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Stabilité et instabilité dans les problèmes inverses

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Abstract

In this thesis we focus on stability and instability issues in some classical inverse problems for the Schrödinger equation and the acoustic equation in dimension $d \geq 2$. The problems considered are the Gel'fand inverse boundary value problem, the near-field and the far-field inverse scattering problems. Stability and instability results presented in the thesis complement each other and contribute to a better understanding of the nature of the aforementioned problems. In particular, we prove new global stability estimates which explicitly depend on coefficient regularity and energy. In addition, we consider the inverse boundary value problem for the Schrödinger equation at fixed energy with boundary measurements represented as the impedance boundary map (or Robin-to-Robin map). We prove global stability estimates for determining potential from boundary measurements in this impedance representation. Moreover, similar techniques also give a global reconstruction procedure for this problem.

Résumé

Dans cette thèse nous nous intéressons aux questions de stabilité et d'instabilité dans certains problèmes inverses classiques pour l'équation de Schrödinger et l'équation acoustique en dimension $d \geq 2$. Les problèmes considérés sont le problème inverse de Gel'fand de valeurs au bord et les problèmes inverses de diffusion en champ proche et en champ lointain. Les résultats de stabilité et d'instabilité présentés dans cette thèse se complètent mutuellement et contribuent à une meilleure compréhension de la nature des problèmes précités. En particulier, nous démontrons des nouvelles estimations de stabilité globale qui dépendent explicitement de la régularité du coefficient et de l'énergie. En outre, nous considérons le problème inverse de valeurs au bord pour l'équation de Schrödinger à l'énergie fixée avec des mesures frontières représentées comme l'opérateur frontière d'impédance (ou l'opérateur Robin-Robin). Nous démontrons des estimations de stabilité globale pour détermination du potentiel à partir de mesures frontières dans cette représentation d'impédance. De plus, des techniques similaires donnent aussi une procédure de reconstruction globale pour ce problème.

Introduction

This thesis consists of several papers in which we explore different aspects of some classical inverse problems for the Schrödinger equation and the acoustic equation in dimension $d \geq 2$. The problems in question are the Gel'fand inverse boundary value problem, the near-field and the far-field inverse scattering problems.

According to problems under consideration, papers are divided into three groups. The first deals with the Gel'fand inverse boundary value problem in the usual formulation in which the boundary measurements are represented as the Dirichlet-to-Neumann map.

A. M.I. Isaev, R.G. Novikov, *Energy and regularity dependent stability estimates for the Gel'fand inverse problem in multidimensions*, J. of Inverse and Ill-posed Probl., Vol. 20(3), 2012, 313–325.

B. M.I. Isaev, *Exponential instability in the Gel'fand inverse problem on the energy intervals*, J. Inverse Ill-Posed Probl., Vol. 19(3), 2011, 453–473.

C. M.I. Isaev, *Instability in the Gel'fand inverse problem at high energies*, Applicable Analysis, 2012, DOI:10.1080/00036811.2012.731501.

In the second group we continue to consider the inverse boundary value problem for the Schrödinger equation but in the case when the boundary measurements are treated as the impedance boundary map (or Robin-to-Robin map).

D. M.I. Isaev, R.G. Novikov, *Stability estimates for determination of potential from the impedance boundary map*, Algebra and Analysis, Vol. 25(1), 2013, 37–63.

E. M.I. Isaev, R.G. Novikov, *Reconstruction of a potential from the impedance boundary map*, Eurasian Journal of Mathematical and Computer Applications, Vol. 1(1), 2013, 5–28.

The third group of papers is devoted to inverse scattering problems.

F. M.I. Isaev, R.G. Novikov, *New global stability estimates for monochromatic inverse acoustic scattering*, SIAM Journal on Mathematical Analysis, Vol. 45(3), 2013, 1495–1504.

G. M.I. Isaev, *Exponential instability in the inverse scattering problem on the energy interval*, Func. Anal. i ego Pril., Vol. 47(3), 2013, 28–36.

H. M.I. Isaev, *Energy and regularity dependent stability estimates for near-field inverse scattering in multidimensions*, Journal of Mathematics, Hindawi Publishing Corp., 2013, DOI:10.1155/2013/318154.

Many physical processes are described by models consisting of systems of partial differential equations. The coefficients of these equations describe the properties of the medium where these processes take place. The so-called inverse problem lies in finding unknown parameters of the model on the basis of the observed data. Most of inverse problems are known to be ill-posed in general, see **[8]**, **[46]** for an introduction to this theory. This weakness constitutes a severe difficulty for the numerical treatment. Theoretical stability and instability results enables us to quantify ill-posedness of such problems.

One important example is the problem of reconstructing from boundary measurements unknown potential v of the equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \tag{0.1}$$

where

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2,\\ &\quad \text{with } \partial D \in C^2, \end{aligned} \tag{0.2}$$

$$v \in \mathbb{L}^\infty(D). \tag{0.3}$$

Equation (0.1) can be considered as the stationary Schrödinger equation of quantum mechanics at fixed energy E. Equation (0.1) at fixed E arises also in acoustics and electrodynamics.

As the observed data we consider the Cauchy data set $\mathcal{C}_v(E)$ defined by

$$\mathcal{C}_v(E) = \left\{ \left(\psi|_{\partial D}, \frac{\partial\psi}{\partial\nu}|_{\partial D}\right) : \begin{array}{r} \text{for all sufficiently regular solutions } \psi \text{ of} \\ \text{equation (0.1) in } \bar{D} = D \cup \partial D \end{array} \right\},$$

where ν is the outward normal to ∂D.

We consider the following inverse boundary value problem:

PROBLEM 1. *Given $\mathcal{C}_v(E)$, find v.*

Problem 1 can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation (see **[30]**, **[54]**). Note that in the initial Gel'fand formulation energy E was not fixed and boundary measurements were considered as an operator relating $\psi|_{\partial D}$ and $\frac{\partial\psi}{\partial\nu}|_{\partial D}$ for ψ satisfying (0.1). At zero energy this problem can be considered also as a generalization of the Calderón problem of the electrical impedance tomography (see **[20]**, **[54]**).

Under additional assumption that

$$E \text{ is not a Dirichlet eigenvalue for operator } -\Delta + v \text{ in } D \tag{0.4}$$

the Cauchy data set $\mathcal{C}_v(E)$ can be represented as the graph of the Dirichlet-to-Neumann map $\hat{\Phi}_v(E)$ defined by

$$\hat{\Phi}_v(E)(\psi|_{\partial D}) = \frac{\partial\psi}{\partial\nu}|_{\partial D}$$

for all sufficiently regular solutions ψ of (0.1) in $\bar{D}$.

The usual formulation of Problem 1 is the following:

PROBLEM 1a. *Given $\hat{\Phi}_v(E)$, find v.*

We consider different variations of this problem: the Dirichlet-to-Neumann map $\hat{\Phi}_v(E)$ can be given either for some fixed energy E satisfying (0.4) or on the union of the energy intervals $S = \bigcup_{j=1}^{K} I_j$ such that condition (0.4) is fulfilled for any $E \in S$.

There is a wide literature on Problem 1 (especially for Problem 1a). This includes, in particular, the following issues: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness for Problem 1a in the case of energy intervals was obtained for the first time by R.G. Novikov (see Theorem 5.4 in [**37**]). Some global reconstruction method for Problem 1 was proposed for the first time in [**37**] also. Global uniqueness theorems and global reconstruction methods in the case of fixed energy were given for the first time in [**54**] in dimension $d \geq 3$ and in [**16**] in dimension $d = 2$.

Global stability estimates for Problem 1a were given for the first time in [**1**] in dimension $d \geq 3$ and in [**67**] in dimension $d = 2$. A principal improvement of the result of [**1**] was given recently in [**65**] (for the zero energy case): stability of [**65**] optimally increases with increasing regularity of v.

Note that for the Calderón problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [**75**] for $d \geq 3$ and in [**50**] for $d = 2$. Global logarithmic stability estimates for this problem were given for the first time in [**1**] for $d \geq 3$ and [**47**] for $d = 2$. Principal increasing of global stability of [**1**], [**47**] for the regular coefficient case was found in [**65**] for $d \geq 3$ and [**72**] for $d = 2$. In addition, for the case of piecewise constant or piecewise real analytic conductivity the first uniqueness results for the Calderón problem in dimension $d \geq 2$ were given in [**26**], [**43**]. Lipschitz stability estimate for the case of piecewise constant conductivity was proved in [**3**], [**6**] and additional studies in this direction were fulfilled in [**11**], [**69**]. An abstract general schema for investigating similar stability questions in different inverse problems is given in [**15**].

The optimality of the logarithmic stability estimates of [**1**], [**47**] with their principal effectivization of [**65**], [**72**] was shown in [**48**] (up to the value of the exponent). Note also that similar instability results for the elliptic inverse problem concerning the determination of inclusions in a conductor by different kinds of boundary measurements and the inverse obstacle acoustic scattering problems were given in [**24**], where some general scheme for investigating questions of this type of instability has been also proposed.

Problem 1 can be also considered for the case when the observed data are given only on a part of the boundary, see, for example, [**2**], [**5**], [**17**], [**25**], [**40**], [**42**] and references therein. In addition, Problem 1 can be also considered in its versions on manifolds, see, for example, [**9**], [**10**], [**35**], [**38**], [**39**], [**73**] and references therein.

On the other hand, it was found in [**58**], [**60**] (see also [**63**], [**68**]) that for inverse problems for the Schrödinger equation at fixed energy E in dimension $d \geq 2$ (like

Problem 1) there is a Hölder stability modulo an error term rapidly decaying as $E \to +\infty$ (at least for the regular coefficient case). This phenomena of increasing stability with respect to some parameter such as energy or wave number has been also observed numerically (see, for example, [**22**] for the inverse obstacle scattering problem).

In addition, for Problem 1a for $d = 3$, global energy dependent stability estimates changing from logarithmic type to Hölder type for high energies were given in [**41**]. However, there is no efficient stability increasing with respect to increasing coefficient regularity in these results of [**41**]. An additional study, motivated by [**41**], [**65**], was given in [**51**].

In Paper **A** we prove new global Hölder-logarithmic stability estimates for Problem 1a in dimension $d \geq 3$ for the regular coefficient case. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity.

In order to formulate the necessary assumptions we consider the Sobolev spaces:

$$W^{m,1}(\mathbb{R}^d) = \{v : \ \partial^J v \in \mathbb{L}^1(\mathbb{R}^d), \ |J| \leq m\}, \ m \in \mathbb{N} \cup \{0\}, \tag{0.5}$$

where $J \in (\mathbb{N} \cup \{0\})^d$, $|J| = \sum_{i=1}^{d} J_i$, $\partial^J v(x) = \dfrac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}$. The norm in the Sovolev space $W^{m,1}(\mathbb{R}^d)$ is defined by $||v||_{m,1} = \max\limits_{|J| \leq m} ||\partial^J v||_{\mathbb{L}^1(\mathbb{R}^d)}$.

Let

$$s_0 = \frac{m-d}{m}, \quad s_1 = \frac{m-d}{d}, \quad s_2 = m - d.$$

Theorem 0.1 (Paper **A**). *Let D satisfy (0.2), where $d \geq 3$. Let $v_j \in W^{m,1}(\mathbb{R}^d)$, $m > d$, $\operatorname{supp} v_j \subset D$ and $||v_j||_{m,1} \leq N$ for some $N > 0$, $j = 1, 2$. Let v_1, v_2 satisfy (0.4) for some fixed real E. Let $\hat{\Phi}_1(E)$ and $\hat{\Phi}_2(E)$ denote the Dirichlet-to-Neumann maps for v_1 and v_2, respectively. Then*

$$||v_2 - v_1||_{\mathbb{L}^\infty(D)} \leq C \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \quad 0 < s \leq s_1, \tag{0.6}$$

where $C = C(N, D, m, s, E) > 0$, $\delta = ||\hat{\Phi}_2(E) - \hat{\Phi}_1(E)||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)}$. In addition, for $E \geq 0$, $\tau \in (0, 1)$ and any $\alpha, \beta \in [0, s_1]$, $\alpha + \beta \leq s_1$,

$$||v_2 - v_1||_{\mathbb{L}^\infty(D)} \leq A(1 + \sqrt{E})\delta^\tau + B(1 + \sqrt{E})^{-\alpha} \left(\ln\left(3 + \delta^{-1}\right)\right)^{-\beta}, \tag{0.7}$$

where $A = A(N, D, m, \tau) > 0$ and $B = B(N, D, m, \tau) > 0$.

Remark 0.1. Estimate (0.6) for $s = s_0$ is a variation of the aforementioned logarithmic stability result of [**1**] (see also [**65**]). This result was improved in [**65**] for $E = 0$ and $d = 3$: estimate (0.6) holds for $s = s_2$. A principal advantage with respect to the result of [**1**] is that

$$s_1 \to +\infty \text{ and } s_2 \to +\infty \quad \text{as} \quad m \to +\infty.$$

In particular cases, Hölder-logarithmic stability estimate (0.7) becomes coherent (although less strong) with respect to results of [**60**], [**63**], [**65**]. Concerning two-dimensional analogs of results of Theorem 0.1, see [**58**], [**68**], [**71**], [**72**].

The proof of Theorem 0.1 is technically very similar to the proof of estimate (0.6) for $s = s_0$, see [**1**], [**65**]. The fundamental object used is a special family of sulutions $\psi(x, k)$ of equation (0.1), depending on a complex parameter $k \in \mathbb{C}^d$ such that $k^2 = k_1^2 + k_2^2 + \ldots + k_d^2 = E \in \mathbb{R}$. These functions were introduced for the first time by L.D. Faddeev [**28**], [**29**] in quantum scattering and are also called *complex geometrical optics solutions*. The Faddeev functions are (non-analytic) continuation to the complex domain of functions of the classical scattering theory for the Schrödinger equation: their main property is an exponential asymptotic condition with a linear phase depending on the complex parameter k.

We would like to mention that in the case of $E = 0$, under the assumptions of Theorem 0.1, according to the aforementioned results of [**48**], estimate (0.6) can not hold with $s > m(2d-1)/d$ for real-values potentials and with $s > m$ for complex potentials.

In Paper **B** we give an extention of the instability estimates of [**48**] to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals.

Let us call interval $I = [a, b] \subset \mathbb{R}$ as a σ-regular interval if for any potential $v \in \mathbb{L}^\infty(D)$ with $||v||_{\mathbb{L}^\infty(D)} \leq \sigma$ and any $E \in I$ condition (0.4) is fulfilled. Let $B(x, r)$ denote the open ball of radius r centred at x.

THEOREM 0.2 ([Paper **B**]). *Let $D = B(0,1) \subset \mathbb{R}^d$, where $d \geq 2$. Let $\sigma > 0$. Let $S = \bigcup_{j=1}^{K} I_j$ be the union of σ-regular intervals. Then for any $m > 0$ and any $\mu \geq 0$ there is a constant $\beta > 0$, such that for any $\epsilon \in (0, \sigma/3)$ and any real-valued $v_0 \in C^m(D)$ with $||v_0||_{\mathbb{L}^\infty(D)} \leq \sigma/3$ and $\operatorname{supp} v_0 \subset B(0, 1/3)$ there exist real-valued potentials $v_1, v_2 \in C^m(D)$, also supported in $B(0, 1/3)$, such that*

$$\begin{aligned}
&\sup_{E \in S} \Big(||\hat{\Phi}_1(E) - \hat{\Phi}_2(E)||_{H^{-\mu} \to H^{\mu}} \Big) \leq \exp\Big(-\epsilon^{-\frac{1}{2m}} \Big), \\
&||v_1 - v_2||_{\mathbb{L}^\infty(D)} \geq \epsilon, \\
&||v_i - v_0||_{C^m(D)} \leq \beta, \quad i = 1, 2, \\
&||v_i - v_0||_{\mathbb{L}^\infty(D)} \leq \epsilon, \quad i = 1, 2,
\end{aligned} \tag{0.8}$$

where $\hat{\Phi}_1(E)$, $\hat{\Phi}_2(E)$ are the Dirichlet-to-Neumann maps for v_1 and v_2, respectively, and $H^\mu = W^{\mu,2}$ denotes the standard Sobolev space on the sphere $S^{d-1} = \partial D$.

The proof of Theorem 0.2 is based on a purely topological argument, which goes back to A.N. Kolmogorov and V.M. Tihomirov [**44**].

In addition to Theorem 0.2, we consider an explicit instability example with a complex potential given in [**48**]. We show that it gives exponential instability even in case of Dirichlet-to-Neumann map given on the energy intervals. Let us consider the cylindrical variables $(r_1, \theta, x') \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^{d-2}$, where $x' = (x_3, \ldots, x_d)$, $r_1 \cos\theta = x_1$ and $r_1 \sin\theta = x_2$. We take some function $\phi \in C^\infty(\mathbb{R}^2)$ supported in $B(0, 1/3) \cap \{x_1 > 1/4\}$ such that $||\phi||_{\mathbb{L}^\infty} = 1$.

THEOREM 0.3 ([Paper **B**]). *Let $D = B(0,1) \subset \mathbb{R}^d$, where $d \geq 2$. For $\sigma > 0$, $m > 0$ and integer $n > 0$ consider the union $S = \bigcup_{j=1}^{K} I_j$ of σ-regular intervals and define the complex potential*

$$v_{nm}(x) = \frac{\sigma}{3} n^{-m} e^{in\theta} \phi(r_1, |x'|).$$

Then $||v_{mn}||_{\mathbb{L}^\infty(D)} = \frac{\sigma}{3} n^{-m}$ and for every $\mu \geq 0$ and $m > 0$ there are constants c, c' such that $||v_{mn}||_{C^m(D)} \leq c$ and for every n

$$\sup_{E \in S} \left(||\hat{\Phi}_{mn}(E) - \hat{\Phi}_0(E)||_{H^{-\mu} \to H^\mu} \right) \leq c' 2^{-n/4},$$

where $\hat{\Phi}_{mn}(E)$, $\hat{\Phi}_0(E)$ are the DtN maps for v_{mn} and $v_0 \equiv 0$, respectively.

In some important sense, this is stronger than Theorem 0.2. Indeed, if we take $\epsilon = \frac{\sigma}{3} n^{-m}$ we obtain (0.8) with $\exp(-C\epsilon^{-1/m})$ in the right-hand side. An explicit real-valued counterexample should be difficult to find from v_{mn}. This is due to nonlinearity of the map $v \to \hat{\Phi}_v(E)$.

REMARK 0.2. For sufficient large μ one can see that

$$||\hat{\Phi}_1(E) - \hat{\Phi}_2(E)||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)} \leq c_\mu \, ||\hat{\Phi}_1(E) - \hat{\Phi}_2(E)||_{H^{-\mu} \to H^\mu}.$$

So Theorems 0.2 and 0.3 imply, in particular, that the estimate

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq C \sup_{E \in S} \left(\ln(3 + \delta(E)^{-1}) \right)^{-s},$$

where $C = C(N, D, m, S)$ and $\delta(E) = ||\hat{\Phi}_1(E) - \hat{\Phi}_2(E)||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)}$, can not hold with $s > 2m$ for real-valued potentials and with $s > m$ for complex potentials. Thus Theorems 0.2 and 0.3 show optimality of logarithmic stability results of [**1**], [**47**], [**65**], [**67**], [**72**] in considerably stronger sense that results of [**48**].

Paper **C** is devoted to the substantiation of optimality of Hölder-logarithmic stability estimate (0.7). The main result is the following theorem:

THEOREM 0.4 ([Paper **C**]). *Let $D = B(0,1) \subset \mathbb{R}^d$, where $d \geq 2$. Then for any fixed constants $A, B, \kappa, \tau, \varepsilon > 0$, $m > d$ and $s_3 > m$ there exist some energy level $E > 0$ and some potential $v \in C^m(D)$ such that condition* (0.4) *holds for potentials v and $v_0 \equiv 0$, simultaneously, supp $v \subset D$, $\|v\|_{\mathbb{L}^\infty(D)} \leq \varepsilon$, $\|v\|_{C^m(D)} \leq c$, where $c = c(d, m) > 0$, but*

$$||v - v_0||_{\mathbb{L}^\infty(D)} > A(1 + \sqrt{E})^\kappa \delta^\tau + B(1 + \sqrt{E})^{2(s - s_3)} \left(\ln\left(3 + \delta^{-1}\right) \right)^{-s}$$

for any $s \in [0, s_3]$, where $\hat{\Phi}(E)$, $\hat{\Phi}_0(E)$ are the Dirichlet-to-Neumann maps for v and v_0, respectively, and $\delta = ||\hat{\Phi}(E) - \hat{\Phi}_0(E)||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)}$.

REMARK 0.3. Theorem 0.4 shows, in particular, that estimate (0.7) can not hold for any $\alpha, \beta \geq 0$, $\alpha + 2\beta > 2m$. In similar sense, as a corollary of Theorem 0.4, one can obtain also an optimality of the stability results of [**58**], [**60**], [**63**], [**68**].

The proof of Theorem 0.4 is based on instability examples with complex potentials. Examples of this type are considered in Theorem 0.3 and were given for the first time in [**48**] for showing exponential instability in Problem 1 in the zero energy case.

Now we consider another representation of the Cauchy data set $\mathcal{C}_v(E)$. Let us define the impedance boundary map $\hat{M}_{\alpha,v}(E)$ by

$$\hat{M}_{\alpha,v}(E)\left([\psi]_\alpha\right) = [\psi]_{\alpha-\pi/2}$$

for all sufficiently regular solutions ψ of equation (0.1) in $\bar{D} = D \cup \partial D$, where

$$[\psi]_\alpha = [\psi(x)]_\alpha = \cos\alpha\,\psi(x) - \sin\alpha\,\frac{\partial\psi}{\partial\nu}|_{\partial D}(x), \quad x \in \partial D,\ \alpha \in \mathbb{R} \tag{0.9}$$

and ν is the outward normal to ∂D. One can show(see Lemma 3.2 in Paper **D**) that there is not more than a countable number of $\alpha \in \mathbb{R}$ such that E is an eigenvalue for the operator $-\Delta + v$ in D with the boundary condition

$$\cos\alpha\,\psi|_{\partial D} - \sin\alpha\,\frac{\partial\psi}{\partial\nu}|_{\partial D} = 0. \tag{0.10}$$

Therefore, for any energy level E we can assume that for some fixed $\alpha \in \mathbb{R}$

$$\begin{aligned} &E \text{ is not an eigenvalue for the operator } -\Delta + v \text{ in } D \\ &\qquad\qquad\qquad\qquad \text{with boundary condition (0.10)} \end{aligned} \tag{0.11}$$

and, as a corollary, $\hat{M}_{\alpha,v}(E)$ can be defined correctly.

We consider $\hat{M}_{\alpha,v}(E)$ as an operator representation of all possible boundary measurements for the physical model described by (0.1). Note that the impedance boundary map $\hat{M}_{\alpha,v}(E)$ is reduced to the Dirichlet-to-Neumann(DtN) map if $\alpha = 0$ and is reduced to the Neumann-to-Dirichlet(NtD) map if $\alpha = \pi/2$. The map $\hat{M}_{\alpha,v}(E)$ can be called also as the Robin-to-Robin map. General Robin-to-Robin map was considered, in particular, in [**31**].

Problem 1 can be formulated as follows:

PROBLEM 1b. *Given $\hat{M}_{\alpha,v}(E)$ for some fixed E and α, find v.*

It should be noted that in most of previous works on inverse boundary value problems for equation (0.1) at fixed E it was assumed in one way or another that E is not a Dirichlet eigenvalue for the operator $-\Delta+v$ in D, see [**1**], [**48**], [**54**], [**65**], [**67**], [**68**], [**72**]. Nevertheless, the results of [**16**] can be considered as global uniqueness and reconstruction results for Problem 1b in dimension $d = 2$ with general α.

In Paper **D** we give global stability estimates for Problem 1b in dimension $d \geq 2$ with general α. The cases $d \geq 3$ and $d = 2$ are treated separately.

For $d \geq 3$ we assume for simplicity that

$$v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d,\ \operatorname{supp} v \subset D, \tag{0.12}$$

where $W^{m,1}$ denotes the Sobolev space of m-times smooth functions in $\mathbb{L}^1$, see (0.5).

THEOREM 0.5 (Paper **D**). *Let D satisfy (0.2), where $d \geq 3$. Let v_1, v_2 satisfy (0.12) and (0.11) for some fixed E and α. Let $||v_j||_{m,1} \leq N$, $j = 1, 2$, for some $N > 0$. Let $\hat{M}_{\alpha,v_1}(E)$, $\hat{M}_{\alpha,v_2}(E)$ denote the impedance boundary maps for v_1 and v_2, respectively. Then*

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s}, \quad 0 < s \leq (m-d)/m, \tag{0.13}$$

where $C_\alpha = C_\alpha(N, D, m, s, E)$, $\delta_\alpha = ||\hat{M}_{\alpha,v_1}(E) - \hat{M}_{\alpha,v_2}(E)||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)}$.

For $d = 2$ we assume for simplicity that

$$v \in C^2(\bar{D}), \quad \operatorname{supp} v \subset D. \tag{0.14}$$

THEOREM 0.6 (Paper **D**). *Let D satisfy (0.2), where $d = 2$. Let v_1, v_2 satisfy (0.14) and (0.11) for some fixed E and α. Let $||v_j||_{C^2(\bar{D})} \leq N$, $j = 1, 2$, for some $N > 0$. Let $\hat{M}_{\alpha,v_1}(E)$, $\hat{M}_{\alpha,v_2}(E)$ denote the impedance boundary maps for v_1 and v_2, respectively. Then*

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s} \left(\ln\left(3 \ln\left(3 + \delta_\alpha^{-1}\right)\right)\right)^2, \quad 0 < s \leq 3/4,$$

where $C_\alpha = C_\alpha(N, D, s, E)$, $\delta_\alpha = ||\hat{M}_{\alpha,v_1}(E) - \hat{M}_{\alpha,v_2}(E)||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)}$.

REMARK 0.4. In the case of $\alpha = 0$ (DtN case) results of Theorems 0.5 and 0.6 are reduced to logarithmic stability estimates for Problem 1a. Estimate (0.13) with $\alpha = 0$ is a variation of the result of [**1**] (see also [**65**]). Theorem 0.6 for $\alpha = 0$ was given in [**67**] with $s = 1/2$ and in [**70**] with $s = 3/4$.

Theorems 0.5 and 0.6 imply the following corollary:

COROLLARY 0.7 (Paper **D**). *Let D satisfy (0.2). Let v_1, v_2 satisfy (0.12) or (0.14) for $d \geq 3$ or $d = 2$, respectively. Then*

- *for $d \geq 3$ and $0 < s \leq (m-d)/m$*

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq \min_{\alpha \in \mathbb{R}} C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s}, \tag{0.15}$$

- *for $d = 2$ and $0 < s \leq 3/4$,*

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq \min_{\alpha \in \mathbb{R}} C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s} \left(\ln\left(3 \ln\left(3 + \delta_\alpha^{-1}\right)\right)\right)^2,$$

where C_α and δ_α at fixed α are the same that in Theorem 0.5 or Theorem 0.6 for $d \geq 3$ or $d = 2$, respectively.

Actually, Corollary 0.7 can be considered as global stability estimate for determining potential v from its Cauchy data set $\mathcal{C}_v(E)$ for equation (0.1) at fixed energy E, where $d \geq 2$.

The proofs of Theorems 0.5 and 0.6 are based on the following identity:

THEOREM 0.8 (Paper **D**). *Let D satisfy (0.2). Let two potentials v_1, v_2 satisfy (0.3), (0.11) for some fixed E and α. Let $\hat{M}_{\alpha,v_1} = \hat{M}_{\alpha,v_1}(E)$, $\hat{M}_{\alpha,v_2} = \hat{M}_{\alpha,v_2}(E)$ denote the impedance boundary maps for v_1, v_2, respectively. Then*

$$\int\limits_D (v_1 - v_2)\,\psi_1\psi_2\,dx = \int\limits_{\partial D} [\psi_1]_\alpha \left(\hat{M}_{\alpha,v_1} - \hat{M}_{\alpha,v_2}\right) [\psi_2]_\alpha dx \tag{0.16}$$

for all sufficiently regular solutions ψ_1 and ψ_2 of equation (0.1) in $\bar{D}$ with $v = v_1$ and $v = v_2$, respectively, where $[\psi]_\alpha$ is defined by (0.9).

Identity (0.16) for $\alpha = 0$ is reduced to Alessandrini's identity (Lemma 1 of [**1**]).

In addition, an important role in the proofs of Theorem 0.5 and 0.6 is played by the aforementioned Faddeev functions and their analogs for $d = 2$ which go back to Bukhgeim's paper [**16**].

REMARK 0.5. The stability estimates of Theorems 0.5 and 0.6 admit principal improvement in the sense described in [**65**], [**66**], [**72**]. In particular, Theorem 0.5 with $s = m - d$ (for $d = 3$ and $E = 0$) follows from results presented in Paper **D** and results presented in Section 8 of [**65**]. In addition, estimates (0.13) and (0.15) for $s = (m - d)/d$ admit a proof technically very similar to the proof of Theorem 0.1.

Furthermore, proceeding from the methods used in the proofs of Theorem 0.5 and 0.6 (in particular, Theorem 0.8), one can obtain the following corollary:

COROLLARY 0.9 (Paper **D**). *Under assumptions (0.2) and (0.3), real-valued potential v is uniquely determined by its Cauchy data $\mathcal{C}_v(E)$ at fixed real energy E.*

Actually, under additional assumptions (0.12), (0.14) for $d \geq 3$ and $d = 2$, respectively, Corollary 0.9 follows from Corollary 0.7 immediately.

To our knowledge the result of Corollary 0.9 for $d \geq 3$ was not yet completely proved in the literature.

We would like to note that the inverse scattering problem, i.e. the reconstruction of a potential in the Schrödinger equation from its (generalised) scattering amplitude, was proposed and studied much earlier then Problem 1. This problem comes initially from quantum mechanics (see [**29**]), but afterwards it appeared in several other context, for instance nonlinear evolution equations (see [**7**], [**37**, Chapter 1], [**32**] for a survey of results).

In Paper **E** we give formulas and equations for finding (generalized) scattering data from the impedance boundary map $\hat{M}_{\alpha,v}(E)$ with general α. Combining these results with results of [**32**], [**33**], [**37**], [**50**], [**55**]-[**59**], [**60**]-[**64**], [**68**], we obtain efficient reconstruction methods for Problem 1 in multidimensions with general α. To our knowledge these results are new already for the Neumann-to-Dirichlet case.

Our approach for finding a potential v in the domain D from its impedance boundary map $\hat{M}_{\alpha,v}(E)$ at fixed E and α is shown by the following schema:

(1) $v^0 \to S_E^0, \hat{M}_{\alpha,v^0}(E)$ via direct problem methods,
(2) $\hat{M}_{\alpha,v^0}(E), \hat{M}_{\alpha,v}(E), S_E^0 \to S_E$ as described in Paper **E**,
(3) $S_E \to v$ as described in [**32**], [**33**], [**37**], [**55**]-[**59**], [**60**]-[**64**], [**68**],

where S_E and S_E^0 denote (generalized) scattering data for the unknown potential v and some known base potential v^0, respectively. Step (2) consists in solving Fredholm linear integral equations of the second type and using explicit formulas (for more detailed information, see Paper **E**).

In particular, in Paper **E** we give the first mathematically justified approach for reconstructing coefficient v from boundary measurements for (0.1) for $d \geq 3$ via inverse scattering without the assumption that E is not a Dirichlet eigenvalue for $-\Delta + v$ in D. In addition, numerical efficiency of related inverse scattering techniques was shown in [**4**], [**14**], [**18**], [**19**]; see also [**13**].

Now we proceed to analyzing of some scattering problems. First, we consider the three-dimensional stationary acoustic equation at frequency ω in an inhomogeneous medium with refractive index n

$$\Delta\psi + \omega^2 n(x)\psi = 0, \quad x \in \mathbb{R}^3,\ \omega > 0, \tag{0.17}$$

where

$$\begin{aligned} (1-n) \in W^{m,1}(\mathbb{R}^3) &\text{ for some } m > 3,\\ \operatorname{Im} n(x) \geq 0, &\quad x \in \mathbb{R}^3,\\ \operatorname{supp}(1-n) \subset B_{r_1} &\text{ for some } r_1 > 0, \end{aligned} \tag{0.18}$$

where $W^{m,1}(\mathbb{R}^3)$ denotes the Sobolev space of m-times smooth functions in $\mathbb{L}^1$ (see for details (0.5)) and $B_r = B(0,r)$ is the open ball of radius r centered at 0.

Let $G^+(x,y,\omega)$ denote the Green function for the operator $\Delta + \omega^2 n(x)$ with the Sommerfeld radiation condition:

$$\begin{aligned} \big(\Delta + \omega^2 n(x)\big)\, G^+(x,y,\omega) &= \delta(x-y),\\ \lim_{|x|\to\infty} |x| \left(\frac{\partial G^+}{\partial |x|}(x,y,\omega) - i\omega G^+(x,y,\omega)\right) &= 0,\\ \text{uniformly for all directions } \hat{x} &= x/|x|,\\ x, y \in \mathbb{R}^3,\ \omega &> 0. \end{aligned} \tag{0.19}$$

It is know that, under assumptions (0.18), the function G^+ is uniquely specified by (0.19), see, for example, [**36**], [**23**].

We consider, in particular, the following near-field inverse scattering problem for equation (0.17):

Problem 2. *Given G^+ on $\partial B_r \times \partial B_r$ for fixed $\omega > 0$ and $r > r_1$, find n on B_{r_1}.*

Consider also the solutions $\psi^+(x,k)$, $x \in \mathbb{R}^3,\ k \in \mathbb{R}^3,\ k^2 = \omega^2$, of equation (0.17) specified by the following asymptotic condition:

$$\begin{gathered}\psi^+(x,k) = e^{ikx} - 2\pi^2 \frac{e^{i|k||x|}}{|x|} f\left(k, |k|\frac{x}{|x|}\right) + o\left(\frac{1}{|x|}\right) \\ \text{as } \ |x| \to \infty \left(\text{uniformly in } \frac{x}{|x|}\right),\end{gathered} \tag{0.20}$$

with some a priory unknown f.

The function f on $\mathcal{M}_\omega = \{k \in \mathbb{R}^3, l \in \mathbb{R}^3 : k^2 = l^2 = \omega^2\}$ arising in (0.20) is the classical scattering amplitude for equation (0.17). For more information on direct scattering for equation (0.17), see, for example, [**29**] and [**52**].

In addition to Problem 2, we consider also the following far-field inverse scattering problem for equation (0.17):

Problem 3. *Given f on $\mathcal{M}_\omega$ for some fixed $\omega > 0$, find n on B_{r_1}.*

In [**12**] it was shown that the near-field data of Problem 2 are uniquely determined by the far-field data of Problem 3 and vice versa.

Global uniqueness for Problems 2 and 3 was proved for the first time in [**54**]; in addition, this proof is constructive. For more information on reconstruction methods for Problems 2 and 3 see [**4**], [**36**], [**49**], [**54**], [**60**], [**68**] and references therein.

The main results of Paper **F** consist of the following two theorems:

Theorem 0.10 (Paper **F**). *Let $N > 0$ and $r > r_1$ be fixed constants. Then there exists a positive constant C (depending only on m, ω, r_1, r and N) such that for all refractive indices n_1, n_2 satisfying $\|1 - n_1\|_{m,1}, \|1 - n_2\|_{m,1} < N$ and $supp\,(1 - n_1), supp\,(1 - n_2) \subset B_{r_1}$, the following estimate holds:*

$$||n_1 - n_2||_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C\left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \quad s = \frac{m-3}{3}, \tag{0.21}$$

where $\delta = ||G_1^+ - G_2^+||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$ and G_1^+, G_2^+ are the near-field scattering data for the refractive indices n_1, n_2, respectively, at fixed frequency ω.

Theorem 0.11 (Paper **F**). *Let $N > 0$ and $0 < \epsilon < \frac{m-3}{3}$ be fixed constants. Then there exists a positive constant C (depending only on m, ϵ, ω, r_1 and N) such that for all refractive indices n_1, n_2 satisfying $\|1 - n_1\|_{m,1}, \|1 - n_2\|_{m,1} < N$, $supp\,(1 - n_1), supp\,(1 - n_2) \subset B_{r_1}$, the following estimate holds:*

$$||n_1 - n_2||_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C\left(\ln\left(3 + \delta^{-1}\right)\right)^{-s+\epsilon}, \quad s = \frac{m-3}{3}, \tag{0.22}$$

where $\delta = ||f_1 - f_2||_{\mathbb{L}^2(\mathcal{M}_\omega)}$ and f_1, f_2 denote the scattering amplitudes for the refractive indices n_1, n_2, respectively, at fixed frequency ω.

REMARK 0.6. For some regularity dependent s but always smaller than 1 the stability estimates of Theorems 0.10 and 0.11 were proved in [**36**]. Possibility of estimates (0.21), (0.22) with $s > 1$ was formulated in [**36**] as an open problem, see page 685 of [**36**]. Our estimates (0.21), (0.22) with $s = \frac{m-3}{3}$ give a solution of this problem. Indeed,

$$s = \frac{m-3}{3} \to +\infty \quad \text{as} \quad m \to +\infty.$$

The proofs of Theorem 0.10 and 0.11 use, in particular, properties of the Faddeev functions for equation (0.17) considered as the Schrödinger equation at fixed energy $E = \omega^2$ and the results of [**36**] consisting in reducing estimates of the form (0.22) for Problem 3 to estimates of the form (0.21) for Problem 2.

We would like to note that logarithmic stability estimates for Problem 3 of the form (0.22) (with the exponent s always smaller than 1) were obtained by P. Stefanov in [**74**] earlier than in [**36**]. However, to guarantee some stability in Problem 3, he used in [**74**] the special norm for the scattering amplitude f on $\mathcal{M}_\omega$ defined as follows:

$$||f||_{\sigma_1,\sigma_2} = \left\{ \sum_{j_1,p_1,j_2,p_2} \left(\frac{2j_1+1}{e\omega}\right)^{2j_1+2\sigma_1} \left(\frac{2j_2+1}{e\omega}\right)^{2j_2+2\sigma_2} |a_{j_1p_1j_2p_2}(\omega)|^2 \right\}^{1/2},$$

where $a_{j_1p_1j_2p_2}(\omega)$, $0 \le j_1,\ 1 \le p_1 \le 2j_1+1,\ 0 \le j_2,\ 1 \le p_2 \le 2j_2+1$, denote the coefficients in the basis of the spherical harmonics $\{Y_{j_1}^{p_1} \times Y_{j_2}^{p_2}\}$ in the space $\mathbb{L}^2(S^2)$:

$$f(k,l) = \sum_{j_1,p_1,j_2,p_2} a_{j_1p_1j_2p_2}(\omega) Y_{j_1}^{p_1}\left(\frac{k}{|k|}\right) Y_{j_2}^{p_2}\left(\frac{l}{|l|}\right).$$

where $S^2 = \partial B_1 = \partial B(0,1)$ is the unit sphere in $\mathbb{R}^3$.

If a function f on $\mathcal{M}_\omega$ is the scattering amplitude for some refractive indice n satisfying (0.18) and supported in $B(0,\rho)$, where $0 < \rho < r_1$, then

$$|a_{j_1p_1j_2p_2}(\omega)| \le C(\omega, ||1-n||_{\mathbb{L}^\infty(D)}) \left(\frac{e\omega\rho}{(2j_1+1)r_1}\right)^{j_1+3/2} \left(\frac{e\omega\rho}{(2j_2+1)r_1}\right)^{j_2+3/2}$$

and, therefore, $||f||_{\sigma_1,\sigma_2} < \infty$, see estimates of Proposition 2.2 of [**74**].

In Paper **G** we prove the following instability result:

THEOREM 0.12 (Paper **G**). *Let $D = B(0,1) \in \mathbb{R}^3$. Let $I = [\omega_1, \omega_2]$ be the interval in $\mathbb{R}$, such that $\omega_1 > 0$. Then for any $m > 0$, $s > 2m$ and any real σ_1, σ_2 there are constants $\beta > 0$ and $N > 0$, such that for any real-valued $v_0 \in C^m(D)$ with $||v_0||_{\mathbb{L}^\infty(D)} \le N$, $\operatorname{supp} v_0 \subset B(0,1/2)$ and any $\epsilon \in (0,N)$, there are real-valued $v_1, v_2 \in C^m(D)$, also supported in $B(0,1/2)$, such that*

$$\begin{aligned}
&\sup_{\omega \in I} (||f_1 - f_2||_{\sigma_1,\sigma_2}) \le \exp\left(-\epsilon^{-1/s}\right),\\
&||v_1 - v_2||_{\mathbb{L}^\infty(D)} \ge \epsilon,\\
&||v_i - v_0||_{\mathbb{L}^\infty(D)} \le \epsilon, \qquad i = 1,2,\\
&||v_i - v_0||_{C^m(D)} \le \beta, \qquad i = 1,2,
\end{aligned}$$

where f_1, f_2 are the scattering amplitudes for refractive indices $n_1 \equiv 1 - v_1$ and $n_2 \equiv 1 - v_2$, respectively, for equation (0.17). In addition, in the case of fixed frequency $\omega_1 = \omega_2$ the condition $s > 2m$ can be replaced by $s > 5m/3$.

The proof of Theorem 0.12 is based on a purely topological argument, which goes back to A.N. Kolmogorov and V.M. Tihomirov [**44**]. Similar ideas was used in the proof of Theorem 0.2, see also [**24**], [**48**].

REMARK 0.7. Theorem 0.12 implies, in particular, that for any real σ_1 and σ_2 the estimate

$$||n_1 - n_2||_{\mathbb{L}^\infty(D)} \leq C \sup_{\omega \in I} \left(\ln(3 + \delta_{\sigma_1,\sigma_2}(\omega)^{-1})\right)^{-s}$$

where $C = C(N, D, m, I)$ and $\delta_{\sigma_1,\sigma_2}(\omega) = ||f_1 - f_2||_{\sigma_1,\sigma_2}$, can not hold with $s > 2m$ in the case of the scattering amplitude given on the interval of frequencies and with $s > 5m/3$ in the case of fixed frequency. Thus Theorem 0.12 shows optimality of the logarithmic stability result of [**74**] (up to the value of the exponent). Taking into account that Stefanov's norm $||f_1 - f_2||_{\sigma_1,\sigma_2}$ is stronger than norm $||f_1 - f_2||_{\mathbb{L}^2(\mathcal{M}_\omega)}$, we obtain also the optimality of estimate (0.22) (up to the value of the exponent s).

Now we focus on inverse scattering problems for the Schrödinger equation

$$L\psi = E\psi, \quad L = -\Delta + v(x), \quad x \in \mathbb{R}^d, \ \ d \geq 2, \tag{0.23}$$

where

$$v \text{ is sufficiently regular real-valued function in } \mathbb{R}^d \text{ with sufficient decay at infinity.}$$

These problems have a long history and there are many important results on this subject, see [**4**], [**7**], [**12**], [**14**], [**16**], [**18**], [**19**], [**21**], [**23**], [**27**]-[**30**], [**32**]-[**34**], [**36**], [**37**], [**49**], [**52**]-[**63**], [**68**], [**74**], [**76**]-[**79**] and references therein.

We consider the resolvent $R(E)$ of the Schrödinger operator L in $\mathbb{L}^2(\mathbb{R}^d)$:

$$R(E) = (L - E)^{-1}, \quad E \in \mathbb{C} \setminus \sigma(L),$$

where $\sigma(L)$ is the spectrum of L in $\mathbb{L}^2(\mathbb{R}^d)$. We assume that $R(x, y, E)$ denotes the Schwartz kernel of $R(E)$ as of an integral operator. We consider also

$$R^+(x, y, E) = R(x, y, E + i0), \quad x, y \in \mathbb{R}^d, \ \ E \in \mathbb{R}_+.$$

We recall that in the framework of equation (0.23) the function $R^+(x, y, E)$ describes scattering of the spherical waves

$$R_0^+(x, y, E) = -\frac{i}{4}\left(\frac{\sqrt{E}}{2\pi|x - y|}\right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}}(\sqrt{E}|x - y|),$$

generated by a source at y (where $H^{(1)}_\mu$ is the Hankel function of the first kind of order μ). We recall also that $R^+(x, y, E)$ is the Green function for $L - E$, $E \in \mathbb{R}_+$, with the Sommerfeld radiation condition at infinity.

In addition, the function

$$S^+(x,y,E) = R^+(x,y,E) - R_0^+(x,y,E),$$
$$x, y \in \partial B_r,\ E \in \mathbb{R}_+,\ r \in \mathbb{R}_+,$$

is considered as near-field scattering data for equation (0.23), where $B_r = B(r,0)$ is the open ball of radius r centered at 0.

We consider, in particular, the following near-field inverse scattering problem for equation (0.23):

PROBLEM 4. *Given S^+ on $\partial B_r \times \partial B_r$ for some fixed $r, E \in \mathbb{R}_+$, find v on B_r.*

This problem can be considered under the assumption that v is a priori known on $\mathbb{R}^d \setminus B_r$. Actually, in the present work we consider Problem 4 under the assumption that $v \equiv 0$ on $\mathbb{R}^d \setminus B_r$ for some fixed $r \in \mathbb{R}_+$.

It is well-known that the near-field scattering data of Problem 4 uniquely and efficiently determine the scattering amplitude f for equation (0.23) at fixed energy E, see [**12**]. Therefore, approaches of [**4**], [**16**], [**18**], [**27**], [**37**], [**54**], [**56**], [**59**], [**60**], [**74**] can be applied to Problem 4 via this reduction.

In addition, it is also known that the near-field data of Problem 4 uniquely determine the Dirichlet-to-Neumann map in the case when E is not a Dirichlet eigenvalue for operator L in B_r, see [**49**], [**54**]. Therefore, approaches of [**1**], [**16**], [**41**], [**48**], [**54**], [**61**]-[**68**], [**70**]-[**72**], [**75**] can be also applied to Problem 4 via this reduction.

However, in some cases it is much more optimal to deal with Problem 4 directly, see, for example, aforementioned logarithmic stability results of [**36**] and Paper **F** in dimension $d = 3$ for Problem 2.

Motivated by results of [**41**], [**51**], [**58**], [**60**], [**63**], [**68**], [**71**] and Paper **A** (see estimate (0.7) of Theorem 0.1), we study in Paper **H** the phenomena of increasing stability for Problem 4 with respect to energy E.

In particular, in dimension $d \geq 3$ we prove the following Hölder-logarithmic stability estimate:

THEOREM 0.13 (Paper **H**). *Let $E > 0$ and $r > r_1 > 0$ be given constants. Let dimension $d \geq 3$ and potentials v_1, v_2 be real-valued such that $v_j \in W^{m,1}(\mathbb{R}^d)$, $m > d$, $\operatorname{supp} v_j \subset B_{r_1}$ and $||v_j||_{m,1} \leq N$ for some $N > 0$, $j = 1, 2$. Let $S_1^+(E)$ and $S_2^+(E)$ denote the near-field scattering data for v_1 and v_2, respectively. Then for $\tau \in (0,1)$ and any $s \in [0, s_1]$ the following estimate holds:*

$$||v_2 - v_1||_{\mathbb{L}^\infty(B_r)} \leq A(1+E)^{\frac{5}{2}}\delta^\tau + B(1+E)^{\frac{s-s_1}{2}}\left(\ln\left(3+\delta^{-1}\right)\right)^{-s}, \tag{0.24}$$

where $s_1 = \frac{m-d}{d}$, $\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$, and constants $A, B > 0$ depend only on N, m, d, r, τ.

REMARK 0.8. The main feature of estimate (0.24) is the explicit dependence on the energy E. This estimate consist of two parts, the first is Hölder and the second is logarithmic; when E increases, the logarithmic part decreases and the Hölder part becomes dominant.

The proof of Theorem 0.13 is based on ideas and formulas used in the proofs of Theorems 0.1, 0.5 and 0.10.

In addition, in dimension $d = 2$ we prove the following logarithmic stability estimate:

THEOREM 0.14. *Let $E > 0$ and $r > r_1 > 0$ be given constants. Let dimension $d = 2$ and and potentials v_1, v_2 be real-valued such that $v_j \in C^2(\mathbb{R}^d)$, $\operatorname{supp} v_j \subset B_{r_1}$ and $||v_j||_{m,1} \leq N$ for some $N > 0$, $j = 1, 2$. Let $S_1^+(E)$ and $S_2^+(E)$ denote the near-field scattering data for v_1 and v_2, respectively. Then*

$$||v_1 - v_2||_{\mathbb{L}^\infty(B_r)} \leq C \left(\ln\left(3 + \delta^{-1}\right)\right)^{-3/4} \left(\ln\left(3 \ln\left(3 + \delta^{-1}\right)\right)\right)^2,$$

where $\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$ and $C > 0$ depends only on N, m, r.

The proof of Theorem 0.14 is based on ideas and formulas used in the proofs of Theorems 0.6 and 0.10.

Note also that the aforementioned Theorem 0.11 of Paper **F** also contributes to the inverse scattering theory for the Schrödinger equation (0.23) at fixed anergy E.

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Paper A

PAPER A

Energy and regularity dependent stability estimates for the Gel'fand inverse problem in multidimensions

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Abstract. We prove new global Hölder-logarithmic stability estimates for the Gel'fand inverse problem at fixed energy in dimension $d \geq 3$. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. Comparisons with preceeding results in this direction are given.

1. Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \tag{1.1}$$

where

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2,\\ &\text{with } \partial D \in C^2, \end{aligned} \tag{1.2}$$

$$v \in \mathbb{L}^\infty(D). \tag{1.3}$$

Consider the map $\hat\Phi = \hat\Phi(E)$ such that

$$\hat\Phi(E)(\psi|_{\partial D}) = \frac{\partial\psi}{\partial\nu}|_{\partial D} \tag{1.4}$$

for all sufficiently regular solutions ψ of (1.1) in $\bar D = D \cup \partial D$, where ν is the outward normal to ∂D. Here we assume also that

$$E \text{ is not a Dirichlet eigenvalue for operator } -\Delta + v \text{ in } D. \tag{1.5}$$

The map $\hat\Phi = \hat\Phi(E)$ is called the Dirichlet-to-Neumann map and is considered as boundary measurements.

