilon)\Big|_{\mathcal{K}_i} &= \Big[-i \ \varepsilon \ \mathrm{sgn}(V_4)\sum_{\text{internal } f} j_f{}^E\Theta_f - i \ \varepsilon \ \mathrm{sgn}(V_4)\sum_{\text{boundary } f} j_f{}^E\Theta_f^B \\ &\quad -i\pi\sum_e n_e\sum_{f\subset t_e} j_f\Big]_{\mathcal{K}_i} \end{aligned} \tag{3.325}$$

where we note that the areas $\gamma j_f$, deficit angles ${}^E\Theta_f$, and dihedral angles ${}^E\Theta_f^B$ are uniquely determined by the discrete metric $g_{\ell_1\ell_2}$. Moreover for each tetrahedron $t$, the sum of face spins $\sum_{f\subset t} j_f$ is an integer. For half-integer spins, $e^{-i\pi\sum_e n_e \sum_{f\subset t_e} j_f} = \pm 1$ gives an overall sign factor. Therefore in general at a type-A degenerate critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ for Lorentzian amplitude,

$$e^{\lambda S}\Big|_{\mathcal{K}_i} = \pm \exp\lambda \left[-i\ \varepsilon\ \mathrm{sgn}(V_4) \sum_{\text{internal } f} j_f{}^E\Theta_f - i\ \varepsilon\ \mathrm{sgn}(V_4) \sum_{\text{boundary } f} j_f{}^E\Theta_f^B\right]_{\mathcal{K}_i}. \tag{3.326}$$

Again there exists two ways to make the overall sign factor disappear: (1) only consider integer spins $j_f$, or (2) modify the embedding from SU(2) unitary irreducible representations to $\mathrm{SL}(2,\mathbb{C})$ unitary irreducible representations by $j_f \mapsto (p_f, k_f) := (2\gamma j_f, 2j_f)$, then the spin foam action $S$ is replaced by $2S$. In these two cases the exponential $e^{\lambda S}$ at the critical configuration is independent of the variable $n_e$.

According to the properties of Euclidean Regge geometry, given a collection of (Euclidean) Regge-like areas $\gamma j_f$, the discrete Euclidean metric ${}^E g_{\ell_1\ell_2}(v)$ is uniquely determined at each vertex $v$. Furthermore since the areas $\gamma j_f$ are Regge-like, There exists a discrete Euclidean metric ${}^E g_{\ell_1\ell_2}$ in the entire bulk of the triangulation, such that the neighboring 4-simplicies are consistently glued together, as we constructed in [67]. This discrete metric ${}^E g_{\ell_1\ell_2}$ is obviously unique by the uniqueness of $g_{\ell_1\ell_2}(v)$. Therefore given the partial-amplitude $A_{j_f}(\mathcal{K})$ in Eq.(3.15) with a specified Euclidean Regge-like $j_f$, all the degenerate critical configurations $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ of type-A corresponds to the same discrete Euclidean metric ${}^E g_{\ell_1\ell_2}$, provided a Regge boundary data. Any two type-A critical configurations $(j_f, g_{ve}, \xi_{ef}, z_{vf}) = (j_f, g^{\pm}_{ve}, \xi_{ef})$ with the same $j_f$ are related by local or global parity transformation in the Euclidean theory, see [67], similar to the Lorentzian nondegenerate case.

As a result, given an Euclidean Regge-like spin configurations $j_f$ and a Regge boundary data, the degenerate critical configurations of type-A give the following asymptotics

$$\begin{aligned} A_{j_f}(\mathcal{K})\Big|_{\text{Deg-A}} \sim\ & \sum_{x_c} a(x_c) \left(\frac{2\pi}{\lambda}\right)^{\frac{r(x_c)}{2} - N(v,f)} \frac{e^{i\mathrm{Ind} H'(x_c)}}{\sqrt{|\det_r H'(x_c)|}} \left[1 + o\left(\frac{1}{\lambda}\right)\right] \times \\ & \times \prod_{i=1}^{n(x_c)} \exp -i\lambda\Big[\varepsilon\ \mathrm{sgn}(V_4) \sum_{\text{internal } f} j_f{}^E\Theta_f + \varepsilon\ \mathrm{sgn}(V_4) \sum_{\text{boundary } f} j_f{}^E\Theta_f^B \\ & + \pi \sum_e n_e \sum_{f\subset t_e} j_f\Big]_{\mathcal{K}_i(x_c)} \end{aligned} \tag{3.327}$$

where $x_c = (j_f, g_{ve}, \xi_{ef}, z_{vf}) = (j_f, g^{\pm}_{ve}, \xi_{ef})$ labels the degenerate critical configurations of type-A, $r(x_c)$ is the rank of the Hessian matrix at $x_c$, and $N(v, f)$ is the number of the pair $(v, f)$ with $v \in \partial f$ (recall Eq.(3.15), there is a factor of $\dim(j_f)$ for each pair of $(v, f)$). $a(x_c)$ is the evaluation of the integration measures at $x_c$, which doesn't scale with $\lambda$. Here ${}^E\Theta_f$ and ${}^E\Theta^B_f$ only depend on the Euclidean metric ${}^E g_{\ell_1\ell_2}$, which is uniquely determined by the Euclidean Regge-like spin configuration $j_f$ and the Regge boundary data.

### 3.7.3 Type-B degenerate critical configuration: vector geometry

Given a type-B degenerate Lorentzian critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, the data $\xi_{ef}$ lead to only one Euclidean solution $(g_{ve}, g_{ve}) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ for Eq.(3.311) in each 4-simplex $\sigma_v$. Then the Euclidean configuration $(j_f, g^{\pm}_{ve}, \xi_{ef})$ is degenerate in $\sigma_v$ in the sense of [66, 74]. Therefore there is no nondegenerate geometric interpretation of a type-B degenerate Lorentzian critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$. It can only be interpreted as a vector geometry in terms of $V_f(v), V_f(e)$ on the triangulation (bulk and boundary), where all the vectors $V_f(v), V_f(e)$ are orthogonal to the unit time-like vector $u = (1, 0, 0, 0)^t$, and $|V_f(v)| = |V_f(e)| = 2\gamma j_f$. The vectors $V_f(v), V_f(e)$ are uniquely determined by $j_f$ and $\xi_{ef}$ by $V_f(e) = 2\gamma j_f \hat{n}_{ef}$ and $V_f(v) = 2\gamma j_f\ g_{ve}\hat{n}_{ef}$, since the group variable $g_{ve}$ is uniquely determined by $\xi_{ef}$. We have the parallel transportation using the Spin-1 representation of $g_{ve}$

$$g_{vv'} \rhd V_f(v') = V_f(v) \quad \text{and} \quad g_{ve} \rhd V_f(e) = V_f(v) \tag{3.328}$$

for all triangles $f$ in the tetrahedron $t_e$ (shared by $v, v'$ if not a boundary tetrahedron). Then the unique group variables $g_{vv'}, g_{ve} \in \mathrm{SU}(2)$ are said to be compatible with the vector geometry $V_f(v), V_f(e)$. Therefore a type-B degenerate Lorentzian critical configuration $(j_f, g_{ve}, \xi_{ef})$ determine uniquely a vector geometry $V_f(v), V_f(e)$. Conversely, given a vector geometry $V_f(v), V_f(e)$, it uniquely determine the SU(2) group variables $g_{ve}$ up to a sign $e^{i\pi n_e}$, due to the 2-to-1 correspondence between SU(2) and SO(3).

Since we have shown from the critical point equations that

$$G_f(e) = e^{i \sum_v \phi_{eve'} \vec{\sigma}\cdot\hat{n}_{ef}} \qquad G_f(e_1, e_0) = g(\xi_{e_1 f})\ e^{i \sum_v \phi_{eve'} \vec{\sigma}\cdot\hat{z}}\ g(\xi_{e_0 f})^{-1}, \tag{3.329}$$

the above SU(2) angle $\sum_v \phi_{eve'}$ is determined uniquely by the group variables $g_{ve}$ (which is uniquely compatible with the vector geometry $V_f(v), V_f(e)$ up to a sign $e^{i\pi n_e}$)

$$\sum_{v\in\partial f}\phi_{eve'}=\frac{1}{2}\Phi_f+\pi\sum_{e\subset\partial f}n_e \quad \text{and} \quad \sum_{v\in p_{e_1e_0}}\phi_{eve'}=\frac{1}{2}\Phi_f^B+\pi\sum_{e\subset p_{e_1e_0}}n_e \tag{3.330}$$

respectively for a internal face and a boundary face, where the SO(3) angle $\Phi_f$ is uniquely determined by the vector geometry $V_f$ only (the factor $\frac{1}{2}$ shows the relation between an SU(2) angle and SO(3) angle). Therefore for the face action (internal face and boundary face)

$$S_f(V_f,n_e)=i\,j_f\Phi_f-2i\pi\sum_{e\subset\partial f}n_e j_f \quad \text{and} \quad S_f(V_f,n_e)=i\,j_f\Phi_f^B-2i\pi\sum_{e\subset\partial f}n_e j_f \tag{3.331}$$

As a result, at a type-B degenerate critical configuration, the Lorentzian spin foam action $S$ is a function of the variables $(V_f,n_e)$:

$$S(V_f,n_e)=-i\sum_{\text{internal }f}j_f\Phi_f-i\sum_{\text{boundary }f}j_f\Phi_f^B-2\pi i\sum_{e\subset\partial f}n_e\sum_{f\subset t_e}j_f \tag{3.332}$$

Moreover for each tetrahedron $t$, the sum of face spins $\sum_{f\subset t}j_f$ is an integer. Therefore in general at a type-B degenerate critical configuration $(j_f,g_{ve},\xi_{ef},z_{vf})$ for Lorentzian amplitude, $e^{\lambda S}$ is a function of vector geometry $V_f$ only:

$$e^{\lambda S}=\exp\lambda\left[-i\sum_{\text{internal }f}j_f\Phi_f-i\sum_{\text{boundary }f}j_f\Phi_f^B\right]. \tag{3.333}$$

where the area $\gamma j_f=\frac{1}{2}|V_f|$ and the angle $\Phi_f$ is uniquely determined by the vector geometry $V_f$.

As a result, given an spin configurations $j_f$ and a boundary data that admit a vector geometry on the triangulation, the degenerate critical configurations of type-B give the following asymptotics

$$\begin{aligned}A_{j_f}(\mathcal{K})\Big|_{\text{Deg-B}} \;\sim\; &\sum_{x_c}a(x_c)\left(\frac{2\pi}{\lambda}\right)^{\frac{r(x_c)}{2}-N(v,f)}\frac{e^{i\mathrm{Ind}H'(x_c)}}{\sqrt{|\det_r H'(x_c)|}}\left[1+o\left(\frac{1}{\lambda}\right)\right]\times\\ &\times\;\exp\lambda\left[-i\sum_{\text{internal }f}j_f\Phi_f-i\sum_{\text{boundary }f}j_f\Phi_f^B\right]\end{aligned} \tag{3.334}$$

where $x_c\equiv(j_f,g_{ve},\xi_{ef},z_{vf})$ labels the degenerate critical configurations of type-B. Note that if we make a suitable gauge fixing for the boundary data, we can always set

$\Phi_f^B = 0$ [66, 74].

## 3.8 Transition between Lorentzian, Euclidean and vector geometry

All the previous analysis assume that on the *entire* triangulation, the critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ is one of the three types: nondegenerate, degenerate of type-A or degenerate of type-B. However they are not the most general case. In principle one should admit the critical configuration that mixes the three types on the triangulation: Given a most general critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ that mixes the three types, one can always make a partition of the triangulation into three regions (maybe disconnected regions) $\mathcal{R}_{\text{Nondeg}}, \mathcal{R}_{\text{Deg-A}}, \mathcal{R}_{\text{Deg-B}}$. Each of the three regions $\mathcal{R}_*$, $* = \text{Nondeg}, \text{Deg-A}, \text{Deg-B}$ is a triangulation with boundary, on which the critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})_{\mathcal{R}_*}$ is of single type $* = \text{Nondeg}, \text{Deg-A}, \text{Deg-B}$.

Therefore for a generic spin configuration $j_f$, the asymptotics of the partial amplitude $A_{j_f}(\mathcal{K})$ is given by

$$\begin{aligned} A_{j_f}(\mathcal{K}) \quad \sim \quad & \sum_{x_c} a(x_c) \left(\frac{2\pi}{\lambda}\right)^{\frac{r(x_c)}{2} - N(v,f)} \frac{e^{i\mathrm{Ind}H'(x_c)}}{\sqrt{|\det_r H'(x_c)|}} \left[1 + o\left(\frac{1}{\lambda}\right)\right] \times \\ & \times \mathcal{A}_{j_f}(\mathcal{R}_{\text{Nondeg}}) \mathcal{A}_{j_f}(\mathcal{R}_{\text{Deg-A}}) \mathcal{A}_{j_f}(\mathcal{R}_{\text{Deg-B}}) \end{aligned} \tag{3.335}$$

where $x_c$ labels the general critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ admitted by the spin configuration $j_f$ and boundary data, and $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ determines the regions $\mathcal{R}_*$, $* = \text{Nondeg}, \text{Deg-A}, \text{Deg-B}$ such that $(j_f, g_{ve}, \xi_{ef}, z_{vf})_{\mathcal{R}_*}$ is of single type. The amplitudes $\mathcal{A}_{j_f}(\mathcal{R}_{\text{Nondeg}}), \mathcal{A}_{j_f}(\mathcal{R}_{\text{Deg-A}}), \mathcal{A}_{j_f}(\mathcal{R}_{\text{Deg-B}})$ are given respectively by

$$\mathcal{A}_{j_f}(\mathcal{R}_{\text{Nondeg}}) = \prod_{i=1}^{n(x_c)} \mathrm{e}^{-i\lambda\left[\varepsilon\ \mathrm{sgn}(V_4) \sum_{\text{internal } f} \gamma j_f \Theta_f + \varepsilon\ \mathrm{sgn}(V_4) \sum_{\text{boundary } f} \gamma j_f \Theta_f^B + \pi \sum_e n_e \sum_{f \subset t_e} j_f\right]_{\mathcal{R}_{\text{Nondeg}}, \mathcal{K}_i(x_c)}}$$

$$\mathcal{A}_{j_f}(\mathcal{R}_{\text{Deg-A}}) = \prod_{j=1}^{n'(x_c)} \mathrm{e}^{-i\lambda\left[\varepsilon\ \mathrm{sgn}(V_4) \sum_{\text{internal } f} j_f{}^E \Theta_f + \varepsilon\ \mathrm{sgn}(V_4) \sum_{\text{boundary } f} j_f{}^E \Theta_f^B + \pi \sum_e n_e \sum_{f \subset t_e} j_f\right]_{\mathcal{R}_{\text{Deg-A}}, \mathcal{K}'_j(x_c)}}$$

$$\mathcal{A}_{j_f}(\mathcal{R}_{\text{Deg-B}}) = \exp -i\lambda \left[\sum_{\text{internal } f} j_f \Phi_f + \sum_{\text{boundary } f} j_f \Phi_f^B\right]_{\mathcal{R}_{\text{Deg-B}}} \tag{3.336}$$

As we discussed previously, given a general critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, the regions $\mathcal{R}_{\text{Nondeg}}$ and $\mathcal{R}_{\text{Deg-A}}$ should be respectively divided into sub-triangulations $\mathcal{K}_1, \cdots, \mathcal{K}_{n(x_c)}$ and $\mathcal{K}'_1, \cdots, \mathcal{K}'_{n(x_c)}$, such that in each $\mathcal{K}_i$ or $\mathcal{K}'_i$, $\text{sgn}(V_4)$ is a constant.

Interestingly, from Eq.(3.335) we find an transition between a nondegenerate Lorentzian geometry and a nondegenerate Euclidean geometry through the boundary shared by $\mathcal{R}_{\text{Nondeg}}$ and $\mathcal{R}_{\text{Deg-A}}$. In $\mathcal{R}_{\text{Nondeg}}$ the asymptotics gives a Regge action in Lorentzian signature (plus an additional term):

$$S_{\text{Nondeg}} = -i\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{internal } f} A_f \Theta_f - i\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{boundary } f} A_f \Theta_f^B - \frac{i\pi}{\gamma} \sum_e n_e \sum_{f \subset t_e} A_f \tag{3.337}$$

where we set the physical area $A_f = \gamma j_f$ (in Planck unit). In $\mathcal{R}_{\text{Deg-A}}$ the asymptotics gives a Euclidean Regge action divided by the Barbero-Immirzi parameter (plus an additional term)

$$S_{\text{Deg-A}} = -\frac{i}{\gamma}\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{internal } f} A_f {}^E\Theta_f - \frac{i}{\gamma}\,\varepsilon\,\text{sgn}(V_4) \sum_{\text{boundary } f} A_f {}^E\Theta_f^B - \frac{i\pi}{\gamma} \sum_e n_e \sum_{f \subset t_e} A_f \tag{3.338}$$

In the case of a single simplex, this asymptotics has been presented in [66, 74]. One might expect the transition between Lorentzian and Euclidean geometry is a quantum tunneling effect. But surprisingly in the large-j regime $e^{S_{\text{Deg-A}}}$ is not damping exponentially but oscillatory. Similarly there is also a transition between a nondegenerate Lorentzian/Euclidean geometry and a vector geometry through the boundary of $\mathcal{R}_{\text{Deg-B}}$, and in the region $\mathcal{R}_{\text{Deg-B}}$, the asymptotics give

$$S_{\text{Deg-B}} = -\frac{i}{\gamma} \sum_{\text{internal } f} A_f \Phi_f - \frac{i}{\gamma} \sum_{\text{boundary } f} A_f \Phi_f^B \tag{3.339}$$

Thus $e^{S_{\text{Deg-B}}}$ is also oscillatory and gives nontrivial transition in the large-j regime. However there are some specialities for the phases $e^{S_{\text{Deg-A}}}$, $e^{S_{\text{Deg-B}}}$. These phases oscillates much more violently than the Regge action part in $e^{S_{\text{Nondeg}}}$ when the Barbero-Immirzi parameter $\gamma$ is small, unless ${}^E\Theta_f, {}^E\Theta_f^B, \Phi_f, \Phi_f^B$ are all vanishing[13]. We expect that when we take into account the sum over spins $j_f$, the violently oscillating phases $e^{S_{\text{Deg-A}}}$ and $e^{S_{\text{Deg-B}}}$ may only have relatively small contribution to the total amplitude

[13]The term $\frac{i\pi}{\gamma}\sum_e n_e \sum_{f\subset t_e} A_f$ in both $S_{\text{Nondeg}}$ and $S_{\text{Deg-A}}$ may need special treatment by imposing the boundary semiclassical state carefully.

$A(\mathcal{K}) = \sum_j A_j(\mathcal{K})$, as is suggested by the the Riemann-Lebesgue lemma[14]. But surely the nontrivial transition between different types of geometries is a interesting phenomena exhibiting in the semiclassical analysis of Lorentzian spin foam amplitude, thus requires further investigation and clarification.

## 3.9 Summary

The present work studies the large-j asymptotics of the Lorentzian EPRL spin foam amplitude on a 4d simplicial complex with an arbitrary number of simplices. The asymptotics of the spin foam amplitude is determined by the critical configurations of the spin foam action. Here we show that, given a critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ in general, there exists a partition of the simplicial complex $\mathcal{K}$ into three types of regions $\mathcal{R}_{\text{Nondeg}}, \mathcal{R}_{\text{Deg-A}}, \mathcal{R}_{\text{Deg-B}}$, where the three regions are simplicial sub-complexes with boundaries. The critical configuration implies different types of geometries in different types of regions, i.e. (1) the critical configuration restricted into $\mathcal{R}_{\text{Nondeg}}$ implies a nondegenerate discrete Lorentzian geometry in $\mathcal{R}_{\text{Nondeg}}$. (2) the critical configuration restricted into $\mathcal{R}_{\text{Deg-A}}$ is degenerate of type-A in our definition of degeneracy, but implies a nondegenerate discrete Euclidean geometry in $\mathcal{R}_{\text{Deg-A}}$, (3) the critical configuration restricted into $\mathcal{R}_{\text{Deg-B}}$ is degenerate of type-B, and implies a vector geometry in $\mathcal{R}_{\text{Deg-B}}$.

With the critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{vf})$, we further make a subdivision of the regions $\mathcal{R}_{\text{Nondeg}}$ and $\mathcal{R}_{\text{Deg-A}}$ into sub-complexes (with boundary) $\mathcal{K}_1(\mathcal{R}_*), \cdots, \mathcal{K}_n(\mathcal{R}_*)$ ($*$=Nondeg,Deg-A) according to their Lorentzian/Euclidean oriented 4-volume $V_4(v)$ of the 4-simplices, such that $\text{sgn}(V_4(v))$ is a constant sign on each $\mathcal{K}_i(\mathcal{R}_*)$. Then in the each sub-complex $\mathcal{K}_i(\mathcal{R}_{\text{Nondeg}})$ or $\mathcal{K}_i(\mathcal{R}_{\text{Deg-A}})$, the spin foam amplitude at the critical configuration gives an exponential of Regge action in Lorentzian or Euclidean signature respectively. However we should note that the Regge action reproduced here contains a sign prefactor $\text{sgn}(V_4(v))$ related to the oriented 4-volume of the 4-simplices. Therefore the Regge action reproduced here is actually a discretized Palatini action with on-shell connection.

Finally the asymptotic formula of the spin foam amplitude is given by a sum of

[14]The Riemann-Lebesgue lemma states that for all complex $L^1$-function $f(x)$ on $\mathbb{R}$,

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\mathrm{d}x = 0 \quad \text{as} \quad \alpha \to \pm\infty. \tag{3.340}$$

the amplitudes evaluated at all possible critical configurations, which are the products of the amplitudes associated to different type of geometries.

# Chapter 4

# Three-point function from LQG

## 4.1 Motivations and outlines

In the previous chapter, we have shown that the spin foam gravity goes back to the Palatini Regge gravity at the large-$j$ semiclassical limit. It implies that in this limit, we could recover the physics in Regge gravity, or even in some physics in perturbative gravity. In this chapter I am going to present the work done with my supervisor Carlo Rovelli, in which we calculated the three-point function from the spin foam gravity.

The difficulty of extracting physical predictions from a background-independent theory is a well-known difficulty of quantum gravity. A strategy to address the problem has been developing in recent years, based on two ideas. The first is to define $n$-point functions over a background by storing the information about the background in the boundary state [83]. In covariant loop gravity [30, 84], this technique yields a definite expression for the theory's $n$-point functions. The second is to explore the expansion of this expression order by order in the number of interaction vertices [85]. Although perhaps counter-intuitive, this expansion has proven effective in certain regimes; for details see [86, 87]. In particular, the low-energy limit of the two-point function (the "graviton propagator") obtained in this way from the improved-Barrett-Crane spin foam dynamics [43, 88–92] (sometime denoted the EPRL/FK model) correctly matches the graviton propagator of pure gravity in a transverse radial gauge (harmonic gauge) [93, 94]. This result has been possible thanks to the introduction of the coherent intertwiner basis [95] and the asymptotic analysis of vertex amplitude [66, 74].

In this chapter we present the computation of the three-point function from the spin foam gravity. As in [94], we work in the Euclidean regime and with the Barbero-Immirzi parameter $0 < \gamma < 1$ where the amplitude defined in [91] and that defined in

[92] coincide.

Our main result is the following. We consider the limit, introduced in [75, 94], where the Barbero-Immirzi parameter is taken to zero $\gamma \to 0$, and the spin of the boundary state is taken to infinity $j \to \infty$, keeping the size of the quantum geometry $A \sim \gamma j$ finite and fixed. This limit corresponds to neglecting Planck scale discreteness effects, at large finite distances. In this limit, the three-point function we obtain exactly matches the one obtained from Regge calculus [96].