We consider the following inverse boundary value problem for equation (1.1):

Problem 1.1. Given $\hat\Phi$ for some fixed E, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at fixed energy (see [**10**], [**23**]). At zero energy this problem can be considered also as a generalization of the Calderon problem of the electrical impedance tomography (see [**6**], [**23**]). Problem 1.1 can be also considered as an example of ill-posed problem: see [**18**], [**4**] for an introduction to this theory.

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness results and global reconstruction methods for Problem 1.1 were given for the first time in [**23**] in dimension $d \geq 3$ and in [**5**] in dimension $d = 2$.

Global logarithmic stability estimates for Problem 1.1 were given for the first time in [**1**] in dimension $d \geq 3$ and in [**30**] in dimension $d = 2$. A principal improvement of the result of [**1**] was given recently in [**29**] (for the zero energy case): stability of [**29**] optimally increases with increasing regularity of v.

For the Calderon problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [**36**] for $d \geq 3$ and in [**21**] for $d = 2$. Global logarithmic stability estimates for this problem were given for the first time in [**1**] for $d \geq 3$ and [**19**] for $d = 2$. Principal increasing of global stability of [**1**], [**19**] for the regular coefficient case was found in [**29**] for $d \geq 3$ and [**34**] for $d = 2$.

In addition, for the case of piecewise constant or piecewise real analytic conductivity the first uniqueness results for the Calderon problem in dimension $d \geq 2$ were given in [**7**], [**16**]. Lipschitz stability estimate for the case of piecewise constant conductivity was proved in [**2**] and additional studies in this direction were fulfilled in [**33**].

Due to [**20**] the logarithmic stability results of [**1**], [**19**] with their principal effectivization of [**29**], [**34**] are optimal (up to the value of the exponent). An extention of the instability estimates of [**20**] to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was given in [**12**].

On the other hand, it was found in [**25**], [**26**] (see also [**28**], [**31**]) that for inverse problems for the Schrödinger equation at fixed energy E in dimension $d \geq 2$ (like Problem 1.1) there is a Hölder stability modulo an error term rapidly decaying as $E \to +\infty$ (at least for the regular coefficient case). In addition, for Problem 1.1 for $d = 3$, global energy dependent stability estimates changing from logarithmic type to Hölder type for high energies were given in [**15**]. However, there is no efficient stability increasing with respect to increasing coefficient regularity in these results of [**15**]. An additional study, motivated by [**15**], [**29**], was given in [**22**].

In the present work we give new global Hölder-logarithmic stability estimates for Problem 1.1 in dimension $d \geq 3$ for the regular coefficient case, see Theorem 2.1 and Remark 2.6. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. In particular cases, our new estimates become coherent (although less strong) with respect to results of **[29]**, **[26]**, see Remarks 2.2, 2.3. In general, our new estimates give some synthesis of several important preceeding results.

2. Stability estimates

In this section we assume for simplicity that

$$v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d,\ \operatorname{supp} v \subset D, \tag{2.1}$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v :\ \partial^J v \in L^1(\mathbb{R}^d),\ |J| \le m\},\ m \in \mathbb{N} \cup 0, \tag{2.2}$$

where

$$J \in (\mathbb{N} \cup 0)^d,\ |J| = \sum_{i=1}^{d} J_i,\ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \ldots \partial x_d^{J_d}}. \tag{2.3}$$

Let

$$||v||_{m,1} = \max_{|J| \le m} ||\partial^J v||_{L^1(\mathbb{R}^d)}. \tag{2.4}$$

Let

$$\begin{array}{c} ||A|| \text{ denote the norm of an operator} \\ A : \mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D). \end{array} \tag{2.5}$$

We recall that if v_1, v_2 are potentials satisfying (1.3), (1.5) for some fixed E, then

$$\hat{\Phi}_2(E) - \hat{\Phi}_1(E) \text{ is a compact operator in } \mathbb{L}^\infty(\partial D), \tag{2.6}$$

where $\hat{\Phi}_1$, $\hat{\Phi}_2$ are the DtN maps for v_1, v_2, respectively, see **[23]**, **[27]**. Note also that (2.1) $\Rightarrow$ (1.3).

Let

$$s_0 = \frac{m-d}{m}, \quad s_1 = \frac{m-d}{d}, \quad s_2 = m - d. \tag{2.7}$$

THEOREM 2.1. *Let D satisfy (1.2), where $d \ge 3$. Let v_1, v_2 satisfy (2.1) and (1.5) for some fixed real E. Let $||v_j||_{m,1} \le N$, $j = 1, 2$, for some $N > 0$. Let $\hat{\Phi}_1(E)$ and $\hat{\Phi}_2(E)$ denote the DtN maps for v_1 and v_2, respectively. Then*

$$||v_2 - v_1||_{\mathbb{L}^\infty(D)} \le C_1 \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \quad 0 < s \le s_1, \tag{2.8}$$

where $C_1 = C_1(N, D, m, s, E) > 0$, $\delta = ||\hat{\Phi}_2(E) - \hat{\Phi}_1(E)||$ is defined according to (2.5). In addition, for $E \ge 0$, $\tau \in (0, 1)$ and any $s \in [0, s_1]$,

$$||v_2 - v_1||_{\mathbb{L}^\infty(D)} \le C_2(1 + \sqrt{E})\delta^\tau + C_3(1 + \sqrt{E})^{s - s_1} \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \tag{2.9}$$

where $C_2 = C_2(N, D, m, \tau) > 0$ and $C_3 = C_3(N, D, m, \tau) > 0$.

REMARK 2.1. Estimate (2.8) for $s = s_0$ is a variation of the result of **[1]** (see also **[29]**, **[13]**). One can see that estimate (2.8), $s = s_1$, of Theorem 2.1 is more strong (as much as s_1 is greater than s_0) than the aforementioned result going back to **[1]**.

Remark 2.2. Estimate (2.8) for $s = s_2$, $E = 0$, $d = 3$ was proved in [**29**]. One can see that this estimate of [**29**] is more strong (as much as s_2 is greater than s_1) than estimate (2.8), $s = s_1$, of Theorem 2.1 for $E = 0$, $d = 3$.

Remark 2.3. Using results of [**26**] one can obtain estimate (2.9) for $s = 0$, $d = 3$, with s_2 in place of s_1, for sufficiently great E with respect to N. One can see that for this particular case the aforementioned corollary of [**26**] is more strong (as much as s_2 is greater than s_1) than estimate (2.9) of Theorem 2.1.

Remark 2.4. In a similar way with results of [**13**], [**14**], estimates (2.8), (2.9) can be extended to the case when we do not assume that condition (1.5) is fulfiled and consider an appropriate impedance boundary map instead of the Dirichlet-to-Neumann map.

Remark 2.5. Concerning two-dimensional analogs of results of Theorem 2.1, see [**25**], [**31**], [**34**], [**35**].

Remark 2.6. Actually, in the proof of Theorem 2.1 we obtain the following estimate (see formula (4.19)):

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \le C_4\sqrt{E+\rho^2}\, e^{2\rho L}\delta + C_5(E+\rho^2)^{-s_1/2}, \tag{2.10}$$

where $L = \max\limits_{x\in\partial D}|x|$, $C_4 = C_4(N,D,m) > 0$, $C_5 = C_5(N,D,m) > 0$ and parameter $\rho > 0$ is such that $E+\rho^2$ is sufficiently large: $E + \rho^2 \ge C_6(N,D,m)$. Estimates of Theorem 2.1 follow from estimate (2.10).

The proof of Theorem 2.1 and estimate (2.10) is given in Section 4 and is based on results recalled in Section 3. Actually, this proof is technically very similar to the proof of estimate (2.8) for $s = s_0$, see [**1**], [**29**], [**13**]. Possibility of such a proof of estimate (2.8) for $s = s_1$, $E = 0$ was mentioned, in particular, in [**32**].

3. Faddeev functions

We consider the Faddeev functions G, ψ, h (see [**8**], [**9**], [**11**], [**23**]):

$$G(x,k) = e^{ikx}g(x,k), \quad g(x,k) = -(2\pi)^{-d}\int\limits_{\mathbb{R}^d}\frac{e^{i\xi x}d\xi}{\xi^2+2k\xi}, \tag{3.1}$$

$$\psi(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^d} G(x-y,k)v(y)\psi(y,k)dy, \tag{3.2}$$

where $x\in\mathbb{R}^d$, $k\in\mathbb{C}^d$, $\operatorname{Im}k \ne 0$, $d\ge 3$,

$$h(k,l) = (2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{-ilx}v(x)\psi(x,k)dx, \tag{3.3}$$

where

$$k, l \in \mathbb{C}^d,\ k^2 = l^2,\ \operatorname{Im} k = \operatorname{Im} l \neq 0. \tag{3.4}$$

One can consider (3.2), (3.3) assuming that

$$\begin{aligned} &v \text{ is a sufficiently regular function on } \mathbb{R}^d \\ &\qquad\qquad \text{with suffucient decay at infinity.} \end{aligned} \tag{3.5}$$

For example, in connection with Problem 1.1, one can consider (3.2), (3.3) assuming that

$$v \in \mathbb{L}^\infty(D), \quad v \equiv 0 \text{ on } \mathbb{R} \setminus D. \tag{3.6}$$

We recall that (see **[8]**, **[9]**, **[11]**, **[23]**):

- The function G satisfies the equation

$$(\Delta + k^2)G(x,k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d; \tag{3.7}$$

- Formula (3.2) at fixed k is considered as an equation for

$$\psi = e^{ikx}\mu(x,k), \tag{3.8}$$

  where μ is sought in $\mathbb{L}^\infty(\mathbb{R}^d)$;
- As a corollary of (3.2), (3.1), (3.7), ψ satisfies (1.1) for $E = k^2$;
- The Faddeev functions G, ψ, h are (non-analytic) continuation to the complex domain of functions of the classical scattering theory for the Schrödinger equation (in particular, h is a generalized "scattering" amplitude).

In addition, G, ψ, h in their zero energy restriction, that is for $E = 0$, were considered for the first time in **[3]**. The Faddeev functions G, ψ, h were, actually, rediscovered in **[3]**.

Let

$$\begin{aligned} \Sigma_E &= \left\{k \in \mathbb{C}^d : k^2 = k_1^2 + \ldots + k_d^2 = E\right\}, \\ \Theta_E &= \left\{k \in \Sigma_E,\ l \in \Sigma_E : \operatorname{Im} k = \operatorname{Im} l\right\}, \\ |k| &= (|\operatorname{Re} k|^2 + |\operatorname{Im} k|^2)^{1/2}. \end{aligned} \tag{3.9}$$

Under the assumptions of Theorem 2.1, we have that:

$$\mu(x,k) \to 1 \quad \text{as} \quad |k| \to \infty \tag{3.10}$$

and, for any $\sigma > 1$,

$$|\mu(x,k)| \leq \sigma \quad \text{for} \quad |k| \geq r_1(N, D, m, \sigma), \tag{3.11}$$

where $x \in \mathbb{R}^d$, $k \in \Sigma_E$;

$$\hat{v}(p) = \lim_{\substack{(k,l) \in \Theta_E,\ k - l = p \\ |\operatorname{Im} k| = |\operatorname{Im} l| \to \infty}} h(k,l) \quad \text{for any } p \in \mathbb{R}^d, \tag{3.12}$$

$$
\begin{aligned}
&|\hat v(p) - h(k,l)| \le \frac{c_1(D,m)N^2}{(E+\rho^2)^{1/2}} \quad \text{for } (k,l)\in\Theta_E, \quad p = k-l,\\
&\qquad |\mathrm{Im}\, k| = |\mathrm{Im}\, l| = \rho, \quad E+\rho^2 \ge r_2(N,D,m),\\
&\qquad p^2 \le 4(E+\rho^2),
\end{aligned} \tag{3.13}
$$

where

$$
\hat v(p) = (2\pi)^{-d}\int\limits_{\mathbb{R}^d} e^{ipx} v(x)dx, \quad p\in\mathbb{R}^d. \tag{3.14}
$$

Results of the type (3.10), (3.11) go back to [**3**]. For more information concerning (3.11) see estimate (4.11) of [**13**]. Results of the type (3.12), (3.13) (with less precise right-hand side in (3.13)) go back to [**11**]. Estimate (3.13) follows, for example, from formulas (3.2), (3.3) and the estimate

$$
\begin{aligned}
&\|\Lambda^{-s} g(k)\Lambda^{-s}\|_{\mathbb{L}^2(\mathbb{R}^d)\to\mathbb{L}^2(\mathbb{R}^d)} = O(|k|^{-1})\\
&\qquad \text{as } \ |k|\to\infty, \quad k\in\mathbb{C}^d\setminus\mathbb{R}^d,
\end{aligned} \tag{3.15}
$$

for $s > 1/2$, where $g(k)$ denotes the integral operator with the Schwartz kernel $g(x-y,k)$ and Λ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$. Estimate (3.15) was formulated, first, in [**17**] for $d\ge 3$. Concerning proof of (3.15), see [**37**].

In addition, we have that:

$$
\begin{aligned}
&h_2(k,l) - h_1(k,l) = (2\pi)^{-d}\int\limits_{\mathbb{R}^d} \psi_1(x,-l)(v_2(x)-v_1(x))\psi_2(x,k)dx\\
&\qquad \text{for } (k,l)\in\Theta_E, \ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \ne 0,\\
&\qquad \text{and } v_1,\ v_2 \text{ satisfying (3.5)},
\end{aligned} \tag{3.16}
$$

$$
\begin{aligned}
&h_2(k,l) - h_1(k,l) = (2\pi)^{-d}\int\limits_{\partial D} \psi_1(x,-l)\left[\left(\hat\Phi_2 - \hat\Phi_1\right)\psi_2(\cdot,k)\right](x)dx\\
&\qquad \text{for } (k,l)\in\Theta_E, \ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \ne 0,\\
&\qquad \text{and } v_1,\ v_2 \text{ satisfying (1.5), (3.6)},
\end{aligned} \tag{3.17}
$$

and, under assumtions of Theorem 2.1,

$$
\begin{aligned}
&|\hat v_1(p) - \hat v_2(p) - h_1(k,l) + h_2(k,l)| \le \frac{c_2(D,m)N\|v_1-v_2\|_{\mathbb{L}^\infty(D)}}{(E+\rho^2)^{1/2}}\\
&\qquad \text{for } (k,l)\in\Theta_E, \quad p=k-l, \quad |\mathrm{Im}\, k| = |\mathrm{Im}\, l| = \rho,\\
&\qquad E+\rho^2 \ge r_3(N,D,m), \quad p^2\le 4(E+\rho^2),
\end{aligned} \tag{3.18}
$$

where h_j, ψ_j denote h and ψ of (3.3) and (3.2) for $v = v_j$, and $\hat\Phi_j$ denotes the Dirichlet-to-Neumann map for $v = v_j$, where $j = 1, 2$.

Formulas (3.16), (3.17) were given in **[24]**, **[27]**. Estimate (3.18) follows from (3.2), (3.15), (3.16) in a similar way as estimate (3.13) follows from (3.2), (3.3), (3.15).

4. Proof of Theorem 2.1

Let

$$\begin{aligned} &\mathbb{L}^\infty_\mu(\mathbb{R}^d) = \{u \in \mathbb{L}^\infty(\mathbb{R}^d) : \|u\|_\mu < +\infty\}, \\ &\|u\|_\mu = \operatorname{ess}\sup_{p\in\mathbb{R}^d} (1+|p|)^\mu |u(p)|, \quad \mu > 0. \end{aligned} \tag{4.1}$$

Note that

$$\begin{aligned} &w \in W^{m,1}(\mathbb{R}^d) \Longrightarrow \hat{w} \in \mathbb{L}^\infty_\mu(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d), \\ &\|\hat{w}\|_\mu \le c_3(m,d)\|w\|_{m,1} \quad \text{for} \quad \mu = m, \end{aligned} \tag{4.2}$$

where $W^{m,1}$, $\mathbb{L}^\infty_\mu$ are the spaces of (2.2), (4.1),

$$\hat{w}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} w(x) dx, \quad p \in \mathbb{R}^d. \tag{4.3}$$

Using the inverse Fourier transform formula

$$w(x) = \int\limits_{\mathbb{R}^d} e^{-ipx} \hat{w}(p) dp, \quad x \in \mathbb{R}^d, \tag{4.4}$$

we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \le \sup_{x\in\bar{D}} \Big| \int\limits_{\mathbb{R}^d} e^{-ipx} \left(\hat{v}_2(p) - \hat{v}_1(p)\right) dp \Big| \le \\ \le I_1(r) + I_2(r) \quad \text{for any} \quad r > 0, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} I_1(r) &= \int\limits_{|p|\le r} |\hat{v}_2(p) - \hat{v}_1(p)| dp, \\ I_2(r) &= \int\limits_{|p|\ge r} |\hat{v}_2(p) - \hat{v}_1(p)| dp. \end{aligned} \tag{4.6}$$

Using (4.2), we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \le 2c_3(m,d) N (1+|p|)^{-m}, \quad p \in \mathbb{R}^d. \tag{4.7}$$

Due to (3.18), we have that

$$
\begin{aligned}
|\hat v_2(p)-\hat v_1(p)| \le |h_2(k,l)-h_1(k,l)| + \frac{c_2(D,m)N\|v_1-v_2\|_{\mathbb{L}^\infty(D)}}{(E+\rho^2)^{1/2}},\\
\text{for } (k,l)\in\Theta_E,\quad p=k-l,\quad |\operatorname{Im} k|=|\operatorname{Im} l|=\rho,\\
E+\rho^2\ge r_3(N,D,m),\quad p^2\le 4(E+\rho^2).
\end{aligned}
\tag{4.8}
$$

Let

$$
\begin{aligned}
c_4=(2\pi)^{-d}\int\limits_{\partial D} dx,\quad L=\max_{x\in\partial D}|x|,\\
\delta=\|\hat\Phi_2(E)-\hat\Phi_1(E)\|,
\end{aligned}
\tag{4.9}
$$

where $\|\hat\Phi_2(E)-\hat\Phi_1(E)\|$ is defined according to (2.5).

Due to (3.17), we have that

$$
\begin{aligned}
|h_2(k,l)-h_1(k,l)|\le c_4\|\psi_1(\cdot,-l)\|_{\mathbb{L}^\infty(\partial D)}\,\delta\,\|\psi_2(\cdot,k)\|_{\mathbb{L}^\infty(\partial D)},\\
(k,l)\in\Theta_E,\ |\operatorname{Im} k|=|\operatorname{Im} l|\ne 0.
\end{aligned}
\tag{4.10}
$$

Using (3.11), we find that

$$
\begin{aligned}
\|\psi(\cdot,k)\|_{\mathbb{L}^\infty(\partial D)}\le\sigma\,\exp\Big(|\operatorname{Im} k|L\Big),\\
k\in\Sigma_E,\ |k|\ge r_1(N,D,m,\sigma).
\end{aligned}
\tag{4.11}
$$

Here and bellow in this section the constant σ is the same that in (3.11).

Combining (4.10) and (4.11), we obtain that

$$
\begin{aligned}
|h_2(k,l)-h_1(k,l)|\le c_4\sigma^2e^{2\rho L}\delta,\quad \text{for } (k,l)\in\Theta_E,\\
\rho=|\operatorname{Im} k|=|\operatorname{Im} l|,\\
E+\rho^2\ge r_1^2(N,D,m,\sigma).
\end{aligned}
\tag{4.12}
$$

Using (4.8), (4.12), we get that

$$
\begin{aligned}
|\hat v_2(p)-\hat v_1(p)|\le c_4\sigma^2e^{2\rho L}\delta+\frac{c_2(D,m)N\|v_1-v_2\|_{\mathbb{L}^\infty(D)}}{(E+\rho^2)^{1/2}},\\
p\in\mathbb{R}^d,\ p^2\le 4(E+\rho^2),\ E+\rho^2\ge\max\{r_1^2,r_3\}.
\end{aligned}
\tag{4.13}
$$

Let

$$
\varepsilon=\left(\frac{1}{2c_2(D,m)Nc_5}\right)^{1/d},\quad c_5=\int\limits_{p\in\mathbb{R}^d,|p|\le 1}dp,
\tag{4.14}
$$

and $r_4(N, D, m, \sigma) > 0$ be such that

$$E + \rho^2 \geq r_4(N, D, m, \sigma) \Longrightarrow \begin{cases} E + \rho^2 \geq r_1^2(N, D, m, \sigma), \\ E + \rho^2 \geq r_3(N, D, m), \\ \left(\varepsilon(E + \rho^2)^{\frac{1}{2d}}\right)^2 \leq 4(E + \rho^2). \end{cases} \tag{4.15}$$

Let

$$c_6 = \int\limits_{p \in \mathbb{R}^d, |p|=1} dp. \tag{4.16}$$

Using (4.6), (4.13), we get that

$$\begin{aligned} I_1(r) \leq c_5 r^d \Big(c_4 \sigma^2 e^{2\rho L} \delta + \frac{c_2(D, m) N \|v_1 - v_2\|_{\mathbb{L}^\infty(D)}}{(E + \rho^2)^{1/2}} \Big), \\ r > 0,\ r^2 \leq 4(E + \rho^2), \\ E + \rho^2 \geq r_4(N, D, m, \sigma). \end{aligned} \tag{4.17}$$

Using (4.6), (4.7), we find that, for any $r > 0$,

$$I_2(r) \leq 2c_3(m, d) N c_6 \int\limits_r^{+\infty} \frac{dt}{t^{m-d+1}} \leq \frac{2c_3(m, D) N c_6}{m - d} \frac{1}{r^{m-d}}. \tag{4.18}$$

Combining (4.5), (4.17), (4.18) for $r = \varepsilon(E + \rho^2)^{\frac{1}{2d}}$ and (4.15), we get that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \leq c_7(N, D, m, \sigma) \sqrt{E + \rho^2}\, e^{2\rho L} \delta + \\ + c_8(N, D, m)(E + \rho^2)^{-\frac{m-d}{2d}} + \frac{1}{2} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)}, \\ E + \rho^2 \geq r_4(N, D, m, \sigma). \end{aligned} \tag{4.19}$$

Let $\tau' \in (0, 1)$ and

$$\beta = \frac{1 - \tau'}{2L}, \quad \rho = \beta \ln\left(3 + \delta^{-1}\right), \tag{4.20}$$

where δ is so small that $E+\rho^2 \geq r_4(N,D,m,\sigma)$. Then due to (4.19), we have that

$$
\begin{aligned}
&\frac{1}{2}\|v_1-v_2\|_{\mathbb{L}^\infty(D)} \leq \\
&\quad \leq c_7(N,D,m,\sigma)\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{1/2}\left(3+\delta^{-1}\right)^{2\beta L}\delta+ \\
&\quad + c_8(N,D,m)\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{-\frac{m-d}{2d}} = \\
&\quad = c_7(N,D,m,\sigma)\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{1/2}\left(1+3\delta\right)^{1-\tau'}\delta^{\tau'}+ \\
&\quad + c_8(N,D,m)\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{-\frac{m-d}{2d}},
\end{aligned}
\tag{4.21}
$$

where τ',β and δ are the same as in (4.20).

Using (4.21), we obtain that

$$
\|v_1-v_2\|_{\mathbb{L}^\infty(D)} \leq c_9(N,D,E,m,\sigma,\tau')\left(\ln\left(3+\delta^{-1}\right)\right)^{-\frac{m-d}{d}} \tag{4.22}
$$

for $\delta = \|\hat{\Phi}_2-\hat{\Phi}_1\| \leq \delta_1(N,D,E,m,\sigma,\tau')$, where δ_1 is a sufficiently small positive constant. Estimate (4.22) in the general case (with modified c_9) follows from (4.22) for $\delta \leq \delta_1(N,D,E,m,\sigma,\tau')$ and the property that

$$
\|v_j\|_{\mathbb{L}^\infty(D)} \leq c_{10}(D,m)N. \tag{4.23}
$$

This completes the proof of (2.8).

If $E \geq 0$ then there is a constant $\delta_2 = \delta_2(N,D,m,\sigma,\tau') > 0$ such that

$$
\delta \in (0,\delta_2) \Longrightarrow \begin{cases} E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2 \geq r_4(N,D,m,\sigma), \\ E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2 \leq \left((1+\sqrt{E})\beta\ln\left(3+\delta^{-1}\right)\right)^2, \\ \beta\ln\left(3+\delta^{-1}\right) \geq 1, \end{cases} \tag{4.24}
$$

where β is the same as in (4.20). Combining (4.21), (4.24), we obtain that for $s \in [0,(m-d)/d]$, $\tau \in (0,\tau')$ and $\delta \in (0,\delta_2)$ the following estimate holds:

$$
||v_2-v_1||_{L^\infty(D)} \leq c_{11}(1+\sqrt{E})\delta^\tau + c_{12}(1+\sqrt{E})^{s-\frac{m-d}{d}}\left(\ln\left(3+\delta^{-1}\right)\right)^{-s}, \tag{4.25}
$$

where constants $c_{11}, c_{12} > 0$ depend only on N, D, m, σ, τ' and τ.

Estimate (4.25) in the general case (with modified c_{11} and c_{12}) follows from (4.25) for $\delta \leq \delta_2(N,D,m,\sigma,\tau')$ and (4.23).

This completes the proof of (2.9)

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Paper B

PAPER B

Exponential instability in the Gel'fand inverse problem on the energy intervals

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Abstract. We consider the Gel'fand inverse problem and continue studies of [Mandache,2001]. We show that the Mandache-type instability remains valid even in the case of Dirichlet-to-Neumann map given on the energy intervals. These instability results show, in particular, that the logarithmic stability estimates of [Alessandrini,1988], [Novikov, Santacesaria,2010] and especially of [Novikov,2010] are optimal (up to the value of the exponent).

1. Introdution

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \tag{1.1}$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2,\ \partial D \in C^2,\ v \in \mathbb{L}^\infty(D). \tag{1.2}$$

Consider the map $\hat{\Phi}(E)$ such that

$$\hat{\Phi}(E)(\psi|_{\partial D}) = \frac{\partial \psi}{\partial \nu}|_{\partial D} \tag{1.3}$$

for all sufficiently regular solutions ψ of (1.1) in $\bar{D} = D \cup \partial D$, where ν is the outward normal to ∂D. Here we assume also that

$$E \text{ is not a Dirichlet eigenvalue for operator } -\Delta + v \text{ in } D. \tag{1.4}$$

The map $\hat{\Phi}(E)$ is called the Dirichlet-to-Neumann map and is considered as boundary measurements.

We consider the following inverse boundary value problem for equation (1.1).

Problem 1.1. Given $\hat{\Phi}$ on the union of the energy intervals $S = \bigcup\limits_{j=1}^{K} I_j$, find v.

Here we suppose that condition (1.4) is fulfilled for any $E \in S$.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation on the energy intervals (see [**5**], [**9**]).

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness for Problem 1.1 was obtained for the first time by Novikov (see Theorem 5.3 in [**6**]). Some global reconstruction method for Problem 1.1 was proposed for the first time in [**6**] also. Global uniqueness theorems and global reconstruction methods in the case of fixed energy were given for the first time in [**9**] in dimension $d \geq 3$ and in [**3**] in dimension $d = 2$.

Global stability estimates for Problem 1.1 were given for the first time in [**2**] in dimension $d \geq 3$ and in [**11**] in dimension $d = 2$. The Alessandrini result of [**2**] was recently improved by Novikov in [**10**]. In the case of fixed energy, Mandache showed in [**8**] that these logarithmic stability results are optimal (up to the value of the exponent). Mandache-type instability estimates for inverse inclusion and scattering problems were given in [**4**], where some general scheme for investigating questions of this type of instability has been also proposed. Although some of the main results of this work can be represented within the general scheme of [**4**], it does not lead to a significant simplification of its complete proof.

In the present work we extend studies of Mandache to the case of Dirichlet-to-Neumann map given on the energy intervals. The stability estimates and our instability results for Problem 1.1 are presented and discussed in Section 2. In Section 5 we prove the main results, using a ball packing and covering by ball arguments. In Section 3 we prove some basic properties of the Dirichlet-to-Neumann map, using some Lemmas about the Bessel functions wich we proved in Section 6.

2. Stability estimates and main results

As in [**10**] we assume for simplicity that

$$\begin{gathered} D \text{ is an open bounded domain in } \mathbb{R}^d,\ \partial D \in C^2, \\ v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d,\ \operatorname{supp} v \subset D,\ d \geq 2, \end{gathered} \tag{2.1}$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v :\ \partial^J v \in \mathbb{L}^1(\mathbb{R}^d),\ |J| \leq m\},\ m \in \mathbb{N} \cup 0, \tag{2.2}$$

where

$$J \in (\mathbb{N} \cup 0)^d,\ |J| = \sum_{i=1}^{d} J_i,\ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}. \tag{2.3}$$

Let

$$||v||_{m,1} = \max_{|J| \leq m} ||\partial^J v||_{\mathbb{L}^1(\mathbb{R}^d)}. \tag{2.4}$$

We recall that if v_1, v_2 are potentials satisfying (1.4),(1.3), where E and D are fixed, then

$$\hat{\Phi}_1 - \hat{\Phi}_2 \text{ is a compact operator in } \mathbb{L}^\infty(\partial D), \tag{2.5}$$

where $\hat{\Phi}_1$, $\hat{\Phi}_2$ are the DtN maps for v_1, v_2 respectively, see [**9**].

Note also that (2.1) $\Rightarrow$ (1.2).

THEOREM 2.1 (variation of the result of [**2**], see [**10**]). *Let conditions (1.4), (2.1) hold for potentials* v_1 *and* v_2, *where* E *and* D *are fixed, dimension* $d \geq 3$. *Let* $||v_j||_{m,1} \leq N,\ j = 1,2,$ *for some* $N > 0$. *Let* $\hat{\Phi}_1,\ \hat{\Phi}_2$ *denote DtN maps for* v_1, v_2 *respectively. Then*

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq c_1(\ln(3 + ||\hat{\Phi}_1 - \hat{\Phi}_2||^{-1}))^{-\alpha_1}, \tag{2.6}$$

where $c_1 = c_1(N, D, m),\ \alpha_1 = (m-d)/m,\ ||\hat{\Phi}_1 - \hat{\Phi}_2|| = ||\hat{\Phi}_1 - \hat{\Phi}_2||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)}$.

An analog of stability estimate of [**2**] for $d = 2$ is given in [**11**].

A disadvantage of estimate (2.6) is that

$$\alpha_1 < 1 \text{ for any } m > d \text{ even if } m \text{ is very great.} \tag{2.7}$$

THEOREM 2.2 (the result of [**10**]). *Let the assumptions of Theorem 2.1 hold. Then*

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq c_2(\ln(3 + ||\hat{\Phi}_1 - \hat{\Phi}_2||^{-1}))^{-\alpha_2}, \tag{2.8}$$

where $c_2 = c_2(N, D, m),\ \alpha_2 = m - d,\ ||\hat{\Phi}_1 - \hat{\Phi}_2|| = ||\hat{\Phi}_1 - \hat{\Phi}_2||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)}$.

A principal advantage of estimate (2.8) in comparison with (2.6) is that

$$\alpha_2 \to +\infty \text{ as } m \to +\infty, \tag{2.9}$$

in contrast with (2.7). Note that strictly speaking Theorem 2.2 was proved in [**10**] for $E = 0$ with the condition that $\operatorname{supp} v \subset D$, so we cant make use of substitution $v_E = v - E$, since condition $\operatorname{supp} v_E \subset D$ does not hold.

We would like to mention that, under the assumptions of Theorems 2.1 and 2.2, according to the Mandache results of [**8**], estimate (2.8) can not hold with $\alpha_2 > m(2d-1)/d$ for real-valued potentials and with $\alpha_2 > m$ for complex potentials.

As in [**8**] in what follows we fix $D = B(0,1)$, where $B(x,r)$ is the open ball of radius r centred at x. We fix an orthonormal basis in $\mathbb{L}^2(S^{d-1}) = \mathbb{L}^2(\partial D)$

$$\begin{aligned} &\{f_{jp} : j \geq 0;\ 1 \leq p \leq p_j\}, \\ &f_{jp} \text{ is a spherical harmonic of degree } j, \end{aligned} \tag{2.10}$$

where p_j is the dimension of the space of spherical harmonics of order j,

$$p_j = \binom{j+d-1}{d-1} - \binom{j+d-3}{d-1}, \tag{2.11}$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad \text{for } n \geq 0 \tag{2.12}$$

and

$$\binom{n}{k} = 0 \quad \text{for } n < 0. \tag{2.13}$$

The precise choice of f_{jp} is irrelevant for our purposes. Besides orthonormality, we only need f_{jp} to be the restriction of a homogeneous harmonic polynomial of degree j to the sphere and so $|x|^j f_{jp}(x/|x|)$ is harmonic. In the Sobolev spaces $H^s(S^{d-1})$ we will use the norm

$$||\sum_{j,p} c_{jp} f_{jp}||^2_{H^s} = \sum_{j,p} (1+j)^{2s} |c_{jp}|^2. \tag{2.14}$$

The notation (a_{jpiq}) stands for a multiple sequence. We will drop the subscript

$$0 \le j,\ 1 \le p \le p_j,\ 0 \le i,\ 1 \le q \le p_i. \tag{2.15}$$

We use notations: $|A|$ is the cardinality of a set A, $[a]$ is the integer part of real number a and $(r,\omega) \in \mathbb{R}_+ \times S^{d-1}$ are polar coordinates for $r\omega = x \in \mathbb{R}^d$.

The interval $I = [a,b]$ will be referred as σ-regular interval if for any potential $v \in \mathbb{L}^\infty(D)$ with $||v||_{\mathbb{L}^\infty(D)} \le \sigma$ and any $E \in I$ condition (1.4) is fulfilled. Note that for any $E \in I$ and any Dirichlet eigenvalue λ for operator $-\Delta$ in D we have that

$$|E - \lambda| \ge \sigma. \tag{2.16}$$

It follows from the definition of σ-regular interval, taking $v \equiv E - \lambda$.

Theorem 2.3. *For $\sigma > 0$ and dimension $d \ge 2$ consider the union $S = \bigcup_{j=1}^{K} I_j$ of σ-regular intervals. Then for any $m > 0$ and any $s \ge 0$ there is a constant $\beta > 0$, such that for any $\epsilon \in (0, \sigma/3)$ and $v_0 \in C^m(D)$ with $||v_0||_{\mathbb{L}^\infty(D)} \le \sigma/3$ and $\operatorname{supp} v_0 \subset B(0,1/3)$ there are real-valued potentials $v_1, v_2 \in C^m(D)$, also supported in $B(0,1/3)$, such that*

$$\begin{aligned}
&\sup_{E\in S}\Big(||\hat\Phi_1(E) - \hat\Phi_2(E)||_{H^{-s}\to H^s}\Big) \le \exp\Big(-\epsilon^{-\frac{1}{2m}}\Big),\\
&||v_1 - v_2||_{\mathbb{L}^\infty(D)} \ge \epsilon,\\
&||v_i - v_0||_{C^m(D)} \le \beta, \quad i = 1,2,\\
&||v_i - v_0||_{\mathbb{L}^\infty(D)} \le \epsilon, \quad i = 1,2,
\end{aligned} \tag{2.17}$$

where $\hat\Phi_1(E)$, $\hat\Phi_2(E)$ are the DtN maps for v_1 and v_2 respectively.

Remark 2.1. We can allow β to be arbitrarily small in Theorem 2.3, if we require $\epsilon \le \epsilon_0$ and replace the right-hand side in the instability estimate by $\exp(-c\epsilon^{-\frac{1}{2m}})$, with $\epsilon_0 > 0$ and $c > 0$, depending on β.

In addition to Theorem 2.3, we consider explicit instability example with a complex potential given by Mandache in [**8**]. We show that it gives exponential instability even in case of Dirichlet-to-Neumann map given on the energy intervals. Consider the cylindrical variables $(r_1, \theta, x') \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^{d-2}$, with $x' = (x_3, \ldots, x_d)$, $r_1 \cos\theta = x_1$ and $r_1 \sin\theta = x_2$. Take $\phi \in C^\infty(\mathbb{R}^2)$ with support in $B(0,1/3) \cap \{x_1 > 1/4\}$ and with $||\phi||_{\mathbb{L}^\infty} = 1$.

THEOREM 2.4. *For $\sigma > 0$, $m > 0$, integer $n > 0$ and dimension $d \geq 2$ consider the union $S = \bigcup_{j=1}^{K} I_j$ of σ-regular intervals and define the complex potential*

$$v_{nm}(x) = \frac{\sigma}{3} n^{-m} e^{in\theta} \phi(r_1, |x'|). \tag{2.18}$$

Then $||v_{mn}||_{\mathbb{L}^\infty(D)} = \frac{\sigma}{3} n^{-m}$ and for every $s \geq 0$ and $m > 0$ there are constants c, c' such that $||v_{mn}||_{C^m(D)} \leq c$ and for every n

$$\sup_{E \in S} \left(||\hat{\Phi}_{mn}(E) - \hat{\Phi}_0(E)||_{H^{-s} \to H^s} \right) \leq c' 2^{-n/4}, \tag{2.19}$$

where $\hat{\Phi}_{mn}(E)$, $\hat{\Phi}_0(E)$ are the DtN maps for v_{mn} and $v_0 \equiv 0$ respectively.

In some important sense, this is stronger than Theorem 2.3. Indeed, if we take $\epsilon = \frac{\sigma}{3} n^{-m}$ we obtain (2.17) with $\exp(-C\epsilon^{-1/m})$ in the right-hand side. An explicit real-valued counterexample should be difficult to find. This is due to nonlinearity of the map $v \to \hat{\Phi}$.

REMARK 2.2. Note that for sufficient large s one can see that

$$||\hat{\Phi}_1 - \hat{\Phi}_2||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)} \leq C ||\hat{\Phi}_1 - \hat{\Phi}_2||_{H^{-s} \to H^s}. \tag{2.20}$$

So Theorem 2.3 and Theorem 2.4 imply, in particular, that the estimate

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq c_3 \sup_{E \in S} \left(\ln(3 + ||\hat{\Phi}_1(E) - \hat{\Phi}_2(E)||^{-1}) \right)^{-\alpha_3}, \tag{2.21}$$

where $c_3 = c_3(N, D, m, S)$ and $||\hat{\Phi}_1(E) - \hat{\Phi}_2(E)|| = ||\hat{\Phi}_1(E) - \hat{\Phi}_2(E)||_{\mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D)}$, can not hold with $\alpha_3 > 2m$ for real-valued potentials and with $\alpha_3 > m$ for complex potentials. Thus Theorem 2.3 and Theorem 2.4 show optimality of logarithmic stability results of Alessandrini and Novikov in considerably stronger sense that results of Mandache.

3. Some basic properties of Dirichlet-to-Neumann map

We continue to consider that $D = B(0, 1)$ and also to use the polar coordinates $(r, \omega) \in \mathbb{R}_+ \times S^{d-1}$, with $x = r\omega$. Solutions of equation $-\Delta\psi = E\psi$ in D can be expressed by the Bessel functions J_α and Y_α with integer or half-integer order α, see definitions of Section 6. We state some lemmas about these functions (Lemma 3.1, Lemma 3.2, Lemma 3.3).

LEMMA 3.1. *Suppose $k \neq 0$ and k^2 is not a Dirichlet eigenvalue for operator $-\Delta$ in D. Then*

$$\psi_0(r, \omega) = r^{-\frac{d-2}{2}} \frac{J_{j+\frac{d-2}{2}}(kr)}{J_{j+\frac{d-2}{2}}(k)} f_{jp}(\omega) \tag{3.1}$$

is the solution of (1.1) with $v \equiv 0$, $E = k^2$ and boundary condition $\psi|_{\partial D} = f_{jp}$.

REMARK 3.1. Note that the assumptions of Lemma 3.1 imply $J_{j+\frac{d-2}{2}}(k) \neq 0$.

LEMMA 3.2. *Let the assumptions of Lemma 3.1 hold. Then system of functions*

$$\{\psi_{jp}(r,\omega) = R_j(k,r) f_{jp}(\omega) : j \geq 0; 1 \leq p \leq p_j\}, \tag{3.2}$$

where

$$R_j(k,r) = r^{-\frac{d-2}{2}} \Big(Y_{j+\frac{d-2}{2}}(kr) J_{j+\frac{d-2}{2}}(k) - J_{j+\frac{d-2}{2}}(kr) Y_{j+\frac{d-2}{2}}(k) \Big), \tag{3.3}$$

is complete orthogonal system (in the sense of L_2) in the space of solutions of equation (1.1) in $D' = B(0,1) \setminus B(0,1/3)$ with $v \equiv 0$, $E = k^2$ and boundary condition $\psi|_{r=1} = 0$.

LEMMA 3.3. *For any $C > 0$ and integer $d \geq 2$ there is a constant $N > 3$ depending on C such that for any integer $n \geq N$ and any $|z| \leq C$*

$$\frac{1}{2} \frac{(|z|/2)^\alpha}{\Gamma(\alpha+1)} \leq |J_\alpha(z)| \leq \frac{3}{2} \frac{(|z|/2)^\alpha}{\Gamma(\alpha+1)}, \tag{3.4}$$

$$|J'_\alpha(z)| \leq 3 \frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.5}$$

$$\frac{1}{2\pi} (|z|/2)^{-\alpha} \Gamma(\alpha) \leq |Y_\alpha(z)| \leq \frac{3}{2\pi} (|z|/2)^{-\alpha} \Gamma(\alpha) \tag{3.6}$$

$$|Y'_\alpha(z)| \leq \frac{3}{\pi} (|z|/2)^{-\alpha-1} \Gamma(\alpha+1) \tag{3.7}$$

where $'$ denotes derivation with respect to z, $\alpha = n + \frac{d-2}{2}$ and $\Gamma(x)$ is the Gamma function.

Proofs of Lemma 3.1, Lemma 3.2 and Lemma 3.3 are given in Section 6.

LEMMA 3.4. *Consider a compact $W \subset \mathbb{C}$. Suppose, that v is bounded,* $\operatorname{supp} v \subset B(0,1/3)$ *and condition (1.4) is fulfilled for any $E \in W$ and potentials v and v_0, where $v_0 \equiv 0$. Denote $\Lambda_{v,E} = \hat{\Phi}(E) - \hat{\Phi}_0(E)$. Then there is a constant $\rho = \rho(W,d)$, such that for any $0 \leq j, 1 \leq p \leq p_j$, $0 \leq i, 1 \leq q \leq p_i$, we have*

$$|\langle \Lambda_{v,E} f_{jp}, f_{iq} \rangle| \leq \rho\, 2^{-\max(j,i)} ||v||_{\mathbb{L}^\infty(D)} ||(-\Delta + v - E)^{-1}||_{\mathbb{L}^2(D)}, \tag{3.8}$$

where $\hat{\Phi}(E)$, $\hat{\Phi}_0(E)$ are the DtN maps for v and v_0 respectively and $(-\Delta + v - E)^{-1}$ is considered with the Dirichlet boundary condition.

PROOF OF LEMMA 3.4. For simplicity we give first a proof under the additional assumtions that $0 \notin W$ and there is a holomorphic germ $\sqrt{E}$ for $E \in W$. Since W is compact there is $C > 0$ such that for any $z \in W$ we have $|z| \leq C$. We take N from Lemma 3.3 for this C. We fix indeces j, p. Consider solutions $\psi(E)$, $\psi_0(E)$ of equation (1.1) with $E \in W$, boundary condition $\psi|_{\partial D} = f_{jp}$ and potentials v and

v_0 respectively. Then $\psi(E)-\psi_0(E)$ has zero boundary values, so it is domain of $-\Delta+v-E$, and since

$$(-\Delta+v-E)\left(\psi(E)-\psi_0(E)\right)=-v\psi_0(E) \text{ in } D, \tag{3.9}$$

we obtain that

$$\psi(E)-\psi_0(E)=-(-\Delta+v-E)^{-1}v\psi_0(E). \tag{3.10}$$

If $j\geq N$ from Lemma 3.1 and Lemma 3.3 we have that

$$\begin{aligned} ||\psi_0(E)||^2_{\mathbb{L}^2(B(0,1/3))} &= ||f_{jp}||^2_{\mathbb{L}^2(S^{d-1})}\int_0^{1/3}\left| r^{-\frac{d-2}{2}}\frac{J_{j+\frac{d-2}{2}}(\sqrt{E}\,r)}{J_{j+\frac{d-2}{2}}(\sqrt{E})}\right|^2 r^{d-1}dr \leq \\ &\leq \int_0^{1/3}\left(\frac{3}{2}\frac{(|E|^{1/2}r/2)^{j+\frac{d-2}{2}}}{\Gamma(j+\frac{d-2}{2}+1)}\right)^2 \Bigg/ \left(\frac{1}{2}\frac{(|E|^{1/2}/2)^{j+\frac{d-2}{2}}}{\Gamma(j+\frac{d-2}{2}+1)}\right)^2 r\,dr = \\ &= 3\int_0^{1/3} r^{2j+d-1}dr=\frac{3}{2j+d}\left(\frac{1}{3}\right)^{2j+d}<2^{-2j}. \end{aligned} \tag{3.11}$$

For $j<N$ we use fact that $||\psi_0(E)||_{\mathbb{L}^2(B(0,1))}$ is continuous function on compact W and, since N depends only on W, we get that there is a constant $\rho_1=\rho_1(W,d)$ such that

$$||\psi_0(E)||_{\mathbb{L}^2(B(0,1/3))}\leq \rho_1 2^{-j}. \tag{3.12}$$

Since v has support in $B(0,1/3)$ from (3.10) we get that

$$||\psi(E)-\psi_0(E)||_{\mathbb{L}^2(B(0,1))}\leq \rho_1 2^{-j}||v||_{\mathbb{L}^\infty(D)}||(-\Delta+v-E)^{-1}||_{\mathbb{L}^2(D)}. \tag{3.13}$$

Note that $\psi(E)-\psi_0(E)$ is the solution of equation (1.1) in $D'=B(0,1)\setminus B(0,1/3)$ with potential $v_0\equiv 0$ and boundary condition $\psi|_{r=1}=0$. From Lemma 3.2 we have that

$$\psi(E)-\psi_0(E)=\sum_{0\leq i,1\leq q\leq p_i} c_{iq}(E)\psi_{iq}(E) \text{ in } D' \tag{3.14}$$

for some c_{iq}, where

$$\psi_{iq}(E)(r,\omega)=R_i(\sqrt{E},r)f_{iq}(\omega). \tag{3.15}$$

Since $R_i(\sqrt{E},1)=0$

$$\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}=\left.\frac{\partial\left(r^{\frac{d-2}{2}}R_i(\sqrt{E},r)\right)}{\partial r}\right|_{r=1}. \tag{3.16}$$

For $i \geq N$ from Lemma 3.3 we have that

$$\left|\frac{\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}}{Y_\alpha(\sqrt{E})J_\alpha(\sqrt{E})}\right| = |E|^{1/2}\left|\frac{Y'_\alpha(\sqrt{E})}{Y_\alpha(\sqrt{E})} - \frac{J'_\alpha(\sqrt{E})}{J_\alpha(\sqrt{E})}\right| \leq \tag{3.17}$$
$$\leq 6|E|^{1/2}\left(\frac{(|E|^{1/2}/2)^{-\alpha-1}\Gamma(\alpha+1)}{(|E|^{1/2}/2)^{-\alpha}\Gamma(\alpha)} + \frac{(|E|^{1/2}/2)^{\alpha-1}\Gamma(\alpha+1)}{(|E|^{1/2}/2)^{\alpha}\Gamma(\alpha)}\right) = 6\alpha,$$

$$\begin{aligned}\left(\frac{\|r^{-\frac{d-2}{2}}Y_\alpha(\sqrt{E}r)\|_{\mathbb{L}^2(\{1/3<|x|<2/5\})}}{|Y_\alpha(\sqrt{E})|}\right)^2 &\geq \int_{1/3}^{2/5}\left(\frac{1}{3}\frac{(|E|^{1/2}r/2)^{-\alpha}\Gamma(\alpha)}{(|E|^{1/2}/2)^{-\alpha}\Gamma(\alpha)}\right)^2 r\,dr \\ &\geq \left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\left(\frac{1}{3}(5/2)^\alpha\right)^2,\end{aligned} \tag{3.18}$$

$$\begin{aligned}\left(\frac{\|r^{-\frac{d-2}{2}}J_\alpha(\sqrt{E}r)\|_{\mathbb{L}^2(\{1/3<|x|<2/5\})}}{|J_\alpha(\sqrt{E})|}\right)^2 &\leq \int_{1/3}^{2/5}\left(3\frac{(|E|^{1/2}r/2)^{\alpha}\Gamma(\alpha)}{(|E|^{1/2}/2)^{\alpha}\Gamma(\alpha)}\right)^2 r\,dr \\ &\leq \left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\left(3(2/5)^\alpha\right)^2,\end{aligned} \tag{3.19}$$

where $\alpha = i + \frac{d-2}{2}$. Since $N > 3$ we have that $\alpha > 3$. Using (3.18) and (3.19) we get that

$$\frac{\|\psi_{iq}(E)\|_{\mathbb{L}^2(\{1/3<|x|<2/5\})}}{\left|Y_\alpha(\sqrt{E})J_\alpha(\sqrt{E})\right|} \geq \left(\left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\right)^{1/2}\left(\frac{1}{3}(5/2)^\alpha - 3(2/5)^\alpha\right) \geq \frac{1}{1000}(5/2)^\alpha. \tag{3.20}$$

For $i \geq N$ we get that

$$\left|\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}\right| \leq 1000\alpha(5/2)^{-\alpha}\|\psi_{iq}(E)\|_{\mathbb{L}^2(\{1/3<|x|<1\})}. \tag{3.21}$$

For $i < N$ we use the fact that $\left|\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}\right| / \|\psi_{iq}(E)\|_{\mathbb{L}^2(\{1/3<|x|<1\})}$ is continuous function on compact W and get that for any $i \geq 0$ there is a constant $\rho_2 = \rho_2(W,d)$ such that

$$\left|\left.\frac{\partial R_i(\sqrt{E},r)}{\partial r}\right|_{r=1}\right| \leq \rho_2\, 2^{-i}\|\psi_{iq}(E)\|_{\mathbb{L}^2(\{1/3<|x|<1\})}. \tag{3.22}$$

Proceeding from (3.14) and using the Cauchy-Schwarz inequality we get that

$$|c_{iq}(E)| = \left|\frac{\left\langle \psi(E)-\psi_0(E), \psi_{iq}(E)\right\rangle_{\mathbb{L}^2(\{1/3<|x|<1\})}}{\|\psi_{iq}(E)\|^2_{\mathbb{L}^2(\{1/3<|x|<1\})}}\right| \leq \frac{\|\psi(E)-\psi_0(E)\|_{\mathbb{L}^2(B(0,1))}}{\|\psi_{iq}(E)\|_{\mathbb{L}^2(\{1/3<|x|<1\})}}. \tag{3.23}$$

Taking into account

$$\langle \Lambda_{v,E} f_{jp}, f_{iq} \rangle = \left\langle \left. \frac{\partial(\psi(E) - \psi_0(E))}{\partial \nu} \right|_{\partial D}, f_{iq} \right\rangle = c_{iq}(E) \left. \frac{\partial R_i(\sqrt{E}, r)}{\partial r} \right|_{r=1} \tag{3.24}$$

and combining (3.22) and (3.23) we obtain that

$$|\langle \Lambda_{v,E} f_{jp}, f_{iq} \rangle| \le \rho_2 2^{-i} ||\psi(E) - \psi_0(E)||_{\mathbb{L}^2(B(0,1))}. \tag{3.25}$$

From (3.13) and (3.25) we get (3.8).

For the general case we consider two compacts

$$W_\pm = W \cap \{z \mid \pm \mathrm{Im} z \ge 0\}. \tag{3.26}$$

Note that $\frac{J_{j+\frac{d-2}{2}}(\sqrt{E}r)}{J_{j+\frac{d-2}{2}}(\sqrt{E})}$ and $\frac{Y_{j+\frac{d-2}{2}}(\sqrt{E}r)}{Y_{j+\frac{d-2}{2}}(\sqrt{E})}$ have removable singularity in $E = 0$ or, more precisely,

$$\begin{gathered} \frac{J_{j+\frac{d-2}{2}}(\sqrt{E}r)}{J_{j+\frac{d-2}{2}}(\sqrt{E})} \longrightarrow r^{j+\frac{d-2}{2}}, \\ \frac{Y_{j+\frac{d-2}{2}}(\sqrt{E}r)}{Y_{j+\frac{d-2}{2}}(\sqrt{E})} \longrightarrow r^{-j-\frac{d-2}{2}} \\ \text{as } E \longrightarrow 0. \end{gathered} \tag{3.27}$$

Considering the limit as $E \to 0$ we get that (3.13), (3.25) and consequently (3.8) are valid for $W_\pm$. To complete proof we can take $\rho = \max\{\rho_+, \rho_-\}$. ■

REMARK 3.2. From (3.1) and (3.10) we get that

$$\langle \Lambda_{v,E} f_{jp}, f_{iq} \rangle \quad \text{is holomorphic function in } W. \tag{3.28}$$

4. A fat metric space and a thin metric space

DEFINITION 4.1. Let $(X, dist)$ be a metric space and $\epsilon > 0$. We say that a set $Y \subset X$ is an ϵ-net for $X_1 \subset X$ if for any $x \in X_1$ there is $y \in Y$ such that $dist(x, y) \le \epsilon$. We call ϵ-entropy of the set X_1 the number $\mathcal{H}_\epsilon(X_1) := \log_2 \min\{|Y| : Y$ is an ϵ-net for $X_1\}$.

A set $Z \subset X$ is called ϵ-discrete if for any distinct $z_1 \in Z$, $z_2 \in Z$, we have that $dist(z_1, z_2) \ge \epsilon$. We call ϵ-capacity of the set X_1 the number $\mathcal{C}_\epsilon := \log_2 \max\{|Z| : Z \subset X_1$ and Z is ϵ-discrete$\}$.

The use of ϵ-entropy and ϵ-capacity to derive properties of mappings between metric spaces goes back to Vitushkin and Kolmogorov (see [**7**] and references therein). One notable application was Hilbert's 13th problem (about representing a function of several variables as a composition of functions of a smaller number of variables). In essence, Lemma 4.1 and Lemma 4.2 are parts of Theorem XIV and Theorem XVII of [**7**], respectively.

LEMMA 4.1. *Let $d \geq 2$ and $m > 0$. For $\epsilon, \beta > 0$, consider the real metric space*

$$X_{m\epsilon\beta} = \{f \in C^m(D) \mid \operatorname{supp} f \subset B(0,1/3),\ ||f||_{\mathbb{L}^\infty(D)} \leq \epsilon,\ ||f||_{C^m(D)} \leq \beta\},$$

with the metric induced by $\mathbb{L}^\infty$. Then there is a $\mu > 0$ such that for any $\beta > 0$ and $\epsilon \in (0, \mu\beta)$, there is an ϵ-discrete set $Z \subset X_{m\epsilon\beta}$ with at least $\exp\left(2^{-d-1}(\mu\beta/\epsilon)^{d/m}\right)$ elements.

Lemma 4.1 was also formulated and proved in [**8**].

LEMMA 4.2. *For the interval $I = [a,b] \in \mathbb{R}$ and $\gamma > 0$ consider ellipse $W_{I,\gamma} \subset \mathbb{C}$*

$$W_{I,\gamma} = \{\frac{a+b}{2} + \frac{a-b}{2}\cos z \mid |Im\, z| \leq \gamma\}. \tag{4.1}$$

Then there is a constant $\nu = \nu(C,\gamma) > 0$, such that for every $\delta \in (0, e^{-1})$, there is a δ-net for the space of functions on I with $\mathbb{L}^\infty$-norm, having holomorphic continuation to $W_{I,\gamma}$ with module bounded above on $W_{I,\gamma}$ by the constant C, with at most $\exp(\nu(\ln\delta^{-1})^2)$ *elements.*

PROOF OF LEMMA 4.2. Theorem XVII in [**7**] provides asymptotic behaviour of the entropy of this space with respect to $\delta \to 0$. Here we get upper estimate of it. Suppose $g(z)$ is holomorphic function in $W_{I,\gamma}$ with module bounded above by the constant C. Consider the function $f(z) = g(\frac{a+b}{2} + \frac{a-b}{2}\cos z)$. By the choise of $W_{I,\gamma}$ we get that $f(z)$ is 2π-periodic holomorphic function in the stripe $|\mathrm{Im}\, z| \leq \gamma$. Then for any integer n

$$|c_n| = \left|\int_0^{2\pi} e^{inx} f(x)dx\right| \leq \int_0^{2\pi} e^{-|n|\gamma} C dx \leq 2\pi C e^{-|n|\gamma}. \tag{4.2}$$

Let n_δ be the smallest natural number such that $2\pi C e^{-n\gamma} \leq 6\pi^{-2}(n+1)^{-2}\delta$ for any $n \geq n_\delta$. Taking natural logarithm and using $\ln\delta^{-1} \geq 1$, we get that

$$n_\delta \leq C' \ln\delta^{-1}, \tag{4.3}$$

where C' depends only on C and γ. We denote $\delta' = 3\pi^{-2}(n_\delta+1)^{-2}\delta$. Consider the set

$$Y_\delta = \delta'\mathbb{Z} \bigcap [-2\pi C, 2\pi C] + i\cdot\delta'\mathbb{Z} \bigcap [-2\pi C, 2\pi C]. \tag{4.4}$$

Using (4.3), we have that

$$|Y_\delta| = (1 + 2[2\pi C/\delta'])^2 \leq C''\delta^{-2}\ln^4\delta^{-1}, \tag{4.5}$$

with C'' depending only on C and γ. We set

$$Y = \left\{\sum_{n=0}^{\infty} d_n \cos\left(n \arccos\frac{x - \frac{a+b}{2}}{\frac{a-b}{2}}\right) \mid d_n \in Y_\delta \text{ for } n \leq n_\delta,\ d_n = 0 \text{ otherwise}\right\}. \tag{4.6}$$

For given $f(z)$ in case of $n \le n_\delta$ we take d_n to be one of the closest elements of Y_δ to c_n. Since $|c_n| \le 2\pi C$, this ensures $|c_n - d_n| \le 2\delta'$. For $n > n_\delta$ we take $d_n = 0$. We have then

$$|c_n - d_n| \le 6\pi^{-2}(n+1)^{-2}\delta. \tag{4.7}$$

For $n > n_\delta$ this is true by the construction of n_δ, otherwise by the choise of δ'. Since $f(x)$ is 2π-periodic even function, we get $g_Y(x) \in Y$ such that

$$||g(x) - g_Y(x)||_{\mathbb{L}^\infty(a,b)} \le \sum_{n=0}^{\infty} |c_n - d_n| \le 6\pi^{-2}\delta \sum_{n=1}^{\infty} \frac{1}{n^2} = \delta. \tag{4.8}$$

We have that $|Y| = |Y_\delta|^{n_\delta}$. Taking into account (4.3),(4.5) and $\ln \delta^{-1} \ge 1$, we get

$$|Y| \le (C''\delta^{-2}\ln^4\delta^{-1})^{C'\ln\delta^{-1}} \le \exp\left(C'''\ln\delta^{-1}C'\ln\delta^{-1}\right) \le \exp(\nu(\ln\delta^{-1})^2). \tag{4.9}$$

■

REMARK 4.1. The assertion is valid even in the case of $a = b$. As δ-net we can take

$$Y = \frac{\delta}{2}\mathbb{Z}\bigcap[-C, C] + i \cdot \frac{\delta}{2}\mathbb{Z}\bigcap[-C, C]. \tag{4.10}$$

Consider an operator $A : H^{-s}(S^{d-1}) \to H^s(S^{d-1})$. We denote its matrix elements in the basis $\{f_{jp}\}$ by $a_{jpiq} = \langle Af_{jp}, f_{iq}\rangle$. From **[8]** we have that

$$||A||_{H^{-s}\to H^s} \le 4 \sup_{j,p,i,q} (1 + \max(j, i))^{2s+d}|a_{jpiq}|. \tag{4.11}$$

Consider system $S = \bigcup_{j=1}^{K} I_j$ of σ-regular intervals. We introduce the Banach space

$$X_{S,s} = \left\{ \Big(a_{jpiq}(E)\Big) \mid \left\|\Big(a_{jpiq}(E)\Big)\right\|_{X_{S,s}} := \sup_{j,p,i,q} \left((1 + \max(j, i))^{2s+d} \sup_{E\in S} |a_{jpiq}(E)| \right) < \infty \right\}.$$

Denote by B^∞ the ball of centre 0 and radius $2\sigma/3$ in $\mathbb{L}^\infty(B(0, 1/3))$. We identify in the sequel an operator $A(E) : H^{-s}(S^{d-1}) \to H^s(S^{d-1})$ with its matrix $\Big(a_{jpiq}(E)\Big)$. Note that the estimate (4.11) implies that

$$\sup_{E\in S} \|A(E)\|_{H^{-s}\to H^s} \le 4 \left\|\Big(a_{jpiq}(E)\Big)\right\|_{X_{S,s}}. \tag{4.12}$$

We consider operator $\Lambda_{v,E}$ from Lemma 3.4 as

$$\Lambda : B^\infty \to \left\{\Big(a_{jpiq}(E)\Big)\right\}, \tag{4.13}$$

where $a_{jpiq}(E)$ are matrix elements in the basis $\{f_{jp}\}$ of operator $\Lambda_{v,E}$.

LEMMA 4.3. *Λ maps B^∞ into $X_{S,s}$ for any s. There is a constant $\eta(S,s,d) > 0$ such that for every $\delta \in (0, e^{-1})$ there is a δ-net Y for $\Lambda(B^\infty)$ in $X_{S,s}$ with at most $\exp(\eta(\ln\delta^{-1})^{2d})$ elements.*

PROOF OF LEMMA 4.3. For simplicity we give first a proof in case of S consists of only one σ-regular interval I. From (4.1) we take $W_I = W_{I,\gamma}$, where constant $\gamma > 0$ is such as for any $E \in W_I$ there is E_I in I such as $|E - E_I| < \sigma/6$. From (2.16) we get that

$$|E - \lambda| \geq |E_I - \lambda| - |E - E_I| \geq 5\sigma/6, \tag{4.14}$$

with λ being Dirichlet eigenvalue for operator $-\Delta$ in D which is closest to E. Then for potential $v \in B^\infty$ and $E \in W_I$ we have that

$$||(-\Delta + v - E)^{-1}||_{\mathbb{L}^2(D)} \leq (|\lambda - E| - 2\sigma/3)^{-1} \leq (5\sigma/6 - 2\sigma/3)^{-1} = 6/\sigma \tag{4.15}$$

and

$$||v||_{\mathbb{L}^\infty(D)}||(-\Delta + v - E)^{-1}||_{\mathbb{L}^2(D)} \leq (2\sigma/3)(6/\sigma) = 4, \tag{4.16}$$

where $(-\Delta + v - E)^{-1}$ is considered with the Dirichlet boundary condition. We obtain from Lemma 3.4 that

$$|a_{jpiq}(E)| \leq 4\rho\, 2^{-\max(j,i)}, \tag{4.17}$$

where $\rho = \rho(W_I, d)$. Hence $||(a_{jpiq}(E))||_{X_{S,s}} \leq \sup_l (1+l)^{2s+d} 4\rho\, 2^{-l} < \infty$ for any s and d and so the first assertion of the Lemma 4.3 is proved.

Let $l_{\delta s}$ be the smallest natural number such that $(1+l)^{2s+d}4\rho\, 2^{-l} \leq \delta$ for any $l \geq l_{\delta s}$. Taking natural logarithm and using $\ln\delta^{-1} \geq 1$, we get that

$$l_{\delta s} \leq C' \ln\delta^{-1}, \tag{4.18}$$

where the constant C' depends only on s, d and I. Denote Y_{jpiq} is δ_{jpiq}-net from Lemma 4.2 with constant $C = \sup_l (1+l)^{2s+d}4\rho\, 2^{-l}$, where $\delta_{jpiq} = (1+\max(j,i))^{-2s-d}\delta$. We set

$$Y = \{(a_{jpiq}(E)) \mid a_{jpiq}(E) \in Y_{jpiq} \text{ for } \max(j,i) \leq l_{\delta s},\ a_{jpiq}(E) = 0 \text{ otherwise}\}. \tag{4.19}$$

For any $(a_{jpiq}(E)) \in \Lambda(B^\infty)$ there is an element $(b_{jpiq}(E)) \in Y$ such that

$$(1+\max(j,i))^{2s+d}|a_{jpiq}(E) - b_{jpiq}(E)| \leq (1+\max(j,i))^{2s+d}\delta_{jpiq} = \delta, \tag{4.20}$$

in case of $\max(j,i) \leq l_{\delta s}$ and

$$(1+\max(j,i))^{2s+d}|a_{jpiq}(E) - b_{jpiq}(E)| \leq (1+\max(j,i))^{2s+d}2\rho\, 2^{-\max(j,i)} \leq \delta, \tag{4.21}$$

otherwise.

It remains to count the elements of Y. Using again the fact that $\ln\delta^{-1} \geq 1$ and (4.18) we get for $\max(j,i) \leq l_{\delta s}$

$$|Y_{jpiq}| \leq \exp(\nu(\ln\delta_{jpiq}^{-1})^2) \leq \exp(\nu'(\ln\delta^{-1})^2). \tag{4.22}$$

From [**8**] we have that $n_{\delta s} \le 8(1+l_{\delta s})^{2d-2}$, where $n_{\delta s}$ is the number of four-tuples (j,p,i,q) with $\max(j,i) \le l_{\delta s}$. Taking η to be big enough we get that

$$\begin{aligned} |Y| &\le \left(\exp(\nu'(\ln\delta^{-1})^2)\right)^{n_{\delta s}} \\ &\le \exp\left(\nu'(\ln\delta^{-1})^2 8(1+C'\ln\delta^{-1})^{2d-2}\right) \\ &\le \exp\left(\eta(\ln\delta^{-1})^{2d}\right). \end{aligned} \tag{4.23}$$

For $S = \bigcup_{j=1}^{K} I_j$ assertion follows immediately, taking η to be in K times more and Y as composition $(Y_1,\dots,Y_K)$ of δ-nets for each interval. ∎

5. Proofs of the main results

In this section we give proofs of Theorem 2.3 and Theorem 2.4.

Proof of Theorem 2.3. Take $v_0 \in \mathbb{L}^\infty(B(0,1/3))$, $||v_0||_{\mathbb{L}^\infty(D)} \le \sigma/3$ and $\epsilon \in (0,\sigma/3)$. By Lemma 4.1, the set $v_0 + X_{m\epsilon\beta}$ has an ϵ-discrete subset v_0+Z. Since for $\epsilon \in (0,\sigma/3)$ we have $v_0 + X_{m\epsilon\beta} \subset B^\infty$, where B^∞ is the ball of centre 0 and radius $2\sigma/3$ in $\mathbb{L}^\infty(B(0,1/3))$. The set Y constructed in Lemma 4.3 is also δ-net for $\Lambda(v_0+X_{m\epsilon\beta})$. We take δ such that $8\delta = \exp\left(-\epsilon^{-\frac{1}{2m}}\right)$. Note that inequalities of (2.17) follow from

$$|v_0+Z| > |Y|. \tag{5.1}$$

In fact, if $|v_0+Z| > |Y|$, then there are two potentials $v_1, v_2 \in v_0+Z$ with images under Λ in the same $X_{S,s}$-ball radius δ centered at a point of Y, so we get from (4.12)

$$\sup_{E\in S} ||\hat\Phi_1(E) - \hat\Phi_2(E)||_{H^{-s}\to H^s} \le 4||\Lambda_{v_1,E} - \Lambda_{v_2,E}||_{X_{S,s}} \le 8\delta = \exp\left(-\epsilon^{-\frac{1}{2m}}\right). \tag{5.2}$$

It remains to find β such as (5.1) is fullfiled. By Lemma 4.3

$$|Y| \le \exp\left(\eta\left(\ln 8 + \epsilon^{-\frac{1}{2m}}\right)^{2d}\right) \le \max\Big(\exp\left((2\ln 8)^{2d}\eta\right), \exp\left(2^{2d}\eta\epsilon^{-d/m}\right)\Big). \tag{5.3}$$

Now we take

$$\beta > \mu^{-1}\max\left(\sigma/3, \eta^{m/d}2^{3m}, \frac{\sigma}{3}\eta^{m/d}2^m(2\ln 8)^{2m}\right) \tag{5.4}$$

This fulfils requirement $\epsilon < \mu\beta$ in Lemma 4.1, which gives

$$\begin{aligned} &|v_0+Z| = |Z| \ge \exp\left(2^{-d-1}(\mu\beta/\epsilon)^{d/m}\right) \overset{(5.4)}{>} \\ &> \max\Big(\exp\left(2^{-d-1}(\eta^{m/d}2^{3m}/\epsilon)^{d/m}\right), \exp\left(2^{-d-1}(\eta^{m/d}2^m(2\ln 8)^{2m})^{d/m}\right)\Big) \overset{(5.3)}{\ge} |Y|. \end{aligned} \tag{5.5}$$

∎

PROOF OF THEOREM 2.4. In a similar way with the proof of Theorem 2 of [**8**] we obtain that

$$\left\langle \left(\hat{\Phi}_{mn}(E) - \hat{\Phi}_0(E) \right) f_{jp}, f_{iq} \right\rangle = 0 \tag{5.6}$$

for $j, i \leq \left[\frac{n-1}{2}\right]$. The only difference is that instead of the operator $-\Delta$ we consider the operator $-\Delta - E$. From (4.11), (4.17) and (5.6) we get

$$||\hat{\Phi}_{mn}(E) - \hat{\Phi}_0(E)||_{H^{-s} \to H^s} \leq 16\rho \sup_{l \geq n/2} (1+l)^{2s+d} 2^{-l} \leq c' 2^{-n/4}. \tag{5.7}$$

The fact that $||v_{mn}||_{C^m(D)}$ is bounded as $n \to \infty$ is also a part of Theorem 2 of [**8**]. ■

6. Bessel functions

In this section we prove Lemma 3.1, Lemma 3.2 and Lemma 3.3 about the Bessel functions. Consider the problem of finding solutions of the form $\psi(r, \omega) = R(r) f_{jp}(\omega)$ of equation (1.1) with $v \equiv 0$. We have that

$$\Delta = \frac{\partial^2}{(\partial r)^2} + (d-1) r^{-1} \frac{\partial}{\partial r} + r^{-2} \Delta_{S^{d-1}}, \tag{6.1}$$

where $\Delta_{S^{d-1}}$ is Laplace-Beltrami operator on S^{d-1}. We have that

$$\Delta_{S^{d-1}} f_{jp} = -j(j+d-2) f_{jp}. \tag{6.2}$$

Then we have the following equation for $R(r)$:

$$-R'' - \frac{d-1}{r} R' + \frac{j(j+d-2)}{r^2} R = ER. \tag{6.3}$$

Taking $R(r) = r^{-\frac{d-2}{2}} \tilde{R}(r)$, we get

$$r^2 \tilde{R}'' + r \tilde{R}' + \left(E r^2 - \left(j + \frac{d-2}{2} \right)^2 \right) \tilde{R} = 0. \tag{6.4}$$

This equation is known as Bessel's equation. For $E = k^2 \neq 0$ it has two linearly independent solutions $J_{j+\frac{d-2}{2}}(kr)$ and $Y_{j+\frac{d-2}{2}}(kr)$, where

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\alpha}}{\Gamma(m+1)\Gamma(m+\alpha+1)}, \tag{6.5}$$

$$Y_\alpha(z) = \frac{J_\alpha(z) \cos \pi\alpha - J_{-\alpha}(z)}{\sin \pi\alpha} \text{ for } \alpha \notin \mathbb{Z}, \tag{6.6}$$

and

$$Y_\alpha(z) = \lim_{\alpha' \to \alpha} Y_{\alpha'}(z) \text{ for } \alpha \in \mathbb{Z}. \tag{6.7}$$

The following Lemma is called the Nielsen inequality. A proof can be found in [**12**]

LEMMA 6.1.

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha+1)}(1+\theta), \quad |\theta| < \exp\left(\frac{|z|^2/4}{|\alpha_0+1|}\right) - 1, \tag{6.8}$$

where $|\alpha_0+1|$ *is the least of numbers* $|\alpha+1|, |\alpha+2|, |\alpha+3|, \dots$.

Lemma 6.1 implies that $r^{-\frac{d-2}{2}} J_{j+\frac{d-2}{2}}(kr)$ has removable singularity at $r = 0$. Then, using the boundary conditions $\acute{R}(1) = 1$ and $R(1) = 0$, one can obtain assertions of Lemma 3.1 and Lemma 3.2, respectively.

PROOF OF LEMMA 3.3 Formula (3.4) follows immediately from Lemma 6.1. We have from **[12]** that

$$J'_\alpha(z) = J_{\alpha-1}(z) - \frac{\alpha}{z} J_\alpha(z). \tag{6.9}$$

Further, taking α big enough we get

$$|J'_\alpha(z)| \le |J_{\alpha-1}(z)| + |\frac{\alpha}{z} J_\alpha(z)| \le \frac{3}{2}\frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)} + \frac{3\alpha}{2|z|}\frac{(|z|/2)^\alpha}{\Gamma(\alpha+1)} \le 3\frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)}. \tag{6.10}$$

For $\alpha = n + 1/2$ we have $Y_\alpha = (-1)^{n+1} J_{-\alpha}$. Consider its series expansion, see (6.5).

$$J_{-\alpha}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m-\alpha}}{m!\,\Gamma(m-\alpha+1)} = \sum_{m=0}^{\infty} c_m (z/2)^{2m-\alpha}. \tag{6.11}$$

Note that $|c_m/c_{m+1}| = (m+1)|m-\alpha+1| \ge n/2$. As corollary we obtain that

$$|Y_\alpha(z)| = \frac{(|z|/2)^{-\alpha}}{|\Gamma(-\alpha+1)|}(1+\theta) = \frac{1}{\pi}(|z|/2)^{-\alpha}\Gamma(\alpha)(1+\theta), \quad |\theta| \le \sum_{m=1}^{\infty}\left(\frac{|z|^2}{2n}\right)^{2m} \le \frac{|z|^2/2n}{1-|z|^2/2n}. \tag{6.12}$$

For $\alpha = n$ we have from **[12]** that

$$\begin{aligned} Y_n(z) &= \frac{2}{\pi} J_n(z) \ln\left(\frac{z}{2}\right) - \frac{1}{\pi}\sum_{m=0}^{n-1}\left(\frac{z}{2}\right)^{2m-n}\frac{(n-m-1)!}{m!} - \\ &-\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m (z/2)^{2m+n}}{m!(m+n)!}\left(\frac{\Gamma'(m+1)}{\Gamma(m+1)} + \frac{\Gamma'(m+n+1)}{\Gamma(m+n+1)}\right) = \\ &= \frac{2}{\pi} J_n(z)\ln\left(\frac{z}{2}\right) - \frac{1}{\pi}\sum_{m=0}^{n-1}\tilde{c}_m (z/2)^{2m-n} - \frac{1}{\pi}\sum_{m=0}^{\infty} b_m (z/2)^{2m+n}. \end{aligned} \tag{6.13}$$

Using well-known equality $\Gamma'(x)/\Gamma(x) < \ln x$, $x > 1$, see **[1]**, we get following estimation for the coefficients b_m are defined in (6.13).

$$|b_m| < \frac{\ln(m+1) + \ln(n+m+1)}{m!(n+m)!} < \frac{2(n+m)}{m!(n+m)!} < \frac{1}{m!}. \tag{6.14}$$

Note also that $|\tilde{c}_m/\tilde{c}_{m+1}| = (m+1)(n-m-1) \geq n/2$. Combining it with (6.13) and (6.14), we obtain that

$$
\begin{gathered}
|Y_n(z)| = \frac{1}{\pi}(|z|/2)^{-n}\Gamma(n)(1+\theta), \\
|\theta| \leq 3\frac{(|z|/2)^{2n}|\ln(z/2)|}{\Gamma(n)} + \sum_{m=1}^{n-1}\left(\frac{|z|^2}{2n}\right)^{2m} + \frac{(|z|/2)^{2n}}{\Gamma(n)}\sum_{m=0}^{\infty}\frac{(|z|/2)^{2m}}{m!} \leq \\
\leq 3\pi\frac{\max\left(1,(|z|/2)^{2n+1}\right)}{\Gamma(n)} + \frac{|z|^2/2n}{1-|z|^2/2n} + \frac{(|z|/2)^{2n}e^{|z|^2/4}}{\Gamma(n)}.
\end{gathered}
\tag{6.15}
$$

Formula (3.6) follows from (6.12) and (6.15). We have from **[12]** that

$$
Y'_\alpha(z) = Y_{\alpha-1}(z) - \frac{\alpha}{z}Y_\alpha(z). \tag{6.16}
$$

Taking n big enough, we get that

$$
\begin{gathered}
|Y'_\alpha(z)| \leq |Y_{\alpha-1}(z)| + |\frac{\alpha}{z}Y_\alpha(z)| \leq \\
\leq \frac{3}{2\pi}\left((|z|/2)^{-\alpha+1}\Gamma(\alpha-1) + \frac{\alpha}{|z|}(|z|/2)^{\alpha}\Gamma(\alpha)\right) \leq \frac{3}{\pi}(|z|/2)^{-\alpha-1}\Gamma(\alpha+1).
\end{gathered}
\tag{6.17}
$$

Combining reqirements for n, stated above, we get that for any $n \geq N+1$ all inequalities of Lemma 3.3 are fullfiled, where N such that

$$
\begin{cases}
N > 3, \\
\exp\left(\dfrac{C^2/4}{N+1}\right) - 1 \leq 1/2, \\
3\pi\dfrac{\max\left(1,(C/2)^{2N+1}\right)}{\Gamma(N)} + \dfrac{C^2}{2N-C^2} + \dfrac{(C/2)^{2N}e^{C^2/4}}{\Gamma(N)} \leq 1/2.
\end{cases}
\tag{6.18}
$$

∎

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PAPER C

PAPER C

Instability in the Gel'fand inverse problem at high energies

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Abstract. We give an instability estimate for the Gel'fand inverse boundary value problem at high energies. Our instability estimate shows an optimality of several important preceeding stability results on inverse problems of such a type.

1. Introduction

In this paper we continue studies on the Gel'fand inverse boundary value problem for the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \tag{1.1}$$

where

$$\begin{aligned} &D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2,\\ &\text{with } \partial D \in C^2, \end{aligned} \tag{1.2}$$

$$v \in \mathbb{L}^\infty(D). \tag{1.3}$$

As boundary data we consider the map $\hat{\Phi} = \hat{\Phi}(E)$ such that

$$\hat{\Phi}(E)(\psi|_{\partial D}) = \frac{\partial \psi}{\partial \nu}|_{\partial D} \tag{1.4}$$

for all sufficiently regular solutions ψ of (1.1) in $\bar{D} = D \cup \partial D$, where ν is the outward normal to ∂D. Here we assume also that

$$E \text{ is not a Dirichlet eigenvalue for operator } -\Delta + v \text{ in } D. \tag{1.5}$$

The map $\hat{\Phi} = \hat{\Phi}(E)$ is known as the Dirichlet-to-Neumann map.

We consider the following inverse boundary value problem for equation (1.1):

Problem 1.1. Given $\hat{\Phi}$ for some fixed E, find v.

This problem is known as the Gel'fand inverse boundary value problem for the Schrödinger equation at fixed energy (see **[7]**, **[19]**). At zero energy this problem can be considered also as a generalization of the Calderon problem of the electrical impedance tomography (see [**5**], [**19**]). Problem 1.1 can be also considered as an example of ill-posed problem: see [**14**], [**3**] for an introduction to this theory.

There is a wide literature on the Gel'fand inverse problem at fixed energy. In a similar way with many other inverse problems, Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global reconstruction methods for Problem 1.1 were obtained for the first time in [**19**] in dimension $d \geq 3$ and in [**4**] in dimension $d = 2$.

Global logarithmic stability estimates for Problem 1.1 were obtained for the first time in [**1**] in dimension $d \geq 3$ and in [**25**] in dimension $d = 2$. A principal improvement of the result of [**1**] was obtained recently in [**24**] (for the zero energy case): stability of [**24**] optimally increases with increasing regularity of v.

Note that for the Calderon problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [**30**] for $d \geq 3$ and in [**17**] for $d = 2$. Global logarithmic stability estimates for this problem were obtained for the first time in [**1**] for $d \geq 3$ and [**15**] for $d = 2$. Principal increasing of global stability of [**1**], [**15**] for the regular coefficient case was found in [**24**] for $d \geq 3$ and [**28**] for $d = 2$. In addition, for the case of piecewise real analytic conductivity the first uniqueness results for the Calderon problem in dimension $d \geq 2$ were given in [**13**]. Lipschitz stability estimate for the case of piecewise constant conductivity was obtained in [**2**] (see [**27**] for additional studies in this direction).

The optimality of the logarithmic stability results of [**1**], [**15**] with their principal effectivizations of [**24**], [**28**] (up to the value of the exponent) follows from [**16**]. An extention of the instability estimates of [**16**] to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was obtained in [**8**].

On the other hand, it was found in [**20**], [**21**] (see also [**23**], [**26**]) that for inverse problems for the Schrödinger equation at fixed energy E in dimension $d \geq 2$ (like Problem 1.1) there is a Hölder stability modulo an error term rapidly decaying as $E \to +\infty$ (at least for the regular coefficient case). In addition, for Problem 1.1 for $d = 3$, global energy dependent stability estimates changing from logarithmic type to Hölder type for high energies were obtained in [**12**], [**11**]. However, there is no efficient stability increasing with respect to increasing coefficient regularity in the results of [**12**]. An additional study, motivated by [**12**], [**24**], was given in [**18**].

The following stability estimate for Problem 1.1 was recently proved in [**11**]:

THEOREM 1.1 (of [**11**]). *Let D satisfy* (1.2), *where $d \geq 3$. Let $v_j \in W^{m,1}(D)$, $m > d$, supp $v_j \subset D$ and $||v_j||_{W^{m,1}(D)} \leq N$ for some $N > 0$, $j = 1, 2$, (where $W^{m,p}$ denotes the Sobolev space of m-times smooth functions in $\mathbb{L}^p$). Let v_1, v_2 satisfy* (1.5) *for some fixed $E \geq 0$. Let $\hat{\Phi}_1(E)$ and $\hat{\Phi}_2(E)$ denote the DtN maps for v_1 and v_2, respectively. Let $s_1 = (m - d)/d$. Then, for any $\tau \in (0, 1)$ and any $\alpha, \beta \in [0, s_1]$, $\alpha + \beta = s_1$,*

$$||v_2 - v_1||_{L^\infty(D)} \leq A(1 + \sqrt{E})\delta^\tau + B(1 + \sqrt{E})^{-\alpha} \left(\ln\left(3 + \delta^{-1}\right)\right)^{-\beta}, \tag{1.6}$$

where $\delta = ||\hat{\Phi}_2(E) - \hat{\Phi}_1(E)||_{\mathbb{L}^\infty(\partial D)\to\mathbb{L}^\infty(\partial D)}$ *and constants* $A, B > 0$ *depend only on* N, D, m, τ.

In particular cases, Hölder-logarithmic stability estimate (1.6) becomes coherent (although less strong) with respect to results of **[21]**, **[23]**, **[24]**. In this connection we refer to **[11]** for more detailed infromation. Concerning two-dimensional analogs of results of Theorem 1.1, see **[20]**, **[26]**, **[28]**, **[29]**.

In a similar way with results of **[9]**, **[10]**, estimate (1.6) can be extended to the case when we do not assume that condition (1.5) is fulfiled and consider an appropriate impedance boundary map (or Robin-to-Robin map) instead of the Dirichlet-to-Neumann map.

In the present work we prove the optimality of estimate (1.6) in the sense that it can not hold with $\alpha, \beta \geq 0$, $\alpha + 2\beta > 2m$. Our related instability results for Problem 1.1 are presented in Section 2, see Theorem 2.1 and Proposition 2.1. Their proofs are given in Section 4 and are based on properties of solutions of the Schrödinger equation in the unit ball given in Section 3.

2. Main results

In what follows we fix $D = B^d(0,1)$, where

$$B^d(x^0, \rho) = \{x \in \mathbb{R}^d : ||x - x^0||_{\mathbb{E}^d} < \rho\}, \quad x_0 \in \mathbb{R}^d, \ \rho > 0. \tag{2.1}$$

Let

$$||F|| \text{ denote the norm of an operator } F : \mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D). \tag{2.2}$$

We recall that if v_1, v_2 are potentials satisfying (1.3), (1.5) for some fixed E, then

$$\hat{\Phi}_2(E) - \hat{\Phi}_1(E) \text{ is a compact operator in } \mathbb{L}^\infty(\partial D), \tag{2.3}$$

where $\hat{\Phi}_1$, $\hat{\Phi}_2$ are the DtN maps for v_1, v_2, respectively, see **[19]**, **[22]**.

Our main result is the following theorem:

THEOREM 2.1. *Let* $D = B^d(0,1)$, *where* $d \geq 2$. *Then for any fixed constants* $A, B, \kappa, \tau, \varepsilon > 0$, $m > d$ *and* $s_2 > m$ *there exist some energy level* $E > 0$ *and some potential* $v \in C^m(D)$ *such that condition* (1.5) *holds for potentials* v *and* $v_0 \equiv 0$, *simultaneously, supp* $v \subset D$, $\|v\|_{\mathbb{L}^\infty(D)} \leq \varepsilon$, $\|v\|_{C^m(D)} \leq C_1$, *where* $C_1 = C_1(d, m) > 0$, *but*

$$||v - v_0||_{L^\infty(D)} > A(1 + \sqrt{E})^\kappa \delta^\tau + B(1 + \sqrt{E})^{2(s - s_2)} \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s} \tag{2.4}$$

for any $s \in [0, s_2]$, *where* $\hat{\Phi}$, $\hat{\Phi}_0$ *are the DtN map for* v *and* v_0, *respectively, and* $\delta = ||\hat{\Phi}(E) - \hat{\Phi}_0(E)||$ *is defined according to* (2.2).

Theorem 2.1 shows, in particular, the optimality (at least for potentials in the neighborhood of zero) of estimate (1.6) (up to the values of the exponents α, β), i.e. Theorem 2.1 shows, in particular, that estimate (1.6) can not hold with $\alpha, \beta \geq 0$,

$\alpha + 2\beta > 2m$. In similar sense, as a corollary of Theorem 2.1, one can obtain also an optimality of the stability results of [**20**], [**21**], [**23**], [**26**].

In the present work Theorem 2.1 is proved by means of instability examples with complex potentials. Examples of this type were considered for the first time in [**16**] for showing the exponential instability in Problem 1.1 in the zero energy case. An extention to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was obtained in [**8**].

More precisely, using explicit potentials v_{nm} of formula (2.6) given below, we obtain that estimate (2.4) holds for $v = v_{nm}$ for appropriate n, m, E depending on $A, B, \kappa, \tau, \varepsilon, m, s_2, d$ (see the proof of Theorem 2.1).

Let us consider the cylindrical variables:

$$\begin{aligned} (r_1, \theta, x') &\in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^{d-2}, \\ r_1 \cos\theta = x_1, &\quad r_1 \sin\theta = x_2, \\ x' &= (x_3, \dots, x_d). \end{aligned} \tag{2.5}$$

Take $\phi \in C^\infty(\mathbb{R}^2)$ with support in $B^2(0, 1/3) \cap \{x_1 > 1/4\}$ and with $\|\phi\|_{\mathbb{L}^\infty} = 1$. For integers $m, n > 0$, define the complex potential

$$v_{nm} = n^{-m} e^{in\theta} \phi(r_1, |x'|). \tag{2.6}$$

We recall that

$$\|v_{nm}\|_{\mathbb{L}^\infty} = n^{-m}, \quad \|v_{nm}\|_{C^m} \le C_1, \tag{2.7}$$

where $C_1 = C_1(d, m) > 0$. Note that C_1 is the same as in Theorem 2.1. Estimates (2.7) were given in [**16**] (see Theorem 2 of [**16**]).

To prove Theorem 2.1 we use, in partucular, the following proposition:

PROPOSITION 2.1. *Let $D = B^d(0,1)$, where $d \ge 2$. Let condition* (1.5) *hold with $v \equiv v_{nm}$ (of (2.6)) and $v \equiv v_0 \equiv 0$ for some $E > 0$ and some integers $m > 0$, $n > 20(1+\sqrt{E})^2$. Then, for any $\sigma > 0$,*

$$\|\hat\Phi_{nm}(E) - \hat\Phi_0(E)\|_{H^{-\sigma}(\mathbb{S}^{d-1}) \to H^{\sigma}(\mathbb{S}^{d-1})} \le C_2(1 + Q + EQ)2^{-n/4}, \tag{2.8}$$

where $\hat\Phi_{nm}$, $\hat\Phi_0$ are the DtN map for v_{nm} and v_0, respectively, $C_2 = C_2(d,\sigma) > 0$,

$$Q = \|(-\Delta + v_0 - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} + \|(-\Delta + v_{nm} - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}, \tag{2.9}$$

where $(-\Delta + v_0 - E)^{-1}$, $(-\Delta + v_{nm} - E)^{-1}$ are considered with the Dirichlet boundary condition in D and $H^{\pm\sigma} = W^{\pm\sigma,2}$ denote the standard Sobolev spaces.

Analogs of estimate (2.8) (but without dependence of the energy) were given in Theorem 2 of [**16**] for the zero energy case and in Theorem 2.4 of [**8**] for the case of the non-zero energy and the case of the energy intervals.

We obtain Theorem 2.1, combining known results on the spectrum of the Laplace operator in the unit ball (see formula (4.9) below), Proposition 2.1, estimates (2.7) and the fact that

$$\|F\|_{L^\infty(\mathbb{S}^{d-1}) \to L^\infty(\mathbb{S}^{d-1})} \le c(d,\sigma) \|F\|_{H^{-\sigma}(\mathbb{S}^{d-1}) \to H^{\sigma}(\mathbb{S}^{d-1})} \tag{2.10}$$

for sufficiently large σ. The detailed proof of Theorem 2.1 and the proof of Proposition 2.1 are given in Section 4. These proofs use, in particular, results, presented in Section 3.

REMARK 2.1. In a similar way with [**16**], [**8**], using a ball packing and covering by ball arguments (see also [**6**]), the instability result of Theorem 2.1 can be extended to the case when only real-valued potentials are considered and in the neighborhood of any potential (not only $v_0 \equiv 0$).

3. Some properties of solutions of the Schrödinger equation in the unit ball

In this section we continue assume that $D = B^d(0,1)$, where $d \geq 2$. We fix an orthonormal basis in $\mathbb{L}^2(\mathbb{S}^{d-1}) = \mathbb{L}^2(\partial D)$

$$\begin{aligned} &\{f_{jp} : j \geq 0,\ 1 \leq p \leq p_j\}, \\ &f_{jp} \text{ is a spherical harmonic of degree } j, \end{aligned} \tag{3.1}$$

where p_j is the dimension of the space of spherical harmonics of order j,

$$p_j = \binom{j+d-1}{d-1} - \binom{j+d-3}{d-1}, \tag{3.2}$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad \text{for } n \geq 0 \tag{3.3}$$

and

$$\binom{n}{k} = 0 \quad \text{for } n < 0. \tag{3.4}$$

The precise choice of f_{jp} is irrelevant for our purposes. Besides orthonormality, we only need f_{jp} to be the restriction of a homogeneous harmonic polynomial of degree j to the sphere and so $|x|^j f_{jp}(x/|x|)$ is harmonic. We use also the polar coordinates $(r,\omega) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$, with $x = r\omega \in \mathbb{R}^d$.

LEMMA 3.1. *Let $D = B^d(0,1)$, where $d \geq 2$. Let potential v satisfy* (1.3) *and* (1.5) *for some fixed E. Let $||v||_{\mathbb{L}^\infty(D)} \leq N$, for some $N > 0$. Then for any solution $\psi \in C(D \cup \partial D)$ of equation* (1.1) *the following inequality holds:*

$$\|\psi\|_{\mathbb{L}^2(D)} \leq \Big(1 + (N + |E|)\|(-\Delta + v - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}\Big)\|f\|_{\mathbb{L}^2(\partial D)}, \tag{3.5}$$

where $f = \psi|_{\partial D}$, $(-\Delta + v - E)^{-1}$ is considered with the Dirichlet boundary condition in D.

PROOF OF LEMMA 3.1. We expand the function f in the basis $\{f_{jp}\}$:

$$f = \sum_{j,p} c_{jp} f_{jp}. \tag{3.6}$$

We have that

$$\|f\|^2_{\mathbb{L}^2(\partial D)} = \sum_{j,p} |c_{jp}|^2. \tag{3.7}$$

Let

$$\psi_0(x) = \sum_{j,p} c_{jp} r^j f_{jp}(\omega). \tag{3.8}$$

Note that

$$\begin{aligned} \|\psi_0\|^2_{\mathbb{L}^2(D)} &= \sum_{j,p} |c_{jp}|^2 \|r^j f_{jp}(\omega)\|^2_{\mathbb{L}^2(D)} = \\ &= \sum_{j,p} |c_{jp}|^2 \int_0^1 r^{2j+d-1} dr \le \sum_{j,p} |c_{jp}|^2 \end{aligned} \tag{3.9}$$

Using (1.1) and the fact that ψ_0 is harmonic, we get that

$$(-\Delta + v - E)(\psi - \psi_0) = (E - v)\psi_0. \tag{3.10}$$

Since $\psi|_{\partial D} = \psi_0|_{\partial D} = f$, using (3.10), we find that

$$\|\psi - \psi_0\|_{\mathbb{L}^2(\partial D)} \le (N + |E|)\|(-\Delta + v - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}\|\psi_0\|_{\mathbb{L}^2(D)}. \tag{3.11}$$

Combining (3.7), (3.9), (3.11), we obtain (3.5). ■

Let $< \cdot, \cdot >$ denote the scalar product in the Hilbert space $\mathbb{L}^2(\partial D)$:

$$< f, g >= \int\limits_{\partial D} f(x)\bar{g}(x)dx, \tag{3.12}$$

where $f, g \in \mathbb{L}^2(\partial D)$.

LEMMA 3.2. *Let $D = B^d(0,1)$, where $d \ge 2$. Let potentials v_1, v_2 satisfy* (1.3) *and* (1.5) *for some fixed E. Let v_1, v_2 be supported in $B^d(0,1/3)$ and $||v_i||_{\mathbb{L}^\infty(D)} \le N$, $i = 1,2$, for some $N > 0$. Then for any $j_1, j_2 \in \mathbb{N}\cup\{0\}$, $1 \le p_1 \le p_{j_1}$, $1 \le p_2 \le p_{j_2}$ and $j_{max} = \max\{j_1, j_2\} \ge 10(1+\sqrt{|E|})^2$ the following inequality holds:*

$$\left|\left\langle f_{j_1p_1}, \left(\hat{\Phi}_1(E) - \hat{\Phi}_2(E)\right) f_{j_2p_2}\right\rangle\right| \le C(d)\Big(1 + (N + |E|)Q\Big)2^{-j_{max}}, \tag{3.13}$$

where

$$Q = \|(-\Delta + v_1 - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} + \|(-\Delta + v_2 - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}, \tag{3.14}$$

$\hat{\Phi}_1$, $\hat{\Phi}_2$ are the DtN map for v_1 and v_2, respectively, and $(-\Delta + v_1 - E)^{-1}$, $(-\Delta + v_2 - E)^{-1}$ are considered with the Dirichlet boundary condition in D.

Analogs of estimate (3.13) (but without dependence of the energy) were given in Lemma 1 of [**16**] for the zero energy case and in Lemma 3.4 of [**8**] for the case of the non-zero energy and the case of the energy intervals.

We prove Lemma 3.2 for $E \neq 0$ in Section 5, using expression of solutions of equation $-\Delta\psi = E\psi$ in $B^d(0,1)\setminus B^d(0,1/3)$ in terms of the Bessel functions J_α and Y_α with integer or half-integer order α.

4. Proofs of Proposition 2.1 and Theorem 2.1

We continue to assume that $D = B^d(0,1)$, where $d \geq 2$ and to use the orthonormal basis $\{f_{jp} : j \in \mathbb{N}\cup\{0\},\ 1 \leq p \leq p_j\}$ in $\mathbb{L}^2(\mathbb{S}^{d-1}) = \mathbb{L}^2(\partial D)$. The Sobolev spaces $H^\sigma(\mathbb{S}^{d-1})$ can be defined by

$$\begin{gathered}\left\{\sum_{j,p} c_{jp} f_{jp} : \Big\|\sum_{j,p} c_{jp} f_{jp}\Big\|_{H^\sigma} < +\infty\right\}, \\ \Big\|\sum_{j,p} c_{jp} f_{jp}\Big\|^2_{H^\sigma} = \sum_{j,p}(1+j)^{2\sigma}|c_{jp}|^2,\end{gathered} \tag{4.1}$$

see, for example, [**16**].

Consider an operator $A : H^{-\sigma}(\mathbb{S}^{d-1}) \to H^\sigma(\mathbb{S}^{d-1})$. We denote its matrix elements in the basis $\{f_{jp}\}$ by

$$a_{j_1p_1j_2p_2} =< f_{j_1p_1}, Af_{j_2p_2} > . \tag{4.2}$$

We identify in the sequel an operator A with its matrix $\{a_{j_1p_1j_2p_2}\}$. In this section we always assume that $j_1, j_2 \in \mathbb{N}\cup\{0\}$, $1 \leq p_1 \leq p_{j_1}$, $1 \leq p_2 \leq p_{j_2}$.

We recall that (see formula (12) of [**16**])

$$\|A\|_{H^{-\sigma}(\mathbb{S}^{d-1})\to H^\sigma(\mathbb{S}^{d-1})} \leq 4 \sup_{j_1,p_1,j_2,p_2} (1+\max\{j_1,j_2\})^{2\sigma+d}|a_{j_1p_1j_2p_2}|. \tag{4.3}$$

Proof of Proposition 2.1. In a similar way with the proof of Theorem 2 of [**16**] we obtain that

$$< f_{j_1p_1}, \left(\hat\Phi_{mn}(E) - \hat\Phi_0(E)\right) f_{j_2p_2} >= 0 \tag{4.4}$$

for $j_{max} = \max\{j_1, j_2\} \leq \left[\frac{n-1}{2}\right]$ (the only difference is that instead of the operator $-\Delta$ we consider the operator $-\Delta-E$), where $[\cdot]$ denotes the integer part of a number. Note that

$$\left[\frac{n-1}{2}\right] + 1 \geq n/2 > 10(1+\sqrt{E})^2, \quad \|v_{nm}\|_{\mathbb{L}^\infty(D)} \leq 1. \tag{4.5}$$

Combining (4.3), (4.4), (4.5) and Lemma 3.2, we get that

$$\begin{aligned}&\|\hat\Phi_{mn}(E)-\hat\Phi_0(E)\|_{H^{-\sigma}(\mathbb{S}^{d-1})\to H^\sigma(\mathbb{S}^{d-1})} \leq \\ &\qquad \leq 4C(d)\Big(1+(1+E)Q\Big) \sup_{j_{max}\geq n/2} (1+j_{max})^{2\sigma+d}2^{-j_{max}} \leq \\ &\qquad \leq C_2(d,\sigma)(1+Q+EQ)2^{-n/4},\end{aligned} \tag{4.6}$$

where

$$Q = \|(-\Delta+v_0-E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} + \|(-\Delta+v_{nm}-E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)}. \tag{4.7}$$

■

Let $N(\rho)$ denote the counting function of the Laplace operator in D

$$N(\rho) = |\{\lambda < \rho^2 : \lambda \text{ is a Dirichlet eigenvalue of } -\Delta \text{ in } D\}|, \tag{4.8}$$

where $|\cdot|$ is the cardinality of the corresponding set. We recall that according to the Weyl formula (of [**31**]):

$$N(\rho) \le c_1(d)\rho^d. \tag{4.9}$$

LEMMA 4.1. *Let $D = B^d(0,1)$, where $d \ge 1$. Then for any $\rho > 1$ there is some $E = E(\rho) \in (\rho^2, 2\rho^2)$ such that the interval*

$$\big(E(\rho) - c_2\rho^{2-d}, E(\rho) + c_2\rho^{2-d}\big) \tag{4.10}$$

does not contain Dirichlet eigenvalues of $-\Delta$ in D, where $c_2 = c_2(d) > 0$.