This implies that the spin foam dynamics is consistent with a discretization of general relativity, not just in the quadratic approximation, but also to the first order in the interaction terms. The same semiclassical limit is considered in detail recently [75] where they showed that in this regime the partition function for a 2-complex takes the form of a path integral over continuous Regge metrics.

The relation between the Regge and Loop three-point function and the three-point function of the weak field perturbation expansion of general relativity around flat space, on the other hand, remains elusive. We compute explicitly the perturbative three-point function in position space in the transverse gauge (harmonic gauge), and we discuss the technical difficulty of comparing this with the Regge/Loop one.

## 4.2 Three-point function in loop gravity

In this section we compute the three-point function of the spin foam amplitude in loop quantum gravity at first order in the vertex expansion. We follow closely the techniques developed for the two-point function in [86, 94] and the calculation of the three-point function for the old Barrett-Crane model in [97]. For previous work in this direction, see also [93, 98, 99].

### 4.2.1 Boundary formalism

The well known difficulty of defining $n$-point functions in a general covariant quantum field theory can be illustrated by the following (naive) argument. If the action $S[g]$ and the measure are invariant under coordinate transformations, then

$$W(x_1, \cdots, x_N) \sim \int \mathcal{D}g \; g(x_1) \cdots g(x_N) \; \mathrm{e}^{\mathrm{i}S[g]} \tag{4.1}$$

is formally independent from $x_n$ (as long as the $x_n$ are distinct), because a change in $x_n$ can be absorbed into a change of coordinates that leaves the integral invariant.

This difficulty is circumvented in the weak field approximation as follows. If we want to study the theory around flat space, we have to impose boundary conditions on Eq.(4.1) demanding that $g$ goes to flat space at infinity. With this choice, the classical solution that dominates the path integral in the weak field limit is flat spacetime. In flat spacetime, we can choose preferred Cartesian coordinates $x$, and write the field insertions in terms of *these* preferred coordinates. Then Eq.(4.1) is well defined: the coordinates $x_n$ are not generally covariant coordinates, but rather Minkowski coordinates giving physical distances and physical time intervals in the background metric picked out by the boundary conditions of the field at infinity. This is the way $n$-point functions are defined for perturbative general relativity. In the full non-perturbative theory, on the other hand, this strategy is not viable, because the integral Eq.(4.1) has formally to be taken over arbitrary geometries, where the notion of preferred Cartesian coordinate loses meaning.

The idea for solving this difficulty was introduced in [83] and is explained in detail in [86]. We give here a short account of this formalism, but we urge the reader to look at the original references for a detailed explanation of the approach. Let us begin by picking a surface $\Sigma$ in flat spacetime, bounding a compact region $\mathcal{R}$, and approximate Eq.(4.1) by replacing $S[g]$ outside $\mathcal{R}$ with the linearized action. Then split Eq.(4.1) into three integrals: the integral on the field variables in $\mathcal{R}$, outside $\mathcal{R}$, and on $\Sigma$. Let $\gamma$ be the value of the field on $\Sigma$. Let $W_\Sigma[\gamma]$ be the result of the internal integration, at fixed value $\gamma$ of the field on $\Sigma$

$$W_\Sigma[\gamma] = \int_{g|_\Sigma=\gamma} \mathcal{D}g \quad \mathrm{e}^{\mathrm{i}S[g]} \ . \tag{4.2}$$

Let $\Psi_\Sigma[\gamma]$ be the result of the outside integral. Then we can write

$$\begin{aligned} W(x_1,...x_N) &\sim \int \mathcal{D}\gamma \ \ W_\Sigma[\gamma]\ \gamma(x_1)...\gamma(x_N)\Psi_\Sigma[\gamma] \\ &\equiv \langle W_\Sigma|\gamma(x_1)...\gamma(x_N)|\Psi_\Sigma\rangle \end{aligned} \tag{4.3}$$

Now observe first that because of the (assumed) diff-invariance of measure and action, $W_\Sigma[\gamma]$ is in fact independent from $\Sigma$. That is $W_\Sigma = W$. Second, since the external integral is that of a free theory, $\Psi_\Sigma[\gamma]$, will be the vacuum state of the free theory on the surface $\Sigma$. This can be shown to be a Gaussian semiclassical state peaked on the intrinsic and extrinsic geometry of $\Sigma$. Inserting the proper normalization we write

$$W(x_1, ...x_N) = \langle\gamma(x_1)...\gamma(x_N)\rangle = \frac{\langle W|\gamma(x_1)...\gamma(x_N)|\Psi_\Sigma\rangle}{\langle W|\Psi_\Sigma\rangle} \tag{4.4}$$

where $W$ is the formal functional integral on a compact region, and $\Psi_\Sigma$ is a semiclassical state peaked on a certain intrinsic and extrinsic geometry. This is the “boundary formalism". For a strictly related approach, see also [100, 101]. The quantities appearing in the formal expression Eq.(4.4) are well defined in loop quantum gravity and this expression can be taken as the starting point for computing $n$-point functions from the background independent theory.

### 4.2.2 The theory

The definition of the non perturbative quantum gravity theory we use is given for instance in [30]. The Hilbert space of the theory is spanned by spin network states $|\Gamma, \psi\rangle$, where $\Gamma$ is a graph with $L$ links $l$ and $N$ nodes $n$ and $\psi$ is in $\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N]$. A convenient basis in $\mathcal{H}_\Gamma$ is given by the coherent states $|j, \vec{n}\rangle$ which are the gauge invariant projections of $SU(2)$ Bloch coherent states [43]. These are labeled by a spin $j_l$ per each link of the graph, and a unit-norm 3-vector $\vec{n}_{nl}$ for each couple node-link of the graph. The dynamics of the theory is determined by the amplitude $W$ defined as a sum over two-complexes, or, equivalently [102], as the limit for $\sigma\tilde{\omega}\infty$ over the two-complexes $\sigma$ bounded by $\Gamma$, of the amplitude (we follow here [103] for the notation)

$$\langle W_\sigma|\Gamma, j, n\rangle = \sum_{j_f} \int dg_{ve} \int d\vec{n}_{ef} \prod_f d_{j_f} \operatorname{Tr}\Big[\prod_{e\in\partial f} P_{ef}\Big] \tag{4.5}$$

where $e \in \partial f$ is the ordered sequence of the oriented edges around the face $f$ and

$$P_{ef} = g_{s_e e} Y|j_f, \vec{n}_{ef}\rangle\langle j_f, \vec{n}_{ef}|Y^\dagger g^{-1}_{t_e e}. \tag{4.6}$$

for an internal edge $e$. For an external edge $e$, namely an edge hitting the boundary $\Gamma$ of $\sigma$,

$$P_{ef} = \langle j_l, \vec{n}_{nl}|Y^\dagger g^{-1}_{t_e e}, \quad \text{or} \quad P_{ef} = g_{s_e e} Y|j_l, \vec{n}_{nl}\rangle \tag{4.7}$$

according to whether the orientation of the edge is incoming or outgoing. Here $l$ is the link bounding the face $f$ and $n$ is the node bounding the edge $e$. In all these formulas, the notation $g$ stands for the matrix elements of the group element $g$ in the appropriate representation.

Here we deal with the Euclidean theory. Then $g_{ev} = (g^+_{ev}, g^-_{ev}) \in Spin(4) \sim SU(2)\times SU(2)$ and $Y$ maps the $SU(2)$ representations $j$ of into the highest weight $SU(2)$ irreducible of the $SO(4)$ representation $(j^+, j^-)$, where $j^\pm = \frac{1}{2}(1 \pm \gamma)j$. The matrix elements of $Y$ are the standard Clebsch-Gordan coefficients.

The amplitude can be written in the form of a path integral by defining the action

$$S = \sum_f S_f = \sum_f \ln \mathrm{Tr}\Big[\prod_{e\in f} P_{ef}\Big]. \tag{4.8}$$

Then

$$\langle W_\sigma|\Gamma, j, n\rangle = \sum_{j_f} \mu \int dg_{ve} \int d\vec{n}_{ef}\; e^S, \tag{4.9}$$

where $\mu = \sum_f d_j$. This is the form which is suitable for the asymptotic expansion that we use below.

Since the coherent states factorize under the Clebsch-Gordan decomposition, and since the scalar product of coherent states in the representation $j$ is the $j$'s power of that in the fundamental representation, we obtain $S = S^+ + S^-$ with

$$S^\pm = \sum_{vf} 2j^\pm_f \ln\langle\vec{n}_{ef}|(g^\pm_{ve})^{-1} g^\pm_{ve'}|\vec{n}_{e'f}\rangle \tag{4.10}$$

where $e$ and $e'$ are the two edges bounding $f$ and $v$.

The last ingredient we need are the gravitational field operators $\gamma(x)$ that enter in Eq.(4.4). The gravitational field operator that corresponds to the metric is expressed in loop quantum gravity by the Penrose operator [30]

$$G^{ab}_l = E^a_l \cdot E^b_l, \tag{4.11}$$

where $E^a_l$ is the left invariant vector field acting on the $h_{la}$ variable of the state vector, namely the $SU(2)$ group element associated to the link $a$ bounded by the node $l$. The key technical observation of [94] is that

$$\Big\langle W\,\Big|G^{ab}_l\Big|\,\Gamma, j, n\Big\rangle = \sum_{j_f} \mu \int dg_{ve} \int dn_{ef}\; q^{ab}_l\; \mathrm{e}^S \tag{4.12}$$

where $q^{ab}_l = A^{la} \cdot A^{lb}$, and $A^{la}_i = A^{la+}_i + A^{la-}_i$,

$$A^{la\pm}_i = \gamma j^\pm_{la} \frac{\langle -\vec{n}_{al}|(g^\pm_a)^{-1} g^\pm_l \sigma^i|\vec{n}_{la}\rangle}{\langle -\vec{n}_{al}|(g^\pm_a)^{-1} g^\pm_l|\vec{n}_{la}\rangle}. \tag{4.13}$$

This is the insertion that we consider below.

### 4.2.3 Vertex expansion

The second idea for computing $n$-point functions is the vertex expansion [85]. This is the idea of studying the approximation to Eq.(4.4) given by the lowest order in the $\sigma\widetilde{\omega}\infty$ limit, namely using small graphs and small two-complexes. Here we only look at the first nontrivial term. That is, we take a minimal two-complex, formed by a single vertex. We consider for simplicity the theory restricted to five-valent vertices and four-valent edges. Then the lowest order is given by a two-complex formed by a single five-valent vertex bounded by the complete graph with 5 nodes $\Gamma_5$. Labeling the nodes with indices $a, b, ... = 1, ..., 5$ the amplitude of this two-complex for the boundary state $|\Gamma_5, j_{ab}, \vec{n}_{ab}\rangle$ (here $j_{ab} = j_{ba}$, but $\vec{n}_{ab} \neq \vec{n}_{ba}$) reads simply

$$\langle W|\Gamma_5, j_{ab}, \vec{n}_{ab}\rangle = \mu(j) \int_{SU(2)^{10}} dg_a^{\pm}\ e^{\sum_{ab} S_{ab}} \tag{4.14}$$

with

$$S_{ab} = \sum_{\pm}\ 2j_{ab}^{\pm} \ln\langle -\vec{n}_{ab}|(g_a^{\pm})^{-1} g_b^{\pm}|\vec{n}_{ba}\rangle \tag{4.15}$$

The $\mu(j)$ term comes from the face amplitude and the measure (and cancels at the tree-level [97]).

The vertex expansion has appeared counterintuitive to some, on the base of the intuition that the large distance limit of quantum gravity could be reached *only* by states defined on very *fine* graphs, and with very *fine* two-complexes. We are not persuaded by this intuition (in spite of the fact that one of the authors is quite responsible for propagandizing it [104–106]) for a number of reasons. The main one is the following. It has been shown that under appropriate conditions Eq.(4.9) can approximates a Regge path integral for large spins [103, 107, 108]. Regge calculus is an approximation to general relativity that is good up to order $\mathcal{O}(l^2/\rho^2)$, where $l$ is the typical Regge discretization length and $\rho$ is the typical curvature radius. This implies that Regge theory on a coarse lattice is good as long as we look at small curvatures scale. In particular, it is obviously perfectly good on flat space, where in fact it is exact, because the Regge simplices are themselves flat, and is good as long as we look at weak field perturbations of long wavelength. This is precisely the limit in which we want to study the theory here. In this limit, it is therefore reasonable to explore whether the vertex expansion give any sensible result.

Reducing the theory to a single vertex is a drastic simplification of the field theory, which reduce the calculation to one for a system with a finite number of degrees of freedom. Is this reasonable? The answer is in noticing that the same drastic simplification occurs in the analog calculation in QED: at the lowest order, an $n$-point function involves only the Hilbert space of a finite number of particles, which are described by a finite number of degrees of freedom in the classical theory. The genuine field theoretical aspects of the problem, such as renormalization, do not show up at the lowest order, of course.

If we regard the calculation from the perspective of the triangulation dual to the two-complex, what is being considered is a region of spacetime with the geometry of a 4-simplex. In the approximation considered the region is flat, but this does not mean that there are no degrees of freedom. In fact, the Hamilton function of general relativity is a nontrivial function of the intrinsic geometry of the boundary, whose variation gives equations that determine the extrinsic geometry as a function of the intrinsic geometry. This relation captures a small finite-dimensional sector of the Einstein-equations dynamics (for a simple example of this, see [109]). This is precisely the component of the dynamics of general relativity captured in this limit. The three-point function in this large wavelength limit describes the correlations between the fluctuations of the boundary geometry of the 4-simplex, governed by the quantum version of this restricted Einstein dynamics.

Let us illustrate this dynamics a bit more in detail, both in second order (metric) and first order (tetrad/connection) variables. In metric variables, the intrinsic geometry of a boundary of a four-simplex (formed by glued flat tetrahedra) is uniquely determined by the 10 areas $A_{ab}$ of their faces. The extrinsic geometry of the boundary four-simplex is determined by the 10 angles $\Phi_{ab}$ between the 4d normals to the tetrahedra. The Einstein equations reduce in the case of a single simplex to the requirement that this is flat. If the four simplex is flat, then the 10 angles $\Phi_{ab}$ are well-defined functions

$$\Phi_{ab} = \Phi_{ab}(A_{ab}) \tag{4.16}$$

of the 10 areas $A_{ab}$ (for comparison, if the four-simplex has constant curvature because of a cosmological constant, then the same $A_{ab}$'s determine different $\Phi_{ab}$'s). This dependence captures the restriction of the Einstein equations to a single simplex. In first order variables, the situation is more complicated. The variables $g$, $j$ and $\vec{n}$ in Eq.(4.8) can be viewed as the discretized version of the connection and the tetrad. The vanishing-torsion equation of the first order formalism, which relates the connection to the tetrad, becomes in the discrete formalism a gluing condition between normals to the faces parallel transported by the group elements.

### 4.2.4 Boundary vacuum state

Following the general strategy described above, we need a boundary state peaked on the intrinsic as well as on the extrinsic geometry. This state cannot be the state $|\Gamma_5, j_{ab}, \vec{n}_{ab}\rangle$ which is an eigenvalue of boundary areas, and therefore is maximally spread in the extrinsic curvature, namely in the 4d dihedral angle between two boundary tetrahedra $\Phi_{ab}$ [110]. Rather, we need a state which is also smeared over spins [111–113].

Following [113], we choose here a boundary state peaked on the intrinsic and extrinsic geometry of a *regular* 4-simplex, and defined as follow. The geometry of a *flat* 4-simplex is uniquely determined by the 10 areas $A_{ab}$ of its 10 faces. Let then $\vec{n}_{ab}(A_{ab})$ be the 20 normals determined up to arbitrary $SO(3)$ rotations of each quadruplet $\vec{n}_{ab_1}, ..., \vec{n}_{ab_4}$ by these areas. By this we mean the following. The flat 4-simplex determined by the given areas is bounded by five tetrahedra. For each such tetrahedron, the four normals to its four faces in the 3-space determined by the tetrahedron determine, up to rotations) the four unit vectors $\vec{n}_{ab_1}, ..., \vec{n}_{ab_4}$. Using this, we define the boundary state as

$$|\Psi_\Sigma\rangle = |\Psi_{j_0}\rangle = \sum_{j_{ab}} c_{j_0}(j_{ab})|\Gamma, j_{ab}, n_{ab}(j_{ab})\rangle \tag{4.17}$$

where the coefficients $c_{j_0}(j)$ in the large $j$ limit are given by [113]

$$c_{j_0}(j_{ab}) = \frac{1}{N} e^{-\sum_{(ab),(cd)} \gamma\alpha^{(ab)(cd)} \frac{j_{ab}-j_0}{\sqrt{j_0}} \frac{j_{cd}-j_0}{\sqrt{j_0}} - \mathrm{i}\sum_{(ab)} \Phi_0 \gamma j_{ab}} \tag{4.18}$$

The coefficients are also given in [85, 86]. $\alpha^{(ab)(cd)}$ is a $10 \times 10$ matrix that has the symmetries of the 4-simplex, that is, it can be written in the form $\alpha^{(ab)(cd)} = \sum_k \alpha_k P_k^{(ab)(cd)}$ where

$$\begin{aligned}
P_0^{(ab)(cd)} &= 1 \quad \text{if} \quad (ab) = (cd) \quad \text{and 0 otherwise,} \\
P_1^{(ab)(cd)} &= 1 \quad \text{if} \quad \{a = c, b \neq d\} \;\; \text{or a permutation,} \\
&\qquad\qquad\qquad\qquad\quad \text{and 0 otherwise,} \\
P_2^{(ab)(cd)} &= 1 \quad \text{if} \quad (ab) \neq (cd) \quad \text{and 0 otherwise.}
\end{aligned}$$

$\Phi_0$ is the background value of the 4d dihedral angles which give the extrinsic curvature of the boundary. $j_0$ is the background value of all the areas. The state is peaked on the areas $j_{ab} = j_0$, which determine a regular 4-simplex. The dihedral angles of a flat tetrahedron is $\Phi_0 = \arccos(-\frac{1}{4})$, and we fix $\Phi_0$ to this value. As a consequence $|\Psi_{j_0}\rangle$ is a semiclassical *physical* state, namely it is peaked on values of intrinsic and extrinsic geometry that satisfy the (Hamilton) equations of motion (4.16) of the theory. See [85, 86, 94, 109] for more details.

### 4.2.5 Three-point function

Let us now choose the operator insertion. We are interested in the connected component of the quantity

$$\tilde{G}^{abcdef}_{lmn} = \langle G^{ab}_l \; G^{cd}_m \; G^{ef}_n \rangle, \tag{4.19}$$

where $G^{ab}_l$ is the Penrose operator associated to the node $l$ of $\Gamma_5$ and the two links of this node going from $l$ to $a$ and from $l$ to $b$ respectively. The connected component is

$$\begin{aligned} G^{abcdef}_{lmn} = & \langle G^{ab}_l \; G^{cd}_m \; G^{ef}_n \rangle + 2\langle G^{ab}_l \rangle \langle G^{cd}_m \rangle \langle G^{ef}_n \rangle \\ & - \langle G^{ab}_l \rangle \langle G^{cd}_m \; G^{ef}_n \rangle - \langle G^{ef}_n \rangle \langle G^{ab}_l G^{cd}_m \rangle \\ & - \langle G^{cd}_m \rangle \langle G^{ab}_l G^{ef}_n \rangle \end{aligned} \tag{4.20}$$

We begin by studying the full three-point function Eq.(4.19), before subtracting the disconnected components. From Eq.(4.4) and Eq.(4.18), and simplifying a bit the notation in a self explicatory way, this is

$$\tilde{G}^{abcdef}_{lmn} = \frac{\sum_j c\,(j) \left\langle W \left| G^{ab}_l \, G^{cd}_m \, G^{ef}_n \right| \Gamma_5, j, n \right\rangle}{\sum_j c\,(j) \, \langle W | \Gamma, j, n \rangle} \tag{4.21}$$

Using Eq.(4.12), this gives

$$\tilde{G}^{abcdef}_{lmn} = \frac{\sum_j c\,(j) \int dg^{\pm}_a q^{ab}_l \, q^{cd}_m \, q^{ef}_n \, e^S}{\sum_j c\,(j) \int dg^{\pm}_a e^S} \tag{4.22}$$

where the sum over spins is only given by the boundary state, since there are no internal faces.

Define the total action as $S_{\text{tot}} = \ln c(j) + S$. Because we want to get the large $j$ limit of the spin foam model, we rescale the spins $j_{ab}$ and $j_0$. Then the action goes to $S_{\text{tot}} \to \lambda S_{\text{tot}}$ and also $q^{ab}_l \to \lambda^2 q^{ab}_n$. In large $\lambda$ limit, the sum over $j$ can be approximated to the integrals over $j$

$$\sum_j \mu \int \mathrm{d}g_a^\pm \, q_l^{ab} \, \mathrm{e}^{\lambda S_{\mathrm{tot}}} \approx \int \mathrm{d}j \mathrm{d}g_a^\pm \mu \, q_l^{ab} \, \mathrm{e}^{\lambda S_{\mathrm{tot}}} \tag{4.23}$$

where $\mu$ is the product of the face amplitudes. Thus (dropping the suffix *tot* from now on)

$$\tilde{G}_{lmn}^{abcdef} = \lambda^6 \; \frac{\int \mathrm{d}j \mathrm{d}g_a^\pm \mu q_l^{ab} q_m^{cd} q_n^{ef} \mathrm{e}^{\lambda S}}{\int \mathrm{d}j \mathrm{d}g_a^\pm \mu \mathrm{e}^{\lambda S}} \tag{4.24}$$

Action, measure and insertions are invariant under a $SO(4)$ symmetry, therefore only four of the five $\mathrm{d}g_a^\pm$ integrals are independent. We can fix the gauge that one of the group element $g^\pm = 1$, and the integral reduced to $\mathrm{d}g = \prod_{a=1}^4 \mathrm{d}g_a^+ \mathrm{d}g_a^-$. This gives the expression

$$\tilde{G}_{lmn}^{abcdef} = \lambda^6 \; \frac{\int \mathrm{d}j \mathrm{d}g \; \mu q_l^{ab} q_m^{cd} q_n^{ef} \mathrm{e}^{\lambda S}}{\int \mathrm{d}j \mathrm{d}g \; \mu \mathrm{e}^{\lambda S}} \tag{4.25}$$

We simplify the notation by writing this in the simple form

$$\tilde{G} = \lambda^6 \; \frac{\int \mathrm{d}j \mathrm{d}g \; \mu \, l \, m \, n \, \mathrm{e}^{\lambda S}}{\int \mathrm{d}j \mathrm{d}g \; \mu \mathrm{e}^{\lambda S}} \equiv \langle lmn \rangle \tag{4.26}$$

where $l = q_l^{ab}, m = q_m^{cd}, n = q_n^{ef}$ are functions of $j$ and $g$. The connected component reads then

$$G = \langle lmn \rangle + 2\langle l \rangle \langle m \rangle \langle n \rangle - \langle lm \rangle \langle n \rangle - \langle nl \rangle \langle m \rangle - \langle mn \rangle \langle l \rangle \tag{4.27}$$

which is the point of departure for the saddle point expansion.

### 4.2.6 Saddle point expansion

To study the asymptotic behavior of Eq.(4.26), we use the saddle point expansion[97, 114, 115]. For this, we need the stationary point of the total action $S_{\mathrm{tot}} = \ln c(j) + S$. Here we briefly review the works in [66] and [94]. They discuss the behavior of the critical point and stationary point of $S = S^+ + S^-$ and $S_{\mathrm{tot}}$. We invite readers to read their articles for full detail discussion.