PROOF OF LEMMA 4.1. We put $c_2 = 2^{d-1}/(c_1(d)+1)$. Then we can select k disjoint intervals of the length $2c_2\rho^{2-d}$ in the interval $(\rho^2, 2\rho^2)$, where

$$k = \left[\frac{\rho^2}{2c_2\rho^{2-d}}\right] = [(c_1(d)+1)\rho^d] > N(\rho). \tag{4.11}$$

Thus, we have that at least one of these intervals does not contain Dirichlet eigenvalues of $-\Delta$ in $D = B^d(0,1)$. ■

PROOF OF THEOREM 2.1. Let $E = E(\rho)$ be the number of Lemma 4.1 for some $\rho > 1$. Using (4.10), we find that the distance from E to the Dirichlet spectrum of the operator $-\Delta$ in D is not less than $c_2\rho^{2-d}$. Using also that $E \in (\rho^2, 2\rho^2)$, we get that

$$\|(-\Delta - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} \le \frac{1}{c_2\rho^{2-d}} \le E^{(d-2)/2}/c_2, \tag{4.12}$$

where $(-\Delta - E)^{-1}$ is considered with the Dirichlet boundary condition in D. Let

$$n = [20(1+\sqrt{E})^2] + 1. \tag{4.13}$$

Using (2.7) and (4.10), we find that the distance from E to the Dirichlet spectrum of the operator $-\Delta + v_{nm}$ in D is not less than $c_2\rho^{2-d} - n^{-m}$, where v_{nm} is defined according to (2.6). Since $m > d$ and $E \in (\rho^2, 2\rho^2)$, using (4.13), we get that

$$\begin{aligned} \|(-\Delta + v_{nm} - E)^{-1}\|_{\mathbb{L}^2(D)\to\mathbb{L}^2(D)} &\le c_3 E^{(d-2)/2}, \\ E = E(\rho), \quad \rho \ge \rho_1(d,m) &> 1, \\ c_3 = c_3(d,m) &> 0, \end{aligned} \tag{4.14}$$

where $(-\Delta + v_{nm} - E)^{-1}$ is considered with the Dirichlet boundary condition in D.

Combining Proposition 2.1 and estimates (2.10), (4.12), (4.14), we find that

$$\begin{aligned} \delta = \|\hat{\Phi}_{nm}(E) - \hat{\Phi}_0(E)\|_{\mathbb{L}^\infty(\mathbb{S}^{d-1})\to\mathbb{L}^\infty(\mathbb{S}^{d-1})} \le c_4 E^{d/2} 2^{-n/4}, \\ E = E(\rho), \quad \rho \ge \rho_1(d,m) > 1, \\ n = [20(1+\sqrt{E})^2] + 1 \\ c_4 = c_4(d,m) > 0. \end{aligned} \tag{4.15}$$

Since $s_2 > m$, taking ρ big enough and using (4.15), we obtain the following inequalities:

$$n^{-m} < \varepsilon, \tag{4.16}$$

$$A(1+\sqrt{E})^\kappa \delta^\tau < \frac{1}{2} n^{-m}, \tag{4.17}$$

$$\begin{aligned} B(1+\sqrt{E})^{2(s-s_2)} \left(\ln\left(3+\delta^{-1}\right)\right)^{-s} < \frac{1}{2} n^{-m}, \\ 0 \le s \le s_2, \end{aligned} \tag{4.18}$$

where

$$E = E(\rho), \quad n = [20(1+\sqrt{E})^2] + 1. \tag{4.19}$$

Combining (2.6), (2.7), (4.16) - (4.19), we get that

$$\begin{aligned} A(1+\sqrt{E})^\kappa \delta^\tau + B(1+\sqrt{E})^{2(s-s_2)} \left(\ln\left(3+\delta^{-1}\right)\right)^{-s} < \\ < \frac{1}{2} n^{-m} + \frac{1}{2} n^{-m} = \|v_{nm} - v_0\|_{\mathbb{L}^\infty(D)} \\ \|v_{nm}\|_{\mathbb{L}^\infty(D)} = n^{-m} < \varepsilon, \\ \|v_{nm}\|_{C^m(D)} < C_1, \\ \operatorname{supp} v_{nm} \subset D. \end{aligned} \tag{4.20}$$

■

5. Proof of Lemma 3.2

To prove Lemma 3.2 we need some preliminaries. Consider the problem of finding solutions of the form $\psi(r,\omega) = R(r) f_{jp}(\omega)$ of equation (1.1) with $v \equiv 0$ and $D = B^d(0,1)$, where $d \ge 2$. We recall that:

$$\Delta = \frac{\partial^2}{(\partial r)^2} + (d-1) r^{-1} \frac{\partial}{\partial r} + r^{-2} \Delta_{S^{d-1}}, \tag{5.1}$$

where $\Delta_{S^{d-1}}$ is Laplace-Beltrami operator on S^{d-1},

$$\Delta_{S^{d-1}} f_{jp} = -j(j+d-2) f_{jp}. \tag{5.2}$$

Then we obtain the following equation for $R(r)$:

$$-R'' - \frac{d-1}{r}R' + \frac{j(j+d-2)}{r^2}R = ER. \tag{5.3}$$

Taking $R(r) = r^{-\frac{d-2}{2}}\tilde{R}(r)$, we get

$$r^2\tilde{R}'' + r\tilde{R}' + \left(Er^2 - \left(j + \frac{d-2}{2}\right)^2\right)\tilde{R} = 0. \tag{5.4}$$

This equation is known as the Bessel equation. For $E = k^2 \neq 0$ it has two linearly independent solutions $J_{j+\frac{d-2}{2}}(kr)$ and $Y_{j+\frac{d-2}{2}}(kr)$, where

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\alpha}}{\Gamma(m+1)\Gamma(m+\alpha+1)}, \tag{5.5}$$

$$Y_\alpha(z) = \frac{J_\alpha(z)\cos\pi\alpha - J_{-\alpha}(z)}{\sin\pi\alpha} \text{ for } \alpha \notin \mathbb{Z}, \tag{5.6}$$

and

$$Y_\alpha(z) = \lim_{\alpha'\to\alpha} Y_{\alpha'}(z) \text{ for } \alpha \in \mathbb{Z}. \tag{5.7}$$

We recall also that the system of functions

$$\begin{gathered}\left\{\psi_{jp}(r,\omega) = R_j(k,r)f_{jp}(\omega) : j \in \mathbb{N}\cup\{0\}, 1 \le p \le p_j\right\},\\ \text{is complete orthogonal system (in the sense of } \mathbb{L}^2\text{) in the space}\\ \text{of solutions of equation (1.1) in } D' = B(0,1)\setminus B(0,1/3)\\ \text{with } v \equiv 0,\ E = k^2 \text{ and boundary condition } \psi|_{r=1} = 0,\end{gathered} \tag{5.8}$$

where

$$R_j(k,r) = r^{-\frac{d-2}{2}}\Big(Y_{j+\frac{d-2}{2}}(kr)J_{j+\frac{d-2}{2}}(k) - J_{j+\frac{d-2}{2}}(kr)Y_{j+\frac{d-2}{2}}(k)\Big). \tag{5.9}$$

For the proof of (5.8) see, for example, [**8**].

LEMMA 5.1. *For any $\rho > 0$, integers $d \ge 2$, $n \ge 10(\rho+1)^2$ and $z \in \mathbb{C}$, $|z| \le \rho$, the following inequalities hold:*

$$\frac{1}{2}\frac{(|z|/2)^\alpha}{\Gamma(\alpha+1)} \le |J_\alpha(z)| \le \frac{3}{2}\frac{(|z|/2)^\alpha}{\Gamma(\alpha+1)}, \tag{5.10}$$

$$|J'_\alpha(z)| \le 3\frac{(|z|/2)^{\alpha-1}}{\Gamma(\alpha)}, \tag{5.11}$$

$$\frac{1}{2\pi}(|z|/2)^{-\alpha}\Gamma(\alpha) \le |Y_\alpha(z)| \le \frac{3}{2\pi}(|z|/2)^{-\alpha}\Gamma(\alpha) \tag{5.12}$$

$$|Y'_\alpha(z)| \le \frac{3}{\pi}(|z|/2)^{-\alpha-1}\Gamma(\alpha+1) \tag{5.13}$$

where $'$ denotes derivation with respect to z, $\alpha = n + \frac{d-2}{2}$ and $\Gamma(x)$ is the Gamma function.

In fact, the proof of Lemma 5.1 is given in [**8**] (see Lemma 3.3 of [**8**]). It was shown in [**8**] that inequalities (5.10) - (5.13) hold for any $n > n_0$, where n_0 is such that

$$\begin{cases} n_0 > 3, \\ \exp\left(\dfrac{\rho^2/4}{n_0+1}\right) - 1 \le 1/2, \\ 3\pi \dfrac{\max\left(1, (\rho/2)^{2n_0+1}\right)}{\Gamma(n_0)} + \dfrac{\rho^2}{2n_0 - \rho^2} + \dfrac{(\rho/2)^{2n_0} e^{\rho^2/4}}{\Gamma(n_0)} \le 1/2, \end{cases} \tag{5.14}$$

(see formula (6.18) of [**8**]). The only thing to check is that $n_0 = [10(\rho+1)^2] - 1$ satisfy (5.14), where $[\cdot]$ denotes the integer part of a number, The first two inequalities are obvious. The third follows from the estimate

$$\Gamma(n_0) = (n_0 - 1)! \ge \left(\frac{n_0 - 1}{e}\right)^{n_0 - 1}. \tag{5.15}$$

The final part of the proof of Lemma 3.2 consists of the following: first, we consider the case when $E = k^2 \neq 0$ and

$$j_1 = \max\{j_1, j_2\} \ge 10(1 + |k|)^2. \tag{5.16}$$

Let ψ_1, ψ_2 denote the solutions of equation (1.1) with boundary condition $\psi|_{\partial D} = f_{j_2 p_2}$ and potentials v_1 and v_2, respectively. Using Lemma 3.1 for v_1 and v_2, we get that

$$\|\psi_1 - \psi_2\|_{\mathbb{L}^2(D)} \le 2\Big(1 + (N + |E|)Q\Big), \tag{5.17}$$

where

$$Q = \|(-\Delta + v_1 - E)^{-1}\|_{\mathbb{L}^2(D) \to \mathbb{L}^2(D)} + \|(-\Delta + v_2 - E)^{-1}\|_{\mathbb{L}^2(D) \to \mathbb{L}^2(D)}, \tag{5.18}$$

Note that $\psi_1 - \psi_2$ is the solution of equation (1.1) in $D' = B(0,1) \setminus B(0,1/3)$ with potential $v \equiv 0$ and boundary condition $\psi|_{r=1} = 0$. According to (5.8), we have that

$$\psi_1 - \psi_2 = \sum_{j,p} c_{jp} \psi_{jp} \ \text{ in } D' \tag{5.19}$$

for some c_{jp}, where

$$\psi_{jp}(r, \omega) = R_j(k, r) f_{jp}(\omega). \tag{5.20}$$

Since $R_j(k, 1) = 0$, we find that

$$\left.\frac{\partial R_j(k,r)}{\partial r}\right|_{r=1} = \left.\frac{\partial\left(r^{\frac{d-2}{2}} R_j(k,r)\right)}{\partial r}\right|_{r=1}. \tag{5.21}$$

For $j \geq 10(1+|k|)^2$, using Lemma 5.1, we have that

$$\left|\frac{\left.\frac{\partial R_i(k,r)}{\partial r}\right|_{r=1}}{Y_\alpha(k)J_\alpha(k)}\right| = |k|\left|\frac{Y_\alpha'(k)}{Y_\alpha(k)} - \frac{J_\alpha'(k)}{J_\alpha(k)}\right| \leq \tag{5.22}$$
$$\leq 6|k|\left(\frac{(|k|/2)^{-\alpha-1}\Gamma(\alpha+1)}{(|k|/2)^{-\alpha}\Gamma(\alpha)} + \frac{(|k|/2)^{\alpha-1}\Gamma(\alpha+1)}{(|k|/2)^{\alpha}\Gamma(\alpha)}\right) = 6\alpha,$$

$$\left(\frac{||r^{-\frac{d-2}{2}}Y_\alpha(kr)||_{\mathbb{L}^2(\{1/3<|x|<2/5\})}}{|Y_\alpha(k)|}\right)^2 \geq \tag{5.23}$$
$$\geq \int_{1/3}^{2/5}\left(\frac{1}{3}\frac{(|k|r/2)^{-\alpha}\Gamma(\alpha)}{(|k|/2)^{-\alpha}\Gamma(\alpha)}\right)^2 r\,dr \geq \left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\left(\frac{1}{3}(5/2)^\alpha\right)^2,$$

$$\left(\frac{||r^{-\frac{d-2}{2}}J_\alpha(kr)||_{\mathbb{L}^2(\{1/3<|x|<2/5\})}}{|J_\alpha(k)|}\right)^2 \leq \tag{5.24}$$
$$\leq \int_{1/3}^{2/5}\left(3\frac{(|k|r/2)^{\alpha}\Gamma(\alpha)}{(|k|/2)^{\alpha}\Gamma(\alpha)}\right)^2 r\,dr \leq \left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\left(3(2/5)^\alpha\right)^2,$$

where $\alpha = j + \frac{d-2}{2}$. Note that if $j \geq 10(1+|k|)^2$ then $j + \frac{d-2}{2} > 3$. Combining (5.23) and (5.24), we get that

$$\frac{||\psi_{jp}||_{L^2(\{1/3<|x|<2/5\})}}{|Y_\alpha(k)J_\alpha(k)|} \geq \tag{5.25}$$
$$\geq \left(\left(\frac{2}{5}-\frac{1}{3}\right)\frac{1}{3}\right)^{1/2}\left(\frac{1}{3}(5/2)^\alpha - 3(2/5)^\alpha\right) > \frac{6}{1000}(5/2)^\alpha.$$

Combining (5.22) and (5.25), we find that

$$\left|\left.\frac{\partial R_j(k,r)}{\partial r}\right|_{r=1}\right| \leq 1000\alpha(5/2)^{-\alpha}||\psi_{jp}(E)||_{\mathbb{L}^2(\{1/3<|x|<1\})}. \tag{5.26}$$

Proceeding from (5.19) and using the Cauchy-Schwarz inequality, we get that

$$|c_{jp}| = \left|\frac{\left\langle \psi_{jp}, \psi_1-\psi_2\right\rangle_{\mathbb{L}^2(\{1/3<|x|<1\})}}{||\psi_{jp}(E)||^2_{\mathbb{L}^2(\{1/3<|x|<1\})}}\right| \leq \frac{||\psi(E)-\psi_0(E)||_{\mathbb{L}^2(B(0,1))}}{||\psi_{jp}(E)||_{\mathbb{L}^2(\{1/3<|x|<1\})}}. \tag{5.27}$$

Using (5.19), we find that

$$\left\langle f_{j_1p_1}, \left(\hat{\Phi}_1(E)-\hat{\Phi}_2(E)\right)f_{j_2p_2}\right\rangle = \left\langle f_{j_1p_1}, \left.\frac{\partial(\psi_1-\psi_2)}{\partial\nu}\right|_{\partial D}\right\rangle = \tag{5.28}$$
$$= \left\langle f_{j_1p_1}, \left.\frac{\partial R_{j_1}(k,r)}{\partial r}\right|_{r=1} f_{j_1p_1}\right\rangle = c_{j_1p_1}\left.\frac{\partial R_{j_1}(k,r)}{\partial r}\right|_{r=1}$$

Combining (5.16), (5.26), (5.27) and (5.28), we obtain that

$$\left\langle f_{j_1p_1}, \left(\hat{\Phi}_1(E) - \hat{\Phi}_2(E)\right) f_{j_2p_2} \right\rangle \le C(d)2^{-j_1}||\psi_1 - \psi_2||_{\mathbb{L}^2(B(0,1))}. \tag{5.29}$$

Combining (5.17) and (5.29), we get (3.13) for $j_1 \ge j_2$ and $E \ne 0$.

For $j_1 < j_2$ we use the fact that $\hat{\Phi}_v^*(E) = \hat{\Phi}_{\bar{v}}(\bar{E})$ in order to swap j_1 and j_2, where $\hat{\Phi}_v^*$ denotes the adjoint operator to $\hat{\Phi}_v$. Thus we complete the proof of Lemma 3.2 for the non-zero energy case.

Estimate (3.13) for the zero energy case follows from Lemma 1 of [**16**].

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Paper D

PAPER D

Stability estimates for determination of potential from the impedance boundary map

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Abstract. We study the impedance boundary map (or Robin-to-Robin map) for the Schrödinger equation in open bounded domain at fixed energy in multidimensions. We give global stability estimates for determining potential from these boundary data and, as corollary, from the Cauchy data set. Our results include also, in particular, an extension of the Alessandrini identity to the case of the impedance boundary map.

1. Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D,\ E \in \mathbb{R}, \tag{1.1}$$

where

$$\begin{gathered} D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2, \\ \text{with } \partial D \in C^2, \end{gathered} \tag{1.2}$$

$$v \in \mathbb{L}^\infty(D). \tag{1.3}$$

We consider the impedance boundary map $\hat{M}_\alpha = \hat{M}_{\alpha,v}(E)$ defined by

$$\hat{M}_\alpha[\psi]_\alpha = [\psi]_{\alpha-\pi/2} \tag{1.4}$$

for all sufficiently regular solutions ψ of equation (1.1) in $\bar{D} = D \cup \partial D$, where

$$[\psi]_\alpha = [\psi(x)]_\alpha = \cos\alpha\, \psi(x) - \sin\alpha\, \frac{\partial\psi}{\partial\nu}|_{\partial D}(x), \quad x \in \partial D,\ \alpha \in \mathbb{R} \tag{1.5}$$

and ν is the outward normal to ∂D. One can show(see Lemma 3.2) that there is not more than a countable number of $\alpha \in \mathbb{R}$ such that E is an eigenvalue for the operator $-\Delta + v$ in D with the boundary condition

$$\cos\alpha\, \psi|_{\partial D} - \sin\alpha\, \frac{\partial\psi}{\partial\nu}|_{\partial D} = 0. \tag{1.6}$$

Therefore, for any energy level E we can assume that for some fixed $\alpha \in \mathbb{R}$

$$E \text{ is not an eigenvalue for the operator } -\Delta + v \text{ in } D \\ \text{with boundary condition (1.6)} \tag{1.7}$$

and, as a corollary, $\hat{M}_\alpha$ can be defined correctly.

Note that the impedance boundary map $\hat{M}_\alpha$ is reduced to the Dirichlet-to-Neumann(DtN) map if $\alpha = 0$ and is reduced to the Neumann-to-Dirichlet(NtD) map if $\alpha = \pi/2$. The map $\hat{M}_\alpha$ can be called also as the Robin-to-Robin map. General Robin-to-Robin map was considered, in particular, in [**11**].

We consider the following inverse boundary value problem for equation (1.1).

PROBLEM 1.1. Given $\hat{M}_\alpha$ for some fixed E and α, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at fixed energy (see [**10**], [**20**]). At zero energy this problem can be considered also as a generalization of the Calderon problem of the electrical impedance tomography (see [**6**], [**20**]).

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness theorems and global reconstruction methods for Problem 1.1 with $\alpha = 0$ were given for the first time in [**20**] in dimension $d \geq 3$ and in [**5**] in dimension $d = 2$.

Global stability estimates for Problem 1.1 with $\alpha = 0$ were given for the first time in [**1**] in dimension $d \geq 3$ and in [**27**] in dimension $d = 2$. A principal improvement of the result of [**1**] was given recently in [**25**] (for the zero energy case). Due to [**18**] these logarithmic stability results are optimal (up to the value of the exponent). An extention of the instability estimates of [**18**] to the case of the non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was given in [**13**].

Note also that for the Calderon problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [**32**] for $d \geq 3$ and in [**19**] for $d = 2$. In addition, for the case of piecewise constant or piecewise real analytic conductivity the first uniqueness results for the Calderon problem in dimension $d \geq 2$ were given in [**7**], [**15**]. Lipschitz stability estimate for the case of piecewise constant conductivity was proved in [**2**] and additional studies in this direction were fulfilled in [**29**].

It should be noted that in most of previous works on inverse boundary value problems for equation (1.1) at fixed E it was assumed in one way or another that E is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in D, see [**1**], [**18**], [**20**], [**25**], [**27**], [**28**], [**30**]. Nevertheless, the results of [**5**] can be considered as global uniqueness and reconstruction results for Problem 1.1 in dimension $d = 2$ with general α.

In the present work we give global stability estimates for Problem 1.1 in dimension $d \geq 2$ with general α. These results are presented in detail in Section 2.

In addition, in the present work we establish some basic properties of the impedance boundary map with general α. In particular, we extend the Alessandrini identity to this general case. These results are presented in detail in Section 3.

In **[14]** we give also global reconstruction method for Problem 1.1 in multidimensions with general α.

2. Stability estimates

In this section we always assume that D satisfies (1.2).

We will use the fact that if v_1, v_2 are potentials satisfying (1.3), (1.7) for some fixed E and α, then

$$\hat{M}_{\alpha,v_1}(E) - \hat{M}_{\alpha,v_2}(E) \text{ is a bounded operator in } \mathbb{L}^\infty(\partial D), \tag{2.1}$$

where $\hat{M}_{\alpha,v_1}(E)$, $\hat{M}_{\alpha,v_2}(E)$ denote the impedance boundary maps for v_1, v_2, respectively. Actually, under our assumptions, $\hat{M}_{\alpha,v_1}(E) - \hat{M}_{\alpha,v_2}(E)$ is a compact operator in $\mathbb{L}^\infty(\partial D)$ (see Corollary 3.1).

Let

$$\begin{gathered} ||A|| \text{ denote the norm of an operator} \\ A : \mathbb{L}^\infty(\partial D) \to \mathbb{L}^\infty(\partial D). \end{gathered} \tag{2.2}$$

Let the Cauchy data set $\mathcal{C}_v$ for equation (1.1) be defined by:

$$\mathcal{C}_v = \left\{ \left(\psi|_{\partial D}, \frac{\partial \psi}{\partial \nu}|_{\partial D} \right) : \begin{array}{r} \text{for all sufficiently regular solutions } \psi \text{ of} \\ \text{equation (1.1) in } \bar{D} = D \cup \partial D \end{array} \right\}. \tag{2.3}$$

In addition, the Cauchy data set $\mathcal{C}_v$ can be represented as the graph of the impedance boundary map $\hat{M}_\alpha = \hat{M}_{\alpha,v}(E)$ defined by (1.4) under assumptions (1.7).

2.1. Estimates for $d \geq 3$. In this subsection we assume for simplicity that

$$v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d, \ \operatorname{supp} v \subset D, \tag{2.4}$$

where

$$W^{m,1}(\mathbb{R}^d) = \{ v : \ \partial^J v \in L^1(\mathbb{R}^d), \ |J| \leq m \}, \ m \in \mathbb{N} \cup 0, \tag{2.5}$$

where

$$J \in (\mathbb{N} \cup 0)^d, \ |J| = \sum_{i=1}^{d} J_i, \ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}. \tag{2.6}$$

Let

$$||v||_{m,1} = \max_{|J| \leq m} ||\partial^J v||_{L^1(\mathbb{R}^d)}. \tag{2.7}$$

Note also that (2.4) $\Rightarrow$ (1.3).

THEOREM 2.1. *Let D satisfy (1.2), where $d \geq 3$. Let v_1, v_2 satisfy (2.4) and (1.7) for some fixed E and α. Let $||v_j||_{m,1} \leq N$, $j = 1,2$, for some $N > 0$. Let $\hat{M}_{\alpha,v_1}(E)$ and $\hat{M}_{\alpha,v_2}(E)$ denote the impedance boundary maps for v_1 and v_2, respectively. Then*

$$||v_1 - v_2||_{L^\infty(D)} \leq C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s}, \quad 0 < s \leq (m-d)/m, \tag{2.8}$$

where $C_\alpha = C_\alpha(N, D, m, s, E)$ and $\delta_\alpha = ||\hat{M}_{\alpha,v_1}(E) - \hat{M}_{\alpha,v_2}(E)||$ is defined according to (2.2).

REMARK 2.1. Estimate (2.8) with $\alpha = 0$ is a variation of the result of [**1**] (see also [**25**]).

Proof of Theorem 2.1 is given in Section 5. This proof is based on results presented in Sections 3, 4.

Theorem 2.1 implies the following corollary:

COROLLARY 2.1. *Let D satisfy (1.2), where $d \geq 3$. Let potentials v_1, v_2 satisfy (2.4). Then*

$$||v_1 - v_2||_{L^\infty(D)} \leq \min_{\alpha \in \mathbb{R}} C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s}, \quad 0 < s \leq (m-d)/m, \tag{2.9}$$

where C_α and δ_α at fixed α are the same that in Theorem 2.1.

Actually, Corollary 2.1 can be considered as global stability estimate for determining potential v from its Cauchy data set $\mathcal{C}_v$ for equation (1.1) at fixed energy E, where $d \geq 3$.

2.2. Estimates for $d = 2$. In this subsection we assume for simplicity that

$$v \in C^2(\bar{D}), \quad \operatorname{supp} v \subset D. \tag{2.10}$$

Note also that (2.10) $\Rightarrow$ (1.3).

THEOREM 2.2. *Let D satisfy (1.2), where $d = 2$. Let v_1, v_2 satisfy (2.10) and (1.7) for some fixed E and α. Let $||v_j||_{C^2(\bar{D})} \leq N$, $j = 1,2$, for some $N > 0$. Let $\hat{M}_{\alpha,v_1}(E)$ and $\hat{M}_{\alpha,v_2}(E)$ denote the impedance boundary maps for v_1 and v_2, respectively. Then*

$$||v_1 - v_2||_{L^\infty(D)} \leq C_\alpha \left(\ln\left(3 + \delta_\alpha^{-1}\right)\right)^{-s} \left(\ln\left(3\ln\left(3 + \delta_\alpha^{-1}\right)\right)\right)^2, \quad 0 < s \leq 3/4, \tag{2.11}$$

where $C_\alpha = C_\alpha(N, D, s, E)$, $\delta_\alpha = ||\hat{M}_{\alpha,v_1}(E) - \hat{M}_{\alpha,v_2}(E)||$ is defined according to (2.2).

REMARK 2.2. Theorem 2.2 for $\alpha = 0$ was given in [**27**] with $s = 1/2$ and in [**30**] with $s = 3/4$.

Proof of Theorem 2.2 is given in Section 7. This proof is based on results presented in Sections 3, 6.

Theorem 2.2 implies the following corollary:

COROLLARY 2.2. *Let D satisfy (1.2), where $d = 2$. Let potentials v_1, v_2 satisfy (2.10). Then*

(2.12)
$$||v_1 - v_2||_{L^\infty(D)} \le \min_{\alpha\in\mathbb{R}} C_\alpha \left(\ln\left(3+\delta_\alpha^{-1}\right)\right)^{-s} \left(\ln\left(3\ln\left(3+\delta_\alpha^{-1}\right)\right)\right)^2, \quad 0 < s \le 3/4,$$

where C_α and δ_α at fixed α are the same that in Theorem 2.2.

Actually, Corollary 2.2 can be considered as global stability estimate for determining potential v from its Cauchy data set $\mathcal{C}_v$ for equation (1.1) at fixed energy E, where $d = 2$.

2.3. Concluding remarks. Furthermore, proceeding from the methods used in the proofs of Theorem 2.1 and 2.2, one can obtain the following corollary:

COROLLARY 2.3. *Under assumptions (1.2), (1.3), real-valued potential v is uniquely determined by its Cauchy data $\mathcal{C}_v$ at fixed real energy E.*

Actually, under additional assumptions (2.4), (2.10) for $d \ge 3$ and $d = 2$, respectively, Corollary 2.3 follows from Corollaries 2.1, 2.2 immediately.

To our knowledge the result of Corollary 2.3 for $d \ge 3$ was not yet completely proved in the literature.

Let $\sigma_{\alpha,v}$ denote the spectrum of the operator $-\Delta+v$ in D with boundary condition (1.6).

REMARK 2.3. In Theorems 2.1 and 2.2 we do not assume that $E \notin \sigma_{\alpha,v_1} \cup \sigma_{\alpha,v_2}$ namely for $\alpha = 0$ in contrast with [**1**], [**25**], [**27**], [**28**], [**30**]. In addition, in fact, in Corollaries 2.1 and 2.2 there are no special assumptions on E and α at all. Actually, the stability estimates of [**1**], [**25**], [**27**], [**28**], [**30**] make no sense for $E \in \sigma_{0,v_1} \cup \sigma_{0,v_2}$ and are too weak if $dist(E, \sigma_{0,v_1} \cup \sigma_{0,v_2})$ is too small.

REMARK 2.4. The stability estimates of Subsections 2.1 and 2.2 admit principal improvement in the sense described in [**25**], [**26**], [**31**]. In particular, Theorem 2.1 with $s = m - d$ (for $d = 3$ and $E = 0$) follows from results presented in Sections 3, 4 of the present work and results presented in Section 8 of [**25**]. In addition, estimates (2.8), (2.9) for $s = (m-d)/d$ admit a proof technically very similar to the proof of Theorem 2.1, presented in Section 5. Possibility of such a proof of estimate (2.8) for $s = (m-d)/d$, $\alpha = 0$, $E = 0$ was mentioned, in particular, in [**35**].

REMARK 2.5. The stability estimates of Subsections 2.1 and 2.2 can be extended to the case when we do not assume that $\operatorname{supp} v \subset D$ or, by other words, that v is zero near the bounadry. In this connection see, for example, [**1**], [**27**].

In the present work we do not develop Remarks 2.4 and 2.5 in detail because of restrictions in time.

Note also that Theorems 2.1 and 2.2 remain valid with complex-valued potentials v_1, v_2 and complex E, α. Finally, we note that in Theorems 2.1, 2.2 and Corollaries 2.1, 2.2 with real α, constant C_α can be considered as independent of α.

3. Some basic properties of the impedance boundary map

LEMMA 3.1. *Let D satisfy (1.2). Let potential v satisfy (1.3) and (1.7) for some fixed E and α. Let $\hat{M}_\alpha = \hat{M}_{\alpha,v}(E)$ denote the impedance boundary map for v. Then*

$$\begin{aligned} \left(\sin\alpha\,\hat{M}_\alpha + \cos\alpha\,\hat{I}\right)[\psi]_\alpha &= \psi|_{\partial D}, \\ \left(\cos\alpha\,\hat{M}_\alpha - \sin\alpha\,\hat{I}\right)[\psi]_\alpha &= \frac{\partial\psi}{\partial\nu}|_{\partial D}, \end{aligned} \tag{3.1}$$

$$\int\limits_{\partial D}[\psi^{(1)}]_\alpha \hat{M}_\alpha [\psi^{(2)}]_\alpha dx = \int\limits_{\partial D}[\psi^{(2)}]_\alpha \hat{M}_\alpha [\psi^{(1)}]_\alpha dx \tag{3.2}$$

for all sufficiently regular solutions ψ, $\psi^{(1)}$, $\psi^{(2)}$ of equation (1.1) in $\bar{D}$, where $\hat{I}$ denotes the identity operator on ∂D and $[\psi]_\alpha$ is defined by (1.5).

Note that identities (3.1) imply that

$$\left(\sin(\alpha_1-\alpha_2)\hat{M}_{\alpha_1} + \cos(\alpha_1-\alpha_2)\hat{I}\right)\left(\sin(\alpha_2-\alpha_1)\hat{M}_{\alpha_2} + \cos(\alpha_2-\alpha_1)\hat{I}\right) = \hat{I}, \tag{3.3}$$

under the assumptions of Lemma 3.1 fulfilled simultaneously for $\alpha = \alpha_1$ and $\alpha = \alpha_2$.

Note also that from (3.2) we have that

$$\int\limits_{\partial D}[\phi^{(1)}]_\alpha \hat{M}_\alpha [\phi^{(2)}]_\alpha dx = \int\limits_{\partial D}[\phi^{(2)}]_\alpha \hat{M}_\alpha [\phi^{(1)}]_\alpha dx \tag{3.4}$$

for all sufficiently regular functions $\phi^{(1)}$, $\phi^{(2)}$ on ∂D.

PROOF OF LEMMA 3.1. Identities (3.1) follow from definition (1.4) of the map $\hat{M}_\alpha$.

To prove (3.2) we use, in particular, the Green formula

$$\int\limits_{\partial D}\left(\phi^{(1)}\frac{\partial\phi^{(2)}}{\partial\nu} - \phi^{(2)}\frac{\partial\phi^{(1)}}{\partial\nu}\right)dx = \int\limits_{D}\left(\phi^{(1)}\Delta\phi^{(2)} - \phi^{(2)}\Delta\phi^{(1)}\right)dx, \tag{3.5}$$

where $\phi^{(1)}$ and $\phi^{(2)}$ are arbitrary sufficiently regular functions in $\bar{D}$. Using (3.5) and the identities

$$\psi^{(1)}\Delta\psi^{(2)} = (v-E)\psi^{(1)}\psi^{(2)} = \psi^{(2)}\Delta\psi^{(1)} \quad \text{in } D, \tag{3.6}$$

we obtain that

$$\int\limits_{\partial D}\left(\psi^{(1)}\frac{\partial\psi^{(2)}}{\partial\nu} - \psi^{(2)}\frac{\partial\psi^{(1)}}{\partial\nu}\right)dx = 0. \tag{3.7}$$

Using (3.7), we get that

$$\int\limits_{\partial D}\left(\cos\alpha\,\psi^{(1)}-\sin\alpha\frac{\partial\psi^{(1)}}{\partial\nu}\right)\left(\sin\alpha\,\psi^{(2)}+\cos\alpha\frac{\partial\psi^{(2)}}{\partial\nu}\right)dx = \tag{3.8}$$
$$=\int\limits_{\partial D}\left(\cos\alpha\,\psi^{(2)}-\sin\alpha\frac{\partial\psi^{(2)}}{\partial\nu}\right)\left(\sin\alpha\,\psi^{(1)}+\cos\alpha\frac{\partial\psi^{(1)}}{\partial\nu}\right)dx.$$

Identity (3.2) follows from (3.8) and definition (1.4) of the map $\hat{M}_\alpha$. ∎

THEOREM 3.1. *Let D satisfy (1.2). Let two potentials v_1, v_2 satisfy (1.3), (1.7) for some fixed E and α. Let $\hat{M}_{\alpha,v_1}=\hat{M}_{\alpha,v_1}(E)$, $\hat{M}_{\alpha,v_2}=\hat{M}_{\alpha,v_2}(E)$ denote the impedance boundary maps for v_1, v_2, respectively. Then*

$$\int\limits_{D}(v_1-v_2)\,\psi_1\psi_2\,dx=\int\limits_{\partial D}[\psi_1]_\alpha\left(\hat{M}_{\alpha,v_1}-\hat{M}_{\alpha,v_2}\right)[\psi_2]_\alpha dx \tag{3.9}$$

for all sufficiently regular solutions ψ_1 and ψ_2 of equation (1.1) in $\bar{D}$ with $v=v_1$ and $v=v_2$, respectively, where $[\psi]_\alpha$ is defined by (1.5).

PROOF OF THEOREM 3.1. As in (3.6) we have that

$$\begin{aligned}\psi_1\Delta\psi_2&=(v_2-E)\psi_1\psi_2,\\ \psi_2\Delta\psi_1&=(v_1-E)\psi_1\psi_2.\end{aligned} \tag{3.10}$$

Combining (3.10) with (3.5), (3.1) and (3.4), we obtain that

$$\begin{aligned}\int\limits_{D}(v_1(x)-v_2(x))\,\psi_1(x)\psi_2(x)dx&=\int\limits_{\partial D}\left(\psi_2\frac{\partial\psi_1}{\partial\nu}-\psi_1\frac{\partial\psi_2}{\partial\nu}\right)dx=\\ =\int\limits_{\partial D}\left(\sin\alpha\,\hat{M}_{\alpha,v_2}+\cos\alpha\,\hat{I}\right)&[\psi_2]_\alpha\left(\cos\alpha\,\hat{M}_{\alpha,v_1}-\sin\alpha\,\hat{I}\right)[\psi_1]_\alpha dx\,-\\ -\int\limits_{\partial D}\left(\sin\alpha\,\hat{M}_{\alpha,v_1}+\cos\alpha\,\hat{I}\right)&[\psi_1]_\alpha\left(\cos\alpha\,\hat{M}_{\alpha,v_2}-\sin\alpha\,\hat{I}\right)[\psi_2]_\alpha dx=\\ &=\int\limits_{\partial D}[\psi_1]_\alpha\left(\hat{M}_{\alpha,v_1}-\hat{M}_{\alpha,v_2}\right)[\psi_2]_\alpha dx.\end{aligned} \tag{3.11}$$

∎

REMARK 3.1. Identity (3.9) for $\alpha=0$ is reduced to Alessandrini's identity (Lemma 1 of [**1**]).

Let $G_\alpha(x,y,E)$ be the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6) under assumptions (1.2), (1.3) and (1.7). Note that

$$G_\alpha(x,y,E) = G_\alpha(y,x,E), \quad x,y \in \bar{D}. \tag{3.12}$$

The symmetry (3.12) is proved in Section 9.

THEOREM 3.2. *Let D satisfy (1.2). Let potential v satisfy (1.3) and (1.7) for some fixed E and α such that $\sin\alpha \neq 0$. Let $G_\alpha(x,y,E)$ be the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6). Then for $x,y \in \partial D$*

$$M_\alpha(x,y,E) = \frac{1}{\sin^2\alpha}\ G_\alpha(x,y,E) - \frac{\cos\alpha}{\sin\alpha}\,\delta_{\partial D}(x-y), \tag{3.13}$$

where $M_\alpha(x,y,E)$ and $\delta_{\partial D}(x-y)$ denote the Schwartz kernels of the impedance boundary map $\hat{M}_\alpha = \hat{M}_{\alpha,v}(E)$ and the identity operator $\hat{I}$ on ∂D, respectively, where $\hat{M}_\alpha$ and $\hat{I}$ are considered as linear integral operators.

PROOF OF THEOREM 3.2. Note that

$$[\phi]_{\alpha-\pi/2} = \frac{1}{\sin^2\alpha}\ \sin\alpha\,\phi|_{\partial D} - \frac{\cos\alpha}{\sin\alpha}\,[\phi]_\alpha. \tag{3.14}$$

for all suffuciently regular functions ϕ in some neighbourhood of ∂D in D. Since G_α is the Green function for equation (1.1) we have that

$$\psi(y) = \int\limits_{\partial D} \left(\psi(x)\frac{\partial G_\alpha}{\partial \nu_x}(x,y,E) - G_\alpha(x,y,E)\frac{\partial\psi}{\partial\nu}(x)\right) dx, \quad y \in D, \tag{3.15}$$

for all suffuciently regular solutions ψ of equation (1.1). Using (3.15) and impedance boundary condition (1.6) for G_α, we get that

$$\begin{aligned}\sin\alpha\,\psi(y) = \sin\alpha \int\limits_{\partial D} \left(\psi(x)\frac{\partial G_\alpha}{\partial \nu_x}(x,y,E) - G_\alpha(x,y,E)\frac{\partial\psi}{\partial\nu}(x)\right) dx = \\ = \int\limits_{\partial D} [\psi(x)]_\alpha G_\alpha(x,y,E)dx, \quad y \in D.\end{aligned} \tag{3.16}$$

Due to (3.4) we have that

$$M_\alpha(x,y,E) = M_\alpha(y,x,E), \quad x,y \in \partial D. \tag{3.17}$$

Combining (1.4), (3.14), (3.16) and (3.17), we obtain (3.13). ∎

COROLLARY 3.1. *Let assumtions of Theorem 3.1 hold. Then*

$$\hat{M}_{\alpha,v_1}(E) - \hat{M}_{\alpha,v_2}(E) \text{ is a compact operator in } \mathbb{L}^\infty(\partial D). \tag{3.18}$$

Scheme of the proof of Corollary 3.1. Let $G_{\alpha,v_1}(x,y,E)$ and $G_{\alpha,v_2}(x,y,E)$ be the Green functions for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6) for $v = v_1$ and $v = v_2$, respectively. Using (3.12), we find that

$$(3.19)\quad \begin{gathered} G_{\alpha,v_1}(x,y,E) = \int\limits_D G_{\alpha,v_1}(x,\xi,E)\,(\Delta_\xi - v_2(\xi) + E)\,G_{\alpha,v_2}(\xi,y,E)\,d\xi, \\ G_{\alpha,v_2}(x,y,E) = \int\limits_D (\Delta_\xi - v_1(\xi) + E)\,G_{\alpha,v_1}(x,\xi,E)G_{\alpha,v_2}(\xi,y,E)\,d\xi, \\ \int\limits_{\partial D}\left(G_{\alpha,v_1}(x,\xi,E)\frac{\partial G_{\alpha,v_2}}{\partial \nu_\xi}(\xi,y,E) - G_{\alpha,v_2}(\xi,y,E)\frac{\partial G_{\alpha,v_1}}{\partial \nu_\xi}(x,\xi,E)\right)d\xi = 0, \\ x, y \in D. \end{gathered}$$

Combining (3.19) with (3.5), we get that

$$(3.20)\quad G_{\alpha,v_1}(x,y,E) - G_{\alpha,v_2}(x,y,E) = \int\limits_D (v_1(\xi) - v_2(\xi))\,G_{\alpha,v_1}(x,\xi,E)G_{\alpha,v_2}(\xi,y,E)\,d\xi, \quad x, y \in D.$$

The proof of (3.18) for the case of $\sin\alpha \neq 0$ can be completed proceeding from (3.3), (3.13), (3.20) and estimates of [**16**] and [**4**] on $G_\alpha(x,y,E)$ for $v \equiv 0$, $E = 0$ (for more detailed information see Section 6 of [**14**]).

Corollary 3.1 for the Dirichlet-to-Neumann case ($\sin\alpha = 0$) was given in [**20**]. ■

Lemma 3.2. *Let D satisfy (1.2). Let v be a real-valued potential satisfying (1.3). Then for any fixed $E \in \mathbb{R}$ there is not more than countable number of $\alpha \in \mathbb{R}$ such that E is an eigenvalue for the operator $-\Delta + v$ in D with boundary condition (1.6).*

Proof of Lemma 3.2. Let $\psi^{(1)}$, $\psi^{(2)}$ be eigenfunctions for the operator $-\Delta + v$ in D with boundary condition (1.6) for $\alpha = \alpha^{(1)}$ and $\alpha = \alpha^{(2)}$, respectively. Then

$$(3.21)\quad \sin\left(\alpha^{(1)} - \alpha^{(2)}\right)\int\limits_{\partial D}\psi^{(1)}\psi^{(2)}dx = \sin\alpha^{(1)}\sin\alpha^{(2)}\int\limits_{\partial D}\left(\psi^{(1)}\frac{\partial\psi^{(2)}}{\partial\nu} - \psi^{(2)}\frac{\partial\psi^{(1)}}{\partial\nu}\right)dx = 0.$$

Since in the separable space $\mathbb{L}^2(\partial D)$ there is not more than countable orthogonal system of functions, we obtain the assertion of Lemma 3.2. ■

Remark 3.2. The assertion of Lemma 3.2 remains valid for the case of $\alpha \in \mathbb{C}$.

4. Faddeev functions

We consider the Faddeev functions G, ψ, h (see **[8]**, **[9]**, **[12]**, **[20]**):

$$\psi(x,k) = e^{ikx} + \int_{\mathbb{R}^d} G(x-y,k)v(y)\psi(y,k)dy, \tag{4.1}$$

$$G(x,k) = e^{ikx}g(x,k), \quad g(x,k) = -(2\pi)^{-d}\int_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2+2k\xi}, \tag{4.2}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d$, $\operatorname{Im} k \neq 0$, $d \geq 3$,

$$h(k,l) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-ilx}v(x)\psi(x,k)dx, \tag{4.3}$$

where

$$k, l \in \mathbb{C}^d,\ k^2 = l^2,\ \operatorname{Im} k = \operatorname{Im} l \neq 0. \tag{4.4}$$

One can consider (4.1), (4.3) assuming that

(4.5) v is a sufficiently regular function on $\mathbb{R}^d$ with suffucient decay at infinity.

For example, in connection with Problem 1.1, one can consider (4.1), (4.3) assuming that

$$v \in \mathbb{L}^\infty(D), \quad v \equiv 0 \text{ on } \mathbb{R} \setminus D. \tag{4.6}$$

We recall that (see **[8]**, **[9]**, **[12]**, **[20]**):

- The function G satisfies the equation
$$(\Delta + k^2)G(x,k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d; \tag{4.7}$$
- Formula (4.1) at fixed k is considered as an equation for
$$\psi = e^{ikx}\mu(x,k), \tag{4.8}$$
where μ is sought in $\mathbb{L}^\infty(\mathbb{R}^d)$;
- As a corollary of (4.1), (4.2), (4.7), ψ satisfies (1.1) for $E = k^2$;
- The Faddeev functions G, ψ, h are (non-analytic) continuation to the complex domain of functions of the classical scattering theory for the Schrödinger equation (in particular, h is a generalized "‘scattering"’ amplitude).

In addition, G, ψ, h in their zero energy restriction, that is for $E = 0$, were considered for the first time in **[3]**. The Faddeev functions G, ψ, h were, actually, rediscovered in **[3]**.

Let

$$\begin{gathered}\Sigma_E = \left\{k \in \mathbb{C}^d : k^2 = k_1^2 + \ldots + k_d^2 = E\right\}, \\ \Theta_E = \{k \in \Sigma_E,\ l \in \Sigma_E : \operatorname{Im} k = \operatorname{Im} l\}.\end{gathered} \tag{4.9}$$

Under the assumptions of Theorem 2.1, we have that:

$$\mu(x,k) \to 1 \quad \text{as} \quad |\mathrm{Im}\, k| \to \infty \tag{4.10}$$

and, for any $\sigma > 1$,

$$|\mu(x,k)| + |\nabla\mu(x,k)| \le \sigma \quad \text{for} \quad |\mathrm{Im}\, k| \ge r_1(N,D,E,m,\sigma), \tag{4.11}$$

where $x \in \mathbb{R}^d$, $k \in \Sigma_E$;

$$\hat{v}(p) = \lim_{\substack{(k,l)\,\in\,\Theta_E,\ k-l=p \\ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \to \infty}} h(k,l) \quad \text{for any } p \in \mathbb{R}^d, \tag{4.12}$$

$$\begin{aligned} |\hat{v}(p) - h(k,l)| \le \frac{c_1(D,E,m)N^2}{\rho} \quad &\text{for } (k,l) \in \Theta_E, \quad p = k - l, \\ &|\mathrm{Im}\, k| = |\mathrm{Im}\, l| = \rho \ge r_2(N,D,E,m), \\ &p^2 \le 4(E + \rho^2), \end{aligned} \tag{4.13}$$

where

$$\hat{v}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} v(x)dx, \quad p \in \mathbb{R}^d. \tag{4.14}$$

Results of the type (4.10) go back to **[3]**. Results of the type (4.12), (4.13) (with less precise right-hand side in (4.13)) go back to **[12]**. In the present work estimate (4.11) is given according to **[22]**, **[24]**. Estimate (4.13) follows, for example, from the estimate

$$\begin{aligned} \|\Lambda^{-s} g(k) \Lambda^{-s}\|_{\mathbb{L}^2(\mathbb{R}^d) \to \mathbb{L}^2(\mathbb{R}^d)} = O(|k|^{-1}) \quad \text{as} \quad |k| \to \infty, \\ k \in \mathbb{C}^d \setminus \mathbb{R}^d,\ |k| = (|\mathrm{Re}\, k|^2 + |\mathrm{Im}\, k|^2)^{1/2}, \end{aligned} \tag{4.15}$$

for $s > 1/2$, where $g(k)$ denotes the integral operator with the Schwartz kernel $g(x-y,k)$ and Λ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$. Estimate (4.15) was formulated, first, in **[17]** for $d \ge 3$. Concerning proof of (4.15), see **[34]**.

In addition, we have that:

$$\begin{aligned} h_2(k,l) - h_1(k,l) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \psi_1(x,-l)(v_2(x) - v_1(x))\psi_2(x,k)dx \\ \text{for } (k,l) \in \Theta_E,\ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \ne 0, \\ \text{and } v_1,\ v_2 \text{ satisfying (4.5)}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} h_2(k,l) - h_1(k,l) = (2\pi)^{-d} \int\limits_{\partial D} [\psi_1(\cdot,-l)]_\alpha \left(\hat{M}_{\alpha,v_2} - \hat{M}_{\alpha,v_1}\right) [\psi_2(\cdot,k)]_\alpha dx \\ \text{for } (k,l) \in \Theta_E,\ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \ne 0, \\ \text{and } v_1,\ v_2 \text{ satisfying (1.7), (4.6)}, \end{aligned} \tag{4.17}$$

where h_j, ψ_j denote h and ψ of (4.3) and (4.1) for $v = v_j$, and $\hat{M}_{\alpha,v_j}$ denotes the impedance boundary map of (1.4) for $v = v_j$, where $j = 1, 2$.