The critical point and stationary point of $\mathrm{Re}(S)$ coincide with each other when $\gamma < 1$. For the real part of the action $S$, the critical points are the group element $\bar{g}^\pm$

satisfy the gluing condition

$$R_a^{\pm} n_{ab} = -R_b^{\pm} n_{ba} \tag{4.28}$$

where $R_a^{\pm} = R(\bar{g}_a^{\pm})$ is the spin-1 irrep. of SU(2). This means that at the critical point the geometry of spacetime goes to a classical one in which all tetrahedra glue perfectly. There are 4 classes of critical points satisfy the condition (4.28). At the critical points of $\mathrm{Re}(S)$, the action $S$ can be written as $S = \mathrm{i}A$, where $A$ is a real function and reduces to Regge like actions. See [66] and [94]. A unique class of critical points is then selected by the stationary point behavior of $S_{\mathrm{tot}}$.

The stationary points of $\mathrm{Re}(S)$ are the critical points of $\mathrm{Re}(S)$, because of the closure constraint, which is satisfied by the boundary state for large $j_0$. We are interested in the stationary points of $S_{\mathrm{tot}}$ are not just with respect to the group variables, but also with respect to the spin $j$ variables. The stationary point $j_{ab} = j_0$ also selects the class of group stationary point. This is because at the stationary point, $S$ must satisfy

$$-\mathrm{i}\gamma\Phi_{ab} + \frac{\partial S(g_0)}{\partial j_{ab}} = 0 \tag{4.29}$$

Therefore it means that only when $S(g_0) = \mathrm{i}S_{\mathrm{Regge}}$ (with a definite sign) this condition can be satisfied. This condition picks the unique class of critical points $g_0^{\pm}$ of $\mathrm{Re}(S)$, which makes $S(g_0) = \mathrm{i}S_{\mathrm{Regge}}$.

We are thus interested in the saddle point expansion of the integrals in Eq.(4.26) around the stationary points $(j_0, g_0^{\pm})$ described above. According to the general theory, the integral

$$F(\lambda) = \int \mathrm{d}x f(x) \mathrm{e}^{\lambda S(x)} \tag{4.30}$$

can be expand for large $\lambda$ around the stationary points as follows

$$F(\lambda) = C(x_0)\left(f(x_0) + \frac{1}{\lambda}\left(\frac{1}{2} f_{ij}(x_0) J^{ij} + D\right)\right) + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \tag{4.31}$$

where $x_0$ is the stationary point, $f_{ij}$ is the Jacobian matrix of $f$, and $J = H^{-1} = (S''(x_0))^{-1}$ is the inverse of the Jacobian matrix of the action $S$. A straightforward application of this formula to Eq.(4.27) shows that

$$G = 0 + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \tag{4.32}$$

This in fact is not surprising, because we are computing a three point function, and this cannot be captured only by the second order of the saddle point expansion. The second order of the saddle point expansion sees only the second derivatives of the action, while the connected component of the three point function depends on the third derivatives of the action. In fact, the 3rd derivative of the action term can be identified with a Feynman vertex, the inverse of the second derivative as the propagator and the insertions as the external legs of a Feynman diagram. Then It is clear that to second order there is no connected component.

Therefore we need the next order of the saddle point expansion. From Eq.(4.30), this is given by

$$F(\lambda) = C(x_0)\left(f(x_0) + \frac{F_1}{\lambda} + \frac{F_2}{\lambda^2}\right) + \mathcal{O}\left(\frac{1}{\lambda^3}\right) \tag{4.33}$$

where

$$F_1 = -\frac{1}{2}f_{ij}J^{ij} + \frac{1}{2}f_iJ^{ij}J^{kl}R_{jkl} - \frac{5}{24}fJ^{il}J^{jm}J^{kn}R_{ijk}R_{lmn} + \frac{1}{8}fJ^{ik}J^{jl}R_{ijkl} \tag{4.34}$$

and

$$\begin{aligned} F_2 = {} & \frac{1}{8}f_{ijkl}J^{ij}J^{kl} - \frac{5}{12}f_{ijk}J^{il}J^{jm}J^{kn}R_{lmn} \\ & - \frac{5}{16}f_{ij}J^{ik}J^{jl}J^{mn}R_{klmn} \\ & + \frac{35}{48}f_{ij}J^{im}J^{jn}J^{ko}J^{lp}R_{mko}R_{nlp} + \cdots \end{aligned} \tag{4.35}$$

Here $R(x) = S(x) - S - \frac{1}{2}H_{ij}(x-x_0)^i(x-\bar{x})^j$, all functions are computed in $x_0$, the stationary point of $S(x)$ and the indices indicate derivatives. In the last two equations we have left understood some symmetrization. For instance the third term of the right hand side of Eq.(4.34) should read .

$$\frac{5}{48}f\left(J^{il}J^{jm}J^{kn} + J^{il}J^{jm}J^{kn}\right)R_{ijk}R_{lmn} \tag{4.36}$$

and so on.

Using this, and recalling that here $f(x) = \mu(x)l(x)m(x)n(x)$, we obtain, up to order $\mathcal{O}(\frac{1}{\lambda^2})$,

$$G^{abcdef}_{lmn} = \lambda^4\Big(-R_{ijk}l_lm_mn_nJ^{il}J^{jm}J^{kn} + (l_{ij}m_kn_l + l_km_{ij}n_l + l_km_ln_{ij})J^{ik}J^{jl}\Big) \tag{4.37}$$

The first term on the right hand side resembles the one vertex diagram with three legs. The second term resembles a 4-point function in which 2 points are identified.

### 4.2.7 Analytical expression

Eq.(4.37) indicates that we need to get the second and third derivatives of the total action, and the first and second derivative of the insertions. Here we compute these terms.

We use Euler angles to parameterize the $SU(2)$ group elements $g_0^{\pm}$ around the stationary point

$$R_a^{\pm} = \mathrm{e}^{\mathrm{i}\theta_i^{\pm} J_i} R_{0a}^{\pm} \tag{4.38}$$

where $i = 1, 2, 3$, $\theta_i$ are Euler angles, $J_i$ are the generators of SU(2), $R_a$ stands for arbitrary irrep. of SU(2). There are 34 independent variables, 10 areas $j_{ab}$ of triangles in the 4 simplex, 24 group element parameters in which 12 for $g^+$ and 12 for $g^-$. Here we give only some steps to get to the result. The whole results can be found in the Appendix. The second order derivative of the total action gives

$$\left.\frac{\partial^2 S_{\text{tot}}}{\partial j_{ab}\partial j_{cd}}\right|_{\theta=0} = -\frac{\gamma\alpha^{(ab)(cd)}}{\sqrt{j_{0ab}}\sqrt{j_{0cd}}} + \mathrm{i}\frac{\partial^2 S_{\text{Regge}}}{\partial j_{ab}\partial j_{cd}} \tag{4.39}$$

$$\left.\frac{\partial^2 S}{\partial\theta_j^{a\pm}\partial\theta_i^{a\pm}}\right|_{\theta=0} = -\frac{\gamma^{\pm}}{2}\sum_{(b\neq a)} j_{ab}\left(\delta_{ij} - \left(n_{ab}^{\pm}\right)_i\left(n_{ab}^{\pm}\right)_j\right) \tag{4.40}$$

$$\left.\frac{\partial^2 S}{\partial\theta_j^{b\pm}\partial\theta_i^{a\pm}}\right|_{\theta=0} = \frac{\gamma^{\pm} j_{ab}}{2}(\delta_{ij} - \left(n_{ab}^{\pm}\right)_i\left(n_{ab}^{\pm}\right)_j - i\varepsilon_{ijk}\left(n_{ab}^{\pm}\right)_k) \tag{4.41}$$

The third order derivative of the total action gives

$$\left.\frac{\partial^3 S_{\text{tot}}}{\partial j_{ab}\partial j_{cd}\partial j_{ef}}\right|_{\theta=0} = \mathrm{i}\frac{\partial^3 S_{\text{Regge}}}{\partial j_{ab}\partial j_{cd}\partial j_{ef}} \tag{4.42}$$

$$\begin{aligned}\left.\frac{\partial^3 S}{\partial\theta_k^{a\pm}\partial\theta_j^{a\pm}\partial\theta_i^{a\pm}}\right|_{\theta=0} = & \sum_{b\neq a}\frac{1}{6} i\gamma^{\pm} j_{ab}(\delta_{jk}\left(n_{ab}^{\pm}\right)_i + \delta_{ki}\left(n_{ab}^{\pm}\right)_j + \delta_{ij}\left(n_{ab}^{\pm}\right)_k \\ & -3\left(n_{ab}^{\pm}\right)_i\left(n_{ab}^{\pm}\right)_j\left(n_{ab}^{\pm}\right)_k)\end{aligned} \tag{4.43}$$

$$\begin{aligned}\frac{\partial^3 S}{\partial\theta_k^{b\pm}\partial\theta_j^{a\pm}\partial\theta_i^{a\pm}}|_{\theta=0} = & -\frac{1}{4} i\gamma^{\pm} j_{ab}(\delta_{jk}\left(n_{ab}^{\pm}\right)_i + \delta_{ki}\left(n_{ab}^{\pm}\right)_j - 2\left(n_{ab}^{\pm}\right)_i\left(n_{ab}^{\pm}\right)_j\left(n_{ab}^{\pm}\right)_k \\ & + i\left(\varepsilon_{kil}\left(n_{ab}^{\pm}\right)_j + \varepsilon_{kjl}\left(n_{ab}^{\pm}\right)_i\right)\left(n_{ab}^{\pm}\right)_l)\end{aligned} \tag{4.44}$$

$$\dots$$

The first derivatives of the insertions

$$\begin{aligned}
\frac{\partial q_c^{ab}}{\partial j_{ef}} &= \gamma^2 \frac{\partial (j_{ca} j_{cb} n_{ca} \cdot n_{cb})}{\partial j_{ef}} = \gamma^2 \frac{\partial (j_{ca} j_{cb} \cos \Theta_{cab})}{\partial j_{ef}} && (4.45)\\
\frac{\partial q_n^{ab}}{\partial \theta_i^{a\pm}}|_{\theta_i^{a\pm}=0} &= -\frac{1}{2} i \gamma^2 \gamma^{\pm} j_{na} j_{nb} (\left(n_{nb}^{\pm}\right)_i - \left(n_{na}^{\pm}\right)_i (n_{nb})_j (n_{na})_j \\
&\quad + i \varepsilon_{ijk} \left(n_{nb}^{\pm}\right)_j \left(n_{na}^{\pm}\right)_k) && (4.46)\\
\frac{\partial q_n^{ab}}{\partial \theta_i^{n\pm}}|_{\theta_i^{n\pm}=0} &= \frac{1}{2} i \gamma^2 \gamma^{\pm} j_{na} j_{nb} \left(\left(n_{na}^{\pm}\right)_i + \left(n_{nb}^{\pm}\right)_i\right) (1 - (n_{na})_j (n_{nb})_j) && (4.47)
\end{aligned}$$

The second derivatives of the insertions

$$\begin{aligned}
\frac{\partial^2 q_n^{ab}}{\partial \theta_j^{a\pm} \partial \theta_i^{a\pm}}|_{\theta=0} &= \frac{1}{4} \gamma^2 \gamma^{\pm} j_{na} j_{nb} (\left(n_{nb}^{\pm}\right)_j \left(n_{na}^{\pm}\right)_i + \left(n_{nb}^{\pm}\right)_i \left(n_{na}^{\pm}\right)_j \\
&\quad -2 (n_{nb})_r (n_{na})_r \left(n_{na}^{\pm}\right)_i \left(n_{na}^{\pm}\right)_j && (4.48)\\
&\quad -i \left(n_{nb}^{\pm}\right)_k \left(n_{na}^{\pm}\right)_m \left(\varepsilon_{kmj} \left(n_{na}^{\pm}\right)_i + \varepsilon_{kmi} \left(n_{na}^{\pm}\right)_j\right)) \\
&\cdots
\end{aligned}$$

### 4.2.8 Numerical results

The derivatives over the spin $j$s can be obtained numerically. For simplicity, we only consider the situation where the boundary is a regular 4-simplex. For the total action $S$, the second derivatives

$$\begin{aligned}
\frac{\partial^2 S}{\partial j_{ab} \partial j_{ab}} &= -\frac{\gamma \alpha_0}{j_0} - \mathrm{i} \frac{\gamma}{j_0} \frac{9}{4} \sqrt{\frac{3}{5}} \\
\frac{\partial^2 S}{\partial j_{ac} \partial j_{ab}} &= -\frac{\gamma \alpha_1}{j_0} + \mathrm{i} \frac{\gamma}{j_0} \frac{8}{7} \sqrt{\frac{3}{5}} \\
\frac{\partial^2 S}{\partial j_{cd} \partial j_{ab}} &= -\frac{\gamma \alpha_2}{j_0} - \mathrm{i} \frac{\gamma}{j_0} \sqrt{\frac{3}{5}}
\end{aligned}$$

For the third derivatives, only seven of them are independent. They are

$$
\begin{aligned}
\frac{\partial^3 S}{\partial j_{ab}\partial j_{ab}\partial j_{ab}} &= -i\frac{\gamma}{j_0^2}\frac{189}{80}\sqrt{\frac{3}{5}}, \quad \frac{\partial^3 S}{\partial j_{ac}\partial j_{ab}\partial j_{ab}} = i\frac{\gamma}{j_0^2}\frac{347}{160}\sqrt{\frac{3}{5}} \\
\frac{\partial^3 S}{\partial j_{cd}\partial j_{ab}\partial j_{ab}} &= -i\frac{\gamma}{j_0^2}\frac{14}{5}\sqrt{\frac{3}{5}}, \quad \frac{\partial^3 S}{\partial j_{ad}\partial j_{ac}\partial j_{ab}} = -i\frac{\gamma}{j_0^2}\frac{453}{160}\sqrt{\frac{3}{5}} \\
\frac{\partial^3 S}{\partial j_{bc}\partial j_{ac}\partial j_{ab}} &= -i\frac{\gamma}{j_0^2}\frac{141}{20}\sqrt{\frac{3}{5}}, \quad \frac{\partial^3 S}{\partial j_{bd}\partial j_{ac}\partial j_{ab}} = i\frac{\gamma}{j_0^2}\frac{39}{20}\sqrt{\frac{3}{5}} \\
\frac{\partial^3 S}{\partial j_{ed}\partial j_{ac}\partial j_{ab}} &= -i\frac{\gamma}{j_0^2}\frac{3}{10}\sqrt{\frac{3}{5}}
\end{aligned}
$$

For the metric quantities $q_n^{ab}$, when $a \neq b$, we can find there are only five of them are independent. They are

$$
\begin{aligned}
&\frac{\partial q_c^{ab}}{\partial j_{ab}} = \frac{4}{3}\gamma^2 j_0, \quad \frac{\partial q_c^{ab}}{\partial j_{ac}} = -\frac{2}{3}\gamma^2 j_0 \\
&\frac{\partial q_c^{ab}}{\partial j_{ad}} = -\frac{2}{3}\gamma^2 j_0, \quad \frac{\partial q_c^{ab}}{\partial j_{cd}} = \frac{1}{3}\gamma^2 j_0 \\
&\frac{\partial q_c^{ab}}{\partial j_{de}} = \frac{4}{3}\gamma^2 j_0
\end{aligned}
$$

The second derivatives

$$
\begin{aligned}
\frac{\partial^2 q_c^{ab}}{\partial j_{gh}\partial j_{ef}} &= \gamma^2 \frac{\partial^2 (j_{ca} j_{cb} n_{ca}\cdot n_{cb})}{\partial j_{gh}\partial j_{ef}} \\
&= \gamma^2 \begin{pmatrix}
\frac{4}{3} & 1 & -2 & -2 & 1 & -2 & -2 & 1 & 1 & 4 \\
1 & -\frac{2}{3} & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \\
-2 & -\frac{1}{2} & -\frac{2}{3} & 4 & -\frac{1}{2} & 4 & -2 & -\frac{1}{2} & -\frac{1}{2} & -2 \\
-2 & -\frac{1}{2} & 4 & -\frac{2}{3} & -\frac{1}{2} & -2 & 4 & -\frac{1}{2} & -\frac{1}{2} & -2 \\
1 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \\
-2 & -\frac{1}{2} & 4 & -2 & -\frac{1}{2} & -\frac{2}{3} & 4 & -\frac{1}{2} & -\frac{1}{2} & -2 \\
-2 & -\frac{1}{2} & -2 & 4 & -\frac{1}{2} & 4 & -\frac{2}{3} & -\frac{1}{2} & -\frac{1}{2} & -2 \\
1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & -1 & 1 \\
1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{3} & 1 \\
4 & 1 & -2 & -2 & 1 & -2 & -2 & 1 & 1 & \frac{4}{3}
\end{pmatrix}
\end{aligned}
$$

Rows and columns are labeled by $j_{gh}$ and $j_{ef}$, respectively. The order is $\{j_{ab}, j_{ac}, j_{ad}, j_{ae}, j_{bc}, j_{bd}, j_{be}, j_{cd}, j_{ce}, j_{de}\}$.

When $a = b$, there is only one non-zero first and second derivatives. They are

$$\frac{\partial q_c^{aa}}{\partial j_{ca}} = 2\gamma^2 j_0, \qquad \frac{\partial^2 q_c^{aa}}{\partial j_{ca}\partial j_{ca}} = 2\gamma^2$$

Now, let us look at the dependence of these quantities from $\gamma$ and $j = j_0$. We obtain

$$\begin{aligned}
&\frac{\partial^3 S}{\partial j \partial j \partial j} \sim \frac{\gamma}{j^2}, \qquad \frac{\partial^2 S}{\partial j \partial j} \sim \frac{\gamma}{j}, \qquad \frac{\partial^3 S}{\partial\theta\partial\theta\partial\theta} \sim \gamma^{\pm} j, \\
&\frac{\partial^2 S}{\partial\theta\partial\theta} \sim \gamma^{\pm} j, \qquad \frac{\partial^3 S}{\partial j\partial\theta\partial\theta} \sim \gamma^{\pm}, \qquad \frac{\partial^2 q}{\partial j \partial j} \sim \gamma^2, \\
&\frac{\partial q}{\partial j} \sim \gamma^2 j, \qquad \frac{\partial^2 q}{\partial\theta\partial\theta} \sim \gamma^2\gamma^{\pm} j^2, \qquad \frac{\partial q}{\partial\theta} \sim \gamma^2\gamma^{\pm} j^2, \\
&\frac{\partial^2 q}{\partial j\partial\theta} \sim \gamma^2\gamma^{\pm} j.
\end{aligned}$$

For the 3-valent term, the scaling is

$$\frac{\partial^3 S}{\partial j \partial j \partial j}\left(\frac{\partial^2 S}{\partial j\partial j}\right)^{-1}\left(\frac{\partial^2 S}{\partial j\partial j}\right)^{-1}\left(\frac{\partial^2 S}{\partial j\partial j}\right)^{-1}\frac{\partial q}{\partial j}\frac{\partial q}{\partial j}\frac{\partial q}{\partial j} \sim \gamma^4 j^4 \tag{4.49}$$

and

$$\begin{aligned}
&\frac{\partial^3 S}{\partial\theta\partial\theta\partial\theta}\left(\frac{\partial^2 S}{\partial\theta\partial\theta}\right)^{-1}\left(\frac{\partial^2 S}{\partial\theta\partial\theta}\right)^{-1}\left(\frac{\partial^2 S}{\partial\theta\partial\theta}\right)^{-1}\frac{\partial q}{\partial\theta}\frac{\partial q}{\partial\theta}\frac{\partial q}{\partial\theta} \sim \gamma^{\pm}\gamma^6 j^4 \to \gamma^6 j^4|_{\gamma\to 0} \\
&\frac{\partial^3 S}{\partial j\partial\theta\partial\theta}\left(\frac{\partial^2 S}{\partial\theta\partial\theta}\right)^{-1}\left(\frac{\partial^2 S}{\partial\theta\partial\theta}\right)^{-1}\left(\frac{\partial^2 S}{\partial j\partial j}\right)^{-1}\frac{\partial q}{\partial\theta}\frac{\partial q}{\partial\theta}\frac{\partial q}{\partial j} \sim \gamma^{\pm}\gamma^5 j^4 \to \gamma^5 j^4|_{\gamma\to 0}
\end{aligned}$$

And for the "4"-point function terms,

$$\frac{\partial^2 q}{\partial j\partial j}\left(\frac{\partial^2 S}{\partial j\partial j}\right)^{-1}\left(\frac{\partial^2 S}{\partial j\partial j}\right)^{-1}\frac{\partial q}{\partial j}\frac{\partial q}{\partial j} \sim \gamma^4 j^4 \tag{4.50}$$

and

$$\begin{aligned}
&\frac{\partial^2 q}{\partial\theta\partial\theta}\left(\frac{\partial^2 S}{\partial\theta\partial\theta}\right)^{-1}\left(\frac{\partial^2 S}{\partial\theta\partial\theta}\right)^{-1}\frac{\partial q}{\partial\theta}\frac{\partial q}{\partial\theta} \sim \gamma^{\pm}\gamma^6 j^4 \to \gamma^6 j^4|_{\gamma\to 0} \\
&\frac{\partial^2 q}{\partial j\partial\theta}\left(\frac{\partial^2 S}{\partial\theta\partial\theta}\right)^{-1}\left(\frac{\partial^2 S}{\partial j\partial j}\right)^{-1}\frac{\partial q}{\partial\theta}\frac{\partial q}{\partial j} \sim \gamma^{\pm}\gamma^5 j^4 \to \gamma^5 j^4|_{\gamma\to 0}
\end{aligned}$$

Consider now the limit which introduced by Bianchi, Magliaro and Perini [94], i.e.

$\gamma \to 0$, $j \to \infty$, with fixed physical area $\gamma j = A$. Then the only term that survives are Eq.(4.49) and Eq.(4.50). These terms are precisely those appearing in the Regge calculus three-point function, given in [97].

Therefore, we can conclude that in the Bianchi-Magliaro-Perini limit the 3 point function of loop quantum gravity matches the Regge calculus one.

With an analogous "dimensional" analysis, we can check that for 4-point function and 5-point function the spin foam model give perturbative Regge calculus result in the same limit. For 4-point function, the Regge part has the scale of $\mathcal{O}(\gamma^5 j^5)$, others have the scale of $\mathcal{O}(\gamma^k j^5)$, $k > 5$. For 4-point function, it is the same. The scale of Regge part is $\mathcal{O}(\gamma^6 j^6)$.

It appears therefore that $\gamma$ scales the amplitude of the "un-gluing" fluctuation. It also measures the difference between area bivectors $A^{IJ}$ and group generators $J^{IJ}$. The $\gamma \to 0$ limit corresponds to $J^{IJ} = A^{IJ}$[91][116].

## 4.3 Three-point function in perturbative quantum gravity

In this section we give for completeness the analytic expression of the three-point function in position space, at tree level, in the harmonic gauge. We will briefly review the main definitions and notations on perturbative quantum general relativity, based on [114][117][118]. We only show the result in this note. More details are in the Appendix.

### 4.3.1 Definitions

perturbative quantum gravity describes the quantum gravitational field as a tensor field in a flat background spacetime. This is a weak field expansion that does not address the problem of the full consistency of the theory, but it gives nevertheless a credible approximation in the very low energy regime. Therefore a consistent full theory of quantum gravity should match the perturbative results in the low energy limit.

Here we focus on the Euclidean spacetime and we take background spacetime to be flat; i.e. the metric of the background is $\delta_{\mu\nu}$. The definition of gravitation field $h_{\mu\nu}(x)$ is

$$h_{\mu\nu}(x) = g_{\mu\nu}(x) - \delta_{\mu\nu} \tag{4.51}$$

where $g_{\mu\nu}(x)$ is the total metric, $x$ is a cartesian coordinate which covers the background spacetime manifold.