Formula (4.16) was given in [**21**]. Formula (4.17) follows from Theorem 3.1 and (4.16). Formula (4.17) for $\alpha = 0$ was given in [**23**].

5. Proof of Theorem 2.1

Let

$$\begin{aligned} &\mathbb{L}^\infty_\mu(\mathbb{R}^d) = \{u \in \mathbb{L}^\infty(\mathbb{R}^d) : \|u\|_\mu < +\infty\}, \\ &\|u\|_\mu = \operatorname{ess}\sup_{p\in\mathbb{R}^d}(1+|p|)^\mu |u(p)|, \quad \mu > 0. \end{aligned} \tag{5.1}$$

Note that

$$\begin{aligned} &w \in W^{m,1}(\mathbb{R}^d) \Longrightarrow \hat{w} \in \mathbb{L}^\infty_\mu(\mathbb{R}^d) \cap C(\mathbb{R}^d), \\ &\|\hat{w}\|_\mu \le c_2(m,d)\|w\|_{m,1} \quad \text{for} \quad \mu = m, \end{aligned} \tag{5.2}$$

where $W^{m,1}$, $\mathbb{L}^\infty_\mu$ are the spaces of (2.5), (5.1),

$$\hat{w}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} w(x)dx, \quad p \in \mathbb{R}^d. \tag{5.3}$$

Using the inverse Fourier transform formula

$$w(x) = \int\limits_{\mathbb{R}^d} e^{-ipx}\hat{w}(p)dp, \quad x \in \mathbb{R}^d, \tag{5.4}$$

we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \le \sup_{x\in\bar{D}} \Big| \int\limits_{\mathbb{R}^d} e^{-ipx}\left(\hat{v}_2(p) - \hat{v}_1(p)\right)dp\Big| \le \\ \le I_1(r) + I_2(r) \quad \text{for any} \quad r > 0, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} I_1(r) &= \int\limits_{|p|\le r} |\hat{v}_2(p) - \hat{v}_1(p)|dp, \\ I_2(r) &= \int\limits_{|p|\ge r} |\hat{v}_2(p) - \hat{v}_1(p)|dp. \end{aligned} \tag{5.6}$$

Using (5.2), we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \le 2c_2(m,d)N(1+|p|)^{-m}, \quad p \in \mathbb{R}^d. \tag{5.7}$$

Due to (4.13), we have that

$$
\begin{aligned}
(5.8)\qquad & |\hat v_2(p) - \hat v_1(p)| \le |h_2(k,l) - h_1(k,l)| + \frac{2c_1(D,E,m)N^2}{\rho},\\
& p\in\mathbb{R}^d,\ p = k-l,\ (k,l)\in\Theta_E,\\
& |\operatorname{Im} k| = |\operatorname{Im} l| = \rho \ge r_2(N,D,E,m),\\
& p^2 \le 4(E+\rho^2).
\end{aligned}
$$

Let

$$
\begin{aligned}
(5.9)\qquad & c_3 = (2\pi)^{-d}\int\limits_{\partial D} dx, \quad L = \max_{x\in\partial D}|x|,\\
& \delta_\alpha = \|\hat M_{\alpha,v_2}(E) - \hat M_{\alpha,v_1}(E)\|,
\end{aligned}
$$

where $\|\hat M_{\alpha,v_2}(E) - \hat M_{\alpha,v_1}(E)\|$ is defined according to (2.2).

Due to (4.16), (4.17), we have that

$$
\begin{aligned}
(5.10)\qquad & |h_2(k,l) - h_1(k,l)| \le c_3 \|[\psi_1(\cdot,-l)]_\alpha\|_{\mathbb{L}^\infty(\partial D)}\,\delta_\alpha\,\|[\psi_2(\cdot,k)]_\alpha\|_{\mathbb{L}^\infty(\partial D)},\\
& (k,l)\in\Theta_E,\ |\operatorname{Im} k| = |\operatorname{Im} l| \ne 0.
\end{aligned}
$$

Using (1.5), (4.11), we find that

$$
\begin{aligned}
(5.11)\qquad & \|[\psi(\cdot,k)]_\alpha\|_{\mathbb{L}^\infty(\partial D)} \le c_4(E)\,\sigma\,\exp\Big(|\operatorname{Im} k|(L+1)\Big),\\
& k\in\Sigma_E,\ |\operatorname{Im} k| \ge r_1(N,D,E,m,\sigma).
\end{aligned}
$$

Here and bellow in this section the constant σ is the same that in (4.11).

Combining (5.10) and (5.11), we obtain that

$$
\begin{aligned}
(5.12)\qquad & |h_2(k,l) - h_1(k,l)| \le c_3\,(c_4(E)\sigma)^2\,\exp\Big(2\rho(L+1)\Big)\delta_\alpha,\\
& (k,l)\in\Theta_E,\ \rho = |\operatorname{Im} k| = |\operatorname{Im} l| \ge r_1(N,D,E,m,\sigma).
\end{aligned}
$$

Using (5.8), (5.12), we get that

$$
\begin{aligned}
(5.13)\qquad & |\hat v_2(p) - \hat v_1(p)| \le c_3\,(c_4(E)\sigma)^2\,\exp\Big(2\rho(L+1)\Big)\delta_\alpha + \frac{2c_1(D,E,m)N^2}{\rho},\\
& p\in\mathbb{R}^d,\ p^2 \le 4(E+\rho^2),\ \rho \ge r_3(N,D,E,m,\sigma),
\end{aligned}
$$

where $r_3(N,D,E,m,\sigma)$ is such that

$$
(5.14)\qquad \rho \ge r_3(N,D,E,m,\sigma) \Longrightarrow \begin{cases} \rho \ge r_1(N,D,E,m,\sigma),\\ \rho \ge r_2(N,D,E,m),\\ \rho^{2/m} \le 4(E+\rho^2). \end{cases}
$$

Let

$$c_5 = \int\limits_{p\in\mathbb{R}^d, |p|\le 1} dp, \quad c_6 = \int\limits_{p\in\mathbb{R}^d, |p|=1} dp. \tag{5.15}$$

Using (5.6), (5.13), we get that

$$\begin{aligned} I_1(r) \le c_5 r^d \left(c_3 \left(c_4(E)\sigma\right)^2 \exp\left(2\rho(L+1)\right)\delta_\alpha + \frac{2c_1(D,E,m)N^2}{\rho}\right), \\ r>0,\ r^2\le 4(\rho^2+E),\ \rho \ge r_3(N,D,E,m,\sigma). \end{aligned} \tag{5.16}$$

Using (5.6), (5.7), we find that for any $r>0$

$$I_2(r) \le 2c_2(m,d)Nc_6 \int\limits_r^{+\infty} \frac{dt}{t^{m-d+1}} \le \frac{2c_2(m,D)Nc_6}{m-d}\frac{1}{r^{m-d}}. \tag{5.17}$$

Combining (5.5), (5.16), (5.17) for $r=\rho^{1/m}$ and (5.14), we get that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \le c_7(D,\sigma)\rho^{d/m}e^{2\rho(L+1)}\delta_\alpha + c_8(N,D,E,m)\rho^{-\frac{m-d}{m}}, \\ \rho \ge r_3(N,D,E,m,\sigma). \end{aligned} \tag{5.18}$$

We fix some $\tau\in(0,1)$ and let

$$\beta = \frac{1-\tau}{2(L+1)}, \quad \rho = \beta\ln\left(3+\delta_\alpha^{-1}\right), \tag{5.19}$$

where δ_α is so small that $\rho\ge r_3(N,D,E,m,\sigma)$. Then due to (5.18), we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \le c_7(D,\sigma)\left(\beta\ln\left(3+\delta_\alpha^{-1}\right)\right)^{d/m}\left(3+\delta_\alpha^{-1}\right)^{2\beta(L+1)}\delta_\alpha + \\ + c_8(N,D,E,m)\left(\beta\ln\left(3+\delta_\alpha^{-1}\right)\right)^{-\frac{m-d}{m}} = \\ = c_7(D,\sigma)\beta^{d/m}\left(1+3\delta_\alpha\right)^{1-\tau}\delta_\alpha^\tau\left(\ln\left(3+\delta_\alpha^{-1}\right)\right)^{d/m} + \\ + c_8(N,D,E,m)\beta^{-\frac{m-d}{m}}\left(\ln\left(3+\delta_\alpha^{-1}\right)\right)^{-\frac{m-d}{m}}, \end{aligned} \tag{5.20}$$

where τ,β and δ_α are the same as in (5.19).

Using (5.20), we obtain that

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(D)} \le c_9(N,D,E,m,\sigma)\left(\ln\left(3+\delta_\alpha^{-1}\right)\right)^{-\frac{m-d}{m}} \tag{5.21}$$

for $\delta_\alpha = \|\hat{M}_{\alpha,v_2} - \hat{M}_{\alpha,v_1}\| \le \delta^{(0)}(N,D,E,m,\sigma)$, where $\delta^{(0)}$ is a sufficiently small positive constant. Estimate (5.21) in the general case (with modified c_9) follows from (5.21) for $\delta_\alpha \le \delta^{(0)}(N,D,E,m,\sigma)$ and the property that $\|v_j\|_{\mathbb{L}^\infty(D)} \le c_{10}(D,m)N$.

Thus, Theorem 2.1 is proved for $s=\frac{m-d}{m}$ and, since $\ln\left(3+\delta_\alpha^{-1}\right)>1$, for any $0<s\le\frac{m-d}{m}$.

6. Buckhgeim-type analogs of the Faddeev functions

In dimension $d = 2$, we consider the functions $G_{z_0}, \psi_{z_0}, \widetilde{\psi}_{z_0}$, δh_{z_0} of [**27**], going back to Buckhgeim's paper [**5**] and being analogs of the Faddeev functions:

$$\begin{aligned}\psi_{z_0}(z,\lambda) &= e^{\lambda(z-z_0)^2} + \int\limits_D G_{z_0}(z,\zeta,\lambda)v(\zeta)\psi_{z_0}(\zeta,\lambda)\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta,\\ \widetilde{\psi}_{z_0}(z,\lambda) &= e^{\bar{\lambda}(\bar{z}-\bar{z}_0)^2} + \int\limits_D \overline{G_{z_0}(z,\zeta,\lambda)}v(\zeta)\widetilde{\psi}_{z_0}(\zeta,\lambda)\,d\mathrm{Re}\zeta\,d\mathrm{Im}\zeta,\end{aligned} \tag{6.1}$$

$$\begin{gathered}G_{z_0}(z,\zeta,\lambda) = \frac{1}{4\pi^2}\int\limits_D \frac{e^{-\lambda(\eta-z_0)^2+\bar{\lambda}(\bar{\eta}-\bar{z}_0)^2}d\mathrm{Re}\eta\,d\mathrm{Im}\eta}{(z-\eta)(\bar{\eta}-\bar{\zeta})}\,e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2},\\ z = x_1+ix_2,\ z_0\in D,\ \lambda\in\mathbb{C},\end{gathered} \tag{6.2}$$

where $\mathbb{R}^2$ is identified with $\mathbb{C}$ and v, D satisfy (1.2), (1.3) for $d = 2$;

$$\delta h_{z_0}(\lambda) = \int\limits_D \widetilde{\psi}_{z_0,1}(z,-\lambda)\,(v_2(z)-v_1(z))\,\psi_{z_0,2}(z,\lambda)\,d\mathrm{Re}z\,d\mathrm{Im}z,\quad \lambda\in\mathbb{C}, \tag{6.3}$$

where v_1, v_2 satisfy (1.3) for $d = 2$ and $\widetilde{\psi}_{z_0,1}$, $\psi_{z_0,2}$ denote $\widetilde{\psi}_{z_0}$, ψ_{z_0} of (6.1) for $v = v_1$ and $v = v_2$, respectively.

We recall that (see [**27**], [**28**]):

$$\begin{aligned}4\frac{\partial^2}{\partial z\partial\bar{z}}G_{z_0}(z,\zeta,\lambda) &= \delta(z-\zeta),\\ 4\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}G_{z_0}(z,\zeta,\lambda) &= \delta(z-\zeta),\end{aligned} \tag{6.4}$$

where $z, z_0, \zeta\in D$, $\lambda\in\mathbb{C}$ and δ is the Dirac delta function; formulas (6.1) at fixed z_0 and λ are considered as equations for ψ_{z_0}, $\widetilde{\psi}_{z_0}$ in $L^\infty(D)$; as a corollary of (6.1), (6.2), (6.4), the functions ψ_{z_0}, $\widetilde{\psi}_{z_0}$ satisfy (1.1) for $E = 0$ and $d = 2$; δh_{z_0} is similar to the right side of (4.16).

Let potentials $v, v_1, v_2\in C^2(\bar{D})$ and

$$\begin{gathered}\|v\|_{C^2(\bar{D})}\le N,\quad \|v_j\|_{C^2(\bar{D})}\le N,\quad j = 1,2,\\ (v_1-v_2)|_{\partial D} = 0,\quad \frac{\partial}{\partial\nu}(v_1-v_2)|_{\partial D} = 0,\end{gathered} \tag{6.5}$$

then we have that:

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_{z_0}(z,\lambda),\quad \widetilde{\psi}_{z_0}(z,\lambda) = e^{\bar{\lambda}(\bar{z}-\bar{z}_0)^2}\widetilde{\mu}_{z_0}(z,\lambda), \tag{6.6}$$

$$\mu_{z_0}(z,\lambda)\to 1,\quad \widetilde{\mu}_{z_0}(z,\lambda)\to 1\quad \text{as } |\lambda|\to\infty \tag{6.7}$$

and, for any $\sigma > 1$,

$$|\mu_{z_0}(z,\lambda)| + |\nabla\mu_{z_0}(z,\lambda)|\le\sigma, \tag{6.8a}$$

$$|\widetilde{\mu}_{z_0}(z,\lambda)| + |\nabla\widetilde{\mu}_{z_0}(z,\lambda)| \le \sigma, \tag{6.8b}$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, $z = x_1 + ix_2$, $z_0 \in D$, $\lambda \in \mathbb{C}$, $|\lambda| \ge \rho_1(N, D, \sigma)$;

$$v_2(z_0) - v_1(z_0) = \lim_{\lambda\to\infty} \frac{2}{\pi}|\lambda|\delta h_{z_0}(\lambda) \quad \text{for any } z_0 \in D, \tag{6.9}$$

$$\left|v_2(z_0) - v_1(z_0) - \frac{2}{\pi}|\lambda|\delta h_{z_0}(\lambda)\right| \le \frac{c_{11}(N, D)\left(\ln(3|\lambda|)\right)^2}{|\lambda|^{3/4}} \quad \text{for } z_0 \in D,\ |\lambda| \ge \rho_2(N, D). \tag{6.10}$$

Formulas (6.6) can be considered as definitions of μ_{z_0}, $\widetilde{\mu}_{z_0}$. Formulas (6.7), (6.9) were given in **[27]**, **[28]** and go back to **[5]**. Estimate (6.10) was obtained in **[27]**, **[30]**. Estimates (6.8) are proved in Section 8.

7. Proof of Theorem 2.2

We suppose that $\widetilde{\psi}_{z_0,1}(\cdot, -\lambda)$, $\psi_{z_0,2}(\cdot, \lambda)$, $\delta h_{z_0}(\lambda)$ are defined as in Section 6 but with $v_j - E$ in place of v_j, $j = 1, 2$. We use the identity

$$\hat{M}_{\alpha,v}(E) = \hat{M}_{\alpha,v-E}(0). \tag{7.1}$$

We also use the notation $N_E = N + E$. Then, using (6.10), we have that

$$\left|v_2(z_0) - v_1(z_0) - \frac{2}{\pi}|\lambda|\delta h_{z_0}(\lambda)\right| \le \frac{c_{11}(N_E, D)\left(\ln(3|\lambda|)\right)^2}{|\lambda|^{3/4}} \quad \text{for } z_0 \in D,\ |\lambda| \ge \rho_2(N_E, D). \tag{7.2}$$

According to Theorem 3.1 and (6.3), we get that

$$\delta h_{z_0}(\lambda) = \frac{1}{4\pi^2}\int\limits_{\partial D} [\widetilde{\psi}_{z_0,1}(\cdot, -\lambda)]_\alpha \left(\hat{M}_{\alpha,v_2}(E) - \hat{M}_{\alpha,v_1}(E)\right) \left[\psi_{z_0,2}(\cdot, \lambda)\right]_\alpha |dz|, \quad \lambda \in \mathbb{C}. \tag{7.3}$$

Let

$$c_{12} = \frac{1}{4\pi^2}\int\limits_{\partial D} |dz|, \quad L = \max_{z\in\partial D}|z|, \qquad \delta_\alpha = \|\hat{M}_{\alpha,v_2}(E) - \hat{M}_{\alpha,v_1}(E)\|, \tag{7.4}$$

where $\|\hat{M}_{\alpha,v_2}(E) - \hat{M}_{\alpha,v_1}(E)\|$ is defined according to (2.2).

Using (7.3), we get that

$$|\delta h_{z_0}(\lambda)| \le c_{12}\|[\widetilde{\psi}_{z_0,1}(\cdot, -\lambda)]_\alpha\|_{\mathbb{L}^\infty(\partial D)}\, \delta_\alpha \,\|\left[\psi_{z_0,2}(\cdot, \lambda)\right]_\alpha\|_{\mathbb{L}^\infty(\partial D)}, \quad \lambda \in \mathbb{C}. \tag{7.5}$$

Using (1.5), (6.8), we find that:

$$\begin{aligned} &\|[\widetilde{\psi}_{z_0,1}(\cdot,-\lambda)]_\alpha\|_{\mathbb{L}^\infty(\partial D)} \le \sigma\ \exp\Big(|\lambda|(4L^2+4L)\Big),\\ &\|[\psi_{z_0,2}(\cdot,\lambda)]_\alpha\|_{\mathbb{L}^\infty(\partial D)} \le \sigma\ \exp\Big(|\lambda|(4L^2+4L)\Big),\\ &\lambda\in\mathbb{C},\quad |\lambda|\ge \rho_1(N_E,D,\sigma). \end{aligned} \tag{7.6}$$

Here and bellow in this section the constant σ is the same that in (6.8).

Combining (7.5), (7.6), we obtain that

$$\begin{aligned} &|\delta h_{z_0}(\lambda)| \le c_{12}\sigma^2\ \exp\Big(|\lambda|(8L^2+8L)\Big)\delta_\alpha,\\ &\lambda\in\mathbb{C},\quad |\lambda|\ge \rho_1(N_E,D,\sigma). \end{aligned} \tag{7.7}$$

Using (7.2) and (7.7), we get that

$$\begin{aligned} &|v_2(z_0)-v_1(z_0)| \le c_{12}\sigma^2\ \exp\Big(|\lambda|(8L^2+8L)\Big)\delta_\alpha + \frac{c_{11}(N_E,D)\left(\ln(3|\lambda|)\right)^2}{|\lambda|^{3/4}},\\ &z_0\in D,\quad \lambda\in\mathbb{C},\quad |\lambda|\ge \rho_3(N_E,D,\sigma)=\max\{\rho_1,\rho_2\}. \end{aligned} \tag{7.8}$$

We fix some $\tau\in(0,1)$ and let

$$\beta = \frac{1-\tau}{8L^2+8L},\qquad \lambda=\beta\ln\left(3+\delta_\alpha^{-1}\right), \tag{7.9}$$

where δ_α is so small that $|\lambda|\ge\rho_3(N_E,D,\sigma)$. Then due to (7.8), we have that

$$\begin{aligned} \|v_1-v_2\|_{\mathbb{L}^\infty(D)} &\le c_{12}\sigma^2\left(3+\delta_\alpha^{-1}\right)^{\beta(8L^2+8L)}\delta_\alpha+\\ &+c_{11}(N_E,D)\frac{\left(\ln\left(3\beta\ln\left(3+\delta_\alpha^{-1}\right)\right)\right)^2}{\left(\beta\ln\left(3+\delta_\alpha^{-1}\right)\right)^{\frac34}}=\\ &=c_{12}\sigma^2\left(1+3\delta_\alpha\right)^{1-\tau}\delta_\alpha^\tau+\\ &+c_{11}(N_E,D)\beta^{-\frac34}\frac{\left(\ln\left(3\beta\ln\left(3+\delta_\alpha^{-1}\right)\right)\right)^2}{\left(\ln\left(3+\delta_\alpha^{-1}\right)\right)^{\frac34}}, \end{aligned} \tag{7.10}$$

where τ,β and δ_α are the same as in (7.9).

Using (7.10), we obtain that

$$\|v_1-v_2\|_{\mathbb{L}^\infty(D)} \le c_{13}(N_E,D,\sigma)\left(\ln\left(3+\delta_\alpha^{-1}\right)\right)^{-\frac34}\left(\ln\left(3\ln\left(3+\delta_\alpha^{-1}\right)\right)\right)^2 \tag{7.11}$$

for $\delta_\alpha=\|\hat{M}_{\alpha,v_2}(E)-\hat{M}_{\alpha,v_1}(E)\|\le\delta^{(0)}(N_E,D,\sigma)$, where $\delta^{(0)}$ is a sufficiently small positive constant. Estimate (5.21) in the general case (with modified c_{13}) follows from (7.11) for $\delta_\alpha\le\delta^{(0)}(N_E,D,\sigma)$ and the property that $\|v_j\|_{\mathbb{L}^\infty(D)}\le c_{14}(D)N$.

Thus, Theorem 2.2 is proved for $s=\frac34$ and, since $\ln\left(3+\delta_\alpha^{-1}\right)>1$, for any $0<s\le\frac34$.

8. Proof of estimates (6.8)

In this section we prove estimate (6.8a). Estimate (6.8b) can be proved a completely similar way. Let

$$C^1_{\bar z}(\bar D) = \left\{u : u, \frac{\partial u}{\partial \bar z} \in C(\bar D)\right\}, \qquad (8.1)$$
$$\|u\|_{C^1_{\bar z}(\bar D)} = \max\left(\|u\|_{C(\bar D)}, \|\frac{\partial u}{\partial \bar z}\|_{C(\bar D)}\right).$$

Due to estimates of Section 3 of [**27**], we have that, for any $\varepsilon_1 > 0$,

$$\mu_{z_0}(\cdot,\lambda) \in C^1_{\bar z}(\bar D), \quad \|\mu_{z_0}(\cdot,\lambda)\|_{C^1_{\bar z}(\bar D)} \le 1 + \varepsilon_1 \quad \text{for } |\lambda| \ge \rho_4(N, D, \varepsilon_1). \qquad (8.2)$$

In view of (8.2), to prove (6.8a) it remains to prove that, for any $\varepsilon_2 > 0$,

$$\partial_z \mu_{z_0}(\cdot,\lambda) \in C(\bar D), \quad \|\partial_z \mu_{z_0}(\cdot,\lambda)\|_{C(\bar D)} \le \varepsilon_2 \quad \text{for } |\lambda| \ge \rho_5(N, D, \varepsilon_2), \qquad (8.3)$$

where $\partial_z \mu_{z_0}(\cdot,\lambda)$ is considered as a function of $z \in \bar D$ and $\partial_z = \partial/\partial z$.

We have that (see Sections 2 and 5 of [**27**]):

$$\partial_z \mu_{z_0} = \frac{1}{4}\Pi\, \bar{\mathrm{T}}_{z_0,\lambda} v \mu_{z_0}, \qquad (8.4)$$

$$\Pi u(z) = -\frac{1}{\pi}\int\limits_D \frac{u(\zeta)}{(\zeta - z)^2} d\mathrm{Re}\zeta\, d\mathrm{Im}\,\zeta, \qquad (8.5)$$

$$\bar{\mathrm{T}}_{z_0,\lambda} u(z) = -\frac{e^{-\lambda(z-z_0)^2 + \bar\lambda(\bar z - \bar z_0)^2}}{\pi}\int\limits_D \frac{e^{\lambda(\zeta - z_0)^2 - \bar\lambda(\bar\zeta - \bar z_0)^2}}{\bar\zeta - \bar z} u(\zeta) d\mathrm{Re}\zeta\, d\mathrm{Im}\,\zeta, \qquad (8.6)$$

where u is a test function, $z \in \bar D$.

In view of (8.2), (8.4) and Theorem 1.33 of [**33**], to prove (8.3) it is sufficient to show that

$$\|\bar{\mathrm{T}}_{z_0,\lambda} u\|_{C_s(\bar D)} \le \frac{A(D,s)}{|\lambda|^{\delta(s)}} ||u||_{C^1_{\bar z}(\bar D)}, \quad |\lambda| \ge 1, \quad z_0 \in \bar D, \qquad (8.7)$$

for some fixed $s \in (0, \frac{1}{2})$ and $\delta(s) > 0$, where $C_s(\bar D)$ is the Hölder space,

$$C_s(\bar D) = \left\{u \in C(\bar D) : \|u\|_{C_s(\bar D)} < +\infty\right\},$$
$$\|u\|_{C_s(\bar D)} = \max\left\{\|u\|_{C(\bar D)}, \|u\|'_{C_s(\bar D)}\right\}, \qquad (8.8)$$
$$\|u\|'_{C_s(\bar D)} = \sup_{z_1, z_2 \in \bar D, 0 < |z_1 - z_2| < 1} \frac{|u(z_1) - u(z_2)|}{|z_1 - z_2|^s}.$$

Due to estimate (5.6) of [**27**], we have that

$$\|\bar{\mathrm{T}}_{z_0,\lambda} u\|_{C(\bar D)} \le \frac{A_0(D)}{|\lambda|^{1/2}} ||u||_{C^1_{\bar z}(\bar D)}, \quad |\lambda| \ge 1, \quad z_0 \in \bar D. \qquad (8.9)$$

Therefore, to prove (8.7) it remains to prove that

$$\|\bar{\mathrm{T}}_{z_0,\lambda}u\|'_{C_s(\bar{D})} \le \frac{A_1(D,s)}{|\lambda|^{\delta(s)}}||u||_{C^1_{\bar{z}}(\bar{D})}, \quad |\lambda| \ge 1, \quad z_0 \in \bar{D}, \tag{8.10}$$

for some fixed $s \in (0, \frac{1}{2})$ and $\delta(s) > 0$.

We will use that

$$\|u_1 u_2\|'_{C_s(\bar{D})} \le \|u_1\|'_{C_s(\bar{D})}\|u_2\|_{C(\bar{D})} + \|u_1\|_{C(\bar{D})}\|u_2\|'_{C_s(\bar{D})}, \quad 0 < s < 1. \tag{8.11}$$

One can see that

$$\bar{\mathrm{T}}_{z_0,\lambda} = \mathrm{F}_{z_0,-\lambda}\bar{\mathrm{T}}\mathrm{F}_{z_0,\lambda}, \tag{8.12}$$

where $\bar{\mathrm{T}} = \bar{\mathrm{T}}_{z_0,0}$ and $\mathrm{F}_{z_0,\lambda}$ is the multiplication operator by the function

$$F(z, z_0, \lambda) = e^{\lambda(z-z_0)^2 - \bar{\lambda}(\bar{z}-\bar{z_0})^2}. \tag{8.13}$$

One can see also that

$$\begin{aligned} &\|F(\cdot, z_0, -\lambda)\|_{C(\bar{D})} = 1, \\ &\|F(\cdot, z_0, -\lambda)\|'_{C_s(\bar{D})} \le A_2(D, s)|\lambda|^s, \quad |\lambda| \ge 1, \quad z_0 \in \bar{D}. \end{aligned} \tag{8.14}$$

In view of (8.9), (8.11) - (8.14), to prove (8.10) it remains to prove that

$$\|\bar{\mathrm{T}}\mathrm{F}_{z_0,\lambda}u\|'_{C_s(\bar{D})} \le \frac{A_3(D,s)}{|\lambda|^{\delta_1(s)}}||u||_{C^1_{\bar{z}}(\bar{D})}, \quad |\lambda| \ge 1, \quad z_0 \in \bar{D}, \tag{8.15}$$

for some fixed $s \in (0, \frac{1}{2})$ and $\delta_1(s) > 0$.

We have that

$$\begin{aligned} \pi\bar{\mathrm{T}}\mathrm{F}_{z_0,\lambda}u(z_1) - \pi\bar{\mathrm{T}}\mathrm{F}_{z_0,\lambda}u(z_2) = \int\limits_D \frac{F(\zeta, z_0, \lambda)u(\zeta)(\bar{z}_2 - \bar{z}_1)}{(\bar{\zeta} - \bar{z}_1)(\bar{\zeta} - \bar{z}_2)} d\mathrm{Re}\zeta\, d\mathrm{Im}\,\zeta = \\ = I_{z_0,\lambda,\varepsilon}(z_1, z_2) + J_{z_0,\lambda,\varepsilon}(z_1, z_2), \end{aligned} \tag{8.16}$$

where

$$I_{z_0,\lambda,\varepsilon}(z_1, z_2) = \int\limits_{D\backslash D_{z_0,z_1,z_2,\varepsilon}} \frac{F(\zeta, z_0, \lambda)u(\zeta)(\bar{z}_2 - \bar{z}_1)}{(\bar{\zeta} - \bar{z}_1)(\bar{\zeta} - \bar{z}_2)} d\mathrm{Re}\zeta\, d\mathrm{Im}\,\zeta, \tag{8.17}$$

$$J_{z_0,\lambda,\varepsilon}(z_1, z_2) = \int\limits_{D_{z_0,z_1,z_2,\varepsilon}} \frac{F(\zeta, z_0, \lambda)u(\zeta)(\bar{z}_2 - \bar{z}_1)}{(\bar{\zeta} - \bar{z}_1)(\bar{\zeta} - \bar{z}_2)} d\mathrm{Re}\zeta\, d\mathrm{Im}\,\zeta, \tag{8.18}$$

where $B_{z,\varepsilon} = \{\zeta \in \mathbb{C} : |\zeta - z| < \varepsilon\}$, $D_{z_0,z_1,z_2,\varepsilon} = D \setminus \left(\bigcup\limits_{j=0}^{2} B_{z_j,\varepsilon}\right)$.

We will use the following inequalities:

$$\left|\frac{z_2 - z_1}{(\zeta - z_1)(\zeta - z_2)}\right| \le n_1|z_2 - z_1|^s \sum_{j=1}^{2} \frac{1}{|\zeta - z_j|^{1+s}}, \tag{8.19}$$

$$\left|\frac{z_2 - z_1}{(\zeta - z_1)(\zeta - z_2)(\zeta - z_0)}\right| \le n_2|z_2 - z_1|^s \sum_{j=0}^{2} \frac{1}{|\zeta - z_j|^{2+s}}, \tag{8.20}$$

$$\left|\frac{\partial}{\partial \zeta}\left(\frac{z_2 - z_1}{(\zeta - z_1)(\zeta - z_2)(\zeta - z_0)}\right)\right| \le n_3|z_2 - z_1|^s \sum_{j=0}^{2} \frac{1}{|\zeta - z_j|^{3+s}}, \tag{8.21}$$

where $s \in (0,1)$, $n_1, n_2, n_3 > 0$, $z_0, z_1, z_2, \zeta \in \mathbb{C}$ and $\zeta \ne z_i$ for $j = 0, 1, 2$.

Using (8.17), (8.19), we obtain that

$$I_{z_0,\lambda,\varepsilon}(z_1, z_2) \le n_4(s)\varepsilon^{1-s}|z_2 - z_1|^s, \tag{8.22}$$

where $n_4(s) > 0$, $z_0, z_1, z_2, \zeta \in \mathbb{C}$ and $\varepsilon \in (0,1)$. Further, we have that

$$\begin{aligned} J_{z_0,\lambda,\varepsilon}(z_1, z_2) = -\frac{1}{2\bar\lambda} \int\limits_{D_{z_0,z_1,z_2,\varepsilon}} \frac{\partial F(\zeta, z_0, \lambda)}{\partial \bar\zeta} \frac{u(\zeta)(\bar z_2 - \bar z_1)}{(\bar\zeta - \bar z_1)(\bar\zeta - \bar z_2)(\bar\zeta - \bar z_0)} d\mathrm{Re}\zeta\, d\mathrm{Im}\,\zeta = \\ = J^1_{z_0,\lambda,\varepsilon}(z_1, z_2) + J^2_{z_0,\lambda,\varepsilon}(z_1, z_2), \end{aligned} \tag{8.23}$$

where

$$\begin{aligned} J^1_{z_0,\lambda,\varepsilon}(z_1, z_2) &= -\frac{1}{4i\bar\lambda} \int\limits_{\partial D_{z_0,z_1,z_2,\varepsilon}} \frac{F(\zeta, z_0, \lambda)u(\zeta)(\bar z_2 - \bar z_1)}{(\bar\zeta - \bar z_1)(\bar\zeta - \bar z_2)(\bar\zeta - \bar z_0)} d\zeta, \\ J^2_{z_0,\lambda,\varepsilon}(z_1, z_2) &= \frac{1}{2\bar\lambda} \int\limits_{D_{z_0,z_1,z_2,\varepsilon}} F(\zeta, z_0, \lambda) \frac{\partial}{\partial \bar\zeta}\left(\frac{u(\zeta)(\bar z_2 - \bar z_1)}{(\bar\zeta - \bar z_1)(\bar\zeta - \bar z_2)(\bar\zeta - \bar z_0)}\right) d\mathrm{Re}\zeta\, d\mathrm{Im}\,\zeta, \end{aligned} \tag{8.24}$$

Using (8.20), (8.21), (8.24), we obtain that

$$\begin{aligned} &J^1_{z_0,\lambda,\varepsilon}(z_1, z_2) \le |\lambda|^{-1} n_5(D, s)\varepsilon^{-1-s}|z_2 - z_1|^s\|u\|_{C(\bar D)}, \\ &J^2_{z_0,\lambda,\varepsilon}(z_1, z_2) \le |\lambda|^{-1} n_6(D, s)\varepsilon^{-1-s}|z_2 - z_1|^s\|u\|_{C(\bar D)} + \\ &\qquad + |\lambda|^{-1} n_7(D, s)\varepsilon^{-s}|z_2 - z_1|^s \left\|\frac{\partial u}{\partial \bar z}\right\|_{C(\bar D)}, \end{aligned} \tag{8.25}$$

where $z_0, z_1, z_2, \lambda \in \mathbb{C}$, $|\lambda| \ge 1$, $\varepsilon \in (0,1)$.

Using (8.16), (8.22), (8.23), (8.25) and putting $\varepsilon = |\lambda|^{-1/2}$ into (8.22), (8.25), we obtain (8.15) with $\delta_1(s) = (1 - s)/2$.

9. Proof of symmetry (3.12)

Let D' be an open bounded domain in $\mathbb{R}^d$ such that

- $D \subset D'$,
- D' satisfies (1.2),
- E is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in D'.

Here and bellow in this section we assume that $v \equiv 0$ on $D' \setminus D$. Let $R(x, y, E)$ denote the Green function for the operator $-\Delta + v - E$ in D' with the Dirichlet boundary condition. We recall that

$$R(x, y, E) = R(y, x, E), \quad x, y \in D'. \tag{9.1}$$

Using (3.5), (9.1), we find that for $x, y \in D$

$$\begin{aligned}
&\int\limits_{\partial D} \left(R(x, \xi, E) \frac{\partial R}{\partial \nu_\xi}(y, \xi, E) - R(y, \xi, E) \frac{\partial R}{\partial \nu_\xi}(x, \xi, E) \right) d\xi = \\
&= \int\limits_{D} \Big(R(x, \xi, E) \left(\Delta_\xi - v + E\right) R(y, \xi, E) - R(y, \xi, E) \left(\Delta_\xi - v + E\right) R(x, \xi, E) \Big) d\xi = \\
&\qquad = -R(x, y, E) + R(y, x, E) = 0.
\end{aligned} \tag{9.2}$$

Note that $W = G_\alpha + R(E)$ is the solution of the equation

$$(-\Delta_x + v - E) W(x, y) = 0, \quad x, y \in D \tag{9.3}$$

with the boundary condition

$$\begin{aligned}
&\left(\cos\alpha\, W(x, y) - \sin\alpha \, \frac{\partial W}{\partial \nu_x}(x, y) \right) \Big|_{x \in \partial D} = \\
&= \left(\cos\alpha\, R(x, y, E) - \sin\alpha \, \frac{\partial R}{\partial \nu_x}(x, y, E) \right) \Big|_{x \in \partial D}, \quad y \in D.
\end{aligned} \tag{9.4}$$

Using (3.5) and (9.3), we find that for $x, y \in D$

$$\begin{aligned}
&\int\limits_{\partial D} \left(W(\xi, x) \frac{\partial W}{\partial \nu_\xi}(\xi, y) - W(\xi, y) \frac{\partial W}{\partial \nu_\xi}(\xi, x) \right) d\xi = \\
&= \int\limits_{D} \Big(W(\xi, x) \left(\Delta_\xi - v + E\right) W(\xi, y) - W(\xi, y) \left(\Delta_\xi - v + E\right) W(\xi, x) \Big) d\xi = 0
\end{aligned} \tag{9.5}$$

Note that

$$W(x, y) = -\int\limits_{D} W(\xi, y) \left(\Delta_\xi - v + E\right) R(\xi, x, E) d\xi, \quad x, y \in D. \tag{9.6}$$

Combining (3.5), (9.3) and (9.6), we obtain that

$$\begin{aligned}
W(x, y) = -\int\limits_{\partial D} \left(W(\xi, y) \frac{\partial R}{\partial \nu_\xi}(\xi, x, E) - R(\xi, x, E) \frac{\partial W}{\partial \nu_\xi}(\xi, y) \right) d\xi, \\
x, y \in D.
\end{aligned} \tag{9.7}$$

Using (9.4) and (9.7), we get that

(9.8)

$$\sin\alpha\, W(x,y) =$$

$$= \int\limits_{\partial D} W(\xi,y)\left(\cos\alpha\, W(\xi,x) - \sin\alpha\, \frac{\partial W}{\partial\nu_\xi}(\xi,x) - \cos\alpha\, R(\xi,x,E)\right) d\xi\ -$$

$$- \int\limits_{\partial D} R(\xi,x,E)\left(\cos\alpha\, R(\xi,y,E) - \sin\alpha\, \frac{\partial R}{\partial\nu_\xi}(\xi,x,E) - \cos\alpha\, W(\xi,y)\right) d\xi,$$

$$x,y\in D.$$

Combining similar to (9.8) formula for $\sin\alpha\, W(y,x)$, (9.2) and (9.5), we obtain that

$$\sin\alpha\, W(x,y) - \sin\alpha\, W(y,x) = 0,\quad x,y\in D. \tag{9.9}$$

In the case of $\sin\alpha = 0$, combining (9.4) and (9.7), we get that

$$W(x,y) = \int\limits_{\partial D}\left(-R(\xi,y,E)\frac{\partial R}{\partial\nu_\xi}(\xi,x,E) + W(\xi,x)\frac{\partial W}{\partial\nu_\xi}(\xi,y)\right) d\xi, \quad x,y\in D. \tag{9.10}$$

Hence, one can get that for any α

$$W(x,y) = W(y,x),\quad x,y\in D. \tag{9.11}$$

Combining (9.1) and (9.11), we obtain (3.12).

We note that symmetry (3.12) for $v\equiv 0$, $E=0$, $d\geq 3$ was proved early, for example, in [**16**].

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Paper **E**

PAPER E

Reconstruction of a potential from the impedance boundary map

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Abstract. We consider the inverse boundary value problem for the Schrödinger equation at fixed energy with boundary measurements represented as the impedance boundary map (or Robin-to-Robin map). We give formulas and equations for finding (generalized) scattering data for the aforementioned equation from boundary measurements in this impedance representation. Combining these results with results of the inverse scattering theory we obtain efficient methods for reconstructing potential from the impedance boundary map. To our knowledge, results of the present work are new already for the case of Neumann-to-Dirichlet map.

1. Introduction

We consider the equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D,\ E \in \mathbb{R}, \tag{1.1}$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d,\ d \geq 2, \text{ with } \partial D \in C^2, \tag{1.2}$$

$$v \in \mathbb{L}^\infty(D), \quad v = \bar{v}. \tag{1.3}$$

Equation (1.1) can be considered as the stationary Schrödinger equation of quantum mechanics at fixed energy E. Equation (1.1) at fixed E arises also in acoustics and electrodynamics.

Following [**19**], [**26**], we consider the impedance boundary map $\hat{M}_\alpha = \hat{M}_{\alpha,v}(E)$ defined by

$$\hat{M}_\alpha[\psi]_\alpha = [\psi]_{\alpha-\pi/2} \tag{1.4}$$

for all sufficiently regular solutions ψ of equation (1.1) in $\bar{D} = D \cup \partial D$, where

$$[\psi]_\alpha = [\psi(x)]_\alpha = \cos\alpha\,\psi(x) - \sin\alpha\,\frac{\partial\psi}{\partial\nu}|_{\partial D}(x), \quad x \in \partial D,\ \alpha \in \mathbb{R} \tag{1.5}$$

and ν is the outward normal to ∂D. Under assumptions (1.2), (1.3), in Lemma 3.2 of [**26**] it was shown that there is not more than a countable number of $\alpha \in \mathbb{R}$ such that E is an eigenvalue for the operator $-\Delta + v$ in D with the boundary condition

$$\cos\alpha\,\psi|_{\partial D} - \sin\alpha\,\frac{\partial\psi}{\partial\nu}|_{\partial D} = 0. \tag{1.6}$$

Therefore, for any fixed E we can assume that for some fixed $\alpha \in \mathbb{R}$

$$\begin{aligned} &E \text{ is not an eigenvalue for the operator } -\Delta + v \text{ in } D \\ &\qquad\qquad\qquad\qquad \text{with boundary condition (1.6)} \end{aligned} \tag{1.7}$$

and, as a corollary, $\hat{M}_\alpha$ can be defined correctly.

We consider $\hat{M}_\alpha = \hat{M}_{\alpha,v}(E)$ as an operator representation of all possible boundary measurements for the physical model described by (1.1). We recall that the impedance boundary map $\hat{M}_\alpha$ is reduced to the Dirichlet-to-Neumann(DtN) map if $\alpha = 0$ and is reduced to the Neumann-to-Dirichlet(NtD) map if $\alpha = \pi/2$. The map $\hat{M}_\alpha$ can be called also as the Robin-to-Robin map.

As in [**26**], we consider the following inverse boundary value problem for equation (1.1):

PROBLEM 1.1. Given $\hat{M}_\alpha$ for some fixed E and α, find v.

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at fixed energy (see [**18**], [**36**]). Note that in the initial Gel'fand formulation energy E was not yet fixed and boundary measurements were considered as an operator relating $\psi|_{\partial D}$ and $\frac{\partial\psi}{\partial\nu}|_{\partial D}$ for ψ satisfying (1.1).

Problem 1.1 for $E = 0$ can be considered also as a generalization of the Calderon problem of the electrical impedance tomography (see [**13**], [**36**]).

Note also that Problem 1.1 can be considered as an example of ill-posed problem: see [**4**], [**31**] for an introduction to this theory.

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness theorems and global reconstruction methods for Problem 1.1 with $\alpha = 0$ (i.e. for the DtN case) were given for the first time in [**36**] in dimension $d \geq 3$ and in [**10**] in dimension $d = 2$.

Global stability estimates for Problem 1.1 with $\alpha = 0$ were given for the first time in [**1**] in dimension $d \geq 3$ and in [**45**] in dimension $d = 2$. A principal improvement of the result of [**1**] was given recently in [**44**] (for $E = 0$). Due to [**32**] these logarithmic stability results are optimal (up to the value of the exponent). An extention of the instability estimates of [**32**] to the case of non-zero energy as well as to the case of Dirichlet-to-Neumann map given on the energy intervals was obtained in [**24**]. An extention of stability estimates of [**44**] to the energy dependent case was given recently in [**27**]. Instability estimates complementing stability results of [**27**] were obtained in [**25**].

Note also that for the Calderon problem (of the electrical impedance tomography) in its initial formulation the global uniqueness was firstly proved in [**51**] for $d \geq 3$ and in [**35**] for $d = 2$. In addition, for the case of piecewise constant or piecewise real analytic conductivity the first uniqueness results for the Calderon problem in dimension $d \geq 2$ were given in [**15**], [**28**]. Lipschitz stability estimate for the case of piecewise constant conductivity was proved in [**2**] and additional studies in this direction were fulfilled in [**48**].

It should be noted that in most of previous works on inverse boundary value problems for equation (1.1) at fixed E it was assumed in one way or another that E is not a Dirichlet eigenvalue for the operator $-\Delta + v$ in D, see [**1**], [**32**], [**36**], [**44**]-[**49**]. Nevertheless, the results of [**10**] can be considered as global uniqueness and reconstruction results for Problem 1.1 in dimension $d = 2$ with general α.

Global stability estimates for Problem 1.1 in dimension $d \geq 2$ with general α were recently given in [**26**].

In the present work we give formulas and equations for finding (generalized) scattering data from the impedance boundary map $\hat{M}_\alpha$ with general α. Combining these results with results of [**21**], [**23**], [**35**], [**37**]-[**39**], [**41**]-[**43**], we obtain efficient reconstruction methods for Problem 1.1 in multidimensions with general α. To our knowledge these results are new already for the NtD case.

In particular, in the present work we give the first mathematically justified approach for reconstructing coefficient v from boundary measurements for (1.1) via inverse scattering without the assumption that E is not a Dirichlet eigenvalue for $-\Delta+v$ in D. In addition, numerical efficiency of related inverse scattering techniques was shown in [**3**], [**9**], [**11**], [**12**]; see also [**8**].

Definitions of (generalized) scattering data are recalled in Section 2. Our main results are presented in Section 3. Proofs of these results are given in Sections 4, 5 and 6.

2. Scattering data

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2 \tag{2.1}$$

where

$$(1 + |x|)^{d+\varepsilon} v(x) \in \mathbb{L}^\infty(\mathbb{R}^d) \text{ (as a function of } x\text{)}, \quad \text{for some } \ \varepsilon > 0. \tag{2.2}$$

For equation (2.1) we consider the functions ψ^+ and f of the classical scattering theory and the Faddeev functions ψ, h, ψ_γ, h_γ (see, for example, [**6**], [**14**], [**16**], [**17**], [**20**], [**23**], [**33**], [**37**]).

The functions ψ^+ and f can be defined as follows:

$$\psi^+(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^d} G^+(x-y,k)v(y)\psi^+(y,k)dy, \tag{2.3}$$

$$G^+(x,k) = -\left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}}{\xi^2 - k^2 - i0}\, d\xi, \qquad x,k \in \mathbb{R}^d,\ \ k^2 > 0, \tag{2.4}$$

where (2.3) at fixed k is considered as an equation for ψ^+ in $\mathbb{L}^\infty(\mathbb{R}^d)$;

$$f(k,l) = \left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} e^{-ilx}\psi^+(x,k)v(x)dx, \qquad k,l \in \mathbb{R}^d,\ k^2 > 0. \tag{2.5}$$

In addition: $\psi^+(x,k)$ satisfies (2.3) for $E = k^2$ and describes scattering of the plane waves e^{ikx}; $f(k,l)$, $k^2 = l^2$, is the scattering amplitude for equation (2.1) for $E = k^2$. Equation (2.3) is the Lippman-Schwinger integral equation.

The functions ψ and h can be defined as follows:

$$\psi(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^d} G(x-y,k)v(y)\psi(y,k)dy, \tag{2.6}$$

$$G(x,k) = -\left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 + 2k\xi}\, e^{ikx}, \qquad x \in \mathbb{R}^d,\ \ k \in \mathbb{C}^d,\ \ \mathrm{Im}\, k \neq 0, \tag{2.7}$$

where (2.6) at fixed k is considered as an equation for $\psi = e^{ikx}\mu(x,k)$, $\mu \in \mathbb{L}^\infty(\mathbb{R}^d)$;

$$h(k,l) = \left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} e^{-ilx}\psi(x,k)v(x)dx, \qquad k,l \in \mathbb{C}^d,\ \mathrm{Im}\, k = \mathrm{Im}\, l \neq 0. \tag{2.8}$$

In addition, $\psi(x,k)$ satisfies (2.1) for $E = k^2$, and ψ, G and h are (nonanalytic) continuations of ψ^+, G^+ and f to the complex domain. In particular, $h(k,l)$ for $k^2 = l^2$ can be considered as the "scattering" amplitude in the complex domain for equation (2.1) for $E = k^2$. The functions ψ_γ and h_γ are defined as follows:

$$\psi_\gamma(x,k) = \psi(x, k + i0\gamma), \qquad h_\gamma(k,l) = h(k + i0\gamma, l + i0\gamma), \qquad x,k,l,\gamma \in \mathbb{R}^d,\ \gamma^2 = 1. \tag{2.9}$$

We recall also that

$$\psi^+(x,k) = \psi_{k/|k|}(x,k), \qquad f(k,l) = h_{k/|k|}(k,l), \qquad x,k,l \in \mathbb{R}^d,\ |k| > 0. \tag{2.10}$$

We consider $f(k,l)$ and $h_\gamma(k,l)$, where $k,l,\gamma \in \mathbb{R}^d$, $k^2 = l^2 = E$, $\gamma^2 = 1$, and $h(k,l)$, where $k,l \in \mathbb{C}^d$, $\operatorname{Im} k = \operatorname{Im} l \neq 0$, $k^2 = l^2 = E$, as scattering data S_E for equation (2.1) at fixed $E \in (0,+\infty)$. We consider $h(k,l)$, where $k,l \in \mathbb{C}^d$, $\operatorname{Im} k = \operatorname{Im} l \neq 0$, $k^2 = l^2 = E$, as scattering data S_E for equation (2.1) at fixed $E \in (-\infty, 0]$.