Since we use a path integral formalism to write the quantum theory of perturbative gravitation field, we need rewrite Einstein-Hilbert (EH) action (without cosmological constant)

$$S = \frac{1}{16\pi G}\int \mathrm{d}x\sqrt{g}R \tag{4.52}$$

in terms of the field $h_{\mu\nu}(x)$. Under general coordinate transform the gravitation field $h_{\mu\nu}$ has a gauge freedom, with a structure similar to the electromagnetic field case. To compute the symmetric three-point function, we choose the harmonic gauge

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h \tag{4.53}$$

where $h \equiv h^\mu_\mu$. We only consider the pure gravity situation, without matter. In this case, the linearization of the Einstein equations reads

$$\partial_\rho\partial^\rho h_{\mu\nu} = \frac{1}{2}\delta_{\mu\nu}\partial_\rho\partial^\rho h. \tag{4.54}$$

Taking the trace for both side, we have

$$\partial_\rho\partial^\rho h = 0, \quad \text{and} \quad \partial_\rho\partial^\rho h_{\mu\nu} = 0 \tag{4.55}$$

Using this and the gauge fixing, the EH action becomes (only keeping the 3-valent terms)

$$S_3 = \frac{1}{64\pi G}\int \mathrm{d}x(h^{\sigma\rho}\partial_\sigma h^{\mu\nu}\partial_\rho h_{\mu\nu} - 2h_{\mu\beta}\partial^\sigma h^{\mu\nu}\partial^\beta h_{\nu\sigma}) \tag{4.56}$$

### 4.3.2 Three-point function

The three-point function at the tree level leading order is defined as follow

$$G_{\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2}(x_1,x_2,x_3) = \frac{1}{Z}\int \mathcal{D}h\ \mathrm{e}^{\mathrm{i}S_2}\ \mathrm{i}S_3\ h_{\mu_1\mu_2}(x_1)\, h_{\nu_1\nu_2}(x_2)\, h_{\sigma_1\sigma_2}(x_3) \tag{4.57}$$

where $Z = \int \mathcal{D}h\ \exp(iS_2)$ and

$$S_2 = \frac{1}{64\pi G}\int \mathrm{d}^4z(\partial^\sigma h^{\mu\beta}\partial_\sigma h_{\beta\mu} - \frac{1}{2}\partial_\rho h\partial^\rho h). \tag{4.58}$$

The terms in $S_3$ are quite analogous; let's focus on the first, namely $h^{\sigma\rho}\partial_\sigma h^{\mu\nu}\partial_\rho h_{\mu\nu}$. Using the Wick contraction method, we obtain

$$\begin{aligned}
&G_{\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2}(x_1,x_2,x_3) = \frac{i}{64\pi G}\frac{1}{Z}\int \mathcal{D}h \exp(iS_2) \\
&\int d^4z\, h^{\sigma\rho}(z)\partial_\sigma h^{\mu\nu}(z)\partial_\rho h_{\mu\nu}(z) h_{\mu_1\mu_2}(x_1)h_{\nu_1\nu_2}(x_2)h_{\sigma_1\sigma_2}(x_3) \\
&= \frac{\kappa}{2}\int d^4z \\
&\quad (D^{\sigma\rho}{}_{,\mu_1\mu_2}(z-x_1)\partial_\sigma D^{\mu\nu}{}_{,\nu_1\nu_2}(z-x_2)\partial_\rho D_{\mu\nu,\sigma_1\sigma_2}(z-x_3) \\
&\quad + D^{\sigma\rho}{}_{,\mu_1\mu_2}(z-x_1)\partial_\sigma D^{\mu\nu}{}_{,\sigma_1\sigma_2}(z-x_3)\partial_\rho D_{\mu\nu,\nu_1\nu_2}(z-x_2) \\
&\quad + D^{\sigma\rho}{}_{,\nu_1\nu_2}(z-x_2)\partial_\sigma D^{\mu\nu}{}_{,\sigma_1\sigma_2}(z-x_3)\partial_\rho D_{\mu\nu,\mu_1\mu_2}(z-x_1) \\
&\quad + D^{\sigma\rho}{}_{,\nu_1\nu_2}(z-x_2)\partial_\sigma D^{\mu\nu}{}_{,\mu_1\mu_2}(z-x_1)\partial_\rho D_{\mu\nu,\sigma_1\sigma_2}(z-x_3) \\
&\quad + D^{\sigma\rho}{}_{,\sigma_1\sigma_2}(z-x_3)\partial_\sigma D^{\mu\nu}{}_{,\mu_1\mu_2}(z-x_1)\partial_\rho D_{\mu\nu,\nu_1\nu_2}(z-x_2) \\
&\quad + D^{\sigma\rho}{}_{,\sigma_1\sigma_2}(z-x_3)\partial_\sigma D^{\mu\nu}{}_{,\nu_1\nu_2}(z-x_2)\partial_\rho D_{\mu\nu,\mu_1\mu_2}(z-x_1))
\end{aligned} \tag{4.59}$$

where $\kappa = \sqrt{32\pi G}$ and $D_{\mu\nu,\rho\sigma}(x-y)$ is graviton propagator in position space, which is

$$\begin{aligned}
D_{\mu\nu,\rho\sigma}(x-y) &= \frac{1}{8\pi^2}\frac{1}{|x-y|^2}(\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\nu}\delta_{\rho\sigma}) \\
&\equiv \frac{1}{8\pi^2}\frac{1}{|x-y|^2}\Delta_{\mu\nu,\rho\sigma}
\end{aligned} \tag{4.60}$$

We do not write the non-connected terms because they equal to zero by gauge symmetry.

Since all the terms in Eq.(4.59) have a similar form, we focus on the first one. This reads

$$\begin{aligned}
&\int d^4z\, D^{\sigma\rho}{}_{,\mu_1\mu_2}(z-x_1)\,\partial_\sigma D^{\mu\nu}{}_{,\nu_1\nu_2}(z-x_2)\,\partial_\rho D_{\mu\nu,\sigma_1\sigma_2}(z-x_3) \\
&= \frac{1}{2(2\pi)^6}\int d^4z \frac{1}{|z-x_1|^2}\frac{z_\sigma-(x_2)_\sigma}{|z-x_2|^4}\frac{z_\rho-(x_3)_\rho}{|z-x_3|^4}\Delta^{\sigma\rho}{}_{,\mu_1\mu_2}\Delta^{\mu\nu}{}_{,\nu_1\nu_2}\Delta_{\mu\nu,\sigma_1\sigma_2}
\end{aligned}$$

The difficulty is to solve the integral in Eq.(4.61). The asymmetric form of the integral comes from the derivatives in the perturbative EH action (4.56). Fortunately we can change the derivative variables and take the derivatives out of the integral, turning it into a three-point function in $\lambda\phi^3$ theory. For Eq.(4.61), it turns into

$$\frac{\Delta^{\sigma\rho}{}_{,\mu_1\mu_2}\Delta^{\mu\nu}{}_{,\nu_1\nu_2}\Delta_{\mu\nu,\sigma_1\sigma_2}}{2(2\pi)^6}\frac{\partial}{\partial x_2^\sigma}\frac{\partial}{\partial x_3^\rho}G_{\lambda\phi^3}(x_1,x_2,x_3) \tag{4.61}$$

where

$$G_{\lambda\phi^3}\left(x_1,x_2,x_3\right)=\int\frac{d^4z}{\left|z-x_1\right|^2\left|z-x_2\right|^2\left|z-x_3\right|^2}\tag{4.62}$$

According to a theorem in [119], for a scalar three-point function, which is rotation, translation and dilation covariant, must have the form $G\left(x_1,x_2,x_3\right)=Cx_{12}^{\alpha}x_{23}^{\beta}x_{31}^{\gamma}$ in general, where $x_{ij}=|x_i-x_j|$, $C$ is a constant. Then

$$G_{\lambda\phi^3}\left(x_1,x_2,x_3\right)=\frac{C}{\left|x_1-x_2\right|^{\frac{2}{3}}\left|x_2-x_3\right|^{\frac{2}{3}}\left|x_3-x_1\right|^{\frac{2}{3}}}\tag{4.63}$$

Then the derivatives outside of the integral give the final results. Let us introduce some notations. Focus on an equilateral 4-simplex. $|x_1-x_2|=|x_2-x_3|=|x_3-x_1|=L$, $x_1^2=x_2^2=x_3^2=\frac{2}{5}L^2$ and $x_i\cdot x_j|_{i\neq j}=-\frac{1}{10}L^2$. Writing

$$I_{ij}^{\mu\nu}=\frac{\partial}{\partial_\mu x_i}\frac{\partial}{\partial_\nu x_j}G_{\lambda\phi^3}\left(x_1,x_2,x_3\right),\tag{4.64}$$

we have, for instance

$$I_{12}^{\mu\nu}=C\frac{4}{9L^6}(x_3^\mu x_3^\nu+x_3^\mu x_1^\nu-2x_3^\mu x_2^\nu-2x_1^\mu x_3^\nu-5x_1^\mu x_1^\nu+7x_1^\mu x_2^\nu+x_2^\mu x_3^\nu+4x_2^\mu x_1^\nu-5x_2^\mu x_2^\nu)\tag{4.65}$$

and similarly for the other components. This allows us to write the three-point function explicitly:

$$\begin{aligned}G_{\mu_1\mu_2\nu_1\nu_2\sigma_1\sigma_2}(x_1,x_2,x_3)=\frac{\kappa}{2}\frac{1}{2\left(2\pi\right)^6}\Bigg(&\left(\begin{array}{c}I_{23}^{\sigma\rho}\Delta_{\sigma\rho,\mu_1\mu_2}\Delta_{\mu\nu,\nu_1\nu_2}\Delta_{\mu\nu,\sigma_1\sigma_2}+I_{32}^{\sigma\rho}\Delta_{\sigma\rho,\mu_1\mu_2}\Delta_{\mu\nu,\sigma_1\sigma_2}\Delta_{\mu\nu,\nu_1\nu_2}\\+I_{31}^{\sigma\rho}\Delta_{\sigma\rho,\nu_1\nu_2}\Delta_{\mu\nu,\sigma_1\sigma_2}\Delta_{\mu\nu,\mu_1\mu_2}+I_{13}^{\sigma\rho}\Delta_{\sigma\rho,\nu_1\nu_2}\Delta_{\mu\nu,\mu_1\mu_2}\Delta_{\mu\nu,\sigma_1\sigma_2}\\+I_{12}^{\sigma\rho}\Delta_{\sigma\rho,\sigma_1\sigma_2}\Delta_{\mu\nu,\mu_1\mu_2}\Delta_{\mu\nu,\nu_1\nu_2}+I_{21}^{\sigma\rho}\Delta_{\sigma\rho,\sigma_1\sigma_2}\Delta_{\mu\nu,\nu_1\nu_2}\Delta_{\mu\nu,\mu_1\mu_2}\end{array}\right)\\&-2\left(\begin{array}{c}I_{23}^{\sigma\beta}\Delta_{\mu\beta,\mu_1\mu_2}\Delta_{\mu\nu,\nu_1\nu_2}\Delta_{\nu\sigma,\sigma_1\sigma_2}+I_{32}^{\sigma\beta}\Delta_{\mu\beta,\mu_1\mu_2}\Delta_{\mu\nu,\sigma_1\sigma_2}\Delta_{\nu\sigma,\nu_1\nu_2}\\+I_{31}^{\sigma\beta}\Delta_{\mu\beta,\nu_1\nu_2}\Delta_{\mu\nu,\sigma_1\sigma_2}\Delta_{\nu\sigma,\mu_1\mu_2}+I_{13}^{\sigma\beta}\Delta_{\mu\beta,\nu_1\nu_2}\Delta_{\mu\nu,\mu_1\mu_2}\Delta_{\nu\sigma,\sigma_1\sigma_2}\\+I_{12}^{\sigma\beta}\Delta_{\mu\beta,\sigma_1\sigma_2}\Delta_{\mu\nu,\mu_1\mu_2}\Delta_{\nu\sigma,\nu_1\nu_2}+I_{21}^{\sigma\beta}\Delta_{\mu\beta,\sigma_1\sigma_2}\Delta_{\mu\nu,\nu_1\nu_2}\Delta_{\nu\sigma,\mu_1\mu_2}\end{array}\right)\Bigg)\end{aligned}\tag{4.66}$$

### 4.3.3 Comparison between the perturbative and loop three-point functions

The comparison of the three-point function computed here with the one computed in the previous section is not easy. In order to compare the expectation values, we need to identify the Penrose operators $G_l^{ab}$ with the metric field. The Penrose operator has a clear geometrical interpretation [110]: it is the scalar product of the flux operator across the boundary triangles $a$ and $b$ of the boundary tetrahedron $l$ of a 4-simplex-like spacetime region. It can therefore immediately compared with quantities well defined in Regge geometry: areas of triangles and angles between triangles.

The direct comparison with the metric operator, on the other hand, is tricky, since areas and angles of simplices are nonlocal functions of the metric. In addition, the $n$-point functions are computed in the linearized theory in a certain gauge. The loop theory defines implicitly a gauge in two steps. First, the boundary operators are naturally defined in a "time" gauge, with respect to the foliation defined by the boundary. Second, the remaining gauge freedom is fixed by the boundary state [87, 120].

Tentatively, we may write

$$G_n^{ab} = E_n^a \cdot E_n^b = \det(q) q^{ij}(x) N_i^{na}(x) N_j^{nb}(x) \tag{4.67}$$

where $N_i^{na}$ is the normal one form to the triangle $(n, a)$ in the plane of the tetrahedron $a$, normalized to the coordinate area of the triangle, in the background geometry, and $q_{ij}$ is the three metric induced on the boundary. More precisely, we can use the two-form $B_{\mu\nu}^{la}$ associated to the $(n, a)$ triangle and write

$$G_n^{ab} = 2 g^{\rho\sigma} g^{\mu\nu} B_{\rho\mu}^{la} B_{\sigma\nu}^{lb}. \tag{4.68}$$

This is the way the loop operator was identified with the perturbative gravitational field in [94]. The same simple minded identification does not appear to work for the three-point function, if we use the numerical values for the boundary state found in [94]. Since the loop calculation matches the Regge one, the inconsistency is not related to the specific of the loop formalism, and is therefore of secondary interest here.

The problem of the consistency between Regge calculus [96] and continuum perturbative quantum gravity field theory has been discussed in [78, 121, 122]. The consistency between Regge calculus and continuum theory is based on the relation between the Regge action $S_{\mathrm{Regge}}$ and EH action $S_{\mathrm{EH}}$. $S_{\mathrm{Regge}}$ can be derived from $S_{\mathrm{EH}}$ [121], and $S_{\mathrm{Regge}}$ yields back $S_{\mathrm{EH}}$ with a correction in the order $\mathcal{O}(l^2/\rho^2)$ [122], where $l$ is the typical length of a four simplex and $\rho$ is Gauss radius which stands for the intrinsic curvature. In the limit $l \to 0$ or $\rho \to \infty$, $S_{\mathrm{Regge}} \to S_{\mathrm{EH}}$. In our calculation, we use the limit $\rho \to \infty$, as we have mentioned in Section 4.2.3. Then we can use the regular way to calculate the graviton $n$-point function, i.e. adding $n$ $h_{\mu\nu}$s into the path integral as insertions and change the action $S_{\rm EH} \rightarrow S_{\rm EH} + \mathcal{O}(l^2/\rho^2)$[122]. Perturbative Regge calculus is given by the strong coupling expansion [78]. The expansion around the saddle point in loop gravity corresponds to the strong coupling expansion in Regge calculus.

We also point out here that in [85][86], the traceless gauge $h^\mu_\mu = 0$ was assumed, but this may not be consistent with the gauge choice implicit in the use of the Penrose field operator. If we take this into account in the definition of two-point function given in [94]

$$G^{abcd}_{mn} = \langle E^a_m \cdot E^b_m \; E^c_n \cdot E^d_n \rangle - \langle E^a_m \cdot E^b_m \rangle \langle E^c_n \cdot E^d_n \rangle \tag{4.69}$$

since $E^a_n$ is a densitized operator, we obtain

$$\begin{aligned} G^{abcd}_{mn} &= \langle \det(g(x_m)) g_{\mu\nu}(x_m) \; \det(g(x_n)) g_{\rho\sigma}(x_n) \rangle \quad (N^a_m)^\mu (N^b_m)^\nu (N^c_n)^\rho (N^d_n)^\sigma \\ &\quad - \langle \det(g(x_m)) g_{\mu\nu}(x_m) \rangle \langle \det(g(x_n)) g_{\rho\sigma}(x_n) \rangle (N^a_m)^\mu (N^b_m)^\nu (N^c_n)^\rho (N^d_n)^\sigma \end{aligned}$$

Then we find at the order $\mathcal{O}(h^2)$

$$G^{abcd}_{mn} = \langle h h_{\rho\sigma} \delta_{\alpha\beta} + h^2 \delta_{\rho\sigma} \delta_{\alpha\beta} + h h_{\alpha\beta} \delta_{\rho\sigma} + h_{\rho\sigma} h_{\alpha\beta} \rangle (N^a_m)^\mu (N^b_m)^\nu (N^c_n)^\rho (N^d_n)^\sigma + \mathcal{O}(h^3)$$

which is certainly not the standard two-point function. For the three-point function case, the relation is even more complicated.

An additional source of uncertainty in the relation between the flux variables $E^a_n$ and $g_{\mu\nu}$ is given by the correct identification of the normals. Above we have assumed

$$E^a_n E^b_n = \det(g) g_{\mu\nu}(x) N^a_n(x) N^b_n(x) \tag{4.70}$$

where the normals $N^a_n$ are those of the background geometry. But in the boundary state used $N^a_n = j_{na} n^a_n(j(h))$, where the normals are determined by the areas of the entire 4-simplex. This gives an extra dependence on the metric: $\det(g) g_{\mu\nu}(x) N^a_n(j(h(x))) N^b_n(j(h(x)))$.

Because of these various technical complications a direct comparison with the weak field expansion in $g_{\mu\nu}$ requires more work. On the other hand, it is not clear that this work is of real interest, since the key result of the consistency of the loop dynamics with the Regge one is already established.

## 4.4 Summary

We have computed the three-point function of loop quantum gravity, starting from the background independent spinfoam dynamics, at the lowest order in the vertex expansion. We have shown that this is equivalent to the one of perturbative Regge calculus in the limit $\gamma \to 0$, $j \to \infty$ and $\gamma j = A$.

Given the good indications on the large distance limit of the $n$-point functions for Euclidean quantum gravity, we think the most urgent open problem is to extend these results to the Lorentzian case, and to the theory with matter [123, 124] and cosmological constant [125–127].

# Chapter 5

# Null geometry from LQG

## 5.1 Motivations and outlines

Null hypersurfaces play a pivotal role in the physical understanding of general relativity. We are interested in understanding how null hypersurfaces can be described within LQG, and their dynamical properties. Research in the dynamics of loop quantum gravity is mostly concerned with the evolution of spacelike hypersurfaces, in the spirit of the ADM (Arnowitt-Deser-Misner) canonical approach it is rooted on. It is commonly described by the spin foam formalism, or its embedding in group field theory. One considers transition amplitudes between fixed graphs, then refines or sums over the graphs. The boundary data assigned on the graphs are typically taken to be spacelike, however, the spin foam formalism is completely covariant, and in principle one can consider arbitrary configurations. Some results on timelike boundaries have appeared in [128, 129], but null configurations have received little attention so far.[1] To extend the description to null boundary data, the first step is to understand what null data mean from the viewpoint of LQG variables on a fixed graph. In this chapter, we point out a natural answer suggested by the recent description of LQG in terms of twistors and twisted geometries [12, 13, 34, 131–136].

Twistors describing LQG in real Ashtekar-Barbero variables satisfy a certain incidence relation [135], determined by the timelike vector used in the $3+1$ splitting of the gravitational action. Such constrained incidence relation is the twistor's version of the discretized (primary) simplicity constraints presenting in the Plebanski action for general relativity. The idea of this work is to describe discrete null hypersurfaces by

[1] For instance, a discussion of admissible null boundaries for spin foams has appeared in [130].

taking the vector appearing in the incidence relation to be null. The first consequence of this choice is that the usual group SU(2) is replaced by ISO(2), the little group of a null vector. Furthermore, the primary simplicity constraints are all first class, and only the SO(2) helicity subgroup survives the symplectic reduction: the translations are pure gauge. This fact has an appealing counterpart in particle theory: as well-known, the representations of massless particles only depend on the spin quantum number, the translations being redundant gauges. In our setting, the gauge orbits have the geometric interpretation of shifts along the null direction of the hypersurface.

In the next section, we briefly review polyhedra with spacelike faces in null hypersurfaces, and how they can be described in terms of bivectors satisfying the closure and simplicity constraints. In particular, we provide a gauge-invariant set of variables allowing us to reconstruct a unique null polyhedron starting from its bivectors. Because of the special isometries present due to the existence of null directions, such gauge-invariant variables are a little more subtle than the scalar products that one may immediately think of by analogy with the Euclidean case. In Sec. 3, we describe the phase space of Lorentzian spin foam models with the null simplicity constraints and its description in terms of twistors, and show how the null polyhedra are endowed in this way with a symplectic structure. We then proceed to study the symplectic reduction, interpret geometrically the orbits of the simplicity constraints and identify the global isometries as well as the transformations changing the shapes of the polyhedra. The latter are also first class; thus the reduced phase describes only an equivalence class of null polyhedra, determined only by the areas and their time orientation.

The geometry of the two-dimensional spacelike surface can be parametrized in purely gauge-invariant terms, and describes a Euclidean singular structure (see e.g. [137]) with scale factors associated with the faces of the graph, instead of the nodes. These data are less than those characterizing a two-dimensional Regge geometry, again a peculiarity of the large amount of symmetry in the system. For planar graphs, the reduced Poisson brackets evaluate to the Laplacian matrix of the dual graph. Therefore proper gauge-invariant action-angle variables can be identified in terms of its eigenvectors. For nonplanar graphs the situation is slightly more complicated, as the matrix of Poisson brackets has off-diagonal elements of both signs. Finally, we comment on the possible role played by secondary constraints that future studies of the dynamics may unveil, in particular, we identify the kinematical degrees of freedom amenable to describing the extrinsic geometry of the foliation.

In Sec. 5, we quantize the system and find an orthonormal basis for the reduced

Hilbert space. Such null spin networks are labeled by SO(2) quantum numbers, and are naturally embedded in the lightlike basis of homogeneous functions used for the unitary, infinite-dimensional representations of the Lorentz group. The basis diagonalizes the oriented areas, and the (complex exponentials of the) deficit angles act as spin-creation operators. This work is only a first, preliminary step toward understanding the dynamics of null surfaces in loop quantum gravity, and in the conclusions we comment on some next steps in the program, as well as desired applications. Finally, the Appendix contains details and conventions on the Lorentz algebra and its ISO(2) subgroup.

## 5.2 Simple bivectors and null polyhedra

In this section, we describe how null polyhedra can be described in terms of bivectors. By null polyhedra, we will mean polyhedra with spacelike faces living in a three-dimensional null hypersurface of Minkowski spacetime. Consider a bivector $B^{IJ}$ in Minkowski spacetime, orthogonal to a given direction $N^I$,

$$N_I B^{IJ} = 0. \tag{5.1}$$

The condition implies that the bivector is simple; namely it can be written in the form $B^{IJ} = 2u^{[I}v^{J]}$. The proof is straightforward, and valid for any signature of $N^I$.[2] Provided $u$ and $v$ are linearly independent, the simple bivector identifies a plane, as well as a scale $B^2 := B^{IJ}B_{IJ}/2$. When $N^I$ is null, the two vectors $u$ and $v$ can then be either null or spacelike. If they are both null, they both must be proportional to $N^I$, and thus the bivector is "degenerate" and does not span a plane. In this work we focus our attention on the case of spacelike bivectors.

Such simple bivectors can always be parametrized as

$$B^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}N^K b^L, \qquad b^2 = 0, \qquad B^2 = (b \cdot N)^2. \tag{5.2}$$

We further denote $A := |B|$, and $b \cdot N = -\varepsilon A$, with $\varepsilon = \pm$.

Next, take a collection of bivectors $B_l$, all lying in the same hypersurface deter-

[2] An arbitrary bivector $B^{IJ}$ can be written as $B^{IJ} = a^{[I}b^{J]} - c^{[I}d^{J]}$. If (5.1) holds, then $(a \cdot N)\, b - (b \cdot N)\, a - (c \cdot N)\, d + (d \cdot N)\, c = 0$, which implies that the four vectors are linearly dependent. Simplicity immediately follows, independent of the signature of $N^I$.

mined by $N^I$, and further constrained by the closure condition

$$\sum_l B_l = 0. \tag{5.3}$$

In the case of a timelike $N^I$, a theorem by Minkowski proves that the set defines a unique, convex and bounded polyhedron, with areas $A_l$ and dihedral angles determined by the scalar products among the bivectors. This fact plays a key role in the interpretation of loop quantum gravity in terms of twisted geometries. See [34] for details and the explicit reconstruction procedure. An application of the same theorem to the case of null $N^I$ implies that the polyhedron now lies in the null hypersurface orthogonal to $N^I$, which includes $N^I$ itself. A null hypersurface has a degenerate induced metric, with signature $(0,+,+)$, and therefore the metric properties of the polyhedron are entirely determined by its projection on the spacelike 2d surface.[3] In fact, one can arbitrarily translate the vertices of the polyhedron along the null direction without changing its intrinsic geometry. Using this symmetry, the polyhedron can always be "squashed" on the two-dimensional spacelike surface, where it will look like a degenerate case of a Euclidean polyhedron. It is indeed often helpful to visualize a null polyhedron as an ordinary polyhedron in coordinate space, endowed with a degenerate metric.

Using the parametrization (5.2) of simple bivectors, the closure condition can be equivalently rewritten as

$$V^I := \sum_l b_l^I = \alpha N^I, \qquad \alpha \in \mathbb{R}. \tag{5.4}$$

These are three independent equations, since $\alpha$ is arbitrary, and therefore the space of $F$ simple, closed bivectors has $3F-3$ dimensions. In particular, contracting both sides with $N_I$ we obtain the "area closure",

$$-N \cdot V = \sum_l \varepsilon_l A_l = 0. \tag{5.5}$$

This condition is also satisfied by a degenerate Euclidean polyhedron squashed on a 2d plane, and it allows us to identify $A_l$ with the areas of the null polyhedron's faces. Furthermore, assuming once and for all $N^I$ to be future pointing, and the normals

[3]This does not mean that the null direction never plays a geometric role: it will acquire a geometrical meaning, if ones embeds the three-dimensional null hypersurface in a nondegenerate ambient space-time.

outgoing to the faces, the sign $\varepsilon_l$ measures whether the face $l$ is future or past pointing. While (5.5) plays a predominant role, one should not forget that the complete closure condition satisfied by the bivectors has two extra equations, contained in (5.3) or (5.4). It is also interesting to note that (5.4) allows us to map the space of null polyhedra with $F$ faces to the space of null polygons with $F+1$ sides, with one direction held fixed, but we will not further pursue this interpretation here.

Another peculiarity of null polyhedra is to have a larger isometry group than their Euclidean brothers. Clearly, global (i.e. acting on all bivectors) Lorentz transformations belonging to the little group of $N^I$, which is the Lie group ISO(2), do not affect the intrinsic geometry. But there is an additional isometry due to the degeneracy of the induced metric: boosts along the $N^I$ direction do not change the intrinsic geometry of the polyhedron, because the induced metric is degenerate along that direction. Therefore, the isometry group has four dimensions, and the space of shapes of null polyhedra has $3F-7$ dimensions.

An interesting question is how to parametrize the intrinsic shapes of null polyhedra. In the Euclidean case, we are used to do so using the scalar products between the normals within the hypersurface, which fully respect the isometries. However, this is not the case for null polyhedra, where it is the common normal $N^I$ to lie in the hypersurface, while the null normals $b^I_l$ characterizing the individual faces do not lie in the hypersurface, and need not respect the isometries. For instance, translating a vertex of the polyhedron along the null direction is an isometry, but this transformation does not preserve the scalar product between the null normals $b^I_l$. Conversely, while individual simple bivectors define planes, the intersection of planes cannot be defined in a degenerate metric. Therefore, the characterization of the intrinsic shapes cannot be done solely in terms of the $b_l$; one must resort to the full Minkowski spacetime and its nondegenerate metric. To fix ideas, consider the foliation of Minkowski spacetime generated by $\mathcal{N}$ and $\widehat{\mathcal{N}}$, the null hypersurfaces defined , respectively by $N^I$ and its parity transformed $\widehat{N}^I = \mathcal{P}N^I$, satisfying $\widehat{N}\cdot N = -1$. See Fig. 5.1.

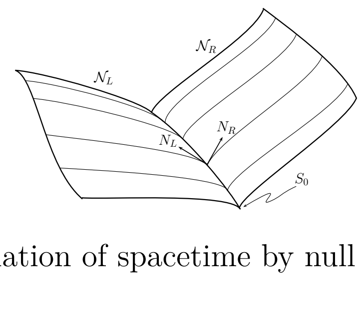


Fig. 5.1 A foliation of spacetime by null hypersurfaces.

Using both normals, one can make sense of the intersection of two faces, say $l$ and $l'$, within $\mathcal{N}$, and characterize it by the (pseudo)vector

$$\widetilde{E}^I_{ll'} = \epsilon^{IJKL} N_J (\epsilon_{KMPQ} \widehat{N}^M B^{PQ}_l)(\epsilon_{LRST} \widehat{N}^R B^{ST}_{l'}). \tag{5.6}$$

With this formula, one can explicitly reconstruct the intrinsic shape of the null polyhedron starting from the bivectors. To show this, let us first consider the case of a tetrahedron, and then a general polyhedron.

The simplicity of the tetrahedral case lies in its trivial adjacency matrix: any two faces identify an edge of the tetrahedron, and the intrinsic shapes can be described by any three edge vectors meeting at one vertex, by providing the lengths and the angles among them. The existence of a null direction will show up explicitly in the fact that only two of the angles are linearly independent, thus the intrinsic shape is characterized by only five quantities. Consider then three faces, say $l = 1, 2, 3$, and the three edges determined by their intersections. Let us first assume that the three edge vectors are not coplanar in $\mathcal{N}$ (the degenerate case will be dealt with later). Then, we define

$$V_c(B)^4 := -\frac{1}{6^4} \epsilon_{IJKL} \widehat{N}^I \widetilde{E}^J_{13}(B) \widetilde{E}^K_{21}(B) \widetilde{E}^L_{32}(B). \tag{5.7}$$

The right-hand side is always positive, and defines a coordinate volume of the tetrahedron, analogous to the definition of the Euclidean volume in terms of the triple product. We can then normalize (5.6) and obtain the proper edge vectors of the tetrahedron as

$$E^I_{ll'} := \frac{1}{6V_c} \widetilde{E}^I_{ll'} = -\frac{1}{6V_c} \epsilon^I{}_{JKL} N^J b^K_l b^L_{l'}, \tag{5.8}$$

where we used (5.2). Finally, the edge lengths and angles of the triple evaluate to

$$E^2_{ll'} = -\frac{2}{(6V_c)^2} (b_l \cdot N)(b_{l'} \cdot N)(b_l \cdot b_{l'}), \tag{5.9a}$$

$$E_{ll'} \cdot E_{l'l''} = \frac{1}{(6V_c)^2} \Big[ (b_l \cdot N)(b_{l'} \cdot N)(b_{l'} \cdot b_{l''}) + (b_{l'} \cdot N)(b_{l''} \cdot N)(b_l \cdot b_{l'}) + (b_{l'} \cdot N)^2 (b_l \cdot b_{l''}) \Big]. \tag{5.9b}$$

It is easy to check that we can always consistently pick $B^{IJ}_l = 2E^{[I}_{ll'} E^{J]}_{l''l}$, and that the triangles' areas computed from the edge vectors coincide with $A_l$. Furthermore, the oriented sum of the angles defined by (5.9b) vanishes, so that only five quantities out of the six defined in (5.9) are independent.

The formulas (5.9) provide the intrinsic shape of the null tetrahedron in terms of simple bivectors. They are valid for any time orientation of the faces and, as promised, are left invariant when any of the vectors is translated along the null direction $N^I$. In particular, this makes the expressions for edges and angles valid also in the special case when the isometry is used to "squash" the tetrahedron down to the spacelike surface $S_0$. When this happens, the $b_l^I$ are all parallel, so their scalar products vanish, but also $V_c$ vanishes, and the ratio $(b_l \cdot b_{l'})/V_c^2$ remains finite. Hence (5.9) are well defined also in the limit case when the edge vectors are coplanar. We conclude that the intrinsic geometry can be characterized in terms of the null vectors $b_l^I$, using the scalar products $b_l \cdot N$ as well as the ratios $(b_l \cdot b_{l'})/V_c^2$, of which only two out of three are independent. On the other hand, notice that the scalar products $b_l \cdot b_m$ are not good variables: they are not preserved by the isometries, and different values can correspond to the same intrinsic geometry.

The main difficulty to extend this construction to higher polyhedra comes from the fact that the adjacency matrix is not trivial anymore: the explicit values of the bivectors themselves will determine whether two faces are adjacent or not. A strategy to deal with this case is to use the reconstruction algorithm already developed for the Euclidean signature. To that end, we work in light-cone coordinates defined by $N^I$ and $\widehat{N}^I$. In these coordinates, the closure constraint (5.13) identifies a closure condition for 3d vectors in a space with a degenerate metric of signature $(0, +, +)$. If we replace this metric by an auxiliary Euclidean metric, we can apply the reconstruction procedure of [34] to the resulting Euclidean polyhedron. In particular, compute its adjacency matrix, and once this is known, apply (5.9) to the existing edges to determine the null geometry of the polyhedron. It would be interesting to know whether the adjacency matrix of a null polyhedron can be reconstructed directly from the $b_l^I$, without passing through the auxiliary Euclidean reconstruction, but this is not needed for the rest of the work, and we leave it as an open question.

Finally, recall that the space of shapes of 3d Euclidean polyhedra has dimensions $3F - 6$, and the $2F - 6$ space of shapes at fixed areas is a phase space [138], a result used in the twisted geometry parametrization [34]. This turns out not to be the case for null polyhedra, because as we show below, the closure condition does not generate all the isometries. While it is an interesting open question to construct a phase space of shapes for null polyhedra, we will see below that the phase space of loop gravity on a null hypersurface does include a description of polyhedra, but rather as equivalence classes, defined by their areas only.

## 5.3 Null simplicity constraints in LQG

Spin foams are based on the nonchiral Plebanski action for general relativity,

$$S(\omega^{IJ}, B, \psi) = \int \mathrm{Tr}\Big(\star + \frac{\mathbb{1}}{\gamma}\Big) B \wedge F(\omega^{IJ}) + \psi_{IJKL} B^{IJ} \wedge B^{KL}, \tag{5.10}$$

where the fundamental variables are a Lorentz connection $\omega_\mu^{IJ}$, and a 2-form valued in the Lorentz algebra $B^{IJ}$, constrained by $\psi_{IJKL}$ to be simple, that is $B^{IJ} = e^I \wedge e^J$. Here $\gamma$ is the Immirzi parameter, and we assumed a vanishing cosmological constant. The canonical analysis of this action has been studied in a number of papers (e.g. [139]), and we refer the reader to the living review [10] for details and an introduction to the spin foam formalism. The phase space is described by the pullback of the Lorentz connection and its conjugate momentum, that is the pullback of the 2-form

$$M^{IJ} = \Big(\star + \frac{\mathbb{1}}{\gamma}\Big) B^{IJ}, \qquad B^{IJ} = \frac{\gamma}{\gamma^2+1}\Big(\mathbb{1} - \gamma \star\Big) M^{IJ}. \tag{5.11}$$

In the following, we are interested in a discretized version of this canonical structure, which is commonly used in the construction of spin foam models [10]. The discrete variables are distributional smearings along an oriented graph $\Gamma$, say with $L$ links and $N$ nodes, where the gravitational connection is replaced by holonomies $h_l$ along the links, and the conjugate momentum by algebra elements $M_l$, referred to as fluxes. The phase space associated with a graph is

$$P_\Gamma = T^*\mathrm{SL}(2,\mathbb{C})^L, \qquad (M_l, h_l) \in T^*\mathrm{SL}(2,\mathbb{C}), \tag{5.12}$$

which notably comes with a noncommutativity of the fluxes. This kinematical phase space appears in Lorentzian spin foam models [140], as well as in covariant loop quantum gravity [141]. We then consider two sets of constraints on the $B$ variables. The first is a discrete Gauss law, or closure condition,

$$G_n^{IJ} = \sum_{l \in n} B_l^{IJ} = 0. \tag{5.13}$$

It is local on the nodes of the graph, and it imposes gauge invariance. The second is a discrete version of the simplicity constraints,

$$S_{nl}^J = N_{nI} B_l^{IJ} = 0, \qquad \forall l \in n, \tag{5.14}$$

where $N_n^I$ is a unit vector assigned independently to each node $n$. This linear version of the discrete simplicity constraints was introduced in [40], with $N^I$ timelike and related to the hypersurface normal used in the $3+1$ decomposition of the action. We denote $\mathcal{S}_\Gamma$ the reduced phase space obtained imposing the constraints (5.13) and (5.14),

$$\mathcal{S}_\Gamma = T^*\mathrm{SL}(2,\mathbb{C})^L//F_{nl}//G_n. \tag{5.15}$$

When $N^I$ is timelike, it was shown in [135] that $\mathcal{S}_\Gamma \equiv T^*\mathrm{SU}(2)^L//\mathrm{SU}(2)^N$, where for any finite $\gamma \neq 0$, the relevant SU(2) subgroup is not the canonical subgroup of the Lorentz group, but a group manifold nontrivially embedded in $T^*\mathrm{SL}(2,\mathbb{C})$, capable in particular of probing boosts degree of freedom. The interpretation of $\mathcal{S}_\Gamma$ is that of a truncation of general relativity to a finite number of degrees of freedom [142], whose geometry can be described by twisted geometries [12].

In this work we investigate the consequences of taking vector $N^I$ in (5.14) to be null, and derive a geometric description for the reduced space (5.15), in the spirit of twisted geometries. Ideally, this should be related to a formulation of the Plebanski action in which we perform a standard $3+1$ splitting, and use the internal Minkowski space to induce a noninvertible 3d metric with signature $(0++)$. The continuum canonical analysis of (5.10) in this null setup, as well as studying the resulting dynamical structure, will be investigated elsewhere.[4] Our goal here is simply to study (5.15) when $N^2=0$, its geometrical interpretation, and its quantization.

We will proceed in two steps, motivated by the structure of (5.15). First, we focus on a single link, studying the phase space $T^*\mathrm{SL}(2,\mathbb{C})$ and the pair of simplicity constraints (5.14), which are local on the links. At a second stage, we consider the full graph structure and the closure condition (5.13).

### 5.3.1 Phase space structure

We saw in Sec. 1 that a set of bivectors satisfying closure and simplicity defines polyhedra. The polyhedra can be endowed with the symplectic structure of $T^*\mathrm{SL}(2,\mathbb{C})$ via (5.11) and (5.12), as follows. Picking a specific time direction $t^I=(1,0,0,0)$, we identify boosts, rotations and chiral left-handed generators, respectively, as

$$K^i := M^{0i}, \qquad L^i = -\frac{1}{2}\epsilon^i{}_{jk}M^{jk}, \qquad \Pi^i = \frac{1}{2}(L^i + \mathrm{i}K^i) = \mathrm{i}\sigma^{iA}{}_B\Pi^B{}_A.$$

[4] In particular, the analysis is expected to reveal the presence of secondary constraints, which should play an important role in the identification of the extrinsic geometry, as we will discuss below.

Here $A, B = 0, 1$ are spinorial indices, raised and lowered with the antisymmetric symbol $\epsilon^{AB}$, and $\sigma^A{}_B$ the Pauli matrices. See Appendix for a complete list of conventions, notations and background material. We parametrize $T^*\mathrm{SL}(2,\mathbb{C})$ via the pair $(\Pi^A{}_B, h^A{}_B)$, with $h$ a group element in the fundamental $(\mathbf{1/2},\mathbf{0})$ representation, and symplectic potential $\Theta = \mathrm{Tr}(\Pi h dh) + cc$. The $\Pi$ are left-invariant vector fields, and $\widetilde{\Pi} = -h\Pi h^{-1}$ right-invariant ones. We can equivalently use the parametrization $(\Pi, \widetilde{\Pi})$ and the complex angle $\mathrm{Tr}(h)$. In this way, we can associate a generator, and thus a bivector $B$ through (5.11), with both source and target nodes of a link. Hence, we can consider the topological polyhedra defined by a cellular decomposition dual to the graph, and associate a bivector $B$ with each face within each frame. By construction, a face inherits two bivectors, and unique norm, $B^2 = \widetilde{B}^2$, and we notice that the closure condition (5.13) is equivalent to closure for the generators.

The simplicity conditions (5.1) introduce a preferred direction via $N^I$, thus reducing the initial Lorentz symmetry to its little group. For a null vector, the Lie group ISO(2). To fix ideas, we take from now on the specific null vector $N^I = (1,0,0,1)/\sqrt{2}$, with the normalization chosen for later convenience. Its little group ISO(2) is generated by

$$L^3, \qquad P^1 := L^1 - K^2, \qquad P^2 := L^2 + K^1,$$

and the simplicity constraints (5.14) read

$$\gamma L^3 + K^3 = 0, \qquad P^a = 0, \qquad a = 1, 2. \tag{5.16}$$

There are two important differences with respect to the timelike case. First of all, the constraints impose the vanishing of part of the little group itself, thus effectively selecting its helicity SO(2) subgroup. Second, by themselves they form a completely first class system, unlike in the timelike case, as can be verified trivially. These facts have important consequences for the geometric interpretation of the reduced phase space. To study the symplectic reduction and its geometric interpretation, we use the twistorial parametrization introduced and studied in [13, 133–135, 143].

### 5.3.2 Twistorial description

A twistor can be described as a pair of spinors,[5] $Z^\alpha = (\omega^A, i\bar{\pi}_{\dot{A}}) \in \mathbb{C}^2 \oplus \bar{\mathbb{C}}^{2*} =: \mathbb{T}$. The space then carries a representation of the Lorentz algebra, which preserves the

[5] The presence of an $i$ differs from the standard Penrose notation, and it is just a matter of convenience to bridge with the conventions used in loop quantum gravity.

complex bilinear $\pi_A\omega^A \equiv \pi\omega$. To describe the symplectic manifold $T^*\mathrm{SL}(2,\mathbb{C})$ on an oriented link, we consider a pair $(Z,\widetilde{Z})$ associated , respectively, with the source and target nodes of the link, and equip each twistor with canonical Poisson brackets,

$$\{\pi_A, \omega^B\} = \delta_A^B = \{\tilde{\pi}_A, \tilde{\omega}^B\}. \tag{5.17}$$

We then impose the following area-matching condition,

$$C = \pi\omega - \tilde{\omega}\tilde{\pi} = 0. \tag{5.18}$$

This is a first class complex constraint generating the scale transformations $(\omega, \pi, \tilde{\omega}, \tilde{\pi}) \mapsto (\mathrm{e}^z\omega, \mathrm{e}^{-z}\pi, \mathrm{e}^z\tilde{\omega}, \mathrm{e}^{-z}\tilde{\pi})$. The 12d manifold obtained by symplectic reduction by (5.18) coincides with $T^*\mathrm{SL}(2,\mathbb{C})$, with holonomies and fluxes that can be parametrized as

$$\Pi^{AB} = \frac{1}{2}\omega^{(A}\pi^{B)}, \qquad h^A{}_B = \frac{\tilde{\omega}^A\pi_B + \tilde{\pi}^A\omega_B}{\sqrt{\pi\omega}\sqrt{\tilde{\omega}\tilde{\pi}}}, \tag{5.19}$$

and

$$\widetilde{\Pi}^A{}_B = \frac{1}{2}\tilde{\omega}^{(A}\tilde{\pi}^{B)} \equiv -h^A{}_C\Pi^C{}_D h^{-1D}{}_B. \tag{5.20}$$

As it is apparent from (5.19), the parametrization is valid provided $\pi\omega$ and $\tilde{\pi}\tilde{\omega}$ do not vanish. The submanifold where this occurs can be safely excluded: it would correspond to null bivectors, whereas we are restricting attention to spacelike bivectors. Notice also that the parametrization is 2-to-1, as it is invariant under the exchange of spinors,

$$(\omega, \pi, \tilde{\omega}, \tilde{\pi}) \mapsto (\pi, \omega, \tilde{\pi}, \tilde{\omega}). \tag{5.21}$$

See [135] for further details. To write the simplicity constraints, we introduce a canonical basis in $\mathbb{C}^2$, $(o^A = \delta_0^A, \iota^A = \delta_1^A)$. The chosen null vector reads $N^{A\dot{A}} = io^A\bar{o}^{\dot{A}}$, and (5.1) becomes

$$N_{A\dot{A}}\Pi^{AB}\epsilon^{\dot{A}\dot{B}} = \mathrm{e}^{i\theta}N_{A\dot{A}}\epsilon^{AB}\overline{\Pi}^{\dot{A}\dot{B}}, \qquad \mathrm{e}^{\mathrm{i}\theta} \equiv (\gamma + \mathrm{i})/(\gamma - \mathrm{i}). \tag{5.22}$$

Notice that the matrix $\delta^{oA\dot{A}} := o^A\bar{o}^{\dot{A}}$ defines an Hermitian scalar product, $\|\omega\|^2 = |\omega^1|^2$, preserved by the little group ISO(2). The above conditions can be conveniently separated as

$$F_1 = \mathrm{Re}(\pi\omega) - \gamma\,\mathrm{Im}(\pi\omega) = 0, \qquad F_2 = o_A\bar{o}_{\dot{A}}\omega^A\bar{\pi}^{\dot{A}} = \omega^1\bar{\pi}^1 = 0, \tag{5.23}$$

where $F_1$ is real and Lorentz invariant, whereas $F_2$ is complex and only ISO(2) invariant. In particular, $F_2$ imposes $P^a = 0$, and on-shell of this condition $F_1$ reduces to the first condition in (5.16). The structure is very similar to the timelike case of [135]: in particular, the Lorentz-invariant part $F_1$ is the same, and can be solved posing

$$\pi\omega = (\gamma + i)\varepsilon j, \qquad \varepsilon = \pm, \qquad j \in \mathbb{R}^+. \tag{5.24}$$

With this parametrization, $\varepsilon$ determines the sign of the twistor's helicity: $\varepsilon = +$ for positive helicity. Notice that the $\mathbb{Z}_2$ symmetry (5.21) of the twistorial parametrization flips this sign, therefore it is possible to fix $\varepsilon = 1$ without loss of generality in parametrizing $T^*\mathrm{SL}(2,\mathbb{C})$. $F_2 = 0$ has two solutions, $\omega^1 = 0$ and $\pi^1 = 0$. Both branches are needed to describe the reduced phase space, introducing a slightly awkward notation, where the reduced phase space is parametrized partly by $\omega^A$ and partly by $\pi^A$. It is convenient to avoid this by exploiting the $\mathbb{Z}_2$ symmetry, since (5.21) switches between the two branches. It then turns out to be convenient to keep the $\varepsilon$ sign in (5.24) free, and pick a single branch of $F_2 = 0$. Let us assume $\omega^1 \neq 0$, and pick the solution $\pi^1 = 0$.