We consider also the sets $\mathcal{E}$, $\mathcal{E}_\gamma$, $\mathcal{E}^+$ defined as follows:

$$\mathcal{E} = \left\{ \begin{array}{c} \zeta \in \mathbb{C}^d \setminus \mathbb{R}^d \ : \ \text{equation (2.6) for } k = \zeta \text{ is not} \\ \text{uniquely solvable for } \psi = e^{ikx}\mu \text{ with } \mu \in \mathbb{L}^\infty(\mathbb{R}^d) \end{array} \right\}, \tag{2.11a}$$

$$\mathcal{E}_\gamma = \left\{ \begin{array}{c} \zeta \in \mathbb{R}^d \setminus \{0\} \ : \ \text{equation (2.6) for } k = \zeta + i0\gamma \\ \text{is not uniquely solvable for } \psi = \mathbb{L}^\infty(\mathbb{R}^d) \end{array} \right\}, \quad \gamma \in \mathbb{S}^{d-1}, \tag{2.11b}$$

$$\mathcal{E}^+ = \left\{ \begin{array}{c} \zeta \in \mathbb{R}^d \setminus \{0\} \ : \ \text{equation (2.6) for } k = \zeta \text{ is not} \\ \text{uniquely solvable for } \psi = \mathbb{L}^\infty(\mathbb{R}^d) \end{array} \right\}. \tag{2.11c}$$

In addition, $\mathcal{E}^+$ is a well-known set of the classical scattering theory for equation (2.1) and $\mathcal{E}^+ = \emptyset$ for real-valued v satisfying (2.2) (see, for example, [**6**], [**33**]). Note also that $\mathcal{E}^+$ is spherically symmetric. The sets $\mathcal{E}$, $\mathcal{E}_\gamma$ were considered for the first time in [**16**], [**17**]. Concerning the properties of $\mathcal{E}$ and $\mathcal{E}_\gamma$, see [**17**], [**22**], [**23**], [**30**], [**33**], [**35**], [**38**], [**52**].

We consider also the functions R, R_γ, R^+ defined as follows:

$$R(x,y,k) = G(x-y,k) + \int_{\mathbb{R}^d} G(x-z,k)v(z)R(z,y,k)dz, \quad x,y \in \mathbb{R}^d,\ k \in \mathbb{C}^d,\ \operatorname{Im} k \neq 0, \tag{2.12}$$

where G is defined by (2.7) and formula (2.12) at fixed y, k is considered as an equation for

$$R(x,y,k) = e^{ik(x-y)} r(x,y,k), \tag{2.13}$$

where r is sought with the properties

$$r(\cdot,y,k) \text{ is continuous on } \mathbb{R}^d \setminus \{y\} \tag{2.14a}$$

$$r(x,y,k) \to 0 \text{ as } |x| \to \infty, \tag{2.14b}$$

$$\begin{aligned} &r(x,y,k) = O(|x-y|^{2-d}) \text{ as } x \to y \text{ for } d \geq 3, \\ &r(x,y,k) = O(|\ln|x-y||) \text{ as } x \to y \text{ for } d = 2; \end{aligned} \tag{2.14c}$$

$$R_\gamma(x,y,k) = R(x,y,k+i0\gamma), \quad x,y \in \mathbb{R}^d,\ k \in \mathbb{R}^d \setminus \{0\},\ \gamma \in \mathbb{S}^{d-1}; \tag{2.15}$$

$$\begin{aligned} R^+(x,y,k) = R_{k/|k|}(x,y,k), \\ x,y\in\mathbb{R}^d,\ k\in\mathbb{R}^d\setminus\{0\}. \end{aligned} \tag{2.16}$$

In addition, the functions $R(x,y,k)$, $R_\gamma(x,y,k)$ and $R^+(x,y,k)$ (for their domains of definition in k and γ) satisfy the following equations:

$$\begin{aligned} (\Delta_x + E - v(x))R(x,y,k) = \delta(x-y), \\ (\Delta_y + E - v(y))R(x,y,k) = \delta(x-y), \\ x,y\in\mathbb{R}^d,\ E=k^2. \end{aligned} \tag{2.17}$$

The function $R^+(x,y,k)$ (defined by means of (2.12) for $k\in\mathbb{R}^d\setminus\{0\}$ with G replaced by G^+ of (2.4)) is well-known in the scattering theory for equations (2.1), (2.17) (see, for example, [**7**]). In particular, this function describes scattering of the spherical waves $G^+(x-y,k)$ generated by a source at y. In addition $R^+(x,y,k)$ is a radial function in k, i.e.

$$R^+(x,y,k) = \mathrm{R}^+(x,y,|k|), \qquad x,y\in\mathbb{R}^d,\ k\in\mathbb{R}^d\setminus\{0\}. \tag{2.18}$$

Apparently, the functions R and R_γ were considered for the first time in [**38**].

In addition, under the assumption (2.2): equation (2.12) at fixed y and k is uniquely solvable for R with the properties (2.13), (2.14) if and only if $k\in\mathbb{C}^d\setminus(\mathbb{R}^d\cup\mathcal{E})$; equation (2.12) with $k=\zeta+i0\gamma$, $\zeta\in\mathbb{R}^d\setminus\{0\}$, $\gamma\in\mathbb{S}^{d-1}$, at fixed y, ζ and γ is uniquely solvable for R_γ if and only if $\zeta\in\mathbb{R}^d\setminus(\{0\}\cup\mathcal{E}_\gamma)$; equation (2.12) with $k=\zeta+i0\zeta/|\zeta|$, $\zeta\in\mathbb{R}^d\setminus 0$, at fixed y and ζ is uniquely solvable for R^+ if and only if $\zeta\in\mathbb{R}^d\setminus(\{0\}\cup\mathcal{E}^+)$.

3. Main results

Let v and v^0 satisfy (1.3), (1.7) for some fixed E and α. Let $M_{\alpha,v}(x,y,E)$, $M_{\alpha,v^0}(x,y,E)$, $x,y\in\partial D$, denote the Schwartz kernels of the impedance boundary maps $\hat M_{\alpha,v}$, $\hat M_{\alpha,v^0}$, for potentials v and v^0, respectively, where $\hat M_{\alpha,v}$, $\hat M_{\alpha,v^0}$ are considered as linear integral operators. In addition, we consider v^0 as some known background potential.

Let h, ψ, f, ψ^+, h_γ, ψ_γ, $\mathcal{E}$, $\mathcal{E}^+$, $\mathcal{E}_\gamma$ and h^0, ψ^0, f^0, $\psi^{+,0}$, h^0_γ, ψ^0_γ, $\mathcal{E}^0$, $\mathcal{E}^{+,0}$, $\mathcal{E}^0_\gamma$ denote the functions and sets of (2.3), (2.5), (2.6), (2.8), (2.9), (2.11) for potentials v and v^0, respectively. Here and bellow in this section we always assume that $v\equiv 0$, $v^0\equiv 0$ on $\mathbb{R}^d\setminus D$.

THEOREM 3.1. *Let D satisfy* (1.2) *and potentials v, v^0 satisfy (1.3), (1.7) for some fixed E and α. Then:*

$$h(k,l) - h^0(k,l) =$$

$$= \left(\frac{1}{2\pi}\right)^d \int\limits_{\partial D}\int\limits_{\partial D} [\psi^0(x,-l)]_\alpha \left(M_{\alpha,v} - M_{\alpha,v_0}\right)(x,y,E)[\psi(y,k)]_\alpha dx\,dy, \tag{3.1}$$

$$k,l \in \mathbb{C}^d \setminus (\mathcal{E} \cup \mathcal{E}^0),\ k^2 = l^2 = E,\ \operatorname{Im} k = \operatorname{Im} l \neq 0,$$

$$[\psi(x,k)]_\alpha = [\psi^0(x,k)]_\alpha + \int\limits_{\partial D} A_\alpha(x,y,k)[\psi(y,k)]_\alpha dy, \tag{3.2}$$

$$x \in \partial D,\ k \in \mathbb{C}^d \setminus (\mathcal{E} \cup \mathcal{E}^0),\ Im\,k \neq 0,\ k^2 = E$$

where

$$A_\alpha(x,y,k) = \lim_{\varepsilon \to +0} \int\limits_{\partial D} D_{\alpha,\varepsilon} R^0(x,\xi,k)\left(M_{\alpha,v} - M_{\alpha,v^0}\right)(\xi,y,E)d\xi, \tag{3.3}$$

$$D_{\alpha,\varepsilon} R^0(x,\xi,k) = [[R^0(x+\varepsilon\nu_x,\xi,k)]_{\xi,\alpha}]_{x,\alpha} =$$

$$= \left(\cos^2\alpha - \sin\alpha\cos\alpha\left(\frac{\partial}{\partial\nu_x} + \frac{\partial}{\partial\nu_\xi}\right) + \sin^2\alpha\frac{\partial^2}{\partial\nu_x\partial\nu_\xi}\right) R^0(x+\varepsilon\nu_x,\xi,k), \tag{3.4}$$

$$x,\xi,y \in \partial D,$$

where R^0 denotes the Green function of (2.12) *for potential v^0, ν_x is the outward normal to ∂D at x. In addition, formulas completely similar to* (3.1) - (3.4) *are also valid for the classical scattering functions f, ψ^+, f^0, $\psi^{+,0}$ and sets $\mathcal{E}^+$, $\mathcal{E}^{+,0}$ of* (2.3), (2.5), (2.11c) *for v and v^0, respectively, but with $R^{+,0}$ in place of R^0 in* (3.3), (3.4), *where $R^{+,0}$ denotes the Green function of* (2.16) *for potential v^0.*

Theorem 3.1 is proved in Section 4.

Note that formula of the type (3.1) for h_γ is not completely similar to (3.1): see formula (3.6) given below. In this formula (3.6), in addition to expected $\psi_\gamma(x,k)$, we use also $\psi_\gamma(x,k,l)$ defined as follows:

$$\psi_\gamma(x,k,l) = e^{ilx} + \int\limits_{\mathbb{R}^d} G_\gamma(x-y,k)v(y)\psi_\gamma(y,k,l)dy, \tag{3.5}$$

$$G_\gamma(x,k) = G(x,k+i0\gamma),$$

$$\gamma \in \mathbb{S}^{d-1},\ x,k,l \in \mathbb{R}^d, k^2 = l^2 > 0,$$

where (3.5) at fixed γ, k, l is considered as an equation for $\psi_\gamma(\cdot,k,l)$ in $\mathbb{L}^\infty(\mathbb{R}^d)$, G is defined by (2.7).

Proposition 3.1. *Let the asssumptions of Theorem 3.1 hold. Let $\psi_\gamma(x,k)$ correspond to v according to* (2.9) *and $\psi^0_{-\gamma}(\cdot,k,l)$ correspond to v^0 according to* (3.5). *Then*

$$h_\gamma(k,l) - h^0_\gamma(k,l) =$$

$$= \left(\frac{1}{2\pi}\right)^d \int\limits_{\partial D}\int\limits_{\partial D} [\psi^0_{-\gamma}(x,-k,-l)]_\alpha \left(M_{\alpha,v} - M_{\alpha,v^0}\right)(x,y,E)[\psi_\gamma(y,k)]_\alpha dx\, dy, \tag{3.6}$$

$$\gamma \in \mathbb{S}^{d-1},\ k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}_\gamma \cup \mathcal{E}^0_\gamma),\ l \in \mathbb{R}^d,\ k^2 = l^2 = E.$$

In addition, formulas completely similar to (3.2) - (3.4) *are also valid for the functions $\psi_\gamma(x,k)$, $\psi^0_\gamma(x,k)$ and sets $\mathcal{E}_\gamma$, $\mathcal{E}^0_\gamma$ of* (2.9), (2.11b) *for v and v^0, respectively, but with R^0_γ in place of R^0 in* (3.3), (3.4), *where R^0_γ denotes the Green function of* (2.15) *for potential v^0.*

Proposition 3.1 is proved in Section 4.

Note that (3.2) is considered as a linear integral equation for finding $[\psi(x,k)]_\alpha$, $x \in \partial D$, at fixed k, from $\hat{M}_{\alpha,v} - \hat{M}_{\alpha,v^0}$ and $[\psi^0(x,k)]_\alpha$, whereas (3.1) is considered as an explicit formula for finding h from h^0, $\hat{M}_{\alpha,v} - \hat{M}_{\alpha,v^0}$, $[\psi^0(x,k)]_\alpha$ and $[\psi(x,k)]_\alpha$. In addition, we use similar interpretation for similar formulas for ψ^+, f and for ψ_γ, h_γ, mentioned in Theorem 3.1 and Proposition 3.1.

Under the assumptions of Theorem 3.1, the following propositions are valid:

Proposition 3.2. *Equation* (3.2) *for $[\psi(x,k)]_\alpha$ at fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$ is a Fredholm linear integral equation of the second kind in the space of bounded functions on ∂D. In addition, the same is also valid for the equation for $[\psi^+(x,k)]_\alpha$ at fixed $k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^{+,0})$, mentioned in Theorem 3.1, and for the equation for $[\psi_\gamma(x,k)]_\alpha$ at fixed $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^0_\gamma)$, mentioned in Proposition 3.1.*

Proposition 3.2 is proved in Section 4.

Proposition 3.3. *For $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$ equation* (3.2) *is uniquely solvable in the space of bounded functions on ∂D if and only if $k \notin \mathcal{E}$. In addition, the aforementioned equations for $[\psi^+(x,k)]_\alpha$, $k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^{+,0})$, and $[\psi_\gamma(x,k)]_\alpha$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^0_\gamma)$, are uniquely solvable in the space of bounded functions on ∂D if and only if $k \notin \mathcal{E}^+$ and $k \notin \mathcal{E}_\gamma$, respectively.*

Proposition 3.3 is proved in Section 5.

Proposition 3.4. *Let $\phi_\alpha(x,y)$ be the solution of the Dirichlet boundary value problem at fixed $y \in \partial D$, $\lambda \in \mathbb{C}$:*

$$\begin{aligned} &-\Delta_x \phi_\alpha(x,y) = \lambda \phi_\alpha(x,y), \qquad x \in D,\\ &\phi_\alpha(x,y) = \left(M_{\alpha,v} - M_{\alpha,v^0}\right)(x,y,E), \quad x \in \partial D, \end{aligned} \tag{3.7}$$

where we assume that λ is not a Dirichlet eigenvalue for $-\Delta$ in D. Then

$$A_\alpha(x,y,k) = \lim_{\varepsilon\to+0} \int\limits_{\partial D} [R^0(x+\varepsilon\nu_x,\xi,k)]_{x,\alpha}[\phi_\alpha(\xi,y)]_{\xi,\alpha} d\xi - \\ - \sin\alpha \int\limits_{D} [R^0(x,\xi,k)]_{x,\alpha}(v^0(\xi)-E+\lambda)\phi_\alpha(\xi,y)d\xi, \quad x,y\in\partial D, \tag{3.8}$$

where

$$[R^0(x+\varepsilon\nu_x,\xi,k)]_{x,\alpha} = \left(\cos\alpha - \sin\alpha\frac{\partial}{\partial\nu_x}\right)R^0(x+\varepsilon\nu_x,\xi,k), \quad x\in\partial D,\ \xi\in\bar{D}, \tag{3.9}$$

$$[\phi_\alpha(\xi,y)]_{\xi,\alpha} = \left(\cos\alpha - \sin\alpha\frac{\partial}{\partial\nu_\xi}\right)\phi_\alpha(\xi,y) = \\ = \cos\alpha\,\phi_\alpha(\xi,y) - \sin\alpha\left(\hat{\Phi}(\lambda)\phi_\alpha(\cdot,y)\right)(\xi), \quad \xi,y\in\partial D, \tag{3.10}$$

where A_α is defined in (3.3), *$\hat{\Phi}(\lambda) = \hat{M}_{0,0}(\lambda)$ is the Dirichlet-to-Neumann map for* (3.7)*. In addition, formulas completely similar to* (3.8) *are also valid for the kernels A^+_α (but with R^+_0 in place of R_0) and $A_{\alpha,\gamma}$ (but with R^0_γ in place of R^0), arising in the equations for $[\psi^+]_\alpha$ and $[\psi_\gamma]_\alpha$, mentioned in Theorem 3.1 and Proposition 3.1.*

Proposition 3.4 is proved in Section 4.

Note that, for the case when $\sin\alpha = 0$, formula (3.8) coincides with (3.3). However, for $\sin\alpha \neq 0$, formula (3.8) does not contain $\partial^2 R^0/\partial\nu_x\partial\nu_\xi$ in contrast with (3.3) and is more convenient than (3.3) in this sense.

Theorem 3.1, Propositions 3.1 - 3.4 and the reconstruction results from generalized scattering data (see **[20]**, **[21]**, **[23]**, **[37]**-**[39]**, **[41]**-**[43]**, **[47]**) imply the following corollary:

Corollary 3.1. To reconstruct a potential v in the domain D from its impedance boundary map $\hat{M}_{\alpha,v}(E)$ at fixed E and α one can use the following schema:

(1) $v^0 \to \{S^0_E\}, \{R^0\}, \{[\psi^0]_\alpha\}, \hat{M}_{\alpha,v^0}$ via direct problem methods,
(2) $\{R^0\}, \hat{M}_{\alpha,v^0}, \hat{M}_{\alpha,v} \to \{A_\alpha\}$ as described in Theorem 3.1 and Propositions 3.1, 3.4,
(3) $\{A_\alpha\}, \{[\psi^0]_\alpha\} \to \{[\psi]_\alpha\}$ as described in Theorem 3.1 and Proposition 3.1,
(4) $\{S^0_E\}, \{[\psi^0]_\alpha\}, \{[\psi]_\alpha\}, \hat{M}_{\alpha,v^0}, \hat{M}_{\alpha,v} \to \{S_E\}$ as described in Theorem 3.1 and Proposition 3.1,
(5) $\{S_E\} \to v$ as described in **[20]**, **[21]**, **[23]**, **[37]**-**[39]**, **[41]**-**[43]**, **[47]**,

where $\{S^0_E\}$ and $\{S_E\}$ denote some appropriate part of h^0, f^0, h^0_γ and h, f, h_γ, respectively, $\{[\psi^0]_\alpha\}$ and $\{[\psi]_\alpha\}$ denote some appropriate part of $[\psi^0]_\alpha$, $[\psi^{+,0}]_\alpha$, $[\psi^0_\gamma]_\alpha$ and $[\psi]_\alpha$, $[\psi^+]_\alpha$, $[\psi_\gamma]_\alpha$, respectively, $\{R^0\}$, $\{A_\alpha\}$ denote some appropriate part of R^0, $R^{+,0}$, R^0_γ, A_α, A^+_α, $A_{\alpha,\gamma}$.

REMARK 3.1. For the case when $v^0 \equiv 0$, $\sin\alpha = 0$, Theorem 3.1, Propositions 3.1 - 3.3 and Corollary 3.1 (with available references at that time at step 5) were obtained in [**36**] (see also [**34**], [**35**]). Note that basic results of [**36**] were presented already in the survey given in [**23**]. For the case when $\sin\alpha = 0$ Theorem 3.1, Propositions 3.1 - 3.3 and Corollary 3.1 (with available references at that time at step 5) were obtained in [**40**].

REMARK 3.2. The results of Theorem 3.1, Propositions 3.1 - 3.4 and Corollary 3.1 remain valid for complex-valued v, v^0 and complex E, α, under the condition that (1.7) holds for both v and v^0.

REMARK 3.3. Under the assumptions of Theorem 3.1, the following formula holds:

$$\hat{M}_{\alpha,v}(E) - \hat{M}_{\alpha,v^0}(E) = (D_\alpha \mathrm{R}^{+,0}(E))^{-1} - (D_\alpha \mathrm{R}^{+}(E))^{-1}, \tag{3.11}$$

$$\begin{gathered} D_\alpha \mathrm{R}^{+}(E)u(x) = \lim_{\varepsilon\to+0} \int\limits_{\partial D} D_{\alpha,\varepsilon}\mathrm{R}^{+}(x,y,\sqrt{E})u(y)dy, \\ D_\alpha \mathrm{R}^{+,0}(E)u(x) = \lim_{\varepsilon\to+0} \int\limits_{\partial D} D_{\alpha,\varepsilon}\mathrm{R}^{+,0}(x,y,\sqrt{E})u(y)dy, \\ x \in \partial D, \end{gathered} \tag{3.12}$$

where $D_{\alpha,\varepsilon}$ is defined as in (3.4), $\mathrm{R}^{+}(x,y,\sqrt{E})$, $\mathrm{R}^{+,0}(x,y,\sqrt{E})$, $\sqrt{E} > 0$, are the Green functions of (2.16) written as in (2.18) for potentials v, v^0, respectively, u is the test function. For the case when $\sin\alpha = 0$, $v^0 \equiv 0$, $d \geq 3$, formula (3.11) was given in [**34**]. Using techniques developed in [**26**] and in the present work, we obtain (3.11) in the general case.

4. Proofs of Theorem 3.1 and Propositions 3.1, 3.2, 3.4

In this section we will use formulas and equations for impedance boundary map from [**26**]. These results are presented in detail in Subsection 4.1. Proofs of Theorem 3.1 and Propositions 3.1, 3.2, 3.4 are given in Subsections 4.2, 4.3.

4.1. Preliminaries. Let $G_{\alpha,v}(x,y,E)$ be the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6) under assumptions (1.2), (1.3) and (1.7). We recall that (see formulas (3.12), (3.13) of [**26**]):

$$G_{\alpha,v}(x,y,E) = G_{\alpha,v}(y,x,E), \quad x,y \in \bar{D}, \tag{4.1}$$

and, for $\sin\alpha \neq 0$,

$$M_{\alpha,v}(x,y,E) = \frac{1}{\sin^2\alpha} G_{\alpha,v}(x,y,E) - \frac{\cos\alpha}{\sin\alpha}\delta_{\partial D}(x-y), \quad x,y \in \partial D, \tag{4.2}$$

where $M_\alpha(x,y,E)$ and $\delta_{\partial D}(x-y)$ denote the Schwartz kernels of the impedance boundary map $\hat{M}_{\alpha,v}(E)$ and the identity operator $\hat{I}$ on ∂D, respectively, where $\hat{M}_\alpha$ and $\hat{I}$ are considered as linear integral operators.

We recall also that (see, for example, formula (3.16) of [**26**]):

$$\psi(x) = \frac{1}{\sin\alpha}\int\limits_{\partial D}(\cos\alpha\,\psi(\xi) - \sin\alpha\frac{\partial}{\partial\nu}\psi(\xi))G_{\alpha,v}(x,\xi,E)d\xi, \quad x \in D, \tag{4.3}$$

for all sufficiently regular solutions ψ of equation (1.1) in $\bar{D}$ and $\sin\alpha \neq 0$.

We will use the following properties of the Green function $G_\alpha(x,y,E)$:

$$G_{\alpha,v}(x,y,E) \ \text{ is continuous in } \ x,y \in \bar{D},\ x \neq y, \tag{4.4}$$

$$\begin{aligned} |G_{\alpha,v}(x,y,E)| \le c_1(|x-y|^{2-d}), \quad x,y\in\bar{D},\ \text{for}\ \ d\ge 3,\\ |G_{\alpha,v}(x,y,E)| \le c_1(|\ln|x-y||), \quad x,y\in\bar{D},\ \text{for}\ \ d=2, \end{aligned}\tag{4.5}$$

where $c_1 = c_1(D,E,v,\alpha) > 0$.

Actually, properties (4.4), (4.5) are well-known for $\sin\alpha = 0$ (the case of the Direchlet boundary condition) and for $\cos\alpha = 0$ (the case of the Neumann boundary condition). Properties (4.4), (4.5) with $d \ge 3$, $\frac{\sin\alpha}{\cos\alpha} < 0$, $v \equiv 0$ and $E = 0$ were proven in [**29**]. For $d = 2$ see also [**5**]. In Section 6 we give proofs of (4.4), (4.5) for the case of general α, v and E.

In addition, under assumptions of Theorem 3.1, the following identity holds (see formula (3.9) of [**26**]):

$$\int\limits_D (v-v^0)\psi\psi^0 dx = \int\limits_{\partial D}[\psi]_\alpha\left(\hat{M}_{\alpha,v} - \hat{M}_{\alpha,v^0}\right)[\psi^0]_\alpha dx \tag{4.6}$$

for all sufficiently regular solutions ψ, ψ^0 of equation (1.1) in $\bar{D}$ for potentials v, v^0, respectively, where $[\psi]_\alpha$, $[\psi^0]_\alpha$ are defined according to (1.5).

Identity (4.6) for $\sin\alpha = 0$ is reduced to the Alessandrini identity (Lemma 1 of [**1**]).

We will use also that:

$$\begin{aligned} \|\hat{R}(k)u\|_{C^{1+\delta}(\Omega)} \le c_2(D,\Omega,v,k,\delta)\|u\|_{\mathbb{L}^\infty(D)},\\ \hat{R}(k)u(x) = \int\limits_D R(x,y,k)u(y)dy, \quad x\in\Omega,\\ k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}), \end{aligned}\tag{4.7a}$$

$$\begin{aligned} \|\hat{R}_\gamma(k)u\|_{C^{1+\delta}(\Omega)} \le c_3(D,\Omega,v,k,\gamma,\delta)\|u\|_{\mathbb{L}^\infty(D)},\\ \hat{R}_\gamma(k)u(x) = \int\limits_D R_\gamma(x,y,k)u(y)dy, \quad x\in\Omega,\\ \gamma\in\mathbb{S}^{d-1},\ k\in\mathbb{R}^d\setminus(\{0\}\cup\mathcal{E}_\gamma), \end{aligned}\tag{4.7b}$$

for $u \in \mathbb{L}^\infty(D)$, $\delta \in [0,1)$, where Ω is such an open bounded domain in $\mathbb{R}^d$ that $\bar{D} \subset \Omega$ and $C^{1+\delta}$ denotes C^1 with the first derivatives belonging to the Hölder space C^δ.

We will use also the Green formula:

$$\int\limits_{\partial D} \left(\phi_1 \frac{\partial \phi_2}{\partial \nu} - \phi_2 \frac{\partial \phi_1}{\partial \nu} \right) dx = \int\limits_D (\phi_1 \Delta \phi_2 - \phi_2 \Delta \phi_1)\, dx, \tag{4.8}$$

where ϕ_1 and ϕ_2 are arbitrary sufficiently regular functions in $\bar{D}$.

4.2. Proof of Theorem 3.1 and Proposition 3.1. For the case when $\sin\alpha = 0$, Theorem 3.1 and Proposition 3.1 were proved in [**40**]. In this subsection we generalize the proof of [**40**] to the case $\sin\alpha \neq 0$. We proceed from the following formulas and equations (being valid under assumption (2.2) on v^0 and v):

$$\begin{gathered} h(k,l) - h^0(k,l) = \left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} \psi^0(x,-l)(v(x) - v^0(x))\psi(x,k)dx, \\ k,l \in \mathbb{C}^d \setminus (\mathcal{E}^0 \cup \mathcal{E}),\ k^2 = l^2,\ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \neq 0, \end{gathered} \tag{4.9}$$

$$\begin{gathered} \psi(x,k) = \psi^0(x,k) + \int\limits_{\mathbb{R}^d} R^0(x,y,k)(v(y) - v^0(y))\psi(y,k)dy, \\ x \in \mathbb{R}^d,\ k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0), \end{gathered} \tag{4.10}$$

where (4.10) at fixed k is considered as an equation for $\psi = e^{ikx}\mu(x,k)$ with $\mu \in \mathbb{L}^\infty(\mathbb{R}^d)$;

$$\begin{gathered} f(k,l) - f^0(k,l) = \left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} \psi^{+,0}(x,-l)(v(x) - v^0(x))\psi^+(x,k)dx, \\ k,l \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^{+,0} \cup \mathcal{E}^+),\ k^2 = l^2, \end{gathered} \tag{4.11}$$

$$\begin{gathered} \psi^+(x,k) = \psi^{+,0}(x,k) + \int\limits_{\mathbb{R}^d} R^{+,0}(x,y,k)(v(y) - v^0(y))\psi^+(y,k)dy, \\ x \in \mathbb{R}^d,\ k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^{+,0}), \end{gathered} \tag{4.12}$$

where (4.12) at fixed k is an equation for $\psi^+ \in \mathbb{L}^\infty(\mathbb{R}^d)$;

$$\begin{gathered} h_\gamma(k,l) - h^0_\gamma(k,l) = \left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbb{R}^d} \psi^0_{-\gamma}(x,-k,-l)(v(x) - v^0(x))\psi_\gamma(x,k)dx, \\ \gamma \in \mathbb{S}^{d-1}, k \in \mathbb{R}^d \setminus (\mathcal{E}^0_\gamma \cup \mathcal{E}_\gamma),\ l \in \mathbb{R}^d,\ k^2 = l^2, \end{gathered} \tag{4.13}$$

$$\psi_\gamma(x,k) = \psi^0_\gamma(x,k) + \int\limits_{\mathbb{R}^d} R^0_\gamma(x,y,k)(v(y)-v^0(y))\psi_\gamma(y,k)dy, \tag{4.14}$$
$$x \in \mathbb{R}^d,\ \gamma \in \mathbb{S}^{d-1},\ k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^0_\gamma),$$

where (4.14) at fixed γ and k is considered as an equation for $\psi_\gamma \in \mathbb{L}^\infty(\mathbb{R}^d)$.

We recall that ψ^+, f, ψ, h, ψ_γ, h_γ were defined in Sections 2, 3 by means of (2.3) - (2.9), (3.5). Equation (4.12) is well-known in the classical scattering theory for the Schrödinger equation (2.1). Formula (4.11) was given, in particular, in [**50**]. To our knowledge formula and equations (4.9), (4.10), (4.14) were given for the first time in [**38**], whereas formula (4.13) was given for the first time in [**40**].

In addition, under assumption (2.2) on v^0 and v:

(4.15a) equation (4.10) at fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$ is uniquely solvable for $\psi = e^{ikx}\mu(x,k)$ with $\mu \in \mathbb{L}^\infty(\mathbb{R}^d)$ if and only if $k \notin \mathcal{E}$;

(4.15b) equation (4.12) at fixed $k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^{+,0})$ is uniquely solvable for $\psi^+ \in \mathbb{L}^\infty(\mathbb{R}^d)$ if and only if $k \notin \mathcal{E}^+$;

(4.15c) equation (4.14) at fixed $\gamma \in \mathbb{S}^{d-1}$ and $k \in \mathbb{R}^d \setminus (\{0\} \cup \mathcal{E}^+_\gamma)$ is uniquely solvable for $\psi_\gamma \in \mathbb{L}^\infty(\mathbb{R}^d)$ if and only if $k \notin \mathcal{E}_\gamma$.

Let us prove Theorem 3.1 for the case of the Faddeev functions ψ, h. The proof of Theorem 3.1 for the case of ψ^+, f and the proof of Proposition 3.1 are similar.

Note that formula (3.1) follows directly from (4.6) and (4.9).

Using (2.17) and applying (4.6) for equation (4.10), we get that

$$\psi(x,k) - \psi^0(x,k) = \int\limits_{\partial D}\int\limits_{\partial D} [R^0(x,\xi,k)]_{\xi,\alpha} \left(M_{\alpha,v} - M_{\alpha,v^0}\right)(\xi,y,E)[\psi(y,k)]_\alpha d\xi dy, \quad x \in \mathbb{R}^d \setminus \bar{D}, \tag{4.16}$$

where

$$[R^0(x,\xi,k)]_{\xi,\alpha} = \left(\cos\alpha - \sin\alpha \frac{\partial}{\partial \nu_\xi}\right) R^0(x,\xi,k). \tag{4.17}$$

Equation (3.2) follows from formula (4.16), definition (1.5) and the property that

$$\lim_{\varepsilon\to+0} \left(\cos\alpha - \sin\alpha \frac{\partial}{\partial \nu_x}\right) u(x+\varepsilon\nu_x) = [u(x)]_\alpha, \quad x \in \partial D, \tag{4.18}$$

for $u(x) = \psi(x,k) - \psi^0(x,k)$.

4.3. Proofs of Propositions 3.2 and 3.4. In this subsection we prove Propositions 3.2, 3.4 for the case of equation (3.2) for $[\psi]_\alpha$. The proofs of Propositions 3.2 and 3.4 for the cases of ψ^+ and ψ_γ are absolutely similar.

Proof of Proposition 3.2. The proof of Proposition 3.2 for the case of $\sin\alpha = 0$ was given in [**40**]. Let us assume that $\sin\alpha \neq 0$.

Using (4.2), we find that

$$(M_{\alpha,v} - M_{\alpha,v^0})(\xi, y, E) = \frac{1}{\sin^2\alpha}(G_{\alpha,v} - G_{\alpha,v^0})(\xi, y, E), \quad \xi, y \in \partial D. \tag{4.19}$$

Using (2.17), (4.1), (4.8) and the impedance boundary condition (1.6) for $G_{\alpha,v}$, G_{α,v^0}, we get that

$$\begin{aligned}
&\int\limits_{\partial D} [R^0(x,\xi,k)]_{\alpha,\xi}(G_{\alpha,v} - G_{\alpha,v^0})(\xi, y, E)d\xi = \\
&= \int\limits_{\partial D}\Big([R^0(x,\xi,k)]_{\alpha,\xi}(G_{\alpha,v} - G_{\alpha,v^0})(\xi, y, E)d\xi - \\
&\qquad - R^0(x,\xi,k)[(G_{\alpha,v} - G_{\alpha,v^0})(\xi, y, E)]_{\alpha,\xi}\Big)d\xi = \\
&= \sin\alpha\int\limits_{D}\Big(R^0(x,\xi,k)\Delta_\xi(G_{\alpha,v} - G_{\alpha,v^0})(\xi, y, E)d\xi - \\
&\qquad - (G_{\alpha,v} - G_{\alpha,v^0})(\xi, y, E)\Delta_\xi R^0(x,\xi,k)\Big)d\xi = \\
&= \sin\alpha\int\limits_{D} R^0(x,\xi,k)\big(v(\xi) - v^0(\xi)\big)G_{\alpha,v}(\xi, y, E)d\xi, \quad x \in \mathbb{R}^d \setminus \bar{D},\ y \in \partial D.
\end{aligned} \tag{4.20}$$

Combining (4.16), (4.19) and (4.20), we obtain that

$$A_\alpha(x, y, k) = \lim_{\varepsilon\to+0}\left(\cos\alpha - \sin\alpha\frac{\partial}{\partial\nu_x}\right)B_\alpha(x + \varepsilon\nu_x, y, k), \quad x, y \in \partial D, \tag{4.21}$$

where

$$\begin{aligned}
B_\alpha(x, y, k) &= \int\limits_{\partial D}[R^0(x,\xi,k)]_{\alpha,\xi}(M_{\alpha,v} - M_{\alpha,v^0})(\xi, y, E)d\xi = \\
&= \frac{1}{\sin\alpha}\int\limits_{D} R^0(x,\xi,k)\big(v(\xi) - v^0(\xi)\big)G_{\alpha,v}(\xi, y, E)d\xi, \\
&\qquad x \in \mathbb{R}^d \setminus \bar{D},\ y \in \partial D.
\end{aligned} \tag{4.22}$$

Thus, we have that the limit in (4.21) (and, hence, in (3.3)) is well defined and

(4.23)
$$A_\alpha(x,y,k) = \frac{1}{\sin\alpha}\int\limits_D [R^0(x,\xi,k)]_{x,\alpha}\left(v(\xi)-v^0(\xi)\right)G_{\alpha,v}(\xi,y,E)d\xi, \quad x,y\in\partial D.$$

Let $\hat{A}_\alpha(k)$ denote the linear integral operator on ∂D with the Schwartz kernel $A_\alpha(x,y,k)$ of (3.3), (4.23). Using (4.5), (4.7), (4.23), we obtain that

$$\begin{gathered}\hat{A}_\alpha(k):\ \mathbb{L}^\infty(\partial D)\to C^\delta(\partial D)\\ \text{is a bounded linear operator.}\end{gathered} \tag{4.24}$$

As a corollary of (4.24), $\hat{A}_\alpha(k)$ is a compact operator in $\mathbb{L}^\infty(D)$. ■

Proof of Proposition 3.4. Using (2.17), (3.7) and (4.8), we get that

$$\begin{aligned}&\int\limits_{\partial D}\left(\phi_\alpha(\xi,y)\frac{\partial}{\partial\nu_\xi}R^0(x,\xi,k)-R^0(x,\xi,k)\frac{\partial}{\partial\nu_\xi}\phi_\alpha(\xi,y)\right)d\xi=\\ &=\int\limits_D\left(\phi_\alpha(\xi,y)\Delta_\xi R^0(x,\xi,k)-R^0(x,\xi,k)\Delta_\xi\phi_\alpha(\xi,y)\right)d\xi=\\ &=\int\limits_D R^0(x,\xi,k)(v^0(\xi)-E+\lambda)\phi_\alpha(\xi,y)d\xi,\\ &\qquad x\in\mathbb{R}^d\setminus\bar{D},\ y\in\partial D.\end{aligned} \tag{4.25}$$

Combining (3.7), (4.22) and (4.25), we find that

(4.26)
$$\begin{aligned}B_\alpha(x,y,k)&=\int\limits_{\partial D}[R^0(x,\xi,k)]_{\xi,\alpha}\phi_\alpha(\xi,y)d\xi=\\ &=\int\limits_{\partial D}R^0(x,\xi,k)[\phi_\alpha(\xi,y)]_{\xi,\alpha}d\xi-\sin\alpha\int\limits_D R^0(x,\xi,k)(v^0(\xi)-E+\lambda)\phi_\alpha(\xi,y)d\xi,\\ &\qquad x\in\mathbb{R}^d\setminus\bar{D},\ y\in\partial D.\end{aligned}$$

Combining (4.21) and (4.26), we obtain (3.8).

Formula (3.10) follows from (3.7) and the definition of $\hat{\Phi}$.

■

5. Proof of Proposition 3.3

For the case when $\sin\alpha=0$, Proposition 3.3 was proved in [**40**]. In this section we prove Proposition 3.3 for $\sin\alpha\neq 0$. We will prove Proposition 3.3 for the case of equation (3.2) for $[\psi]_\alpha$. The proofs for the cases of ψ^+ and ψ_γ are similar.

According to (4.15), to prove Proposition 3.3 (for the case of ψ) it is sufficient to show that equation (3.2) (at fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$) is uniquely solvable in the space of bounded functions on ∂D if and only if equation (4.10) is uniquely solvable for $\psi = e^{ikx}\mu(x,k)$ with $\mu \in \mathbb{L}^\infty(\mathbb{R}^d)$.

Let equation (4.10) have several solutions. Then, repeating the proof of Theorem 3.1 separately for each solution, we find that $[\psi]_\alpha$ on ∂D for each of these solutions satisfies equation (3.2). Thus, using also (1.7) we obtain that equation (3.2) has at least as many solutions as equation (4.10).

To prove the converse (and thereby to prove Proposition 3.3) it remains to show that any solution $[\psi]_\alpha$ of (3.2) can be continued to a continuos solution of (4.10).

Let ψ be the solution of (1.1) with the impedance boundary data $[\psi]_\alpha$, satisfying (3.2). Let

$$\psi_1(x) = \psi^0(x,k) + \int\limits_D R^0(x,y,k)(v(y) - v^0(y))\psi(y)dy, \quad x \in \mathbb{R}^d. \tag{5.1}$$

Using (4.7), we obtain that

$$\psi_1 \text{ defined by (5.1) belongs to } C^{1+\delta}(\mathbb{R}^d),\ \delta \in [0,1). \tag{5.2}$$

We have that

$$(-\Delta + v^0(x) - E)\psi(x) = (v^0(x) - v(x))\psi(x), \quad x \in D, \tag{5.3}$$

$$\begin{aligned}(-\Delta + v^0(x) - E)\psi_1(x) = \int\limits_D -\delta(x-y)(v(y) - v^0(y))\psi(y)dy = \\ = (v^0(x) - v(x))\psi(x), \quad x \in D.\end{aligned} \tag{5.4}$$

Combining (4.6) and (4.22), we get that

$$\int\limits_D R^0(x,y,k)(v(y) - v^0(y))\psi(y)dy = \int\limits_{\partial D} B_\alpha(x,y,k)[\psi(y)]_\alpha dy, \quad x \in \mathbb{R}^d \setminus \bar{D}. \tag{5.5}$$

Using (3.2), (4.21), (5.2), (5.5), we find that

$$[\psi_1(x)]_\alpha = [\psi^0(x,k)]_\alpha + \int\limits_{\partial D} A_\alpha(x,y,k)[\psi(y)]_\alpha dy = [\psi(x)]_\alpha, \quad x \in \partial D. \tag{5.6}$$

Using (5.3), (5.4) and (5.6), we obtain that

$$\begin{aligned}(-\Delta + v^0(x) - E)(\psi_1(x) - \psi(x)) = 0, \quad x \in D,\\ [\psi_1(x) - \psi(x)]_\alpha = 0, \quad x \in \partial D.\end{aligned} \tag{5.7}$$

Since v^0 satisfies (1.7), we get that

$$\psi_1(x) = \psi(x), \quad x \in \bar{D}. \tag{5.8}$$

Combining (5.1), (5.2) and (5.8), we find that ψ_1 is a continuos solution of (4.10).

6. Proofs of properties (4.4), (4.5)

As it was mentioned in Subsection 4.1, properties (4.4), (4.5) are well-known for $\cos\alpha = 0$ (the case of the Neumann boundary condition). To extend these properties to the case of general α, v, E, we use the following schema:

(1) $G_{\alpha_1,v} \to G_{\alpha_2,v}$ by means of Lemma 6.1 given bellow (with $\sin\alpha_1 \neq 0$ and $\sin\alpha_2 \neq 0$),
(2) $G_{\alpha,v_1} \to G_{\alpha,v_2}$ by means of Lemma 6.2 given bellow.

The proofs of steps 1, 2 are based on the theory of Fredholm linear integral equations of the second kind.

Starting from (4.4), (4.5) for $\cos\alpha = 0$ and combining steps 1, 2 and the property

$$G_{\alpha,v}(\cdot,\cdot,E) = G_{\alpha,v-E}(\cdot,\cdot,0), \tag{6.1}$$

we obtain these properties for the case when $\sin\alpha \neq 0$.

As it was already mentioned in Section 4, properties (4.4), (4.5) are well-known for $\sin\alpha = 0$ (the case of the Dirichlet boundary condition).

LEMMA 6.1. *Let D satisfy* (1.2) *and potential v satisfy (1.3), (1.7) for some fixed E and for $\alpha = \alpha_1$, $\alpha = \alpha_2$ simultaneously, where* $\sin\alpha_1 \neq 0$ *and* $\sin\alpha_2 \neq 0$. *Let G_j denote the Green function $G_{\alpha_j,v}$, $j = 1, 2$. Let G_1 satisfy:*

$$G_1(x,y,E) \ \text{ is continuous in } \ x,y \in \bar{D},\ x \neq y, \tag{6.2}$$

$$\begin{aligned} |G_1(x,y,E)| &\le a_1|x-y|^{2-d} \quad \text{for } \ d \geq 3,\\ |G_1(x,y,E)| &\le a_1|\ln|x-y|| \quad \text{for } \ d = 2,\\ & \qquad\qquad x,y \in \bar{D}. \end{aligned} \tag{6.3}$$

Then:

$$G_2(x,y,E) \ \text{ is continuous in } \ x,y \in \bar{D},\ x \neq y, \tag{6.4}$$

$$\begin{aligned} |G_2(x,y,E)| &\le a_2|x-y|^{2-d} \quad \text{for } \ d \geq 3,\\ |G_2(x,y,E)| &\le a_2|\ln|x-y|| \quad \text{for } \ d = 2,\\ & \qquad\qquad x,y \in \bar{D}, \end{aligned} \tag{6.5}$$

where $a_2 = a_2(D,E,a_1,v,\alpha_1,\alpha_2) > 0$.

PROOF OF LEMMA 6.1. First, we derive formally some formulas and equations relating the Green functions G_1 and G_2. Then, proceeding from these formulas and equations, we obtain, in particular, estimates (6.4), (6.5).

Consider $W = G_2 - G_1$. Using definitions of G_1, G_2 and formula (4.3), we find that:

$$(-\Delta_x + v(x) - E)W(x,y) = 0, \quad x,y \in D, \tag{6.6}$$

(6.7)
$$\begin{aligned}\Big(\cos\alpha_2\, W(x,y) - \sin\alpha_2\, \frac{\partial W}{\partial \nu_x}(x,y)\Big)\Big|_{x\in\partial D} &=\\ = -\Big(\cos\alpha_2\, G_1(x,y,E) - \sin\alpha_2\, \frac{\partial G_1}{\partial \nu_x}(x,y,E)\Big)\Big|_{x\in\partial D} &=\\ = -\Big(\cos\alpha_2\, G_1(x,y,E) - \sin\alpha_2\, \frac{\cos\alpha_1}{\sin\alpha_1} G_1(x,y,E)\Big)\Big|_{x\in\partial D} &=\\ = \frac{\sin(\alpha_2-\alpha_1)}{\sin\alpha_1} G_1(x,y,E)\Big|_{x\in\partial D}&, \quad y\in D,\end{aligned}$$

(6.8)
$$W(x,y) = \frac{1}{\sin\alpha_1}\int\limits_{\partial D}\Big(\cos\alpha_1 W(\xi,y) - \sin\alpha_1 \frac{\partial W}{\partial \nu_\xi}(\xi,y)\Big) G_1(\xi,x,E)d\xi, \quad x,y\in D.$$

Using (6.7) and (6.8), we find the following linear integral equation for $W(\cdot,y)$ on ∂D:

$$W(\cdot,y) = W_0(\cdot,y) + \hat K_1 W(\cdot,y), \quad y\in D, \tag{6.9}$$

where

$$W_0(x,y) = \frac{\sin(\alpha_2-\alpha_1)}{\sin\alpha_2}\int\limits_{\partial D} G_1(\xi,x,E)G_1(\xi,y,E)d\xi, \tag{6.10}$$

$$\begin{gathered}\hat K_1 u(x) = \frac{\sin(\alpha_2-\alpha_1)}{\sin\alpha_2\sin\alpha_1}\int\limits_{\partial D} G_1(\xi,x,E)u(\xi)d\xi,\\ x\in\partial D,\ y\in D,\ u \text{ is a test function.}\end{gathered} \tag{6.11}$$

In addition, for

$$\delta_n W = W - \sum_{j=1}^{n} (\hat K_1)^{j-1} W_0 \tag{6.12}$$

equation (6.9) takes the form

$$\delta_n W = (\hat K_1)^n W_0 + \hat K_1 \delta_n W. \tag{6.13}$$

Our analysis based on (6.6)-(6.13) is given bellow.
Using (6.2), (6.3), we obtain that

$$(\hat K_1)^n W_0 \in C(\partial D\times \bar D) \text{ for sufficiently great } n \text{ with respect to } d, \tag{6.14}$$

$$\hat K_1 \text{ is a compact operator in } C(\partial D). \tag{6.15}$$

Let us show that the homogeneous equation

$$u = \hat K_1 u, \quad u\in C(\partial D), \tag{6.16}$$

has only trivial solution $u \equiv 0$.

Using the fact that the potential v satisfy (1.7) for $\alpha = \alpha_1$, we define ψ by

$$\begin{gathered}(-\Delta + v(x) - E)\psi(x) = 0, \quad x \in D,\\ \cos\alpha_1 \psi|_{\partial D} - \sin\alpha_1 \frac{\partial\psi}{\partial\nu}|_{\partial D} = u.\end{gathered} \tag{6.17}$$

Due to (4.3), we have that

$$\psi(x) = \frac{1}{\sin\alpha_1}\int\limits_{\partial D}(\cos\alpha_1\psi(\xi) - \sin\alpha_1\frac{\partial\psi}{\partial\nu}(\xi))G_1(\xi,x,E)d\xi, \quad x \in D. \tag{6.18}$$

Using (6.16), (6.18), we find that

$$\frac{\sin(\alpha_2-\alpha_1)}{\sin\alpha_2}\psi(x) = \hat{K}_1 u(x) = u(x), \quad x \in \partial D. \tag{6.19}$$

Therefore, we have that

$$\cos\alpha_1\psi(x) - \sin\alpha_1\frac{\partial\psi}{\partial\nu}(x) = \frac{\sin(\alpha_2-\alpha_1)}{\sin\alpha_2}\psi(x), \quad x \in \partial D. \tag{6.20}$$

Since $\sin\alpha_1 \neq 0$ and $\sin\alpha_2 \neq 0$, using (6.20), we obtain that

$$\cos\alpha_2\psi(x) - \sin\alpha_2\frac{\partial\psi}{\partial\nu}(x) = 0 \tag{6.21}$$

Taking into account the fact that the potential v satisfy (1.7) for $\alpha = \alpha_2$, we get that $\psi \equiv 0$ and $u \equiv 0$.