The five-dimensional surface of *simple* twistor solutions of (5.23) can be parametrized by $(\omega^A, j)$, and

$$\pi^A = -re^{i\frac{\theta}{2}}\delta^{oA\dot{A}}\bar{\omega}_{\dot{A}}, \qquad r = \frac{\varepsilon j\sqrt{1+\gamma^2}}{\|\omega\|^2}. \tag{5.25}$$

On this surface, the simplicity constraints generate the following gauge transformations,

$$\{F_1, \omega^A\} = \frac{1+\mathrm{i}\gamma}{2}\omega^A, \qquad \{F_2, \omega^A\} = 0, \qquad \{\bar{F}_2, \omega^A\} = -\delta^A_0\bar{\omega}^1, \qquad \{F_1, j\} = \{F_2, j\} = 0. \tag{5.26}$$

For the nontivial ones, the finite action is

$$e^{\{\alpha F_1,\cdot\}}\omega^A = \mathrm{e}^{\frac{1+\mathrm{i}\gamma}{2}\alpha}\omega^A, \qquad e^{\{\alpha \bar{F}_2,\cdot\}}\omega^A = \omega^A - \alpha\delta^A_0\bar{\omega}^1. \tag{5.27}$$

We see that $\omega^0$ is pure gauge and that $\omega^1$ contains a dependence on the gauge generated by $F_1$. The gauge invariant reduced space has two dimensions, and can be parametrized by the following complex variable,

$$z = \frac{\sqrt{2j}}{\|\omega\|^{i\gamma+1}}\omega^1, \qquad |z|^2 = 2j, \tag{5.28}$$

plus the sign $\varepsilon$. Notice that shifting the phase of $z$ by $\pi$ has the same effect as switching the sign of $\varepsilon$. Hence, with our choice of parametrization $\arg(z) \in [0, \pi)$, to avoid covering twice the same space. In this way we identify the positive complex half-plane with positive helicities, and the negative half-plane with negative helicities. The reduced symplectic potential evaluates to

$$\Theta_{\text{red}} = -\frac{i}{2}\varepsilon z d\bar{z} + cc, \qquad \{z, \bar{z}\} = i\varepsilon, \tag{5.29}$$

so the sign of the helicity determines the sign of the Poisson brackets. In conclusion, the symplectic reduction gives $\mathbb{T}//F = T^*S^1$, with the circle parametrized by two half-circles via $\arg(z) \in [o, \pi), \varepsilon = \pm$.

To better understand the geometric meaning of the orbits of the simplicity constraints, it is useful to look at the bivectors $B^{IJ}$. These are given by (5.11) in terms of the algebra generators $M^{IJ}$, whose spinorial form reads, from (5.19), $M^{IJ} = -\omega^{(A}\pi^{B)}\epsilon^{\dot{A}\dot{B}} + cc$. Introducing the following doubly null reference frame,

$$\ell^I = i\omega^A\bar{\omega}^{\dot{A}}, \quad k^I = i\pi^A\bar{\pi}^{\dot{A}}, \quad m^I = i\omega^A\bar{\pi}^{\dot{A}}, \quad \bar{m}^I = i\pi^A\bar{\omega}^{\dot{A}}, \quad \ell\cdot k = -|\pi\omega|^2 = -m\cdot\bar{m}, \tag{5.30}$$

we can rewrite the bivectors as

$$B^{IJ} = \frac{\gamma}{1+\gamma^2}\frac{2}{|\pi\omega|^2}\left[(\gamma I - R)\ell^{[I}k^{J]} + \mathrm{i}(\gamma R + I)m^{[I}\bar{m}^{J]}\right] \approx \frac{2\mathrm{i}\varepsilon\gamma}{j(1+\gamma^2)}m^{[I}\bar{m}^{J]}, \tag{5.31}$$

where $\approx$ means that the equality holds on the constraint surface. The last equation defines a spacelike plane, and a scale $B^2 = \gamma^2 j^2$, which represent the spacelike projection of the polyhedron's face. Comparing (5.31) and (5.2), we derive a parametrization of the normal null vector $b^I$ in terms of spinors,

$$b^I = \frac{\varepsilon\gamma j}{\|\omega\|^2}\ell^I, \qquad b\cdot N = -\epsilon\gamma j. \tag{5.32}$$

Hence, we can also identify the helicity sign in (5.24) with the sign of the time component of the face normal in (5.5), and since we are doing this identification for the "untilded" variables, it means that it holds provided the link is oriented outgoing from the node.

It is straightforward to see that the orbits of $F_1$ leave the bivector $B^{IJ}$ as well as $b^I$ invariant. On the other hand, $F_2$ changes $b^I$, and its action can be used to always align this null vector with $\widehat{N}^I = 1/\sqrt{2}(1, 0, 0, -1)$. Hence, the orbits of $F_2$ allow us to project the face on the spacelike surface $S_0$ orthogonal to both $N^I$ and $\widehat{N}^I$. This action becomes even clearer if we look at the spacelike vectors spanning the triangle,

$$e^{\{-\alpha\bar{F}_2-\bar{\alpha}F_2,\cdot\}}\mathrm{Re}(m)^I \approx \mathrm{Re}(m)^I + \varepsilon j[\gamma\mathrm{Re}(\alpha)+\mathrm{Im}(\alpha)]N^I, \tag{5.33a}$$

$$e^{\{-\alpha\bar{F}_2-\bar{\alpha}F_2,\cdot\}}\mathrm{Im}(m)^I \approx \mathrm{Im}(m)^I + \varepsilon j[\mathrm{Re}(\alpha)-\gamma\mathrm{Im}(\alpha)]N^I. \tag{5.33b}$$

If we do this globally on all links around a node, that is we take $\alpha_l \equiv \alpha, \forall l$, we obtain the isometry corresponding to shifting the vectors along the null direction, and this action can be used to project all the faces to $S_0$. On the other hand, acting independently on each link will genuinely deform the polyhedron, and can in principle break it open. We will come back to this important point below in Sec. 4. The geometric meaning of the action of $F_1$ will become clear next, when we discuss the reduction on the holonomy.

Let us conclude this section with a side comment, on the exact relation between the null simplicity constraints, and the usual twistor incidence relation. To that end, it is more convenient to look at the other solution of $F_2=0$, that is $\omega^1=0$. This solution is equivalent to the one $\pi^1=0$ in the sense that this solution can be obtained from the $\mathbb{Z}_2$ symmetry 5.21. In this case, the simplicity conditions can then be packaged as the following constrained incidence relation,

$$\omega^A = iX^{A\dot{A}}\bar{\pi}^\gamma_{\dot{A}}, \qquad X^{A\dot{A}} = -\frac{\varepsilon j\sqrt{1+\gamma^2}}{\|\pi\|^2}n^{A\dot{A}}, \qquad \bar{\pi}^\gamma_{\dot{A}} = e^{i\frac{\theta}{2}}\bar{\pi}_{\dot{A}}. \tag{5.34}$$

From the point of view of twistor theory, (5.34) implies that (i) the twistor is $\gamma$-null, namely that it is isomorphic to a null twistor, the $\gamma$-dependent isomorphism being $(\omega,\pi)\mapsto(\omega,\pi^\gamma := e^{-i\theta/2}\pi)$; and that (ii) the null ray $X^{A\dot{A}}$ described by the associated null twistor is aligned with $n^I$ and "truncated": a simple twistor describes a specific null vector, and not anymore a null ray.

### 5.3.3 Symplectic reduction, $T^*\mathrm{ISO}(2)$ and $T^*\mathrm{SO}(2)$

To study the symplectic reduction on the link phase space, we consider two twistors $Z$ and $\widetilde{Z}$, and impose the simplicity constraints (5.23) on both, in agreement with (5.14), as well as the area-matching condition (5.18). The complete system is first class, and partially redundant: $C=0=F_1$ implies $\widetilde{F}_1=0$. The simplicity constraints in the "tilded" sector can be solved in the same way,

$$\tilde{\pi}^A = -\tilde{r}e^{i\frac{\theta}{2}}\delta^{oA\dot{A}}\bar{\tilde{\omega}}_{\dot{A}}, \qquad \tilde{r} = \frac{\tilde{\varepsilon}\tilde{\jmath}\sqrt{1+\gamma^2}}{\|\tilde{\omega}\|^2}. \tag{5.35}$$

The area matching (5.18) then imposes $\tilde{\varepsilon}\tilde{\jmath} = -\varepsilon j$, which we solve fixing $\tilde{\jmath} = j$ and $\tilde{\varepsilon} = -\varepsilon$. The opposite sign between $\varepsilon$ and $\tilde{\varepsilon}$ keeps track of the sign difference between $\Pi$ and $\tilde{\Pi}$ in (5.20). As a consequence, a face which is future pointing in the frame of the source node is past pointing in the frame of the target node: following the same steps leading to (5.32), we find $\tilde{b}\cdot\tilde{N} = -\tilde{\varepsilon}\gamma j = \varepsilon\gamma j$. In other words, $\varepsilon$ coincides with the time orientation in the frame of the source node, and with its opposite in the frame of the target node.

On the seven-dimensional surface $\mathcal{C} \subset T^*\mathrm{SL}(2,\mathbb{C})$, where the simplicity constraints hold, fluxes and holonomies are

$$\Pi^A{}_B \approx \frac{(\gamma+i)\varepsilon j}{4}\begin{pmatrix} -1 & 2\omega^0/\omega^1 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{\Pi}^A{}_B \approx -\frac{(\gamma+i)\varepsilon j}{4}\begin{pmatrix} -1 & 2\tilde{\omega}^0/\tilde{\omega}^1 \\ 0 & 1 \end{pmatrix}, \tag{5.36a}$$

$$h^A{}_B \approx \begin{pmatrix} \omega^1/\tilde{\omega}^1 & \tilde{\omega}^0/\omega^1 - \omega^0/\tilde{\omega}^1 \\ 0 & \tilde{\omega}^1/\omega^1 \end{pmatrix}. \tag{5.36b}$$

As expected, the generators are restricted to those of the little group (up to the phase introduced by the Immirzi angle). The group element is also restricted, to a form which includes the little group ISO(2) as well as the extra isometry generated by a boost along the null direction ($K_3$ with our gauge choice for $N^I$). We can conveniently parametrize it as

$$h \approx e^{\frac{1}{2}\Xi\sigma_3}\, u, \qquad u = e^{\frac{1}{2}\Xi\sigma_3}\, e^{-\mathrm{i}\frac{1}{2}(\xi-\gamma\Xi)\sigma_3}\, T(\omega^0,\tilde{\omega}^0) \;\in\; \mathrm{ISO}(2), \tag{5.37}$$

where the boost rapidity is

$$\Xi := \ln\frac{\|\omega\|^2}{\|\tilde{\omega}\|^2}, \tag{5.38}$$

and we also defined

$$\xi := -2\arg(z) - 2\arg(\tilde{z}) \in [0,4\pi). \tag{5.39}$$

Finally, the translational part

$$T(\omega^0, \widetilde{\omega}^0) = \begin{pmatrix} 1 & \widetilde{\omega}^0/\omega^1 - \omega^0/\widetilde{\omega}^1 \\ 0 & 1 \end{pmatrix} \tag{5.40}$$

vanishes when $\omega^0$ and $\widetilde{\omega}^0$ do, a fact that plays an important role below.

A key aspect of this result is that the boost rapidity $\Xi$ enters also the rotational part of $h$. This is a consequence of the mixing between rotations and boosts introduced by the Immirzi parameter [see (5.11)], and it is presented also in the timelike case [135]: it is the discrete equivalent of the mixing in the real Ashtekar-Barbero connection defined by $A^i_a = \omega^i_a + (\gamma - \mathrm{i})K^i_a$, where $\omega^i_a$ is the anti-self-dual part of the Lorentz connection and $K^i_a$ the (triad projection of the) extrinsic curvature. Loosely speaking, the mixing allows us to probe the Lorentzian phase space through a smaller subgroup, SU(2) in the timelike case and ISO(2) here. But while in the timelike case the holonomy on the constraint surface is still a generic $\mathrm{SL}(2,\mathbb{C})$ element [135], in the present null case it is a restricted group element, missing the algebra directions $\widehat{P}^a$ capable of changing the direction of the vector $N^I$, a fact whose consequences will show up below. Concerning the Poissonian structure of $\mathcal{C}$, the symplectic potential of $T^*\mathrm{SL}(2,\mathbb{C})$ restricted by the simplicity constraints contains a piece generating the canonical Poisson brackets of $T^*\mathrm{ISO}(2)$ between $\Pi$ and $u$, and a degenerate direction. Therefore, $\mathcal{C}$ contains a proper symplectic submanifold, and can be identified at least locally with the Cartesian product $T^*\mathrm{ISO}(2) \times \mathbb{R}$, where the additional dimension corresponds to boosts along $N^I$. The cotangent bundle of the little group thus appears at the level of the constraint surface. However, a good part of it is just gauge, as we now show.

The next stage of the symplectic reduction is to divide by the gauge orbits. The gauge orbits of $F_1$ and $F_2$ have been studied in the previous sections: they amount to linear shifts of $\|\omega\|$ and $\omega^0$ , respectively. The latter are thus good coordinates along the orbits, and the gauge invariant part is the complex variable $z$ introduced in (5.28). The situation is analogous for the tilded variables, corresponding to the twistor associated with the second half of the link. In this case, we parametrize the reduced variable as

$$\bar{\tilde{z}} = \frac{\sqrt{2j}}{\|\widetilde{\omega}\|^{i\gamma+1}}\widetilde{\omega}^1, \qquad |\tilde{z}|^2 = 2j, \qquad \{\tilde{z}, \bar{\tilde{z}}\} = i\varepsilon. \tag{5.41}$$

Notice the extra complex conjugation appearing here, a convention taken to preserve the same sign of the brackets of $\tilde{z}$ as for $z$. Proceeding in this way we have reduced by both $F_1$ and $\widetilde{F}_1$, and thus by part of the area-matching constraint (5.18). The remaining part is $C_{\rm red} := |z|^2 - |\tilde{z}|^2 = 0$, which is already satisfied by the fact that we took in (5.41) the same $j$ as in (5.28). Its gauge transformations generate opposite phase shifts,

$$\{C_{\rm red}, \arg(z)\} = -\varepsilon = -\{C_{\rm red}, \arg(\tilde{z})\}. \tag{5.42}$$

Hence, $\arg(z) - \arg(\tilde{z})$ is a good coordinate along the orbits, and $\xi = -2\arg(z) - 2\arg(\tilde{z})$ previously defined is gauge invariant. The two-dimensional reduced phase space on a link is thus spanned by the pair $(\varepsilon j, \xi)$, which turns out to be canonical,

$$\{\varepsilon j, \xi\} = 1. \tag{5.43}$$

Eliminating the gauges from (5.36), we see that the reduced link phase space coincides with $T^*\mathrm{SO}(2)$,

$$X^A{}_B = \frac{(\gamma+i)\varepsilon j}{4}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g^A{}_B = \begin{pmatrix} e^{-i\xi/2} & 0 \\ 0 & e^{i\xi/2} \end{pmatrix}, \qquad \widetilde{X}^A{}_B = -\frac{(\gamma+i)\varepsilon j}{4}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.44}$$

We notice that the translations are removed dividing by the $F_2$ orbits. The same happens in the representation of massless particles, and here it has the nice geometric interpretation of being shifts along a null direction. The remaining algebra consists of the helicity generator $L^3$, which coincides with the oriented area of the bivector,

$$L^3 = \varepsilon j = -\widetilde{L}^3, \qquad \{L^3, \xi\} = 1 = -\{\widetilde{L}^3, \xi\}. \tag{5.45}$$

We conclude that $\mathbb{T}_2//C//F = T^*\mathrm{SO}(2)$, parametrized by its holonomies and fluxes, or directly by $(\varepsilon j, \xi)$. After symplectic reduction, the initial Lorentz algebra has collapsed to the helicity subgroup SO(2) of $N^I$. In particular, $\varepsilon$ is the sign of the helicity, consistent with its initial twistorial definition, (5.24).

Let us also discuss the covariance of our construction. Above we have fixed the same null vector for both source and target nodes, $N^I = \widetilde{N}^I = (1,0,0,1)/\sqrt{2}$, and the reduction has led to the canonical little group. Any different choice, say for the source, can be written as $VN$, where $V$ is a group element in the complement of the little group, and similarly $\widetilde{V}\widetilde{N}$ for the target normal. In this general case, the resulting reduced phase space would be of the form $(VXV^{-1}, Vg\widetilde{V}^{-1})$, that is the canonical little group embedded by the conjugate action. In this sense, our construction is completely covariant.

## 5.4 Null twisted geometries

We have so far described the constraint structure and the symplectic reduction on a given link. We now move on to consider the full graph, and include the closure condition (5.13) in the analysis. For simplicity, we take the same canonical null vector $N^I$ on each node. The case of arbitrary $N^I$ can be dealt with via the adjoint action as explained above, and does not change the geometric interpretation which is covariant by construction. The results of the previous section show that the twistor phase space on the graph, reduced by the null simplicity conditions (5.14) and the area matching (5.18), is $\mathbb{T}^{2L}//C_l//F_{nl} = T^*\mathrm{SO}(2)^L$, a phase space of dimensions $2L$, parametrized by $(\varepsilon_l j_l, \xi_l)$. This result used the fact that the simplicity constraints are all first class by themselves. The situation slightly changes when the closure condition(5.13) is included. On shell of the simplicity and area-matching constraints, (5.13) reduces to

$$G_n = \sum_{l\in n} L^3 = 0, \qquad \widehat{I}^a_n = \sum_{l\in n} \widehat{P}^a = 0, \qquad a = 1, 2. \tag{5.46}$$

Here $\widehat{P}^a$ are the translation generators of the little group of $\widehat{N}^I = \mathcal{P}N^I$, the only generators changing $N^I$.

These three conditions are equivalent to (5.4), in particular the first is the area closure (5.5), as follows immediately from (5.32) and (5.45). Taking into account the link orientations, we have

$$G_n = \sum_{l^+\in n} L^3 + \sum_{l^-\in n} \widetilde{L}^3 = \sum_{l^+\in n} \varepsilon_l j_l - \sum_{l^-\in n} \varepsilon_l j_l = 0, \tag{5.47}$$

where $l^+$ are the links outgoing from the node, and $l^-$ the incoming ones. This expression coincides with the area closure (5.5), once we take into account that $\varepsilon_l$ coincides with the time orientation for an outgoing link, and its opposite for an incoming link, as discussed below (5.35). Therefore, we can interpret the reduced phase space as a collection of null polyhedra, dual to the nodes of the graph. The polyhedra are glued along faces, sharing the same area $A_l \propto j_l$, and with opposite time orientation.

Notice that out of the closure conditions (5.46), only $G_n$ generates an isometry of the null plane. The other isometries of the null hypersurface are not generated by the closure condition, but by combinations of the simplicity constraints, as can be deduced from their action investigated in the previous section, and to which we will come back below. As it turns out, $\widehat{I}^a$ do not generate symmetries at all, as they form a second class system with part of the $F_2$ simplicity constraints.[6] To study the structure of the constraints and bring this fact to the surface, we compute the Dirac matrix associated with the graph. As variables on different links commute, the matrix has a block structure, in which each block is associated with a node. Since the Lorentz-invariant constraints $F_1$ commute with everything, we leave them out of the analysis. Then for a node of valence $m$, the $F_2$ and closure constraints form a $(2m+3)$-dimensional system. On shell of the $F_1$ constraints, it is possible and convenient to replace for each link the complex $F_2$ constraints by the two real $P^a$. We then take the basis of node constraints

$$\phi_\mu = \{P^1_1, P^2_2, \ \dots \ , P^1_m, P^2_m, \widehat{I}^1, \widehat{I}^2, G\}. \tag{5.48}$$

On the constraint surface, the node's block of the Dirac matrix evaluates to

$$D_{\mu\nu} \equiv \{\phi_\mu, \phi_\nu\} \approx \left(\begin{array}{ccccc|ccc} 0 & 0 & \cdots & 0 & 0 & -2\gamma L^3_1 & 2L^3_1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -2L^3_1 & -2\gamma L^3_1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -2\gamma L^3_m & 2L^3_m & 0 \\ 0 & 0 & \cdots & 0 & 0 & -2L^3_m & -2\gamma L^3_m & 0 \\ \hline 2\gamma L^3_1 & 2L^3_1 & \cdots & 2\gamma L^3_m & 2L^3_m & 0 & 0 & 0 \\ -2L^3_1 & 2\gamma L^3_1 & \cdots & -2L^3_m & 2\gamma L^3_m & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{5.49}$$

The rank of this matrix is always 4, independent of the valence of the node. Hence, the node algebra contains $2m-1$ first class constraints and two pairs of second class constraints. Using this result, and reintroducing the $F_1$'s (one independent first class constraint per link), the counting of dimensions of the reduced phase space $\mathcal{S}_\Gamma$ defined in (5.15) gives

$$12L - 2L - 4N - 2\sum_n (2\,\text{valence}_n - 1) = 2L - 2N. \tag{5.50}$$

It is much smaller than in the timelike case, where one obtains $6L-6N$, which we recall to the reader that it represents a collection of Euclidean polyhedra plus an angle ($\xi$ in the literature) associated with each shared face. In the null case, the

[6]Notice that in the timelike case, the covariant closure condition is a first class constraint in the discrete theory, whereas the continuous Gauss law in the time gauge has a second class part corresponding to the complement to the little group. In this sense, the null case considered here bears some interesting similarities with the continuum theory.

reduced space is much smaller. Since we proved at the beginning of the chapter that a geometric interpretation in terms of null polyhedra is still possible, we must conclude that information on the intrinsic shapes of the polyhedra is being lost in the reduction. In fact, recall from (5.33) that on each face the orbit of $F_2$ changes the value of $b^I$. These transformations can be distinguished in three types. First, those corresponding to translations of the vertices in the null direction, which correspond to isometries. Second, those corresponding to translations of the vertices changing the reconstructed angles (5.9b), and thus the intrinsic geometry of the polyhedron. Third, those incompatible with the closure condition (5.46) and thus breaking the polyhedron apart. The first two types turn out to be first class, while the third type is second class. Therefore, while the interpretation in terms of closed polyhedra is valid, because of the closure condition, the intrinsic shapes at fixed areas are pure gauge, the variables $\omega^0_l$ drop out, and the reduced phase space contains only the conjugated variables $(\epsilon_l j_l, \xi_l)$, constrained by the first class constraint $G_n$. Hence,

$$\mathcal{S}_\Gamma = T^*\mathrm{SO}(2)^L//G_n. \tag{5.51}$$

We now prove these statements.

To diagonalize the Dirac matrix on each node, we first observe that the combinations

$$C^a_{ij} := L^3_i P^a_j - L^3_j P^a_i = 0, \tag{5.52}$$

$$I^a := \sum_{l\in n} P^a = 0 \tag{5.53}$$

are first class. Second, the set

$$C^a_{1i}, \quad i = 2, 3, \cdots, m-1, \qquad P^a_m, \qquad I^a \tag{5.54}$$

is equivalent to all of the $F_2$'s. Therefore, we can take out of (5.48) the two pairs $(P^a_m, \widehat{I}^a)$ as the four second class constraints, and the rest are first class, with $P^a_1, \ldots, P^a_{m-1}$ replaced by (5.52) and (5.53). In particular, the first class constraints contain the global isometry ISO(2) generated by $I^a$ and $G_n$,[7] as well as $2m-4$ additional first class constraints. Their orbits can be used, together with the four second class constraints, to eliminate all of the $\omega^0_l$ from the reduced phase space.

[7] The remaining isometry of the null hypersurface, the boosts $\sum_l K^3_l$, is generated by the $F_1$'s.

To see this explicitly, we compute the action of the first class generators on the spinors, obtaining

$$\mathrm{e}^{\{-\alpha_j(\mathrm{i}C^1_{1j}-C^2_{1j}),\bullet\}}\omega^0_i = \omega^0_i+\delta_{ij}\lambda_j\omega^1_i, \qquad \lambda_i := \alpha_i(\gamma+\mathrm{i})\varepsilon_i j_i, \qquad i=2,\cdots,m-1, \tag{5.55}$$

and

$$\mathrm{e}^{\{-\beta(\mathrm{i}I^1-I^2),\bullet\}}\omega^0_i = \omega^0_i + \beta\omega^1_i. \tag{5.56}$$

Therefore, we can always set to zero all $\omega^0_l$, except when $l = m$. The remaining variable is, however, constrained by the second class closure constraint in (5.46),

$$\omega^0_m = -\frac{z_m|\omega^1_m|^{\mathrm{i}\gamma+1}}{\varepsilon_m j_m^{3/2}} \sum_{i=1}^{m-1} \varepsilon_i j_i^{3/2} \frac{\omega^0_i}{z_i|\omega^1_i|^{\mathrm{i}\gamma+1}}, \tag{5.57}$$

and it is thus automatically vanishing with the previous gauge choice.

Going back to the picture of the null tetrahedron, we see that there are some constraints which generate the global isometries, and others which can arbitrarily move around the vertices of the polyhedron, while preserving the closure and the individual areas. In doing so, we can squash the polyhedron on the spacelike surface and wash away as gauge all information on the intrinsic shapes. This becomes manifest if we rewrite the null polyhedra in terms of the reduced variables. To see this, we fix the $F_1$ gauge $|\omega^1| = 1$ and write the spinors in terms of $z_l$ and the orbits of $C^a_{1i}$ and $I^a$,

$$\omega^A_i = \left((\lambda_i+\beta)\mathrm{e}^{\mathrm{i}\arg(z_i)}, \mathrm{e}^{\mathrm{i}\arg(z_i)}\right), \qquad i \neq 1, m, \tag{5.58}$$

and the $\pi^A_i$ are given by (5.25), assuming all the links are outgoing. Let us consider the case of a 4-valent node, so we do not have to deal with the reconstruction procedure, and we can immediately apply the formulas (5.9). A straightforward calculation then gives

$$E^2_{12} = \gamma\frac{j_1 j_2}{3j_3}\frac{|2\lambda_2+\lambda_3|^2}{\mathrm{Im}(\lambda_2\bar\lambda_3)}, \qquad E_{12}\cdot E_{23} = -2\gamma\varepsilon_1\varepsilon_3 j_2\frac{|\lambda_2|^2+|\lambda_3|^2+\mathrm{Re}\lambda_2\bar\lambda_3}{\mathrm{Im}(\lambda_2\bar\lambda_3)}. \tag{5.59}$$

The intrinsic shape of the null tetrahedron is determined by the independent areas and also the gauge orbits of $C_{1i}$, while being invariant under action of the isometries, in particular $\beta$ drops out.

### 5.4.1 Intrinsic geometry: Euclidean singular structures

We have seen above that the first-class constraints eliminate the intrinsic shapes at fixed areas and we are left with an Abelian reduced phase space $T^*\mathrm{SO}(2)$. The remaining closure condition (5.47) can be solved explicitly, and we are able to provide a complete set of gauge-invariant observables, unlike in the non-Abelian case. This leads to a very simple geometric picture, where the polyhedra give way to a continuous, albeit singular, metric structure.

Consider a closed graph, the extension to an open graph being straightforward. The dimension of the reduced phase space is $2(L-N+1)$, where we took into account the fact that on a closed graph one of the closure conditions is redundant. The gauge invariant information can be associated with the faces of the graph, up to moduli taking into account the possible nonplanarity of the graph. Consider first a planar graph. Its genus being zero, $2(L-N+1)=2(F-1)$, so it is enough to remove the pair of variables associated with a specified face, say for instance the external one in the Schlegel representation of the graph. Denoting $f=1,\dots F-1$, we trade the $\xi_l$ for the gauge-invariant traces of the holonomies,

$$\Phi_f := 2\arccos\left[\frac{1}{2}\mathrm{Tr}\left(\prod_{l\in\partial f} h_l\right)\right] \approx \sum_{l\in\partial f}\eta_l\xi_l, \qquad \{G_n,\Phi_f\}=0, \tag{5.60a}$$

where $\eta_l=\pm$ depending on the consistency of the orientation between the face and the link. The same faces can be used to define an independent set of spins,

$$J_f := \sum_{l\in\partial f}\eta_l j_l. \tag{5.60b}$$

The reason to weigh the sum with the same signs is to have a nice Poisson structure. In fact, for a planar graph the faces can be consistently oriented so that each link is traversed in opposite directions by the sharing faces. A moment of reflection then reveals that the coordinates (5.60) of the gauge-invariant phase space satisfy the brackets

$$\{J_f,\Phi_{f'}\}=L_{ff'}, \tag{5.61}$$

where $L_{ff'}$ is the Laplacian of the dual graph.[8] Proper action-angle variables can then be readily found diagonalizing the Laplacian.

[8]Notice that this graph is open, because of the redundancy of a global closure condition and associated gauge.

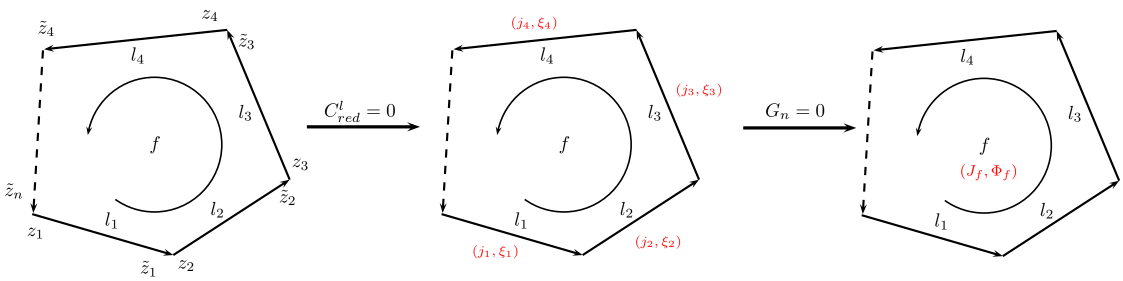


Fig. 5.2 From half links $(z,\tilde{z})$ to links $(j,\xi)$ and to loops $(J,\Phi)$

Since the intrinsic shapes of the polyhedra have been gauged away, the reduced variables describe equivalence classes characterized uniquely by the areas. However, the same variables can be given a simpler and more direct geometric interpretation. Recall that the intrinsic geometry is fully determined by the projection on $S_0$. One can then describe a spacelike 2d geometry using the reduced variables. First of all, we observe that the reduced gauge-invariant holonomies describe an SO(2) transformation on each face. For simplicity, consider first the case of a trivalent graph dual to a triangulation. This structure alone defines the conformal structure of a 2d Regge geometry, that is a collection of deficit angles $2\pi-\Phi_f$ associated with the vertices dual to the faces. Then, the positive real number $J_f$ associates a scale with each face, thus picking a representative of the conformal class. If we pick a local complex

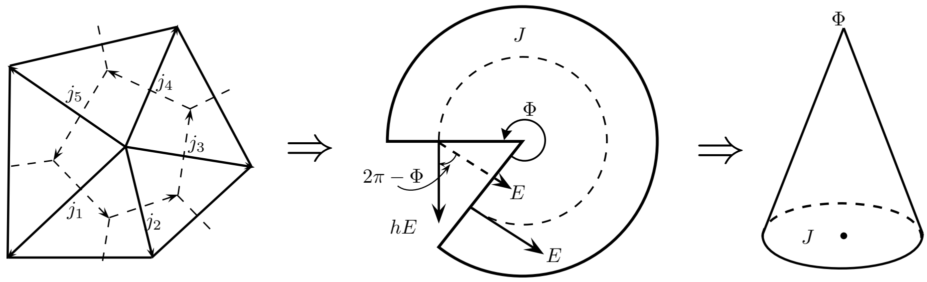


Fig. 5.3 The deficit angle $(2\pi-\Phi)$ and the scale $J$ of the cone

coordinate on each face, say $\zeta_f$, chosen so that the origin is the location of the vertex, we can write the face metric as

$$ds^2 = J_f\, |\zeta_f|^{-\Phi_f/\pi}\, d\zeta \otimes d\bar{\zeta}. \tag{5.62}$$

The resulting geometry is a singular Euclidean structure (e.g. [137]) on $S_0$.

Notice that by assigning these variables we are specifying fewer data than those required by a 2d Regge triangulation, which would be $L = 3(F-2)$. A Regge geometry would be specified uniquely if instead of assigning a scale factor to each dual face, we would do so to each triangle. Since a triangulation has more triangles than vertices, our data are fewer and do not specify a unique 2d Regge geometry. On the other hand, it is more general than a Regge geometry in the sense that it can be extended to any graph and not just a dual to a triangulation, and furthermore because the special case $\Phi_f = 2\pi$, which in Regge would be a pathological infinite spike, is a perfectly regular configuration, which can be interpreted as hyperbolic triangles [137]. Finally, the description has the pleasant features of a natural split into a conformal metric plus scale factors, locally conjugated.

For non-planar graphs, the situation is slightly different, because more than the faces, one should look at the independent cycles, and these cannot be oriented in such a way that each link is traversed at most twice, in opposite directions. Therefore evaluation of Poisson brackets gives a matrix whose off-diagonal entries can have both signs. This can *a priori* still be interpreted as a weighted Laplacian of some dual graph, but one in which the weights have indefinite signature. For instance, in the case of the 4-simplex, the six independent cycles can be chosen so that there is a single $-1$ entry in the adjacency matrix.[9]

### 5.4.2 Extrinsic geometry: $\Xi$ and the role of the embedding

The above description concerns the intrinsic geometry of the hypersurface, which being null is equivalent to a 2d one. However the 3d nature should show up in the study of the extrinsic geometry. As the reader familiar with loop quantum gravity knows, information on the extrinsic geometries is also contained in the reduced phase space, but it is mixed with the intrinsic one. This is the trade-off for the use of real Ashtekar-Barbero variables. It can be extracted once the solution to the secondary simplicity constraint is known, for this provides a specific (in general, nontivial) embedding of

[9]The cycles are e.g. $012, 103, 132, 402, 430, 413$, and the Poisson brackets evaluate to the following matrix,

$$\begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & -1 \\ -1 & -1 & 3 & 0 & 0 & 1 \\ -1 & 0 & 0 & 3 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & -1 & 1 & 0 & -1 & 3 \end{pmatrix}. \tag{5.63}$$

It can still be casted in the form $D - A$ of a certain dual graph, where $D$ and $A$ are respectively the degree and weighted adjacency matrix, with the latter having also negative entries.

the reduced phase space into the Lorentzian one. The same has been argued to happen in the discrete theory in [135], and indeed shown at least for flat dynamics. A similar situation should happen in the present null case, and in order to talk about extrinsic geometry, we need to first understand the dynamics of our null twisted geometries, which we plan to do in future work.

Here we limit ourselves to characterizing the kinematical degrees of freedom suitable to describing the extrinsic geometry. In the timelike case, this was identified on the constraint surface as the (boost) dihedral angle between the normals $N^I$ in adjacent nodes. However, as we stressed above in (5.36b), in the null case the holonomy is a restricted group element already at the level of the constraints surface, and as a consequence, the angle between the normals $N^I$ and $\widetilde{N}^I$ on adjacent nodes vanishes,

$$\widetilde{N} \cdot \Lambda(h) N = 0. \tag{5.64}$$

The vanishing of this scalar product is consistent with the fact that we are dealing with a null hypersurface, and in order to specify a notion of extrinsic geometry, we need an embedding in some nondegenerate four-dimensional spacetime. Indeed, considering also the null hypersurface spanned by the parity transformed vector $\widehat{N}^I$, we can evaluate a nonzero scalar product, given by

$$\mathcal{P}\widetilde{N} \cdot \Lambda(h) N = -\mathrm{e}^{\Xi}, \tag{5.65}$$

where $\Xi$ is the boost rapidity previously defined, and $\Lambda(h)N = \mathrm{e}^{\Xi} N$. The equation above suggests that $\Xi$ should be related to a discretization of a certain free coordinate (denoted $\lambda$ in [144]) used in the null formulation of general relativity [144–146]. We postpone the comparison of our discrete data to a discretization thereof to future work.

We expect that $\Xi$ plays an important role in characterizing the extrinsic geometry, as well as possibly the intrinsic shapes of the null polyhedra. The fact that these quantities have disappeared from the reduced phase space has do to with the fact that in the constrained system considered so far, the simplicity constraints were all first class. Future studies of the dynamics may reveal the presence of secondary constraints, that could turn some or all of the simplicity constraints into second class, e.g. [147]. If that happens, the solutions to the secondary constraints can be interpreted as providing specific, nontivial gauge fixing for the orbits, thus restoring a geometric interpretation for $\Xi$ and the intrinsic shapes through the dynamical embedding.

## 5.5 Quantization and null spin networks

Quantizing the above phase space and its Poisson algebra introduces a notion of spin networks for null hypersurfaces. The reduced phase space $T^*\mathrm{SO}(2)$ with its canonical algebra $\{m, \xi\} = 1$, $m = \varepsilon j$, can immediately be quantized on the Hilbert space $L_2[\mathrm{SO}(2)]$, the space of SO(2) unitary irreducible representations with eigenvalues $m \in \mathbb{Z}/2$, and operator algebra

$$\psi[\xi], \qquad [\hat{m}, e^{\mathrm{i}\hat{\xi}/2}] = \frac{1}{2} e^{\mathrm{i}\hat{\xi}/2}. \tag{5.66}$$

Since $\xi \in [0, 4\pi)$, the eigenvalues of $\hat{m}$ are half-integers, and $e^{\mathrm{i}\hat{\xi}}$ acts as a raising operator,

$$\hat{m}|m\rangle = m|m\rangle, \qquad e^{\mathrm{i}\hat{\xi}/2}|m\rangle = |m + 1/2\rangle, \tag{5.67}$$

the Abelian version of the holonomy-flux algebra. Finally, a basis is given by Fourier modes on the (double cover of the) circle,

$$\psi_m[\xi] = \langle \xi | m \rangle = e^{\mathrm{i}m\xi}. \tag{5.68}$$

This Hilbert space bears similarities with the more familiar one of the harmonic oscillator in action-angle variables, the main difference being that the “Hamiltonian" $\hat{m}$ is not bounded from below, and $m \in \mathbb{Z}/2$.

The gauge-invariant Hilbert space $\mathcal{H}_\Gamma$, corresponding to $\mathcal{S}_\Gamma$, is obtained by taking the tensor product of the states on the links and imposing the closure condition (5.47) on the nodes. The results are Abelian SO(2) spin networks, with trivial intertwiners and flux conservation on the nodes,

$$\Psi_{\Gamma, m_l}[\xi_l] = \otimes_l \psi_{m_l}[\xi_l] \prod_n \delta\Big( \sum_{l^+ \in n} m_l - \sum_{l^- \in n} m_l \Big). \tag{5.69}$$

To appreciate how these simple states can represent quantized null hypersurfaces, it is instructive to derive $\mathcal{H}_\Gamma$ following Dirac's procedure, starting from a Hilbert space for the twistor phase space and its algebra, and then implement the quantized constraints. This procedure will show how such Abelian spin networks are to be embedded in the Lorentz group, and identify $m$ as the helicity quantum number. While being necessary for future studies of dynamics, it will also expose some of the covariance properties of the states, as well as their integrability properties with respect to the $\mathrm{SL}(2,\mathbb{C})$ Haar measure. As in the classical reduction, we proceed in two steps: we first consider the quantization of a single twistor phase space, and the simplicity constraints it satisfies; then, we look at the link phase space and impose the area-matching condition.

For the twistorial Hilbert space we take wave functions $f(\omega) \in L^2[\mathbb{C}^2, \mathrm{d}^4\omega]$, where

$$\mathrm{d}^4\omega = \frac{1}{16}\mathrm{d}\omega_A \wedge \mathrm{d}\omega^A \wedge cc, \tag{5.70}$$

and a Schrödinger representation of the canonical Poisson algebra (5.17),

$$[\hat{\pi}_A, \hat{\omega}^B] = -\mathrm{i}\hbar\delta_A^B, \qquad (\hat{\omega}^A f)(\omega^A) = \omega^A f(\omega^A), \qquad (\hat{\pi}_A f)(\omega^A) = -\mathrm{i}\hbar\frac{\partial}{\partial\omega^A} f(\omega^A). \tag{5.71}$$

A convenient basis for these is provided by homogeneous functions, since they diagonalize the dilatation operator appearing in $F_1$, and carry a unitary, infinite-dimensional representation of the Lorentz group. In particular, since the simplicity constraints are the vanishing of the ISO(2) translation generators $P^a$, it is convenient to take a basis diagonalizing the latter, called the null basis, instead of the canonical basis labeled by the rotational subgroup SU(2). Denoting $p^a$ the eigenvalues, and $p := -p^2 + \mathrm{i}p^1$, the null basis element are the wave functions

$$f_p^{(\rho,k)}(\omega^A) = \frac{1}{2\pi}(\omega^1)^{-k-1+\mathrm{i}\rho}(\bar{\omega}^{\dot{1}})^{k-1+\mathrm{i}\rho} \exp\left[\frac{\mathrm{i}}{2}\left(\frac{\bar{\omega}^{\dot{0}}}{\bar{\omega}^{\dot{1}}}p + \frac{\omega^0}{\omega^1}\bar{p}\right)\right] \tag{5.72}$$

where $(\rho \in \mathbb{R}, k \in \mathbb{Z}/2)$. Details about the $\mathrm{SL}(2,\mathbb{C})$ and ISO(2) representations can be found in the Appendix.

To represent quadratic operators, we introduce the normal ordering

$$:\widehat{\pi\omega} := \frac{1}{2}(\hat{\pi}_A\hat{\omega}^A + \hat{\omega}^A\hat{\pi}_A) = -\mathrm{i}\hbar\left(\omega^A\frac{\partial}{\partial\omega^A} + 1\right). \tag{5.73}$$

With this ordering, the spinorial simplicity constraints (5.23) read

$$\hat{F}_1 = \frac{\hbar}{2}\left((\gamma - \mathrm{i})\omega^A\frac{\partial}{\partial\omega^A} - (\gamma + \mathrm{i})\bar{\omega}^{\dot{A}}\frac{\partial}{\partial\bar{\omega}^{\dot{A}}} - 2\mathrm{i}\right), \qquad \hat{F}_2 = \mathrm{i}\hbar\bar{\omega}^1\frac{\partial}{\partial\omega^0}, \qquad \hat{\bar{F}}_2 = \hat{F}_2^\dagger = \mathrm{i}\hbar\omega^1\frac{\partial}{\partial\bar{\omega}^0}. \tag{5.74}$$

Since on each link these constraints are first class, they can be imposed as operator equations on states. An immediate calculation then gives

$$\hat{F}_1 f_p^{(\rho,k)}(\omega_A) = 0 \quad \Rightarrow \quad \rho = \gamma k, \tag{5.75}$$

$$\hat{F}_2 f_p^{(\rho,k)}(\omega_A) = \hat{\bar{F}}_2 f_p^{(\rho,k)}(\omega_A) = 0 \quad \Rightarrow \quad p = 0, \tag{5.76}$$

so the solutions are the functions

$$f_k(\omega^A) \equiv f_0^{(\gamma k,k)}(\omega^A) = \frac{1}{2\pi}(\omega^1)^{(\mathrm{i}\gamma-1)k-1}(\bar{\omega}^1)^{(\mathrm{i}\gamma+1)k-1}. \tag{5.77}$$

The formula (5.77) defines a state also for $k = 0$, but this case corresponds classically to $\pi\omega = 0$, for which the twistorial description of $T^*\mathrm{SL}(2,\mathbb{C})$ breaks down. To complete the quantization, we need to provide independently the missing state. If we extrapolate (5.77) to $k = 0$ we get a nontivial state, $|\omega^1|^{-2}$, which could pose problems with cylindrical consistency. Hence, we fix instead

$$f_0(\omega^A) = 1. \tag{5.78}$$

The first thing to notice is that in the $p = 0$ sector $P^a$ and $L^3$ commute, thus these functions are also eigenfunctions of $L^3$, with

$$\hat{L}^3 f_k(\omega^A) = \hbar k f_k(\omega^A), \tag{5.79}$$

and thus $k$ is the helicity eigenvalue. Next, the solutions can be expressed in terms of the reduced phase space variable $z$ using (5.28), obtaining

$$f_k(\omega^A) = \frac{1}{2\pi|\omega^1|^2}\left(\frac{\bar{z}}{z}\right)^k. \tag{5.80}$$

Notice the leftover dependence on the non-$F_1$-invariant term $|\omega^1|$. As the action generated by $F_1$ is noncompact, Dirac's quantization does not lead to a proper subspace of functions on the reduced phase space, but rather distributions. Proper function can be defined taking into account the reduced measure.

The reduced measure can be obtained starting from (5.70), imposing the constraints and dividing by the gauge orbits generated by their Hamiltonian vector fields $h_{F_i}$,

$$\mathrm{d}\mu(z) := 4\pi\mathrm{i}\, \iota_{h_{F_i}}(\mathrm{d}^4\omega)\Big|_{F_i=0}, \tag{5.81}$$

where $\iota$ denotes the interior product and $4\pi\mathrm{i}$ is a normalization motivated *a posteriori*.

The Hamiltonian vector fields are

$$h_{F_1} := \{F_1, \bullet\} \approx \frac{1}{2}(1+i\gamma)\omega^0\frac{\partial}{\partial\omega^0} + \mathrm{i}\gamma\omega^1\frac{\partial}{\partial\omega^1} + cc. \qquad h_{F_2} := \{F_2, \bullet\} \approx -2\omega^1\frac{\partial}{\partial\omega^0}. \tag{5.82}$$

Evaluating the interior products gives

$$\iota_{h_{F_2}}\iota_{h_{\bar{F}_2}}[(\mathrm{d}\omega_A \wedge \mathrm{d}\omega^A) \wedge cc.] \approx -4|\omega^1|^2\, \mathrm{d}\omega^1 \wedge \mathrm{d}\bar{\omega}^1, \tag{5.83}$$

and

$$\iota_{h_{F_1}}(\mathrm{d}\omega^1 \wedge \mathrm{d}\bar{\omega}^1) \approx \mathrm{i}\gamma(\omega^1\mathrm{d}\bar{\omega}^1 - \bar{\omega}^1\mathrm{d}\omega^1). \tag{5.84}$$

Putting these results together, and reintroducing $z$, we get

$$\mathrm{d}\mu(z) = -\pi\mathrm{i}|\omega^1|^4\left(\frac{\mathrm{d}\bar{z}}{\bar{z}} - \frac{\mathrm{d}z}{z}\right). \tag{5.85}$$

Notice that the dependence on $\gamma$ has disappeared, and the measure factor $|\omega^1|^4$ perfectly compensates the one in the reduced functions (5.80).