Proceeding from

$$\begin{aligned}F = W(x,y) \ \text{ and } \ F' = \frac{\cos\alpha_2}{\sin\alpha_2}W(x,y) - \frac{\sin(\alpha_2-\alpha_1)}{\sin\alpha_1\sin\alpha_2}G_1(x,y,E),\\ x \in \partial D,\ y \in \bar{D},\end{aligned} \tag{6.22}$$

found from (6.9), (6.13) and (6.7) (with F' substituted in place of $\partial W/\partial\nu_x$), we consider

$$W(x,y) = \frac{1}{\sin\alpha_1}\int\limits_{\partial D}\Big(\cos\alpha_1 F(\xi,y) - \sin\alpha_1 F'(\xi,y)\Big)G_1(\xi,x,E)d\xi, \quad x,y \in \bar{D}. \tag{6.23}$$

Using (6.9) and properties of G_1 (including formula (4.3)), we subsequently obtain that

$$\lim_{\varepsilon\to+0} W(x-\varepsilon\nu_x,y) = F(x,y), \quad x \in \partial D,\ y \in \bar{D}, \tag{6.24}$$

$$W \text{ satisfies (6.6)}, \tag{6.25}$$

$$\lim_{\varepsilon\to+0}\frac{\partial}{\partial\nu_x}W(x-\varepsilon\nu_x,y) = F'(x,y), \quad x \in \partial D,\ y \in \bar{D}. \tag{6.26}$$

From (6.2), (6.3), (6.10)-(6.16), (6.24)-(6.26) it follows that G_2 defined as $G_2 = G_1 + W$ is the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6) for $\alpha = \alpha_2$ and that G_2 satisfies (6.4), (6.5). ■

LEMMA 6.2. *Let D satisfy (1.2) and potentials v_1, v_2 satisfy (1.3), (1.7) for some fixed E and α. Let G_j denote the Green function G_{α,v_j}, $j = 1, 2$. Let G_1 satisfy:*

$$G_1(x, y, E) \text{ is continuous in } x, y \in \bar{D},\ x \neq y, \tag{6.27}$$

$$\begin{aligned} &|G_1(x, y, E)| \le a_3 |x - y|^{2-d} \quad \text{for } d \ge 3,\\ &|G_1(x, y, E)| \le a_3 |\ln |x - y|| \quad \text{for } d = 2,\\ &\qquad x, y \in \bar{D}. \end{aligned} \tag{6.28}$$

Then:

$$G_2(x, y, E) \text{ is continuous in } x, y \in \bar{D},\ x \neq y, \tag{6.29}$$

$$\begin{aligned} &|G_2(x, y, E)| \le a_4 |x - y|^{2-d} \quad \text{for } d \ge 3,\\ &|G_2(x, y, E)| \le a_4 |\ln |x - y|| \quad \text{for } d = 2,\\ &\qquad x, y \in \bar{D}, \end{aligned} \tag{6.30}$$

where $a_4 = a_4(D, E, a_3, v_1, v_2, \alpha) > 0$.

PROOF OF LEMMA 6.2. First, we derive formally some formulas and equations relating the Green functions G_1 and G_2. Then, proceeding from these formulas and equations, we obtain, in particular, estimates (6.29), (6.30).

Using (4.1), the impedance boundary condition for G_1, G_2, we find that

$$\begin{aligned} G_1(x, y, E) &= \int_D G_1(x, \xi, E)\Big(\Delta_\xi - v_2(\xi) + E\Big) G_2(\xi, y, E)\, d\xi,\\ G_2(x, y, E) &= \int_D G_2(\xi, y, E)\Big(\Delta_\xi - v_1(\xi) + E\Big) G_1(x, \xi, E)\, d\xi,\\ \int_{\partial D} &\left(G_1(x, \xi, E)\frac{\partial G_2}{\partial \nu_\xi}(\xi, y, E) - G_2(\xi, y, E)\frac{\partial G_1}{\partial \nu_\xi}(x, \xi, E)\right) d\xi = 0,\\ &\qquad x, y \in D. \end{aligned} \tag{6.31}$$

Combining (6.31) with (4.8), we get that

$$G_2(\cdot, y, E) - G_1(\cdot, y, E) = \hat{K}_2 G_2(\cdot, y, E), \quad y \in D, \tag{6.32}$$

where

$$\hat{K}_2 u(x) = \int_D (v_2(\xi) - v_1(\xi))\, G_1(x, \xi, E) u(\xi) d\xi. \tag{6.33}$$

In addition, for

$$\delta_n G = G_2 - \sum_{j=1}^{n} (\hat{K}_2)^{j-1} G_1 \tag{6.34}$$

equation (6.32) takes the form

$$\delta_n G = (\hat{K}_2)^n G_1 + \hat{K}_2 \delta_n G. \tag{6.35}$$

Our analysis based on (6.31)-(6.35) is given bellow.

Using (6.27), (6.28), we find that

$$(\hat{K}_2)^n G_1 \in C(\bar{D} \times \bar{D}) \text{ for sufficiently great } n \text{ with respect to } d, \tag{6.36}$$

$$\hat{K}_2 \text{ is a compact operator in } C(\bar{D}). \tag{6.37}$$

Let us show that the homogeneous equation

$$u = \hat{K}_2 u, \quad u \in C(\bar{D}), \tag{6.38}$$

has only trivial solution $u \equiv 0$. Using (6.33), (6.38) and properties of the Green function G_1, we find that

$$\begin{aligned} (-\Delta + v_1(x) - E)u(x) &= \int_D -\delta(x-\xi)\,(v_2(\xi) - v_1(\xi))\,u(\xi)d\xi = \\ &= (v_1 - v_2)u(x), \quad x \in D, \\ \cos\alpha\, u(x) - \sin\alpha \frac{\partial u}{\partial \nu}(x) &= 0, \quad x \in \partial D. \end{aligned} \tag{6.39}$$

Using (6.27), (6.28), we find that $u \in C(\bar{D})$. Taking into account the fact that the potential v_2 satisfy (1.7), we get that $u \equiv 0$.

Proceeding from (6.27), (6.28), (6.36), (6.37) it follows that G_2 found from (6.32), (6.35) is the Green function for the operator $\Delta - v + E$ in D with the impedance boundary condition (1.6) for $v = v_2$ and that G_2 satisfies (6.29), (6.30). ■

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Paper F

PAPER F

New global stability estimates for monochromatic inverse acoustic scattering

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Abstract. We give new global stability estimates for monochromatic inverse acoustic scattering. These estimates essentially improve estimates obtained in [P. Hähner, T. Hohage, SIAM J. Math. Anal., 33(3), 2001, 670–685] and can be considered as a solution of an open problem formulated in the aforementioned work.

1. Introduction

We consider the equation

$$\Delta\psi + \omega^2 n(x)\psi = 0, \quad x \in \mathbb{R}^3,\ \omega > 0, \tag{1.1}$$

where

$$\begin{aligned} (1-n) \in W^{m,1}(\mathbb{R}^3) &\text{ for some } m > 3,\\ \operatorname{Im} n(x) \geq 0, &\quad x \in \mathbb{R}^3,\\ \operatorname{supp}(1-n) \subset B_{r_1} &\text{ for some } r_1 > 0, \end{aligned} \tag{1.2}$$

where $W^{m,1}(\mathbb{R}^3)$ denotes the standard Sobolev space on $\mathbb{R}^3$ (see formula (2.11) of Section 2 for details), $B_r = \{x \in \mathbb{R}^3 : |x| < r\}$.

We interpret (1.1) as the stationary acoustic equation at frequency ω in an inhomogeneous medium with refractive index n.

In addition, we consider the Green function $G^+(x,y,\omega)$ for the operator $\Delta + \omega^2 n(x)$ with the Sommerfeld radiation condition:

$$\begin{aligned} \left(\Delta + \omega^2 n(x)\right) G^+(x,y,\omega) &= \delta(x-y),\\ \lim_{|x|\to\infty} |x| \left(\frac{\partial G^+}{\partial |x|}(x,y,\omega) - i\omega G^+(x,y,\omega)\right) &= 0,\\ \text{uniformly for all directions } \hat{x} &= x/|x|,\\ x, y \in \mathbb{R}^3,\ \omega &> 0. \end{aligned} \tag{1.3}$$

It is know that, under assumptions (1.2), the function G^+ is uniquely specified by (1.3), see, for example, **[9]**, **[6]**.

We consider, in particular, the following near-field inverse scattering problem for equation (1.1):

Problem 1.1. Given G^+ on $\partial B_r \times \partial B_r$ for some fixed $\omega > 0$ and $r > r_1$, find n on B_{r_1}.

We consider also the solutions $\psi^+(x,k)$, $x \in \mathbb{R}^3$, $k \in \mathbb{R}^3$, $k^2 = \omega^2$, of equation (1.1) specified by the following asymptotic condition:

$$\begin{aligned} &\psi^+(x,k) = e^{ikx} - 2\pi^2 \frac{e^{i|k||x|}}{|x|} f\left(k, |k|\frac{x}{|x|}\right) + o\left(\frac{1}{|x|}\right) \\ &\qquad \text{as } \ |x| \to \infty \left(\text{uniformly in } \frac{x}{|x|}\right), \end{aligned} \tag{1.4}$$

with some a priory unknown f.

The function f on $\mathcal{M}_\omega = \{k \in \mathbb{R}^3, l \in \mathbb{R}^3 : k^2 = l^2 = \omega^2\}$ arising in (1.4) is the classical scattering amplitude for equation (1.1).

In addition to Problem 1.1, we consider also the following far-field inverse scattering problem for equation (1.1):

Problem 1.2. Given f on $\mathcal{M}_\omega$ for some fixed $\omega > 0$, find n on B_{r_1}.

In **[4]** it was shown that the near-field data of Problem 1.1 are uniquely determined by the far-field data of Problem 1.2 and vice versa.

Global uniqueness for Problems 1.1 and 1.2 was proved for the first time in **[17]**; in addition, this proof is constructive. For more information on reconstruction methods for Problems 1.1 and 1.2 see **[2]**, **[9]**, **[16]**, **[17]**, **[19]**, **[23]** and references therein.

Problems 1.1 and 1.2 can be also considered as examples of ill-posed problems: see **[15]**, **[5]** for an introduction to this theory.

The main results of the present article consist of the following two theorems:

Theorem 1.1. *Let $C_n > 0$, $r > r_1$ be fixed constants. Then there exists a positive constant C (depending only on m, ω, r_1, r and C_n) such that for all refractive indices n_1, n_2 satisfying $\|1-n_1\|_{W^{m,1}(\mathbb{R}^3)}, \|1-n_2\|_{W^{m,1}(\mathbb{R}^3)} < C_n$, $\mathrm{supp}\,(1-n_1), \mathrm{supp}\,(1-n_2) \subset B_{r_1}$, the following estimate holds:*

$$||n_1 - n_2||_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C\left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \quad s = \frac{m-3}{3}, \tag{1.5}$$

where $\delta = ||G_1^+ - G_2^+||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$ and G_1^+, G_2^+ are the near-field scattering data for the refractive indices n_1, n_2, respectively, at fixed frequency ω.

Remark 1.1. We recall that if n_1, n_2 are refractive indices satisfying (1.2), then $G_1^+ - G_2^+$ is bounded in $\mathbb{L}^2(\partial B_r \times \partial B_r)$ for any $r > r_1$, where G_1^+ and G_2^+ are the near-field scattering data for the refractive indices n_1 and n_2, respectively, at fixed frequency ω, see, for example, Lemma 2.1 of **[9]**.

THEOREM 1.2. *Let $C_n > 0$ and $0 < \epsilon < \frac{m-3}{3}$ be fixed constants. Then there exists a positive constant C (depending only on m, ϵ, ω, r_1 and C_n) such that for all refractive indices n_1, n_2 satisfying $\|1-n_1\|_{W^{m,1}(\mathbb{R}^3)}, \|1-n_2\|_{W^{m,1}(\mathbb{R}^3)} < C_n$, $\operatorname{supp}(1-n_1), \operatorname{supp}(1-n_2) \subset B_{r_1}$, the following estimate holds:*

$$||n_1 - n_2||_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C \left(\ln\left(3+\delta^{-1}\right)\right)^{-s+\epsilon}, \quad s = \frac{m-3}{3}, \tag{1.6}$$

where $\delta = ||f_1 - f_2||_{\mathbb{L}^2(\mathcal{M}_\omega)}$ and f_1, f_2 denote the scattering amplitudes for the refractive indices n_1, n_2, respectively, at fixed frequency ω.

For some regularity dependent s but always smaller than 1 the stability estimates of Theorems 1.1 and 1.2 were proved in [**9**]. Possibility of estimates (1.5), (1.6) with $s > 1$ was formulated in [**9**] as an open problem, see page 685 of [**9**]. Our estimates (1.5), (1.6) with $s = \frac{m-3}{3}$ give a solution of this problem. Indeed,

$$s = \frac{m-3}{3} \to +\infty \quad \text{as} \quad m \to +\infty. \tag{1.7}$$

Apparently, using the methods of [**21**], [**22**] estimates (1.5), (1.6) can be proved for $s = m-3$. For more information on stability estimates for Problems 1.1 and 1.2 see [**9**], [**11**], [**24**] and references therein. In particular, as a corollary of [**11**] estimates (1.5), (1.6) can not be fulfilled, in general, for $s > \frac{5m}{3}$.

The proofs of Theorem 1.1 and 1.2 are given in Section 3. These proofs use, in particular:

(1) Properties of the Faddeev functions for equation (1.1) considered as the Schrödinger equation at fixed energy $E = \omega^2$, see Section 2.
(2) The results of [**9**] consisting in Lemma 3.1 and in reducing (via Lemma 3.2) estimates of the form (1.6) for Problem 1.2 to estimates of the form (1.5) for Problem 1.1.

In addition in the proofs of Theorem 1.1 and 1.2 we combine some of the aforementioned ingredients in a similar way with the proof of stability estimates of [**13**].

2. Faddeev functions

We consider (1.1) as the Schrödinger equation at fixed energy $E = \omega^2$:

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^3, \tag{2.1}$$

where $v = \omega^2(1-n)$, $E = \omega^2$.

For equation (2.1) we consider the Faddeev functions G, ψ, h (see [**7**], [**8**], [**10**], [**17**]):

$$G(x,k) = e^{ikx} g(x,k), \quad g(x,k) = -(2\pi)^{-3} \int\limits_{\mathbb{R}^3} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \tag{2.2}$$

$$\psi(x,k) = e^{ikx} + \int\limits_{\mathbb{R}^3} G(x-y,k)v(y)\psi(y,k)dy, \tag{2.3}$$

where $x \in \mathbb{R}^3$, $k \in \mathbb{C}^3$, $k^2 = E$, $\operatorname{Im} k \neq 0$,

$$h(k,l) = (2\pi)^{-3} \int\limits_{\mathbb{R}^3} e^{-ilx} v(x)\psi(x,k)dx, \tag{2.4}$$

where

$$k, l \in \mathbb{C}^3,\ k^2 = l^2 = E,\ \operatorname{Im} k = \operatorname{Im} l \neq 0. \tag{2.5}$$

One can consider (2.3), (2.4) assuming that

$$\begin{aligned} &v \text{ is a sufficiently regular function on } \mathbb{R}^3 \\ &\qquad\qquad \text{with suffucient decay at infinity.} \end{aligned} \tag{2.6}$$

For example, in connection with Problems 1.1 and 1.2, one can consider (2.3), (2.4) assuming that

$$v \in \mathbb{L}^\infty(B_{r_1}), \quad v \equiv 0 \text{ on } \mathbb{R}^3 \setminus B_{r_1}. \tag{2.7}$$

We recall that (see **[7]**, **[8]**, **[10]**, **[17]**):

- The function G satisfies the equation

$$(\Delta + E)G(x,k) = \delta(x), \quad x \in \mathbb{R}^3, \quad k \in \mathbb{C}^3 \setminus \mathbb{R}^3, \quad E = k^2; \tag{2.8}$$

- Formula (2.3) at fixed k is considered as an equation for

$$\psi = e^{ikx}\mu(x,k), \tag{2.9}$$

  where μ is sought in $\mathbb{L}^\infty(\mathbb{R}^3)$;
- As a corollary of (2.3), (2.2), (2.8), ψ satisfies (2.1) for $E = k^2$;
- The Faddeev functions G, ψ, h are (non-analytic) continuation to the complex domain of functions of the classical scattering theory for the Schrödinger equation (in particular, h is a generalized "'scattering"' amplitude).

In addition, G, ψ, h in their zero energy restriction, that is for $E = k^2 = 0$, were considered for the first time in **[3]**. The Faddeev functions G, ψ, h were, actually, rediscovered in **[3]**.

Let

$$\begin{aligned} \Sigma_E &= \left\{k \in \mathbb{C}^3 : k^2 = k_1^2 + k_2^2 + k_3^2 = E\right\}, \\ \Theta_E &= \{k \in \Sigma_E,\ l \in \Sigma_E : \operatorname{Im} k = \operatorname{Im} l\}, \\ |k| &= (|\operatorname{Re} k|^2 + |\operatorname{Im} k|^2)^{1/2}. \end{aligned} \tag{2.10}$$

Let

$$W^{m,q}(\mathbb{R}^3) = \{w:\ \partial^J w \in \mathbb{L}^q(\mathbb{R}^3),\ |J| \le m\}, \quad m \in \mathbb{N}\cup 0,\ q \ge 1,$$
$$J \in (\mathbb{N}\cup 0)^3,\ |J| = \sum_{i=1}^{3} J_i,\ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1}\partial x_2^{J_2}\partial x_3^{J_3}}, \tag{2.11}$$
$$||w||_{m,q} = \max_{|J|\le m} ||\partial^J w||_{\mathbb{L}^q(\mathbb{R}^3)}.$$

Let the assumptions of Theorems 1.1 and 1.2 be fulfilled:

$$(1-n) \in W^{m,1}(\mathbb{R}^3) \text{ for some } m > 3,$$
$$\operatorname{Im} n(x) \ge 0, \quad x \in \mathbb{R}^3, \tag{2.12}$$
$$\operatorname{supp}(1-n) \subset B_{r_1},$$
$$\|1-n\|_{m,1} \le C_n.$$

Let

$$v = \omega^2(1-n), \quad N = \omega^2 C_n, \quad E = \omega^2. \tag{2.13}$$

Then we have that:

$$\mu(x,k) \to 1 \quad \text{as} \quad |k| \to \infty \tag{2.14}$$

and, for any $\sigma > 1$,

$$|\mu(x,k)| \le \sigma \quad \text{for} \quad |k| \ge \lambda_1(N, m, \sigma, r_1), \tag{2.15}$$

where $x \in \mathbb{R}^3$, $k \in \Sigma_E$;

$$\hat{v}(p) = \lim_{\substack{(k,l)\,\in\,\Theta_E,\ k-l=p \\ |\operatorname{Im} k| = |\operatorname{Im} l| \to \infty}} h(k,l) \quad \text{ for any } p \in \mathbb{R}^3, \tag{2.16}$$

$$|\hat{v}(p) - h(k,l)| \le \frac{c_1(m,r_1)N^2}{(E+\rho^2)^{1/2}} \quad \text{for } (k,l) \in \Theta_E, \quad p = k-l,$$
$$|\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \quad E + \rho^2 \ge \lambda_2(N, m, r_1), \tag{2.17}$$
$$p^2 \le 4(E+\rho^2),$$

where

$$\hat{v}(p) = (2\pi)^{-3} \int\limits_{\mathbb{R}^3} e^{ipx} v(x)dx, \quad p \in \mathbb{R}^3. \tag{2.18}$$

Results of the type (2.14), (2.15) go back to **[3]**. For more information concerning (2.15) see estimate (4.11) of **[12]**. Results of the type (2.16), (2.17) (with less precise right-hand side in (2.17)) go back to **[10]**. Estimate (2.17) follows, for example, from formulas (2.3), (2.4) and the estimate

$$\|\Lambda^{-s} g(k) \Lambda^{-s}\|_{\mathbb{L}^2(\mathbb{R}^d)\to\mathbb{L}^2(\mathbb{R}^d)} = O(|k|^{-1})$$
$$\text{as } \ |k| \to \infty, \quad k \in \mathbb{C}^3 \setminus \mathbb{R}^3, \tag{2.19}$$

for $s > 1/2$, where $g(k)$ denotes the integral operator with the Schwartz kernel $g(x-y,k)$ and Λ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$. Estimate (2.19) was formulated, first, in **[14]**. This estimate generilizes, in particular, some related estimate of **[25]** for $k^2 = E = 0$. Concerning proof of (2.19), see **[26]**.

In addition, we have that:

$$h_2(k,l) - h_1(k,l) = (2\pi)^{-3} \int_{\mathbb{R}^3} \psi_1(x,-l)(v_2(x) - v_1(x))\psi_2(x,k)dx \tag{2.20}$$
$$\text{for } (k,l) \in \Theta_E,\ |\mathrm{Im}\, k| = |\mathrm{Im}\, l| \neq 0, \quad \text{and } v_1,\ v_2 \text{ satisfying (2.6)},$$

and, under the assumptions of Theorems 1.1 and 1.2,

$$|\hat{v}_1(p) - \hat{v}_2(p) - h_1(k,l) + h_2(k,l)| \le \frac{c_2(m,r_1)N\|v_1 - v_2\|_{\mathbb{L}^\infty(B_{r_1})}}{(E+\rho^2)^{1/2}} \tag{2.21}$$
$$\text{for } (k,l) \in \Theta_E, \quad p = k - l, \quad |\mathrm{Im}\, k| = |\mathrm{Im}\, l| = \rho,$$
$$E + \rho^2 \ge \lambda_3(N,m,r_1), \quad p^2 \le 4(E+\rho^2),$$

where h_j, ψ_j denote h and ψ of (2.4) and (2.3) for $v_j = \omega^2(1-n_j)$, $j = 1,2$, $N = \omega^2 C_n$, $E = \omega^2$.

Formula (2.20) was given in **[18]**, **[20]**. Estimate (2.21) was given e.g. in **[13]**.

3. Proofs of Theorem 1.1 and Theorem 1.2

3.1. Preliminaries. In this section we always assume for simplicity that $r_1 = 1$.

We consider the operators $\hat{\mathrm{S}}_j$, $j = 1,2$, defined as follows

$$(\hat{\mathrm{S}}_j\phi)(x) = \int_{\partial B_r} G_j^+(x,y,\omega)\phi(y)dy, \quad x \in \partial B_r,\ j = 1,2. \tag{3.1}$$

Note that

$$\|\hat{\mathrm{S}}_1 - \hat{\mathrm{S}}_2\|_{\mathbb{L}^2(\partial B_r)} \le \|G_1^+ - G_2^+\|_{\mathbb{L}^2(\partial B_r)\times\mathbb{L}^2(\partial B_r)}. \tag{3.2}$$

To prove Theorems 1.1 and 1.2 we use, in particular, the following lemmas (see Lemma 3.2 and proof of Theorem 1.2 of **[9]**):

Lemma 3.1. *Assume $r_1 = 1 < r < r_2$. Moreover, n_1, n_2 are refractive indices with* $\mathrm{supp}\,(1-n_1), \mathrm{supp}\,(1-n_2) \subset B_1$. *Then, there exists a postive constant c_3 (depending only on ω, r, r_2) such that for all solutions $\psi_1 \in C^2(B_{r_2}) \cap \mathbb{L}^2(B_{r_2})$ to $\Delta\psi + \omega^2 n_1\psi = 0$ in B_{r_2} and all solutions $\psi_2 \in C^2(B_{r_2}) \cap \mathbb{L}^2(B_{r_2})$ to $\Delta\psi + \omega^2 n_2\psi = 0$ in B_{r_2} the following estimate holds:*

$$\left| \int_{B_1} (n_1 - n_2)\psi_1\psi_2 dx \right| \le c_3\|\hat{S}_1 - \hat{S}_2\|_{\mathbb{L}^2(\partial B_r)}\|\psi_1\|_{\mathbb{L}^2(B_{r_2})}\|\psi_2\|_{\mathbb{L}^2(B_{r_2})}. \tag{3.3}$$

Note that estimate (3.3) is derived in **[9]** using an Alessandrini type identity, where instead of the Dirichlet-to-Neumann maps the operators $\hat{\mathrm{S}}_1$, $\hat{\mathrm{S}}_2$ are used, see **[1]**, **[9]**.

LEMMA 3.2. *Let $r > r_1 = 1$, $\omega > 0$, $C_n > 0$, $\mu > 3/2$ and $0 < \theta < 1$. Let n_1, n_2 be refractive indices such that $\|(1 - n_j)\|_{H^\mu(\mathbb{R}^3)} \leq C_n$, $\mathrm{supp}(1 - n_j) \subset B_1$, $j = 1, 2$, where $H^\mu = W^{\mu,2}$. Then there exist positive constants T and η such that*

$$\|G_1^+ - G_2^+\|^2_{\mathbb{L}^2(\partial B_{2r} \times \partial B_{2r})} \leq \eta^2 \exp\left(-\left(-\ln \frac{\|f_1 - f_2\|_{\mathbb{L}^2(\mathcal{M}_\omega)}}{T\eta}\right)^\theta\right) \tag{3.4}$$

for sufficiently small $\|f_1 - f_2\|_{\mathbb{L}^2(\mathcal{M}_\omega)}$, where G_j^+, f_j are near and far field scattering data for n_j, $j = 1, 2$, at fixed frequency ω.

3.2. Proof of Theorem 1.1. Let

$$\begin{aligned} &\mathbb{L}^\infty_\mu(\mathbb{R}^3) = \{u \in \mathbb{L}^\infty(\mathbb{R}^3) : \|u\|_\mu < +\infty\}, \\ &\|u\|_\mu = \mathrm{ess} \sup_{p \in \mathbb{R}^3} (1 + |p|)^\mu |u(p)|, \quad \mu > 0. \end{aligned} \tag{3.5}$$

Note that

$$\begin{aligned} w \in W^{m,1}(\mathbb{R}^3) \Longrightarrow \hat{w} \in \mathbb{L}^\infty_\mu(\mathbb{R}^3) \cap C(\mathbb{R}^3), \\ \|\hat{w}\|_\mu \leq c_4(m)\|w\|_{m,1} \quad \text{for} \quad \mu = m, \end{aligned} \tag{3.6}$$

where $W^{m,1}$, $\mathbb{L}^\infty_\mu$ are the spaces of (2.11), (3.5),

$$\hat{w}(p) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ipx} w(x) dx, \quad p \in \mathbb{R}^3. \tag{3.7}$$

Let

$$N = \omega^2 C_n, \quad E = \omega^2, \quad v_j = \omega^2(1 - n_j),\ j = 1, 2. \tag{3.8}$$

Using the inverse Fourier transform formula

$$w(x) = \int_{\mathbb{R}^3} e^{-ipx} \hat{w}(p) dp, \quad x \in \mathbb{R}^3, \tag{3.9}$$

we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)} \leq \sup_{x \in \bar{B}_1} \left| \int_{\mathbb{R}^3} e^{-ipx} \left(\hat{v}_2(p) - \hat{v}_1(p)\right) dp \right| \leq \\ \leq I_1(\kappa) + I_2(\kappa) \quad \text{for any} \quad \kappa > 0, \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} I_1(\kappa) &= \int\limits_{|p|\le\kappa} |\hat v_2(p) - \hat v_1(p)|dp, \\ I_2(\kappa) &= \int\limits_{|p|\ge\kappa} |\hat v_2(p) - \hat v_1(p)|dp. \end{aligned} \tag{3.11}$$

Using (3.6), we obtain that

$$|\hat v_2(p) - \hat v_1(p)| \le 2c_4(m)N(1+|p|)^{-m}, \quad p \in \mathbb{R}^3. \tag{3.12}$$

Using (3.11), (3.12), we find that, for any $\kappa > 0$,

$$I_2(\kappa) \le 8\pi c_4(m)N \int\limits_{\kappa}^{+\infty} \frac{dt}{t^{m-2}} \le \frac{8\pi c_4(m)N}{m-3}\frac{1}{\kappa^{m-3}}. \tag{3.13}$$

Due to (2.21), we have that

$$\begin{aligned} |\hat v_2(p) - \hat v_1(p)| \le |h_2(k,l) - h_1(k,l)| + \frac{c_2(m)N\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)}}{(E+\rho^2)^{1/2}}, \\ \text{for } (k,l)\in\Theta_E, \quad p = k-l, \quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \\ E+\rho^2 \ge \lambda_3(N,m), \quad p^2 \le 4(E+\rho^2). \end{aligned} \tag{3.14}$$

Let

$$\begin{aligned} r_2 \text{ be some fixed constant such that } r_2 > r, \\ \delta = ||G_1^+ - G_2^+||_{\mathbb{L}^2(\partial B_r\times\partial B_r)}, \\ c_5 = (2\pi)^{-3}\int\limits_{B_{r_2}} dx. \end{aligned} \tag{3.15}$$

Combining (2.20), (3.2), (3.3) and (3.8), we get that

$$\begin{aligned} |h_2(k,l) - h_1(k,l)| \le \\ \le c_3c_5\omega^2\|\psi_1(\cdot,-l)\|_{\mathbb{L}^\infty(B_{r_2})}\,\delta\,\|\psi_2(\cdot,k)\|_{\mathbb{L}^\infty(B_{r_2})}, \\ (k,l)\in\Theta_E,\ |\operatorname{Im} k| = |\operatorname{Im} l| \ne 0. \end{aligned} \tag{3.16}$$

Using (2.15), we find that

$$\begin{aligned} \|\psi_j(\cdot,k)\|_{\mathbb{L}^\infty(B_{r_2})} \le \sigma \exp\Big(|\operatorname{Im} k| r_2\Big), \quad j = 1,2, \\ k\in\Sigma_E,\ |k| \ge \lambda_1(N,m,\sigma). \end{aligned} \tag{3.17}$$

Here and bellow in this section the constant σ is the same that in (2.15).

Combining (3.16) and (3.17), we obtain that

$$\begin{aligned} &|h_2(k,l) - h_1(k,l)| \le c_3 c_5 \omega^2 \sigma^2 e^{2\rho r_2} \delta, \\ &\text{for } (k,l) \in \Theta_E, \quad \rho = |\operatorname{Im} k| = |\operatorname{Im} l|, \\ &\qquad E + \rho^2 \ge \lambda_1^2(N,m,\sigma). \end{aligned} \tag{3.18}$$

Using (3.14), (3.18), we get that

$$\begin{aligned} &|\hat v_2(p) - \hat v_1(p)| \le c_3 c_5 \omega^2 \sigma^2 e^{2\rho r_2} \delta + \frac{c_2(m) N \|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)}}{(E+\rho^2)^{1/2}}, \\ &\qquad p \in \mathbb{R}^3,\ p^2 \le 4(E+\rho^2),\ E+\rho^2 \ge \max\{\lambda_1^2, \lambda_3\}. \end{aligned} \tag{3.19}$$

Let

$$\varepsilon = \left(\frac{3}{8\pi c_2(m) N}\right)^{1/3} \tag{3.20}$$

and $\lambda_4(N,m,\sigma) > 0$ be such that

$$E + \rho^2 \ge \lambda_4(N,m,\sigma) \Longrightarrow \begin{cases} E+\rho^2 \ge \lambda_1^2(N,m,\sigma), \\ E+\rho^2 \ge \lambda_3(N,m), \\ \left(\varepsilon (E+\rho^2)^{\frac{1}{6}}\right)^2 \le 4(E+\rho^2). \end{cases} \tag{3.21}$$

Using (3.11), (3.19), we get that

$$\begin{aligned} &I_1(\kappa) \le \frac{4}{3}\pi\kappa^3 \Big(c_3 c_5 \omega^2 \sigma^2 e^{2\rho r_2}\delta + \frac{c_2(m) N \|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)}}{(E+\rho^2)^{1/2}}\Big), \\ &\qquad \kappa > 0,\ \kappa^2 \le 4(E+\rho^2), \\ &\qquad E + \rho^2 \ge \lambda_4(N,m,\sigma). \end{aligned} \tag{3.22}$$

Combining (3.10), (3.13), (3.22) for $\kappa = \varepsilon(E+\rho^2)^{\frac{1}{6}}$ and (3.21), we get that

$$\begin{aligned} &\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)} \le c_6(N,m,\omega,\sigma)\sqrt{E+\rho^2}\, e^{2\rho r_2}\delta + \\ &\quad + c_7(N,m)(E+\rho^2)^{-\frac{m-3}{6}} + \frac{1}{2}\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)}, \\ &\qquad E+\rho^2 \ge \lambda_4(N,m,\sigma). \end{aligned} \tag{3.23}$$

Let $\tau \in (0,1)$ and

$$\beta = \frac{1-\tau}{2r_2}, \quad \rho = \beta \ln\left(3 + \delta^{-1}\right), \tag{3.24}$$

where δ is so small that $E+\rho^2 \geq \lambda_4(N,m,\sigma)$. Then due to (3.23), we have that

$$
\begin{aligned}
&\frac{1}{2}\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)} \leq \\
&\leq c_6(N,m,\omega,\sigma)\left(E + \left(\beta \ln\left(3+\delta^{-1}\right)\right)^2\right)^{1/2}\left(3+\delta^{-1}\right)^{2\beta r_2}\delta + \\
&\qquad + c_7(N,m)\left(E + \left(\beta \ln\left(3+\delta^{-1}\right)\right)^2\right)^{-\frac{m-3}{6}} = \\
&= c_6(N,m,\omega,\sigma)\left(E + \left(\beta \ln\left(3+\delta^{-1}\right)\right)^2\right)^{1/2}(1+3\delta)^{1-\tau}\delta^{\tau} + \\
&\qquad\qquad + c_7(N,m)\left(E + \left(\beta \ln\left(3+\delta^{-1}\right)\right)^2\right)^{-\frac{m-3}{6}},
\end{aligned}
\tag{3.25}
$$

where τ, β and δ are the same as in (3.24).

Using (3.25), we obtain that

$$
\|v_1 - v_2\|_{\mathbb{L}^\infty(B_1)} \leq c_8(N,m,\omega,\sigma,\tau)\left(\ln\left(3+\delta^{-1}\right)\right)^{-\frac{m-3}{3}}
\tag{3.26}
$$

for $\delta = ||G_1^+ - G_2^+||_{\mathbb{L}^2(\partial B_r \times \partial B_r)} \leq \delta_1(N,m,\omega,\sigma,\tau)$, where δ_1 is a sufficiently small positive constant. Estimate (3.26) in the general case (with modified c_8) follows from (3.26) for $\delta \leq \delta_1(N,m,\omega,\sigma,\tau)$ and the property that

$$
\|v_j\|_{\mathbb{L}^\infty(B_1)} \leq c_9(m)N, \quad j = 1,2.
\tag{3.27}
$$

Taking into account (3.8), we obtain (1.5).

3.3. Proof of Theorem 1.2. According to the Sobolev embedding theorem, we have that

$$
W^{m,1}(\mathbb{R}^3) \subset H^{m-3/2}(\mathbb{R}^3),
\tag{3.28}
$$

where $H^\mu = W^{\mu,2}$.

Combining (1.2), (1.5), (3.4) with θ satisfying $\theta\frac{m-3}{3} = \frac{m-3}{3} - \epsilon$, and (3.28), we obtain (1.6) for sufficiently small $\|f_1 - f_2\|_{\mathbb{L}^2(\mathcal{M}_\omega)}$ (analogously with the proof of Theorem 1.2 of [**9**]). Using also (3.27) and (3.8), we get estimate (1.6) in the general case.

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Paper G

PAPER G

Exponential instability in the inverse scattering problem on the energy interval

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Abstract. We consider the inverse scattering problem on the energy interval in three dimensions. We are focused on stability and instability questions for this problem. In particular, we prove an exponential instability estimate which shows optimality of the logarithmic stability result of [Stefanov, 1990] (up to the value of the exponent).

1. Introdution

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^3, \tag{1.1}$$

where

$$\begin{aligned} &v \text{ is real-valued}, \quad v \in \mathbb{L}^\infty(\mathbb{R}^3),\\ &v(x) = O(|x|^{-3-\epsilon}), \quad |x| \to \infty, \quad \text{for some } \epsilon > 0. \end{aligned} \tag{1.2}$$

Under conditions (1.2), for any $k \in \mathbb{R}^3 \setminus 0$ equation (1.1) with $E = k^2$ has a unique continuous solution $\psi^+(x,k)$ with asymptotics of the form

$$\begin{aligned} &\psi^+(x,k) = e^{ikx} - 2\pi^2 \frac{e^{i|k||x|}}{|x|} f\left(\frac{k}{|k|}, \frac{x}{|x|}, |k|\right) + o\left(\frac{1}{|x|}\right)\\ &\text{as } |x| \to \infty \left(\text{uniformly in } \frac{x}{|x|}\right), \end{aligned} \tag{1.3}$$

where $f(k/|k|, \omega, |k|)$ with fixed k is a continuous function of $\omega \in S^2$.

The function $f(\theta, \omega, s)$ arising in (1.3) is refered to as the scattering amplitude for the potential v for equation (1.1). (For more information on direct scattering for equation (1.1), under condition (1.2), see, for example, [**6**] and [**11**].)

It is well known that for equation (1.1), under conditions (1.2), the scattering amplitude f in its high-energy limit uniquely determines $\hat{v}$ on $\mathbb{R}^3$, where

$$\hat{v}(p) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ipx} v(x)dx, \quad p \in \mathbb{R}^3, \tag{1.4}$$

via the Born formula. As a mathematical theorem this result was given for the first time in [**5**] (see, for example, Section 2.1 of [**11**] and Theorem 1.1 of [**14**] for details).

We consider the following inverse problem for equation (1.1).

Problem 1.1. Given f on the energy interval I, find v.

In [**7**] it was shown that for equation (1.1), under conditions (1.2), for any $E > 0$ and $\delta > 0$ the scattering amplitude $f(\theta, \omega, s)$ on

$$\{(\theta, \omega, s) \in S^2 \times S^2 \times R_+,\ E \le s^2 \le E + \delta\}$$

uniquely determines $\hat{v}(p)$ on

$$\{p \in \mathbb{R}^3 \mid |p| \le 2\sqrt{E}\}.$$

This determination is based on solving linear integral equations and on an analytic continuation. This result of [**7**] was improved in [**14**]. On the other hand, if v satisfies (1.2) and, in addition, is compactly supported or exponentially decaying at infinity, then $\hat{v}(p)$ on

$$\{p \in \mathbb{R}^3 \mid |p| \le 2\sqrt{E}\}$$

uniquely determines $\hat{v}(p)$ on $\{p \in \mathbb{R}^3 \mid |p| > 2\sqrt{E}\}$ by an analytic continuation and, therefore, uniquely determines v on $\mathbb{R}^3$.

In the case of fixed energy and potential v, satisfying (1.2) and, in addition, being compactly supported or exponentially decaying at infinity, global uniqueness theorems and precise reconstructions were given for the first time in [**12**], [**13**].

An approximate but numerically efficient method for finding potential v from the scattering amplitude f in the case of fixed energy was devoloped in [**15**]. Related numerical implementation was given in [**2**].

Global stability estimates for Problem 1.1 were given by Stefanov in [**17**] (at fixed energy for compactly supported potentials), see Theorem 2.1 in Section 2 of the present paper. In [**17**], using a special norm for the scattering amplitude f, it was shown that the stability estimates for Problem 1.1 follow from the Alessandrini stability estimates of [**1**] for the Gel'fand-Calderon inverse problem of finding potential v in bounded domain from the Direchlet-to-Neumann map. The Alessandrini stability estimates were recently improved by Novikov in [**16**].

In the case of fixed energy, the Mandache results of [**10**] show that logarithmic stability estimates of Alessandrini of [**1**] and especially of Novikov of [**16**] are optimal (up to the value of the exponent). In [**8**] studies of Mandache were extended to the case of Direchlet-to-Neumann map given on the energy intervals. Note also that Mandache-type instability estimates for the elliptic inverse problem concerning the determination of inclusions in a conductor by different kinds of boundary measurements and the inverse obstacle acoustic scattering problems were given in [**3**], where some general scheme for investigating questions of this type of instability has been also proposed. Although the main result of this work can be represented within the general scheme of [**3**], it does not lead to a significant simplification of its complete proof.

In the present work we apply to Problem 1.1 the approach of [**10**],[**8**] and show that the Stefanov logarithmic stability estimates of [**17**] are optimal (up to the value of the exponent). The Stefanov stability estimates and our instability result for Problem 1.1 are presented and discussed in Section 2. In Section 3 we prove some basic analytic properties of the scattering amplitude. Finally, in Section 5 we prove the main result, using a ball packing and covering by ball arguments.

2. Stability and instability estimates

In what follows we suppose

$$\operatorname{supp} v(x) \subset D = B(0,1), \tag{2.1}$$

where $B(x,r)$ is the open ball of radius r centered at x. We consider the orthonormal basis of the spherical harmonics in $\mathbb{L}^2(S^2) = \mathbb{L}^2(\partial D)$:

$$\{Y_j^p : j \geq 0;\ 1 \leq p \leq 2j+1\}. \tag{2.2}$$

The notation $(a_{j_1p_1j_2p_2})$ stands for a multiple sequence. We will drop the subscript

$$\begin{aligned} &0 \leq j_1,\ 1 \leq p_1 \leq 2j_1+1,\\ &0 \leq j_2,\ 1 \leq p_2 \leq 2j_2+1. \end{aligned} \tag{2.3}$$

We expand function $f(\theta,\omega,s)$ in the basis $\{Y_{j_1}^{p_1} \times Y_{j_2}^{p_2}\}$:

$$f(\theta,\omega,s) = \sum_{j_1,p_1,j_2,p_2} a_{j_1p_1j_2p_2}(s) Y_{j_1}^{p_1}(\theta) Y_{j_2}^{p_2}(\omega). \tag{2.4}$$

As in [**17**] we use the norm

$$||f(\cdot,\cdot,s)||_{\sigma_1,\sigma_2} = \left\{ \sum_{j_1,p_1,j_2,p_2} \left(\frac{2j_1+1}{es}\right)^{2j_1+2\sigma_1} \left(\frac{2j_2+1}{es}\right)^{2j_2+2\sigma_2} |a_{j_1p_1j_2p_2}(s)|^2 \right\}^{1/2}. \tag{2.5}$$

If a function f is the scattering amplitude for some potential $v \in \mathbb{L}^\infty(D)$ supported in $B(0,\rho)$, where $0 < \rho < 1$, then

$$|a_{j_1p_1j_2p_2}(s)| \leq C(s, ||v||_{\mathbb{L}^\infty(D)}) \left(\frac{es\rho}{2j_1+1}\right)^{j_1+3/2} \left(\frac{es\rho}{2j_2+1}\right)^{j_2+3/2} \tag{2.6}$$

and, therefore, $||f(\cdot,\cdot,s)||_{\sigma_1,\sigma_2} < \infty$, see estimates of Proposition 2.2 of [**17**].

THEOREM 2.1 (see [**17**]). *Let v_1, v_2 be real-valued such that $v_i \in \mathbb{L}^\infty(D) \cap H^q(\mathbb{R}^3)$, $\operatorname{supp} v_i \subset B(0,\rho)$, $||v_i||_{\mathbb{L}^\infty(D)} \leq N$ for $i = 1,2$ and some $N > 0$, $q > 3/2$ and $0 < \rho < 1$. Let f_1 and f_2 denote the scattering amplitudes for v_1 and v_2, respectively, in the framework of equation (1.1) with $E = s^2$, $s > 0$, then*

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \leq c(N,\rho)\phi_\delta(||f_1(\cdot,\cdot,s) - f_2(\cdot,\cdot,s)||_{3/2,-1/2}), \tag{2.7}$$

where $\phi_\delta(t) = (-\ln t)^{-\delta}$ for some fixed δ, where, in particular, $0 < \delta < 1$, and for sufficiently small $t > 0$.

The main result of the present work is the following theorem:

THEOREM 2.2. *For the interval $I = [s_1, s_2]$, such that $s_1 > 0$, and for any $m > 0$, $\delta > 2m$ and any real σ_1, σ_2 there are constants $\beta > 0$ and $N > 0$, such that for any $v_0 \in C^m(D)$ with $||v_0||_{\mathbb{L}^\infty(D)} \le N$, $\operatorname{supp} v_0 \subset B(0, 1/2)$ and any $\epsilon \in (0, N)$, there are real-valued potentials $v_1, v_2 \in C^m(D)$, also supported in $B(0, 1/2)$, such that*

$$\begin{aligned} &\sup_{s\in I} (||f_1(\cdot,\cdot,s) - f_2(\cdot,\cdot,s)||_{\sigma_1,\sigma_2}) \le \exp\left(-\epsilon^{-\frac{1}{\delta}}\right), \\ &||v_1 - v_2||_{\mathbb{L}^\infty(D)} \ge \epsilon, \\ &||v_i - v_0||_{\mathbb{L}^\infty(D)} \le \epsilon, \qquad i = 1, 2, \\ &||v_i - v_0||_{C^m(D)} \le \beta, \qquad i = 1, 2, \end{aligned} \tag{2.8}$$

where f_1, f_2 are the scattering amplitudes for v_1, v_2, respectively, for equation (1.1).

REMARK 2.1. In the case of fixed energy $s_1 = s_2$ we can replace the condition $\delta > 2m$ in Theorem 2.2 by $\delta > 5m/3$.

REMARK 2.2. We can allow β to be arbitrarily small in Theorem 2.2 if we require $\epsilon \le \epsilon_0$ and replace the right-hand side in the first inequality in (2.8) by $\exp(-c\epsilon^{-\frac{1}{\delta}})$, with $\epsilon_0 > 0$ and $c > 0$ depending on β.

REMARK 2.3. Note that Theorem 2.2 and Remark 2.1 imply, in particular, that for any real σ_1 and σ_2 the estimate

$$||v_1 - v_2||_{\mathbb{L}^\infty(D)} \le \tilde{c}(N, \rho, m, I) \sup_{s\in I} \phi_\delta(||f_1(\cdot,\cdot,s) - f_2(\cdot,\cdot,s)||_{\sigma_1,\sigma_2}) \tag{2.9}$$

can not hold with $\delta > 2m$ in the case of the scattering amplitude given on the energy interval and with $\delta > 5m/3$ in the case of fixed energy. Thus Theorem 2.2 and Remark 2.1 show optimality of the Stefanov logarithmic stability result (up to the value of the exponent).

REMARK 2.4. A disadvantage of estimate (2.7) is that

$$\delta < 1 \text{ even if } m \text{ is very great.} \tag{2.10}$$

Apparently, proceeding from results of **[16]**, it is not difficult to improve estimate (2.7) for

$$\delta = m + o(m) \text{ as } m \to \infty. \tag{2.11}$$

3. Some basic analytic properties of the scattering amplitude

Consider the solution $\psi^+(x, k)$ of equation 1.1, see formula (1.3). We have that

$$\psi^+(x, k) = e^{ikx}\mu^+(x, \theta, s), \tag{3.1}$$

where $\theta \in S^2$, $k = s\theta$ and $\mu^+(x, \theta, s)$ solves the equation

$$\mu^+(x, \theta, s) = 1 - \int_{\mathbb{R}^3} G^+(x, y, s)e^{-is\theta(x-y)}v(y)\mu^+(y, \theta, s)dy, \tag{3.2}$$

where

$$G^+(x,y,s) = \frac{e^{is|x-y|}}{4\pi|x-y|}. \tag{3.3}$$

We suppose that condition (2.1) holds and, in addition, for some $h > 0$ we have that

$$|\mathrm{Im}\, s| \leq h, \tag{3.4}$$

$$c_1(h,D)||v||_{\mathbb{L}^\infty(D)} \leq 1/2, \tag{3.5}$$

where $D = B(0,1)$,

$$c_1(h,D) = \sup_{x\in D} \int_D \frac{e^{2h|x-y|}}{4\pi|x-y|} dy. \tag{3.6}$$

Then, in particular,

$$\left| e^{-is\theta(x-y)} e^{is|x-y|} \right| \leq e^{2h|x-y|}. \tag{3.7}$$

Solving (3.2) by the method of succesive approximations in $\mathbb{L}^\infty(D)$, we obtain that

$$|\mu^+(x,\theta,s)| \leq \frac{1}{1 - c_1||v||_{\mathbb{L}^\infty(D)}}, \quad \theta \in S^2, \quad x \in D. \tag{3.8}$$

LEMMA 3.1. *Let f be the scattering amplitude for potential $v \in \mathbb{L}^\infty(D)$ such that conditions (2.1) and (3.5) hold for some $h > 0$ and*

$$f(\theta,\omega,s) = \sum_{j_1,p_1,j_2,p_2} a_{j_1p_1j_2p_2}(s) Y_{j_1}^{p_1}(\theta) Y_{j_2}^{p_2}(\omega) \tag{3.9}$$

be its expansion in the basis of the spherical harmonics $\{Y_{j_1}^{p_1} \times Y_{j_2}^{p_2}\}$. Then $a_{j_1p_1j_2p_2}(s)$ is a holomorphic function in $W_h = \{s \mid |Im\, s| \leq h\}$ and

$$|a_{j_1p_1j_2p_2}(s)| \leq c_2(h,D) s \in W_h. \tag{3.10}$$

PROOF OF LEMMA 3.1. We start with the well-known formula

$$f(\theta,\omega,s) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{is(\theta-\omega)x} v(x) \mu^+(x,\theta,s) dx. \tag{3.11}$$

Note that, since $\theta, \omega \in S^2$,

$$|e^{is(\theta-\omega)x}| \leq e^{2|Im\, s||x|}. \tag{3.12}$$

Combining it with (2.1), (3.5), (3.8) and (3.11) we obtain that

$$|f(\theta,\omega,s)| \leq \tilde{c}_2(h,D) \text{ for } s \in W_h. \tag{3.13}$$

Using also that

$$a_{j_1p_1j_2p_2}(s) = \int_{S^2\times S^2} f(\theta,\omega,s) Y_{j_1}^{p_1}(\theta) Y_{j_2}^{p_2}(\omega) d\theta d\omega \tag{3.14}$$

we obtain the result of Lemma 3.1. ∎

4. A fat metric space and a thin metric space

DEFINITION 4.1. Let $(X, dist)$ be a metric space and $\epsilon > 0$. We say that a set $Y \subset X$ is an ϵ-net for $X_1 \subset X$ if for any $x \in X_1$ there is $y \in Y$ such that $dist(x, y) \leq \epsilon$. We call ϵ-entropy of the set X_1 the number $\mathcal{H}_\epsilon(X_1) := \log_2 \min\{|Y| : Y$ is an ϵ-net fot $X_1\}$.

A set $Z \subset X$ is called ϵ-discrete if for any distinct $z_1, z_2 \in Z$, we have $dist(z_1, z_2) \geq \epsilon$. We call ϵ-capacity of the set X_1 the number $\mathcal{C}_\epsilon := \log_2 \max\{|Z| : Z \subset X_1$ and Z is ϵ-discrete$\}$.

The use of ϵ-entropy and ϵ-capacity to derive properties of mappings between metric spaces goes back to Vitushkin and Kolmogorov (see **[9]** and references therein). One notable application was Hilbert's 13th problem (about representing a function of several variables as a composition of functions of a smaller number of variables). In essence, Lemma 4.1 and Lemma 4.2 are parts of Theorem XIV and Theorem XVII of **[9]**, respectively.