Denoting $\arg(z) = -2\phi$, we have $\mathrm{d}\mu(z) = 4\pi|\omega^1|^4\mathrm{d}\phi$, and the proper reduced Hilbert space is given by

$$f_k(\phi) = \langle\phi|k\rangle = \frac{1}{2\pi}e^{2\mathrm{i}k\phi}, \qquad \langle k'|k\rangle = \frac{1}{\pi}\int_0^\pi \mathrm{d}\phi\, e^{2\mathrm{i}(k-k')\phi} = \delta_{kk'}, \tag{5.86}$$

with $k \in \mathbb{Z}/2$. This half-link Hilbert space already coincides with $L_2[\mathrm{SO}(2)]$, with operator algebra

$$\hat{m}|k\rangle = k|k\rangle, \qquad \exp\left(\mathrm{i}\frac{\hat{\phi}}{2}\right)|k\rangle = |k+\frac{1}{2}\rangle. \tag{5.87}$$

The next step is to consider the two copies of this Hilbert space associated with a link, and impose the area-matching condition, but this procedure will lead trivially to an equivalent Hilbert space.[10] In fact, the quantum version of the area-matching condition on one link corresponding to (5.18) is

$$\hat{C} \equiv :\widehat{\pi\omega}: + :\widehat{\tilde{\pi}\tilde{\omega}}: \tag{5.88}$$

[10] This should not come as a surprise: the whole point of the twistorial parametrization is to encode a nonlinear space (the group manifold) into the solution to a quadratic equation of a linear space (twistor space). But if the starting point is already linear, as in this Abelian case, the procedure is clearly trivial.

and imposing it strongly on a tensor product state $f_k(\omega^A)\otimes f_{\widetilde{k}}(\widetilde{\omega}^A)$ gives immediately $k=-\widetilde{k}$. The state simplifies to

$$F_k(\xi)=\frac{1}{(2\pi)^2}e^{\mathrm{i}k\xi},\qquad \xi\in[0,4\pi). \tag{5.89}$$

The appropriate link measure is also obtained trivially. We have thus recovered the initial $L_2[\mathrm{SO}(2)]$, with holonomy-flux algebra (5.66), and further we can identify the oriented area operator $\hat{m}$ with the helicity and its eigenvalues with the label $k$ of the Lorentz irreps.

Finally, gauge invariance can easily be implemented, and the results are the Abelian spin networks (5.69). Just as ordinary SU(2) spin networks can be interpreted as quantized twisted geometries, the null spin networks represent quantized null twisted geometries.[11]

The embedding allows us to define and evaluate generic Lorentz operators on the reduced Hilbert space. For instance, the first Casimir, classically the oriented area

$$A^2=\frac{1}{2}B_{IJ}B^{IJ}=\frac{\gamma^2}{2(\gamma^2+1)^2}\left[(\gamma-\mathrm{i})^2(\pi\omega)^2+(\gamma+\mathrm{i})^2(\bar{\pi\omega})^2\right]\approx\gamma^2 j^2, \tag{5.90}$$

is the last equality holding onto the constraint surface. The corresponding operator is

$$\hat{A}^2\equiv\frac{-\gamma^2\hbar^2}{2(\gamma^2+1)^2}\left[(\gamma-\mathrm{i})^2\left(\omega^A\frac{\partial}{\partial\omega^A}+1\right)^2+(\gamma+\mathrm{i})^2\left(\bar{\omega}^{\dot{A}}\frac{\partial}{\partial\bar{\omega}^{\dot{A}}}+1\right)^2\right], \tag{5.91}$$

and on the solution space spanned by (5.89) gives

$$\hat{A}^2F_k=\hbar^2\gamma^2k^2F_k. \tag{5.92}$$

## 5.6 Summary

In this chapter, we have exploited the parametrization of LQG on a fixed graph in terms of twistors to describe null hypersurfaces and their quantization in terms of spin networks. Our construction is based on the fact that the twistors appearing in LQG satisfy a restricted incidence relation, in turn determined by the timelike vector appearing in the $3+1$ decomposition of the Plebanski action. Taking this vector to be null forces the geometric interpretation of the theory to lie on a null hypersurface,

[11] In other words, coherent states of (5.69) are peaked on a null twisted geometry.

and the result is a collection of null polyhedra with spacelike faces.

The first result concerns properties of the geometry of null polyhedra. We provided a characterization of the intrinsic shapes in terms of simple bivectors, and showed that the space of shapes at fixed external areas is not a phase space obtained from bivectors and the action generated by the closure constraint, as it is the case for spacelike and timelike polyhedra, because in the null case the reduced closure condition does not generate all of the isometries, but only the helicity part of it. The rest of the closure is second class. The remaining isometries are in turn generated by the (global) action of the simplicity constraints around a node. However, all the simplicity constraints (compatible with the closure condition) are first class, not just their total sum on a node, and their action changes the intrinsic shapes of the null polyhedron. Therefore, the phase space obtained by symplectic reduction is much smaller, algebraically described just by the helicity subgroup, and geometrically an equivalence class of null polyhedra determined only by the areas and their time orientation.

The second result concerns the description of the gauge-invariant phase space. As the helicity subgroup is Abelian, the remaining closure condition can be solved explicitly, and proper action-angle variables given. For planar graphs, these are given by the eigenvectors of the Laplacian of the dual graph. The action-angle variables have a compelling geometric interpretation, as a Euclidean singular structure on the two-dimensional spacelike surface determined by a null foliation of spacetime. In particular, it is naturally decomposed into deficit angles and scale factors, locally conjugated. We are not in a condition to discuss the extrinsic geometry and thus the three-dimensional picture of the null twisted geometries, because this requires the discrete analogue of the secondary simplicity constraints, and it is thus referred to future work on the dynamics. However, we identified the variables in the phase space susceptible of carrying such information.

Finally, we quantized the phase space and its algebra, introducing a notion of null spin networks. They are Abelian spin networks, whose embedding the Lorentz group permits one to identify the Abelian quantum number with the helicity along the null direction of the hypersurface. We derived the spin networks by directly quantizing the reduced phase space, and also by following Dirac's procedure starting from a Hilbert space for twistors. Notice that a loop-inspired quantization of null hypersurfaces has appeared some time ago in [148]. The main difference is that the approach of [148] is based on asymptotic quantities defined at null infinity, whereas here we look at local quantities associated with a fixed graph. Notwithstanding this important difference, a comparison of the two approaches would be valuable.

# Chapter 6

# Conclusion

The first key result is that we studied the large-$j$ asymptotics of the Lorentzian EPRL spin foam amplitude on a 4d simplicial complex with an arbitrary number of simplices. The asymptotics of the spin foam amplitude is determined by the critical configurations. Here we have shown that, given a critical configuration in general, there exists a partition of the simplicial complex into three type of regions $\mathcal{R}_{\text{Nondeg}}, \mathcal{R}_{\text{Deg-A}}, \mathcal{R}_{\text{Deg-B}}$, where the three regions are simplicial sub-complexes with boundaries. The critical configuration implies different types of geometries in different types of regions, i.e. (1) the critical configuration restricted into $\mathcal{R}_{\text{Nondeg}}$ implies a nondegenerate discrete Lorentzian geometry, (2) the critical configuration restricted into $\mathcal{R}_{\text{Deg-A}}$ is degenerate of type-A in our definition of degeneracy, but implies a nondegenerate discrete Euclidean geometry in $\mathcal{R}_{\text{Deg-A}}$, (3) the critical configuration restricted into $\mathcal{R}_{\text{Deg-B}}$ is degenerate of type-B, and implies a vector geometry in $\mathcal{R}_{\text{Deg-B}}$.

With the critical configuration, we subdivided the regions $\mathcal{R}_{\text{Nondeg}}$ and $\mathcal{R}_{\text{Deg-A}}$ into sub-complexes (with boundary) according to their Lorentzian/Euclidean oriented four-volume $V_4(v)$ of the 4-simplices, such that $\text{sgn}(V_4(v))$ is a constant sign on each sub-complex. Then in the each sub-complex, the spin foam amplitude at the critical configuration gives the Regge action in Lorentzian or Euclidean signature respectively in $\mathcal{R}_{\text{Nondeg}}$ or $\mathcal{R}_{\text{Deg-A}}$. The Regge action reproduced here contains a sign prefactor $\text{sgn}(V_4(v))$ related to the oriented 4-volume of the 4-simplices. Therefore the Regge action reproduced here can be viewed a discretized Palatini action with on-shell connection. The asymptotic formula of the spin foam amplitude is given by a sum of the amplitudes evaluated at all possible critical configurations, which are the products of the amplitudes associated to different type of geometries.

The second key result is the calculation of the three-point function from LQG.

We compute the leading order of the three-point function in loop quantum gravity, using the vertex expansion of the Euclidean version of the new spin foam dynamics, in the region of $\gamma < 1$. We find results consistent with Regge calculus in the limit $\gamma \to 0$, $j \to \infty$. I also present the computation of the tree-level three-point function of perturbative quantum general relativity in position space, and discuss the possibility of directly comparing the two results.

Among the problem that we leave open, are the following. (i) We have computed the three-point function in position space from perturbative quantum gravity, treated as a flat-space quantum field theory. We have found that we cannot use here the techniques of [85, 86, 94, 98] to compare this with the loop calculation, because of technical complications in comparing the expansion. These can be traced to the different gauges in which the calculations are performed, to the traceless condition $h^\mu_\mu = 0$ which in general is not satisfied and to the fact that the normals have a non-trivial relation with the field $N^a_n = N^a_n(h)$. (ii) The boundary vacuum state and the parameters $\alpha^{(ab)(cd)}$ introduced in Eq.(4.18) should be better understood and checked. A possibility is to compute them from the first principle, using the unitary condition $\langle W|\Psi_\gamma\rangle = 1$.[1]. (iii) The gauge implicit in the use of the loop formalism is not completely clear. In weak field expansion, the De Donder-like (harmonic) gauge, turns out to be consistent for the lattice graviton propagator [78, 149], and with the radial structure of the loop calculation [150]. But the extension of this to higher $n$-point functions in not clear.

The third key result presented in this thesis is a definition and investigation of a quantization of null hypersurfaces in the context of loop quantum gravity on a fixed graph. The main tool we use is the parametrization of the theory in terms of twistors, which has already proved useful in discussing the interpretation of spin networks as the quantization of twisted geometries. The classical formalism can be extended in a natural way to null hypersurfaces, with the Euclidean polyhedra replaced by null polyhedra with spacelike faces, and SU(2) by the little group ISO(2). The main difference is that the simplicity constraints present in the formalism are all first class, and the symplectic reduction selects only the helicity subgroup of the little group. As a consequence, information on the shapes of the polyhedra is lost, and the result is a much simpler, Abelian geometric picture. It can be described by a Euclidean singular structure on the two-dimensional spacelike surface defined by a foliation of spacetime by null hypersurfaces. This geometric structure is naturally decomposed into a conformal metric and scale factors, forming locally conjugate pairs. Proper action-

[1]private communication with Simone Speziale

angle variables on the gauge-invariant phase space are described by the eigenvectors of the Laplacian of the dual graph. We also identify the variables of the phase space amenable to characterize the extrinsic geometry of the foliation. Finally, we quantize the phase space and its algebra using Dirac's algorithm, obtaining a notion of spin networks for null hypersurfaces. Such spin networks are labeled by SO(2) quantum numbers, and are embedded nontrivially in the unitary, infinite-dimensional irreducible representations of the Lorentz group.

As such, our result are only a first, kinematical step toward our goal of understanding the dynamics of null surfaces in LQG. The applications are many and furnish important motivations to our research program, from the possibility of including dynamical effects in black hole physics and isolated horizons [151], describing the near horizon quantum geometry, to the use in the constraint-free formulation of general relativity on null hypersurfaces. To that end, many nontivial steps are needed. First of all, our analysis needs to be complemented with a continuum canonical analysis of the Plebanski action on a null hypersurface. Second, our geometric description should be compared with the null formulations of general relativity [144–146, 148], and suitable discretizations thereof, in particular, identifying the shear degrees of freedom, and completing the geometric picture developed here with its extrinsic geometry. On a complementary level, one should also investigate what type of spin foams can support the boundary data here studied (see e.g. [130]). We expect this line of research to bring new tools and results to LQG, and to show us how deep the connection with twistors goes.

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# Appendix A

# Conventions

I use $A, B, C, \ldots$ for spinor indices in the left-handed representation; $\dot{A}, \dot{B}, \dot{C}, \ldots$ in the right-handed representation; $I, J, K, \ldots$ the Minkowski indices; and $i, j, k, \ldots$ space indices running from 1 to 3. A bijection between Minkowski space and spinors is given by

$$M^{A\dot{A}} = \frac{\mathrm{i}}{\sqrt{2}} M^I \sigma_I^{A\dot{A}}, \tag{A.1}$$

where $\sigma_I^{A\dot{A}} = (1, \vec{\sigma})$ and $\sigma_{jB}^{A} = \sigma_j^{A\dot{A}} \delta_{B\dot{A}}$ are Pauli matrices. Notice that we are mapping vectors to *anti*-Hermitian matrices consistently with Minkowski metric signature $(-,+,+,+)$. The normalization of the Levi-Civita tensor is $\epsilon^{0123} = 1$. We raise and lower spinor indices with

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{AB}, \quad \epsilon^{AB}\epsilon_{AC} = \delta_C^B, \quad \omega^A = \epsilon^{AB}\omega_B, \quad \omega_A = \epsilon_{BA}\omega^B. \tag{A.2}$$

For the Lorentz algebra, we define

$$[L^i, L^j] = -\mathrm{i}\epsilon^{ijk} L^k, \qquad [L^i, K^j] = -\mathrm{i}\epsilon^{ijk} K^k, \qquad [K^i, K^j] = \mathrm{i}\epsilon^{ijk} L^k \tag{A.3}$$

in terms of rotations $L^i \equiv -\frac{1}{2}\epsilon^{0i}{}_{jk} M^{jk}$ and boosts $K^i \equiv M^{0i}$. We also introduce left-handed ($-$, anti-self-dual) and right-handed ($+$, self-dual) projectors $P_{(\pm)}$, as

$$P_{(\pm)}{}^{IJ}{}_{KL} = \frac{1}{2}\Big(\delta_K^{[I}\delta_L^{J]} \mp \frac{\mathrm{i}}{2}\epsilon^{IJ}{}_{KL}\Big), \tag{A.4}$$

and the left-handed generators are defined as

$$\Pi^i := \mathrm{i} P^{0i}_{(-)IJ} M^{IJ} = \frac{1}{2}(L^i + \mathrm{i} K^i). \tag{A.5}$$

In general the spinorial form of a bivector is

$$B^{IJ} = B^{AB} \epsilon^{\dot{A}\dot{B}} + cc, \tag{A.6}$$

where the left-handed and right-handed parts are

$$B^i = P^{0i}_{(-)IJ} B^{IJ} = \frac{1}{2} B^{AB} \sigma^i_{AB}, \qquad \bar{B}^i = P^{0i}_{(+)IJ} B^{IJ} = \frac{1}{2} \bar{B}^{\dot{A}\dot{B}} \bar{\sigma}^i_{\dot{A}\dot{B}}. \tag{A.7}$$

In terms of the self-dual quantities, the Immirzi shift (5.11) reads

$$\Pi^i = \frac{\gamma + \mathrm{i}}{\gamma} B^i, \qquad \Pi^{AB} = -\frac{1}{2} \frac{\gamma + \mathrm{i}}{\mathrm{i}\gamma} B^{AB}. \tag{A.8}$$

# Appendix B

# Null little group and its representation

## B.1 Null little group

The group ISO(2), sometimes denoted as E(2), is the symmetry group of two-dimensional Euclidean space $\mathbb{R}^2$. It is not compact, nor semisimple. Its Lie algebra $\mathfrak{iso}(2)$ has three generators, $J$, $P^1$ and $P^2$, satisfying

$$[J, P^a] = \mathrm{i}\epsilon^{ab} P^b, \quad [P^a, P^b] = 0, \quad (a, b = 1, 2). \tag{B.1}$$

$J$ is the generator of rotations in $\mathbb{R}^2$, and $P^a$ generate the translations.

This Lie group appears as the little group of a null direction $N^I$ in Minkowski space, with generators related to the Lorentz generators $M^{IJ}$ by

$$X^I = \frac{1}{\sqrt{2}} \epsilon^I{}_{JKL} N^J M^{KL} \tag{B.2}$$

Two canonical choices are $N^I_\pm = (1, 0, 0, \pm 1)/\sqrt{2}$. In this two cases, the generators are,

$$L^3, \qquad P^1_+ \equiv P^1 = L^1 - K^2, \qquad P^2_+ \equiv P^2 = L^2 + K^1, \tag{B.3}$$

$$L^3, \qquad P^1_- \equiv \widehat{P}^1 = L^1 + K^2, \qquad P^2_- \equiv \widehat{P}^2 = L^2 - K^1, \tag{B.4}$$

and satisfy

$$[L^3, P^a_\pm] = \mathrm{i}\epsilon^{ab} P^b_\pm, \qquad [P^a_\pm, P^b_\pm] = 0, \qquad [P^a_\pm, P^b_\mp] = 2\mathrm{i}(\epsilon^{ab} L^3 \pm \delta^{ab} K^3). \tag{B.5}$$

On the fundamental representation $(\mathbf{1/2}, \mathbf{0})$ of $\mathfrak{sl}(2,\mathbb{C})$, the generators are

$$L^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P^1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0 & \mathrm{i} \\ 0 & 0 \end{pmatrix}, \quad \widehat{P}^1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \widehat{P}^2 = \begin{pmatrix} 0 & 0 \\ -\mathrm{i} & 0 \end{pmatrix} \tag{B.6}$$

Exponentiating the generators we get the respective group elements,

$$g^A{}_B = \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\theta} & -p \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta} \end{pmatrix}, \qquad \widehat{g}^A{}_B = \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\theta} & 0 \\ \bar{p} & \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta} \end{pmatrix}, \qquad p := -p^2 + \mathrm{i}p^1. \tag{B.7}$$

## B.2 Unitary irreducible representation of $\mathrm{ISO}(2)$ and $\mathrm{SL}(2,\mathbb{C})$

Unitary irreducible representations (irreps) of ISO(2) are complex function $f$ on $\mathbb{C}$, with basis labeled by the eigenvalues $p^a \in \mathbb{R}$ of $P^a$,

$$f_p(z) = \frac{1}{2\pi}\mathrm{e}^{\frac{\mathrm{i}}{2}(\bar{z}p + z\bar{p})}, \qquad z = -z^2 + \mathrm{i}z^1, \qquad p \equiv -p^2 + \mathrm{i}p^1 \tag{B.8}$$

$$[P^a \circ f_p](z) = p^a f_p(z), \qquad [L^3 \circ f_p](z) = (z\partial_z - \bar{z}\partial_{\bar{z}}) f_p(z) \tag{B.9}$$

The basis is orthogonal,

$$\langle f_p, f_{p'} \rangle = \frac{\mathrm{i}}{2}\int_{\mathbb{C}} \mathrm{d}z \wedge \mathrm{d}\bar{z}\ \overline{f_p(z)} f_{p'}(z) = \frac{\mathrm{i}}{8\pi^2}\int_{\mathbb{C}} \mathrm{d}z \wedge \mathrm{d}\bar{z}\ \mathrm{e}^{\frac{\mathrm{i}}{2}\bar{z}(p'-p) - cc.} = \delta_{\mathbb{C}}(p' - p), \tag{B.10}$$

and complete,

$$\frac{\mathrm{i}}{2}\int_{\mathbb{C}} \mathrm{d}p \wedge \mathrm{d}\bar{p}\ \overline{f_p(z)} f_p(z') = \frac{\mathrm{i}}{8\pi^2}\int_{\mathbb{C}} \mathrm{d}p \wedge \mathrm{d}\bar{p}\ \mathrm{e}^{\frac{\mathrm{i}}{2}\bar{p}(z'-z) - cc.} = \delta_{\mathbb{C}}(z' - z). \tag{B.11}$$

Thanks to these properties, and the induced representations theorem, irreps of $\mathrm{SL}(2,\mathbb{C})$ can be spanned by irreps of ISO(2), with a faithful one-to-one map.

To make the map explicit, recall that irreps of $\mathrm{SL}(2,\mathbb{C})$ are built from homogeneous functions on $\mathbb{C}^2$, $f : \mathbb{C}^2 \to \mathbb{C}$. For the principal series, the homogeneity weights can be conveniently parametrized by the pair $(\rho, k) \in (\mathbb{R}, \mathbb{Z}/2)$ as follows:

$$\forall \lambda \in \mathbb{C}/\{0\},\ f(\lambda\omega^A) = \lambda^{-k-1+\mathrm{i}\rho}\bar{\lambda}^{k-1+\mathrm{i}\rho} f(\omega^A), \tag{B.12}$$

and the unitary irrep $D(g)$ of $g^A{}_B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$ is given by

$$[D(g) \circ f^{(\rho,k)}](\omega^A) = f^{(\rho,k)}(g^A{}_B \omega^B). \tag{B.13}$$

Then, we define $\omega = \omega^0/\omega^1$, and

$$f^{(\rho,k)}(\omega) := f^{(\rho,k)}\left(\frac{\omega^0}{\omega^1}, 1\right) = (\omega^1)^{k+1-\mathrm{i}\rho}(\bar{\omega}^{\bar{1}})^{-k+1-\mathrm{i}\rho} f^{(\rho,k)}(\omega^A). \tag{B.14}$$

By inverting this relation, each homogeneous function $f^{(\rho,k)}(\omega^A) \in H^{(\rho,k)}(\omega^A)$ is uniquely determined by a $f^{(\rho,k)}(\omega)$, and picking in particular the basis (B.8) for the latter, we find

$$f_p^{(\rho,k)}(\omega^A) = (\omega^1)^{-k-1+\mathrm{i}\rho}(\bar{\omega}^{\bar{1}})^{k-1+\mathrm{i}\rho} f_p^{(\rho,k)}(\omega) = \frac{1}{2\pi}(\omega^1)^{-k-1+\mathrm{i}\rho}(\bar{\omega}^{\bar{1}})^{k-1+\mathrm{i}\rho} \mathrm{e}^{\frac{\mathrm{i}}{2}\left(\frac{\bar{\omega}^{\bar{0}}}{\bar{\omega}^{\bar{1}}}p + \frac{\omega^0}{\omega^1}\bar{p}\right)}. \tag{B.15}$$

This defines the null basis for the principal series of SL(2, ℂ) irreps.

The SL(2, ℂ) action is

$$[D(g) \circ f^{(\rho,k)}](\omega) = (c\omega + d)^{-k-1+\mathrm{i}\rho}\overline{(c\omega + d)}^{k-1+\mathrm{i}\rho} f^{(\rho,k)}\left(\frac{a\omega + b}{c\omega + d}\right), \tag{B.16}$$

and the inner product

$$\langle f, h\rangle^{(\rho,k)} = \frac{\mathrm{i}}{2}\int_{\mathbb{C}} \overline{f^{(\rho,k)}(\omega)} h^{(\rho,k)}(\omega) \mathrm{d}\omega \wedge \mathrm{d}\bar{\omega} = \frac{\mathrm{i}}{2}\int_{\mathrm{P}\mathbb{C}^2} \overline{f^{(\rho,k)}(\omega^A)} h^{(\rho,k)}(\omega^A)\omega_A \mathrm{d}\omega^A \wedge \bar{\omega}_{\bar{A}} \mathrm{d}\bar{\omega}^{\bar{A}}. \tag{B.17}$$

In particular,

$$\langle f_p^{(\rho,k)}, f_{p'}^{(\rho,k)}\rangle = \delta_{\mathbb{C}}(p' - p). \tag{B.18}$$