LEMMA 4.1. *Let $d \geq 2$ and $m > 0$. For $\epsilon, \beta > 0$, consider the real metric space*

$$X_{m\epsilon\beta} = \{v \in C^m(\mathbb{R}^d) \mid \operatorname{supp} v \subset B(0, 1/2),\ ||v||_{\mathbb{L}^\infty(\mathbb{R}^d)} \leq \epsilon,\ ||v||_{C^m(\mathbb{R}^d)} \leq \beta\}$$

with the metric induced by $\mathbb{L}^\infty$. Then there is $\mu > 0$ such that for any $\beta > 0$ and $\epsilon \in (0, \mu\beta)$, there is an ϵ-discrete set $Z \subset X_{m\epsilon\beta}$ with at least $\exp\left(2^{-d-1}(\mu\beta/\epsilon)^{d/m}\right)$ *elements.*

LEMMA 4.2. *Let*

$$W_{I,\gamma} = \{\frac{a+b}{2} + \frac{a-b}{2}\cos z \mid |Im\, z| \leq \gamma\}. \tag{4.1}$$

denote the ellipse $\in \mathbb{C}$, where I is some interval $[a, b]$ in $\mathbb{R}$ and $\gamma > 0$. Then there is a constant $\nu = \nu(C, \gamma) > 0$ such that for any $\delta \in (0, e^{-1})$ there is a δ-net for the space of functions on I with $\mathbb{L}^\infty$-norm, having holomorphic continuation to $W_{I,\gamma}$ with module bounded above on $W_{I,\gamma}$ by the constant C, with at most $\exp(\nu(\ln \delta^{-1})^2)$ *elements.*

REMARK 4.1. In the case of $a = b$, taking

$$Y = \frac{\delta}{2}\mathbb{Z} \bigcap [-C, C] + i \cdot \frac{\delta}{2}\mathbb{Z} \bigcap [-C, C], \tag{4.2}$$

we get δ-net with at most $\exp(\nu \ln \delta^{-1})$ elements.

Lemma 4.1 and Lemma 4.2 were also formulated and proved in **[10]** and **[8]**, respectively.

For the interval $I = [s_1, s_2]$ such that $s_1 > 0$ and real σ_1, σ_2 we introduce the Banach space

$$X_{I,\sigma_1,\sigma_2} = \left\{ \left(a_{j_1p_1j_2p_2}(s)\right) \mid \left\| \left(a_{j_1p_1j_2p_2}(s)\right) \right\|_{X_{I,\sigma_1,\sigma_2}} < \infty \right\}, \tag{4.3}$$

where

(4.4)
$$\left\|\left(a_{j_1p_1j_2p_2}(s)\right)\right\|_{X_{I,\sigma_1,\sigma_2}} = \sup_{\substack{j_1,p_1,j_2,p_2\\ s\in I}} \left(\left(\frac{2j_1+1}{es}\right)^{j_1+\sigma_1}\left(\frac{2j_2+1}{es}\right)^{j_2+\sigma_2}|a_{j_1p_1j_2p_2}(s)|\right).$$

We consider the scattering amplitude f for some potential $v\in\mathbb{L}^\infty(D)$ supported in $B(0,\rho)$, where $0<\rho<1$. We identify in the sequel the scattering amplitude $f(s,\theta,\omega)$ with its matrix $\left(a_{j_1p_1j_2p_2}(s)\right)$ in the basis of the spherical harmonics $\{Y_{j_1}^{p_1}\times Y_{j_2}^{p_2}\}$. We have that

$$\sup_{s\in I}||f(\cdot,\cdot,s)||_{\sigma_1,\sigma_2}\le c_3\left\|\left(a_{j_1p_1j_2p_2}(s)\right)\right\|_{X_{I,\tilde\sigma_1,\tilde\sigma_2}}, \tag{4.5}$$

where $\tilde\sigma_1-\sigma_2=\tilde\sigma_2-\sigma_2=3$ and $c_3=c_3(I)>1$. We obtain (4.5) from definitions (2.5), (4.4) and by taking $c_3>1$ in a such a way that

$$\sum_{j_1,p_1,j_2,p_2}\left(\frac{2j_1+1}{es}\right)^{-3}\left(\frac{2j_2+1}{es}\right)^{-3}<c_3. \tag{4.6}$$

For $h>0$ we denote by $\mathcal{A}_h$ the set of the matrices, corresponding to the scattering amplitudes for the potentials $v\in\mathbb{L}^\infty(D)$ supported in $B(0,1/2)$ such that condition (3.5) holds.

Lemma 4.3. *For any $h>0$ and any real σ_1,σ_2, the set $\mathcal{A}_h$ belongs to X_{I,σ_1,σ_2}. In addition, there is a constant $\eta=\eta(I,h,\sigma_1,\sigma_2)>0$ such that for any $\delta\in(0,e^{-1})$ there is a δ-net Y for $\mathcal{A}_h$ in X_{I,σ_1,σ_2} with at most $\exp\left(\eta\,(\ln\delta^{-1})^6\,(1+\ln\ln\delta^{-1})^2\right)$ elements.*

Proof of Lemma 4.3. We can suppose that $\sigma_1,\sigma_2\ge 0$ as the assertion is stronger in this case. If a function f is the scattering amplitude for some potential $v\in\mathbb{L}^\infty(D)$ supported in $B(0,1/2)$, we have from (2.6) that

$$\left(\frac{2j_1+1}{es}\right)^{j_1+\sigma_1}\left(\frac{2j_2+1}{es}\right)^{j_2+\sigma_2}|a_{j_1p_1j_2p_2}(s)|\le c_4\frac{(2j_1+1)^{\sigma_1}(2j_2+1)^{\sigma_2}}{2^{j_1+j_2}}, \tag{4.7}$$

where $c_4=c_4(I,h)>0$. Hence, for any positive σ_1 and σ_2,

$$\left\|\left(a_{j_1p_1j_2p_2}(s)\right)\right\|_{X_{I,\sigma_1,\sigma_2}}\le\sup_{j_1,j_2}\left(c_4\frac{(2j_1+1)^{\sigma_1}(2j_2+1)^{\sigma_2}}{2^{j_1+j_2}}\right)<\infty \tag{4.8}$$

and so the first assertion of the Lemma 4.3 is proved.

Let $l_{\delta,\sigma_1,\sigma_2}$ be the smallest natural number such that $c_4(2l+1)^{\sigma_1+\sigma_2}2^{-l}<\delta$ for any $l\ge l_{\delta,\sigma_1,\sigma_2}$. Taking natural logarithm we have that

$$-\ln c_4-(\sigma_1+\sigma_2)\ln(2l+1)+l\ln 2>\ln\delta^{-1}\text{ for }l\ge l_{\delta,\sigma_1,\sigma_2}. \tag{4.9}$$

Using $\ln\delta^{-1}>1$, we get that

$$l_{\delta,\sigma_1,\sigma_2}\le C'\ln\delta^{-1}, \tag{4.10}$$

where the constant C' depends only on h, σ_1, σ_2 and $I = [s_1, s_2]$. We take $W_I = W_{I,\gamma}$ of (4.1), where the constant $\gamma > 0$ is such that $W_I \subset \{s \mid |\mathrm{Im}\, s| \le h\}$. If $\max(j_1, j_2) \le l_{\delta,\sigma_1,\sigma_2}$, then we denote by $Y_{j_1p_1j_2p_2}$ some $\delta_{j_1p_1j_2p_2}$-net from Lemma 4.2 with the constant $C = c_2$, where the constant c_2 is from Lemma 3.1 and

$$\delta_{j_1p_1j_2p_2} = \left(\frac{es_1}{2j_1+1}\right)^{j_1+\sigma_1}\left(\frac{es_1}{2j_2+1}\right)^{j_2+\sigma_2}\delta. \tag{4.11}$$

Otherwise we take $Y_{j_1p_1j_2p_2} = \{0\}$. We set

$$Y = \left\{\left(a_{j_1p_1j_2p_2}(s)\right) \mid a_{j_1p_1j_2p_2}(s) \in Y_{j_1p_1j_2p_2}\right\}. \tag{4.12}$$

For any $\left(a_{j_1p_1j_2p_2}(s)\right) \in \mathcal{A}_h$ there is an element $\left(b_{j_1p_1j_2p_2}(s)\right) \in Y$ such that

$$\begin{aligned} \left(\frac{2j_1+1}{es}\right)^{j_1+\sigma_1}\left(\frac{2j_2+1}{es}\right)^{j_2+\sigma_2} &|a_{j_1p_1j_2p_2}(s) - b_{j_1p_1j_2p_2}(s)| \le \\ \le \left(\frac{2j_1+1}{es}\right)^{j_1+\sigma_1}\left(\frac{2j_2+1}{es}\right)^{j_2+\sigma_2} &\delta_{j_1p_1j_2p_2} \le \delta \end{aligned} \tag{4.13}$$

in the case of $\max(j_1, j_2) \le l_{\delta,\sigma_1,\sigma_2}$ and

$$\begin{aligned} &\left(\frac{2j_1+1}{es}\right)^{j_1+\sigma_1}\left(\frac{2j_2+1}{es}\right)^{j_2+\sigma_2} |a_{j_1p_1j_2p_2}(s) - b_{j_1p_1j_2p_2}(s)| \le \\ &\le c_4\frac{(2j_1+1)^{\sigma_1}(2j_2+1)^{\sigma_2}}{2^{j_1+j_2}} \le c_4\frac{(2\max(j_1,j_2)+1)^{\sigma_1+\sigma_2}}{2^{\max(j_1,j_2)}} < \delta, \end{aligned} \tag{4.14}$$

otherwise.

It remains to count the elements of Y. We recall that $|Y_{j_1p_1j_2p_2}| = 1$ in the case of $\max(j_1, j_2) > l_{\delta,\sigma_1,\sigma_2}$. Using again the fact that $\ln\delta^{-1} \ge 1$ and (4.10) we get in the case of $\max(j_1, j_2) \le l_{\delta,\sigma_1,\sigma_2}$:

$$|Y_{j_1p_1j_2p_2}| \le \exp(\nu(\ln\delta^{-1}_{j_1p_1j_2p_2})^2) \le \exp\left(\nu'\left(\ln\delta^{-1}\right)^2\left(1+\ln\ln\delta^{-1}\right)^2\right). \tag{4.15}$$

We have that $n_{\delta,\sigma_1,\sigma_2} \le l^2_{\delta,\sigma_1,\sigma_2}(2l_{\delta,\sigma_1,\sigma_2}+1)^2 \le (2l_{\delta,\sigma_1,\sigma_2}+1)^4$, where $n_{\delta,\sigma_1,\sigma_2}$ is the number of four-tuples (j_1, p_1, j_2, p_2) with $\max(j_1, j_2) \le l_{\delta,\sigma_1,\sigma_2}$. Taking η to be big enough we get that

$$\begin{aligned} |Y| &\le \left(\exp\left(\nu'\left(\ln\delta^{-1}\right)^2\left(1+\ln\ln\delta^{-1}\right)^2\right)\right)^{n_{\delta,\sigma_1,\sigma_2}} \\ &\le \exp\left(\nu'\left(\ln\delta^{-1}\right)^2\left(1+\ln\ln\delta^{-1}\right)^2(1+2C'\ln\delta^{-1})^4\right) \\ &\le \exp\left(\eta\left(\ln\delta^{-1}\right)^6\left(1+\ln\ln\delta^{-1}\right)^2\right). \end{aligned} \tag{4.16}$$

■

REMARK 4.2. In the case of $s_1 = s_2$, taking into account Remark 4.1 and using it in (4.15) and (4.16), we get δ-net Y with at most $\exp\left(\eta\,(\ln\delta^{-1})^5\,(1+\ln\ln\delta^{-1})\right)$ elements.

5. Proof of Theorem 2.2

We take N such that condition (3.5) holds for any $||v||_{\mathbb{L}^\infty(D)} \le 2N$ for some $h > 0$. By Lemma 4.1, the set $v_0 + X_{m\epsilon\beta}$ has an ϵ-discrete subset $v_0 + Z$. Since $\epsilon \in (0, N)$ we have that the set Y constructed in Lemma 4.3 is also δ-net for the set of the matrices, corresponding to the scattering amplitudes for the potentials $v \in v_0 + X_{m\epsilon\beta}$. We take δ such that $2c_3\delta = \exp\left(-\epsilon^{-\frac{1}{\alpha}}\right)$, see (4.5). Note that inequalities of (2.8) follow from

$$|v_0 + Z| > |Y|, \tag{5.1}$$

where the set Y is constructed in Lemma 4.3 with $\tilde{\sigma}_1 = \sigma_1 + 3$ and $\tilde{\sigma}_2 = \sigma_2 + 3$. In fact, if $|v_0 + Z| > |Y|$, then there are two potentials $v_1, v_2 \in v_0 + Z$ with the matrices $\left(a_{j_1p_1j_2p_2}(s)\right)$ and $\left(b_{j_1p_1j_2p_2}(s)\right)$, corresponding to the scattering amplitudes for them, being in the same X_{I,σ_1,σ_2}-ball radius δ centered at a point of Y. Hence, using (4.5) we get that

$$\begin{aligned}\sup_{s\in I}||f_1(\cdot,\cdot,s) - f_2(\cdot,\cdot,s)||_{\sigma_1,\sigma_2} &\le c_3\left\|\left(a_{j_1p_1j_2p_2}(s)\right) - \left(b_{j_1p_1j_2p_2}(s)\right)\right\|_{X_{I,\tilde{\sigma}_1,\tilde{\sigma}_2}} \le\\ &\le 2c_3\delta = \exp\left(-\epsilon^{-\frac{1}{\alpha}}\right).\end{aligned} \tag{5.2}$$

It remains to find β such that (5.1) is fullfiled. By Lemma 4.3 for some $\eta_\alpha = \eta_\alpha(I, \sigma_1, \sigma_2, \alpha) > 0$

$$|Y| \le \exp\left(\eta\left(\ln(2c_3) + \epsilon^{-\frac{1}{\alpha}}\right)^6\left(1 + \ln\left(\ln(2c_3) + \epsilon^{-\frac{1}{\alpha}}\right)\right)^2\right) \le \exp\left(\eta_\alpha\epsilon^{-\frac{3}{m}}\right). \tag{5.3}$$

Now we take

$$\beta > \mu^{-1}\max\left(N, \eta_\alpha^{m/3}2^{2m}\right). \tag{5.4}$$

This fulfils requirement $\epsilon < \mu\beta$ in Lemma 4.1, which gives

$$\begin{aligned}|v_0 + Z| = |Z| &\ge \exp\left(2^{-4}(\mu\beta/\epsilon)^{3/m}\right) \overset{(5.4)}{>}\\ &> \exp\left(2^{-4}(\eta_\alpha^{m/3}2^{2m}/\epsilon)^{3/m}\right) \overset{(5.3)}{\ge} |Y|.\end{aligned} \tag{5.5}$$

This completes the proof of Theorem 2.2.

In the case of fixed energy $s_1 = s_2$, using Remark 4.2 in (5.3), we can replace the condition $\alpha > 2m$ in Theorem 2.2 by $\alpha > 5m/3$.

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Paper **H**

PAPER H

Energy and regularity dependent stability estimates for near-field inverse scattering in multidimensions

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Abstract. We prove new global Hölder-logarithmic stability estimates for the near-field inverse scattering problem in dimension $d \geq 3$. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. In addition, a global logarithmic stability estimate for this inverse problem in dimension $d = 2$ is also given.

1. Introduction

We consider the Schrödinger equation

$$L\psi = E\psi, \quad L = -\Delta + v(x), \quad x \in \mathbb{R}^d, \quad d \geq 2, \tag{1.1}$$

where

$$\begin{aligned} &v \text{ is real-valued}, \quad v \in \mathbb{L}^\infty(\mathbb{R}^d), \\ &v(x) = O(|x|^{-d-\varepsilon}), \quad |x| \to \infty, \quad \text{for some } \varepsilon > 0. \end{aligned} \tag{1.2}$$

We consider the resolvent $R(E)$ of the Schrödinger operator L in $\mathbb{L}^2(\mathbb{R}^d)$:

$$R(E) = (L - E)^{-1}, \qquad E \in \mathbb{C} \setminus \sigma(L), \tag{1.3}$$

where $\sigma(L)$ is the spectrum of L in $\mathbb{L}^2(\mathbb{R}^d)$. We assume that $R(x, y, E)$ denotes the Schwartz kernel of $R(E)$ as of an integral operator. We consider also

$$R^+(x, y, E) = R(x, y, E + i0), \quad x, y \in \mathbb{R}^d, \quad E \in \mathbb{R}_+. \tag{1.4}$$

We recall that in the framework of equation (1.1) the function $R^+(x, y, E)$ describes scattering of the spherical waves

$$R_0^+(x, y, E) = -\frac{i}{4}\left(\frac{\sqrt{E}}{2\pi|x-y|}\right)^{\frac{d-2}{2}} H^{(1)}_{\frac{d-2}{2}}(\sqrt{E}|x-y|), \tag{1.5}$$

generated by a source at y (where $H^{(1)}_\mu$ is the Hankel function of the first kind of order μ). We recall also that $R^+(x, y, E)$ is the Green function for $L - E$, $E \in \mathbb{R}_+$, with the Sommerfeld radiation condition at infinity.

In addition, the function

$$S^+(x,y,E) = R^+(x,y,E) - R_0^+(x,y,E), \quad x,y \in \partial B_r,\ E \in \mathbb{R}_+,\ r \in \mathbb{R}_+, \tag{1.6}$$

is considered as near-field scattering data for equation (1.1), where B_r is the open ball of radius r centered at 0.

We consider, in particular, the following near-field inverse scattering problem for equation (1.1):

PROBLEM 1.1. Given S^+ on $\partial B_r \times \partial B_r$ for some fixed $r, E \in \mathbb{R}_+$, find v on B_r.

This problem can be considered under the assumption that v is a priori known on $\mathbb{R}^d \setminus B_r$. Actually, in the present paper we consider Problem 1.1 under the assumption that $v \equiv 0$ on $\mathbb{R}^d \setminus B_r$ for some fixed $r \in \mathbb{R}_+$. Below in this paper we always assume that this additional condition is fulfilled.

It is well-known that the near-field scattering data of Problem 1.1 uniquely and efficiently determine the scattering amplitude f for equation (1.1) at fixed energy E, see [**4**]. Therefore, approaches of [**2**], [**6**], [**7**], [**9**], [**13**], [**14**], [**23**], [**24**], [**26**], [**27**], [**36**] can be applied to Problem 1.1 via this reduction.

In addition, it is also known that the near-field data of Problem 1.1 uniquely determine the Dirichlet-to-Neumann map in the case when E is not a Dirichlet eigenvalue for operator L in B_r, see [**22**], [**23**]. Therefore, approaches of [**1**], [**6**], [**16**], [**18**], [**21**], [**23**], [**28**]-[**33**], [**37**] can be also applied to Problem 1.1 via this reduction.

However, in some case it is much more optimal to deal with Problem 1.1 directly, see, for example, logarithmic stability results of [**12**] for Problem 1.1 in dimension $d = 3$. A principal improvement of estimates of [**12**] was given recently in [**17**]: stability of [**17**] efficiently increases with increasing regularity of v.

Problem 1.1 can be also considered as an example of ill-posed problem: see [**5**], [**20**] for an introduction to this theory.

In the present paper we continue studies of [**12**], [**17**]. We give new global Hölder-logarithmic stability estimates for Problem 1.1 in dimension $d \geq 3$, see Theorem 2.1. Our estimates are given in uniform norm for coefficient difference and related stability efficiently increases with increasing energy and/or coefficient regularity. Results of such a type for the Gel'fand inverse problem were obtained recently in [**16**] for $d \geq 3$ and in [**35**] for $d = 2$.

The main feature of our new estimates is the explicit dependence on the energy E. These estimates consist of two parts, the first is Hölder and the second is logarithmic; when E increases, the logarithmic part decreases and the Hölder part becomes dominant.

In addition, we give also global logarithmic stability estimates for Problem 1.1 in dimension $d = 2$, see Theorem 2.2.

2. Stability estimates

We recall that if v satisfies (1.2) and $\operatorname{supp} v \subset B_{r_1}$ for some $r_1 > 0$, then

$$S^+(E) \text{ is bounded in } \mathbb{L}^2(\partial B_r \times \partial B_r) \text{ for any } r > r_1, \tag{2.1}$$

where $S^+(E)$ is the near-field scattering data of v for equation (1.1) with $E > 0$, for more details see, for example, Section 2 of **[12]**.

2.1. Estimates for $d \geq 3$. In this subsection we assume for simplicity that

$$\begin{aligned} v \in W^{m,1}(\mathbb{R}^d) &\text{ for some } m > d, \\ v &\text{ is real-valued}, \\ \operatorname{supp} v \subset B_{r_1} &\text{ for some } r_1 > 0, \end{aligned} \tag{2.2}$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v : \ \partial^J v \in \mathbb{L}^1(\mathbb{R}^d), \ |J| \leq m\}, \ m \in \mathbb{N} \cup 0, \tag{2.3}$$

where

$$J \in (\mathbb{N} \cup 0)^d, \ |J| = \sum_{i=1}^{d} J_i, \ \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}. \tag{2.4}$$

Let

$$||v||_{m,1} = \max_{|J| \leq m} ||\partial^J v||_{\mathbb{L}^1(\mathbb{R}^d)}. \tag{2.5}$$

Note that (2.2) $\Rightarrow$ (1.2).

THEOREM 2.1. *Let $E > 0$ and $r > r_1$ be given constants. Let dimension $d \geq 3$ and potentials v_1, v_2 satisfy (2.2). Let $||v_j||_{m,1} \leq N$, $j = 1, 2$, for some $N > 0$. Let $S_1^+(E)$ and $S_2^+(E)$ denote the near-field scattering data for v_1 and v_2, respectively. Then for $\tau \in (0, 1)$ and any $s \in [0, s^*]$ the following estimate holds:*

$$||v_2 - v_1||_{L^\infty(B_r)} \leq C_1(1+E)^{\frac{5}{2}} \delta^\tau + C_2(1+E)^{\frac{s-s^*}{2}} \left(\ln\left(3 + \delta^{-1}\right)\right)^{-s}, \tag{2.6}$$

where $s^* = \frac{m-d}{d}$, $\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$, and constants $C_1, C_2 > 0$ depend only on N, m, d, r, τ.

Proof of Theorem 2.1 is given in Section 5. This proof is based on results presented in Sections 3, 4.

2.2. Estimates for $d = 2$. In this subsection we assume for simplicity that

$$\begin{aligned} &v \text{ is real-valued}, \quad v \in C^2(\overline{B}_{r_1}), \\ &\operatorname{supp} v \subset B_{r_1} \quad \text{for some } r_1 > 0. \end{aligned} \tag{2.7}$$

Note also that (2.7) $\Rightarrow$ (1.2).

THEOREM 2.2. *Let $E>0$ and $r>r_1$ be given constants. Let dimension $d=2$ and potentials v_1, v_2 satisfy* (2.7). *Let $||v_j||_{C^2(B_r)} \le N$, $j=1,2$, for some $N>0$. Let $S_1^+(E)$ and $S_2^+(E)$ denote the near-field scattering data for v_1 and v_2, respectively. Then*

$$||v_1 - v_2||_{L^\infty(B_r)} \le C_3 \left(\ln\left(3+\delta^{-1}\right)\right)^{-3/4} \left(\ln\left(3\ln\left(3+\delta^{-1}\right)\right)\right)^2, \tag{2.8}$$

where $\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}$ and constant $C_3>0$ depends only on N, m, r.

Proof of Theorem 2.2 is given in Section 7. This proof is based on results presented in Sections 3, 6.

2.3. Concluding remarks.

REMARK 2.1. The logarithmic stability estimates for Problem 1.1 of [**12**] and [**17**] follow from estimate (2.6) for $d=3$ and $s=s^*$. Apparently, using the methods of [**29**], [**30**] it is possible to improve estimate (2.6) for $s^* = m-d$.

REMARK 2.2. In the same way as in [**12**] and [**17**] for dimesnsion $d=3$, using estimates (2.6) and (2.8), one can obtain logarithmic stability estimates for the reconstruction of a potential v from the inverse scattering amplitude f for any $d \ge 2$.

REMARK 2.3. Actually, in the proof of Theorem 2.1 we obtain the following estimate (see formula (5.20)):

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \le C_4(1+E)^2\sqrt{E+\rho^2}\, e^{2\rho(r+1)}\delta + C_5(E+\rho^2)^{-\frac{m-d}{2d}}, \tag{2.9}$$

where constants $C_4, C_5 > 0$ depend only on N, m, d, r and the parameter $\rho>0$ is such that $E+\rho^2$ is sufficiently large: $E+\rho^2 \ge C_6(N,r,m)$. Estimate (2.6) follows from estimate (2.9).

3. Alessandrini-type identity for near-field scattering

In this section we always assume that assumptions of Theorems 2.1 and 2.2 are fulfilled (in the cases of dimension $d\ge 3$ and $d=2$, respectively).

Consider the operators $\hat{\mathrm{R}}_j$, $j=1,2$, defined as follows

$$(\hat{\mathrm{R}}_j\phi)(x) = \int\limits_{\partial B_r} R_j^+(x,y,E)\phi(y)dy, \quad x\in\partial B_r, \quad j=1,2. \tag{3.1}$$

Note that

$$||\hat{\mathrm{R}}_1 - \hat{\mathrm{R}}_2||_{\mathbb{L}^2(\partial B_r)} \le \|S_1^+(E) - S_2^+(E)\|_{\mathbb{L}^2(\partial B_r)\times\mathbb{L}^2(\partial B_r)}. \tag{3.2}$$

We recall that (see [**12**]) for any functions $\phi_1, \phi_2 \in C(\mathbb{R}^d)$, sufficiently regular in $\mathbb{R}^d \setminus \partial B_r$ and satisfying

$$\begin{aligned} &-\Delta\phi + v(x)\phi = E\phi, \quad \text{in } \mathbb{R}^d\setminus\partial B_r,\\ &\lim_{|x|\to+\infty} |x|^{\frac{d-1}{2}}\left(\frac{\partial}{\partial|x|}\phi - i\sqrt{E}\phi\right) = 0, \end{aligned} \tag{3.3}$$

with $v = v_1$ and $v = v_2$, respectively, the following identity holds:

$$\int\limits_{B_r} (v_2 - v_1)\phi_1\phi_2 dx = \tag{3.4}$$
$$= \int\limits_{\partial B_r} \left(\frac{\partial \phi_1}{\partial \nu_+} - \frac{\partial \phi_1}{\partial \nu_-}\right) \left[\left(\hat{\mathrm{R}}_1 - \hat{\mathrm{R}}_2\right)\left(\frac{\partial \phi_2}{\partial \nu_+} - \frac{\partial \phi_2}{\partial \nu_-}\right)\right] dx,$$

where where ν_+ and ν_- are the outward and inward normals to ∂B_r, respectively.

Remark 3.1. The identity (3.4) is similar to the Alessandrini identity (see Lemma 1 of [1]), where the Dirichlet-to-Neumann maps are considered instead of operators $\hat{\mathrm{R}}_j$.

To apply identity (3.4) to our considerations, we use also the following lemma:

Lemma 3.1. *Let $E, r > 0$ and $d \geq 2$. Then, there is a positive constant C_7 (depending only on r and d) such that for any $\phi \in C(\mathbb{R}^d \setminus B_r)$ satisfying*

$$-\Delta\phi = E\phi, \quad \text{in } \mathbb{R}^d \setminus \overline{B}_r,$$
$$\lim_{|x|\to+\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial}{\partial |x|}\phi - i\sqrt{E}\phi\right) = 0, \tag{3.5}$$
$$\phi|_{\partial B_r} \in \mathbb{H}^1(\partial B_r),$$

the following inequality holds:

$$\left\| \left.\frac{\partial \phi}{\partial \nu_+}\right|_{\partial B_r} \right\|_{\mathbb{L}^2(\partial B_r)} \leq C_7(1+E)\, \|\phi|_{\partial B_r}\|_{\mathbb{H}^1(\partial B_r)}, \tag{3.6}$$

where $\mathbb{H}^1(\partial B_r)$ denotes the standard Sobolev space on ∂B_r.

The proof of Lemma 3.1 is given in Section 8.

4. Faddeev functions

In dimension $d \geq 3$, we consider the Faddeev functions h, ψ, G (see **[10]**, **[11]**, **[13]**, **[23]**):

$$h(k, l) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{-ilx} v(x)\psi(x, k) dx, \tag{4.1}$$

where $k, l \in \mathbb{C}^d,\ k^2 = l^2,\ \operatorname{Im} k = \operatorname{Im} l \neq 0$,

$$\psi(x, k) = e^{ikx} + \int\limits_{\mathbb{R}^d} G(x - y, k) v(y)\psi(y, k) dy, \tag{4.2}$$

$$G(x, k) = e^{ikx} g(x, k), \quad g(x, k) = -(2\pi)^{-d} \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \tag{4.3}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d$, $\operatorname{Im} k \neq 0$, $d \geq 3$,

One can consider (4.1), (4.2) assuming that

$$v \text{ is a sufficiently regular function on } \mathbb{R}^d \quad \text{with suffucient decay at infinity.} \tag{4.4}$$

For example, in connection with Theorem 2.1, we consider (4.1), (4.2) assuming that

$$v \in \mathbb{L}^\infty(B_r), \quad v \equiv 0 \text{ on } \mathbb{R} \setminus B_r. \tag{4.5}$$

We recall that (see [**10**], [**11**], [**13**], [**23**]):

$$(\Delta + k^2)G(x,k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d; \tag{4.6}$$

formula (4.2) at fixed k is considered as an equation for

$$\psi = e^{ikx}\mu(x,k), \tag{4.7}$$

where μ is sought in $\mathbb{L}^\infty(\mathbb{R}^d)$; as a corollary of (4.2), (4.3), (4.6), ψ satisfies (1.1) for $E = k^2$; h of (4.1) is a generalized "‘scattering"’ amplitude.

In addition, h, ψ, G in their zero energy restriction, that is for $E = 0$, were considered for the first time in [**3**]. The Faddeev functions h, ψ, G were, actually, rediscovered in [**3**].

Let

$$\begin{aligned} \Sigma_E &= \left\{k \in \mathbb{C}^d : k^2 = k_1^2 + \ldots + k_d^2 = E\right\}, \\ \Theta_E &= \left\{k \in \Sigma_E,\ l \in \Sigma_E : \operatorname{Im} k = \operatorname{Im} l\right\}, \\ |k| &= (|\operatorname{Re} k|^2 + |\operatorname{Im} k|^2)^{1/2}. \end{aligned} \tag{4.8}$$

Let

$$v \text{ satisfy (2.2)}, \quad \|v\|_{m,1} \leq N, \tag{4.9}$$

$$\hat{v}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} v(x)dx, \quad p \in \mathbb{R}^d, \tag{4.10}$$

then we have that:

$$\mu(x,k) \to 1 \quad \text{as} \quad |k| \to \infty \tag{4.11}$$

and, for any $\sigma > 1$,

$$|\mu(x,k)| + |\nabla\mu(x,k)| \leq \sigma \quad \text{for} \quad |k| \geq \lambda_1(N,m,d,r,\sigma), \tag{4.12}$$

where $x \in \mathbb{R}^d$, $k \in \Sigma_E$;

$$\hat{v}(p) = \lim_{\substack{(k,l) \in \Theta_E,\ k-l=p \\ |\operatorname{Im} k| = |\operatorname{Im} l| \to \infty}} h(k,l) \quad \text{for any } p \in \mathbb{R}^d, \tag{4.13}$$

$$
\begin{aligned}
&|\hat{v}(p) - h(k,l)| \le \frac{c_1(m,d,r)N^2}{(E+\rho^2)^{1/2}} \quad \text{for } (k,l) \in \Theta_E, \quad p = k-l,\\
&\qquad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \quad E+\rho^2 \ge \lambda_2(N,m,d,r),\\
&\qquad p^2 \le 4(E+\rho^2).
\end{aligned} \tag{4.14}
$$

Results of the type (4.11), (4.12) go back to [**3**]. For more information concerning (4.12) see estimate (4.11) of [**15**]. Results of the type (4.13), (4.14) (with less precise right-hand side in (4.14)) go back to [**13**]. Estimate (4.14) follows, for example, from formulas (4.2), (4.1) and the estimate

$$
\begin{aligned}
&\|\Lambda^{-s} g(k) \Lambda^{-s}\|_{\mathbb{L}^2(\mathbb{R}^d)\to\mathbb{L}^2(\mathbb{R}^d)} = O(|k|^{-1})\\
&\qquad \text{as } |k| \to \infty, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d,
\end{aligned} \tag{4.15}
$$

for $s > 1/2$, where $g(k)$ denotes the integral operator with the Schwartz kernel $g(x-y,k)$ and Λ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$. Estimate (4.15) was formulated, first, in [**19**] for $d \ge 3$. Concerning proof of (4.15), see [**39**].

In addition, we have that:

$$
\begin{aligned}
&h_2(k,l) - h_1(k,l) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \psi_1(x,-l)(v_2(x) - v_1(x))\psi_2(x,k)dx\\
&\qquad \text{for } (k,l) \in \Theta_E, \ |\operatorname{Im} k| = |\operatorname{Im} l| \neq 0,\\
&\qquad \text{and } v_1,\ v_2 \text{ satisfying (4.4)},
\end{aligned} \tag{4.16}
$$

and, under assumtions of Theorem 2.1,

$$
\begin{aligned}
&|\hat{v}_1(p) - \hat{v}_2(p) - h_1(k,l) + h_2(k,l)| \le \frac{c_2(m,d,r)N\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)}}{(E+\rho^2)^{1/2}}\\
&\qquad \text{for } (k,l) \in \Theta_E, \quad p = k-l, \quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho,\\
&\qquad E+\rho^2 \ge \lambda_3(N,m,d,r), \quad p^2 \le 4(E+\rho^2),
\end{aligned} \tag{4.17}
$$

where h_j, ψ_j denote h and ψ of (4.1) and (4.2) for $v = v_j$, $j = 1,2$.

Formula (4.16) was given in [**25**]. Estimate (4.17) was given e.g. in [**16**].

5. Proof of Theorem 2.1

Let

$$
\begin{aligned}
&\mathbb{L}^\infty_\mu(\mathbb{R}^d) = \{u \in \mathbb{L}^\infty(\mathbb{R}^d) : \|u\|_\mu < +\infty\},\\
&\|u\|_\mu = \operatorname{ess}\sup_{p\in\mathbb{R}^d}(1+|p|)^\mu |u(p)|, \quad \mu > 0.
\end{aligned} \tag{5.1}
$$

Note that

$$
\begin{aligned}
&w \in W^{m,1}(\mathbb{R}^d) \Longrightarrow \hat{w} \in \mathbb{L}^\infty_\mu(\mathbb{R}^d) \cap C(\mathbb{R}^d),\\
&\|\hat{w}\|_\mu \le c_3(m,d)\|w\|_{m,1} \quad \text{for} \quad \mu = m,
\end{aligned} \tag{5.2}
$$

where $W^{m,1}$, $\mathbb{L}^\infty_\mu$ are the spaces of (2.3), (5.1),

$$\hat{w}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} w(x) dx, \quad p \in \mathbb{R}^d. \tag{5.3}$$

Using the inverse Fourier transform formula

$$w(x) = \int\limits_{\mathbb{R}^d} e^{-ipx} \hat{w}(p) dp, \quad x \in \mathbb{R}^d, \tag{5.4}$$

we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \leq \sup_{x \in \overline{B}_r} \Big| \int\limits_{\mathbb{R}^d} e^{-ipx} \left(\hat{v}_2(p) - \hat{v}_1(p)\right) dp \Big| \leq \\ \leq I_1(\kappa) + I_2(\kappa) \quad \text{for any} \quad \kappa > 0, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} I_1(\kappa) &= \int\limits_{|p| \leq \kappa} |\hat{v}_2(p) - \hat{v}_1(p)| dp, \\ I_2(\kappa) &= \int\limits_{|p| \geq \kappa} |\hat{v}_2(p) - \hat{v}_1(p)| dp. \end{aligned} \tag{5.6}$$

Using (5.2), we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \leq 2c_3(m,d) N (1 + |p|)^{-m}, \quad p \in \mathbb{R}^d. \tag{5.7}$$

Let

$$c_4 = \int\limits_{p \in \mathbb{R}^d, |p| = 1} dp. \tag{5.8}$$

Combining (5.6), (5.7), we find that, for any $\kappa > 0$,

$$I_2(\kappa) \leq 2c_3(m,d) N c_4 \int\limits_{\kappa}^{+\infty} \frac{dt}{t^{m-d+1}} \leq \frac{2c_3(m,d) N c_4}{m-d} \frac{1}{\kappa^{m-d}}. \tag{5.9}$$

Due to (4.17), we have that

$$\begin{aligned} |\hat{v}_2(p) - \hat{v}_1(p)| \leq |h_2(k,l) - h_1(k,l)| + \frac{c_2(m,d,r) N \|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)}}{(E + \rho^2)^{1/2}}, \\ \text{for } (k,l) \in \Theta_E, \quad p = k - l, \quad |\operatorname{Im} k| = |\operatorname{Im} l| = \rho, \\ E + \rho^2 \geq \lambda_3(N,m,d,r), \quad p^2 \leq 4(E + \rho^2). \end{aligned} \tag{5.10}$$

Let

$$\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}. \tag{5.11}$$

Combining (3.2), (3.4) and (4.16), we get that

$$\begin{aligned}|h_2(k,l)-h_1(k,l)| \le \delta \left\|\frac{\partial\phi_1}{\partial\nu_+}-\frac{\partial\phi_1}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)}\left\|\frac{\partial\phi_2}{\partial\nu_+}-\frac{\partial\phi_2}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)},\\ (k,l)\in\Theta_E,\ |\mathrm{Im}\,k|=|\mathrm{Im}\,l|\neq 0,\end{aligned} \tag{5.12}$$

where ϕ_j, $j=1,2$, denotes the solution of (3.3) with $v=v_j$, satisfying

$$\phi_j(x)=\psi_j(x,k) \quad \text{for} \quad x\in\overline{B}_r. \tag{5.13}$$

Using (3.6), (4.12) and the fact that $C^1(\partial B_r)\subset\mathbb{H}^1(\partial B_r)$, we find that

$$\begin{aligned}\left\|\frac{\partial\phi_j}{\partial\nu_+}-\frac{\partial\phi_j}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)} \le \sigma c_5(r,d)(1+E)\exp\Big(|\mathrm{Im}\,k|(r+1)\Big),\\ k\in\Sigma_E,\ |k|\ge\lambda_1(N,m,d,r,\sigma),\ j=1,2.\end{aligned} \tag{5.14}$$

Here and bellow in this section the constant σ is the same that in (4.12).

Combining (5.12) and (5.14), we obtain that

$$\begin{aligned}|h_2(k,l)-h_1(k,l)| \le c_5^2\sigma^2(1+E)^2e^{2\rho(r+1)}\delta,\\ \text{for } (k,l)\in\Theta_E,\quad \rho=|\mathrm{Im}\,k|=|\mathrm{Im}\,l|,\\ E+\rho^2\ge\lambda_1^2(N,m,d,r,\sigma).\end{aligned} \tag{5.15}$$

Using (5.10), (5.15), we get that

$$\begin{aligned}|\hat v_2(p)-\hat v_1(p)| \le c_5^2\sigma^2(1+E)^2e^{2\rho(r+1)}\delta+\\ +\frac{c_2(m,d,r)N\|v_1-v_2\|_{\mathbb{L}^\infty(B_1)}}{(E+\rho^2)^{1/2}},\\ p\in\mathbb{R}^d,\ p^2\le 4(E+\rho^2),\ E+\rho^2\ge\max\{\lambda_1^2,\lambda_3\}.\end{aligned} \tag{5.16}$$

Let

$$\varepsilon=\left(\frac{1}{2c_2(m,d,r)Nc_6}\right)^{1/d},\quad c_6=\int\limits_{p\in\mathbb{R}^d,|p|\le 1}dp, \tag{5.17}$$

and $\lambda_4(N,m,d,r,\sigma)>0$ be such that

$$E+\rho^2\ge\lambda_4(N,m,d,r,\sigma)\Longrightarrow\begin{cases}E+\rho^2\ge\lambda_1^2(N,m,d,r,\sigma),\\ E+\rho^2\ge\lambda_3(N,m,d,r),\\ \left(\varepsilon(E+\rho^2)^{\frac{1}{2d}}\right)^2\le 4(E+\rho^2).\end{cases} \tag{5.18}$$

Using (5.6), (5.16), we get that

$$
\begin{aligned}
&I_1(\kappa) \le c_6\kappa^d\Big(c_5^2\sigma^2(1+E)^2e^{2\rho(r+1)}\delta + \frac{c_2(m,d,r)N\|v_1-v_2\|_{\mathbb{L}^\infty(B_1)}}{(E+\rho^2)^{1/2}}\Big),\\
&\qquad\qquad \kappa>0,\ \kappa^2\le 4(E+\rho^2),\\
&\qquad\qquad E+\rho^2\ge\lambda_4(N,m,d,r,\sigma).
\end{aligned}
\tag{5.19}
$$

Combining (5.5), (5.9), (5.19) for $\kappa=\varepsilon(E+\rho^2)^{\frac{1}{2d}}$ and (5.18), we get that

$$
\begin{aligned}
&\|v_1-v_2\|_{\mathbb{L}^\infty(B_r)} \le c_7(N,m,d,r,\sigma)(1+E)^2\sqrt{E+\rho^2}\,e^{2\rho(r+1)}\delta+\\
&\qquad + c_8(N,m,d)(E+\rho^2)^{-\frac{m-d}{2d}} + \frac{1}{2}\|v_1-v_2\|_{\mathbb{L}^\infty(B_r)},\\
&\qquad E+\rho^2\ge\lambda_4(N,m,d,r,\sigma).
\end{aligned}
\tag{5.20}
$$

Let $\tau'\in(0,1)$,

$$
\beta=\frac{1-\tau'}{2(r+1)},\qquad \rho=\beta\ln\left(3+\delta^{-1}\right),
\tag{5.21}
$$

and $\delta_1=\delta_1(N,m,d,\sigma,r,\tau')>0$ be such that

$$
\delta\in(0,\delta_1)\Longrightarrow\begin{cases} E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\ge\lambda_4(N,m,d,r,\sigma),\\ E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\le(1+E)\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2,\end{cases}
\tag{5.22}
$$

Then for the case when $\delta\in(0,\delta_1)$, due to (5.20), we have that

$$
\begin{aligned}
&\frac{1}{2}\|v_1-v_2\|_{\mathbb{L}^\infty(B_r)}\le\\
&\le c_7(1+E)^2\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{\frac12}\left(3+\delta^{-1}\right)^{2\beta(r+1)}\delta+\\
&\qquad + c_8\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{-\frac{m-d}{2d}} =\\
&= c_7(1+E)^2\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{\frac12}\left(1+3\delta\right)^{1-\tau'}\delta^{\tau'}+\\
&\qquad + c_8\left(E+\left(\beta\ln\left(3+\delta^{-1}\right)\right)^2\right)^{-\frac{m-d}{2d}}.
\end{aligned}
\tag{5.23}
$$

Combining (5.22) and (5.23), we obtain that for $s\in[0,s^*]$, $\tau\in(0,\tau')$ and $\delta\in(0,\delta_1)$ the following estimate holds:

$$
||v_2-v_1||_{L^\infty(B_r)}\le c_9(1+E)^{\frac52}\delta^\tau + c_{10}(1+E)^{\frac{s-s^*}{2}}\left(\ln\left(3+\delta^{-1}\right)\right)^{-s},
\tag{5.24}
$$

where $s^*=\frac{m-d}{d}$ and $c_9, c_{10}>0$ depend only on N, m, d, r, σ, τ' and τ.

Estimate (5.24) in the general case (with modified c_9 and c_{10}) follows from (5.24) for $\delta\le\delta_1(N,m,d,\sigma,r,\tau')$ and and the property that

$$
\|v_j\|_{\mathbb{L}^\infty(B_r)}\le c_{11}(m,d)N.
\tag{5.25}
$$

This completes the proof of (2.6)

6. Buckhgeim-type analogs of the Faddeev functions

Let us identify $\mathbb{R}^2$ with $\mathbb{C}$ and use coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, where $(x_1, x_2) \in \mathbb{R}^2$. Following **[31]**-**[34]**, we consider the functions $G_{z_0}, \psi_{z_0}, \widetilde{\psi}_{z_0}, \delta h_{z_0}$ going back to Buckhgeim's paper **[6]** and being analogs of the Faddeev functions:

$$\begin{aligned}
\psi_{z_0}(z,\lambda) &= e^{\lambda(z-z_0)^2} + \int\limits_{B_r} G_{z_0}(z,\zeta,\lambda)v(\zeta)\psi_{z_0}(\zeta,\lambda)\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta, \\
\widetilde{\psi}_{z_0}(z,\lambda) &= e^{\bar{\lambda}(\bar{z}-\bar{z}_0)^2} + \int\limits_{B_r} \overline{G_{z_0}(z,\zeta,\lambda)}v(\zeta)\widetilde{\psi}_{z_0}(\zeta,\lambda)\, d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta,
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
G_{z_0}(z,\zeta,\lambda) &= \frac{1}{4\pi^2}\int\limits_{B_r} \frac{e^{-\lambda(\eta-z_0)^2+\bar{\lambda}(\bar{\eta}-\bar{z}_0)^2} d\mathrm{Re}\eta\, d\mathrm{Im}\eta}{(z-\eta)(\bar{\eta}-\bar{\zeta})}\, e^{\lambda(z-z_0)^2-\bar{\lambda}(\bar{\zeta}-\bar{z}_0)^2}, \\
&\qquad\qquad z = x_1 + ix_2,\ z_0 \in B_r,\ \lambda \in \mathbb{C},
\end{aligned} \tag{6.2}$$

where v satisfies (2.7);

$$\delta h_{z_0}(\lambda) = \int\limits_{B_r} \widetilde{\psi}_{z_0,1}(z,-\lambda)\left(v_2(z) - v_1(z)\right)\psi_{z_0,2}(z,\lambda)\, d\mathrm{Re}z\, d\mathrm{Im}z, \quad \lambda \in \mathbb{C}, \tag{6.3}$$

where v_1, v_2 satisfy (2.7) and $\widetilde{\psi}_{z_0,1}$, $\psi_{z_0,2}$ denote $\widetilde{\psi}_{z_0}$, ψ_{z_0} of (6.1) for $v = v_1$ and $v = v_2$, respectively.

We recall that (see **[31]**, **[32]**):

- The function G_{z_0} satisfies the equations
$$\begin{aligned}
4\frac{\partial^2}{\partial z\partial\bar{z}}G_{z_0}(z,\zeta,\lambda) &= \delta(z-\zeta), \\
4\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}G_{z_0}(z,\zeta,\lambda) &= \delta(z-\zeta),
\end{aligned} \tag{6.4}$$
where $z, z_0, \zeta \in B_r$, $\lambda \in \mathbb{C}$ and δ is the Dirac delta function;
- Formulas (6.1) at fixed z_0 and λ are considered as equations for ψ_{z_0}, $\widetilde{\psi}_{z_0}$ in $L^\infty(B_r)$;
- As a corollary of (6.1), (6.2), (6.4), the functions ψ_{z_0}, $\widetilde{\psi}_{z_0}$ satisfy (1.1) in B_r for $E = 0$ and $d = 2$;
- The function δh_{z_0} is similar to the right side of (4.16).

Let potentials $v, v_1, v_2 \in C^2(\overline{B}_r)$ and

$$\begin{aligned}
\|v\|_{C^2(\overline{B}_r)} \le N, &\quad \|v_j\|_{C^2(\overline{B}_r)} \le N, \quad j = 1,2, \\
(v_1 - v_2)|_{\partial B_r} = 0, &\quad \frac{\partial}{\partial\nu}(v_1 - v_2)|_{\partial B_r} = 0,
\end{aligned} \tag{6.5}$$

then we have that:

$$\psi_{z_0}(z,\lambda) = e^{\lambda(z-z_0)^2}\mu_{z_0}(z,\lambda), \quad \widetilde{\psi}_{z_0}(z,\lambda) = e^{\bar{\lambda}(\bar{z}-\bar{z}_0)^2}\widetilde{\mu}_{z_0}(z,\lambda), \tag{6.6}$$

$$\mu_{z_0}(z,\lambda) \to 1, \;\; \widetilde{\mu}_{z_0}(z,\lambda) \to 1 \quad \text{as } |\lambda| \to \infty \tag{6.7}$$

and, for any $\sigma > 1$,

$$|\mu_{z_0}(z,\lambda)| + |\nabla\mu_{z_0}(z,\lambda)| \le \sigma, \tag{6.8a}$$

$$|\widetilde{\mu}_{z_0}(z,\lambda)| + |\nabla\widetilde{\mu}_{z_0}(z,\lambda)| \le \sigma, \tag{6.8b}$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, $z = x_1 + ix_2$, $z_0 \in B_r$, $\lambda \in \mathbb{C}$, $|\lambda| \ge \rho_1(N,r,\sigma)$;

$$v_2(z_0) - v_1(z_0) = \lim_{\lambda\to\infty} \frac{2}{\pi}|\lambda|\delta h_{z_0}(\lambda) \quad \text{for any } z_0 \in B_r, \tag{6.9}$$

$$\left| v_2(z_0) - v_1(z_0) - \frac{2}{\pi}|\lambda|\delta h_{z_0}(\lambda)\right| \le \frac{c_{12}(N,r)\left(\ln(3|\lambda|)\right)^2}{|\lambda|^{3/4}} \quad \text{for } z_0 \in B_r,\ |\lambda| \ge \rho_2(N,r). \tag{6.10}$$

Formulas (6.6) can be considered as definitions of μ_{z_0}, $\widetilde{\mu}_{z_0}$. Formulas (6.7), (6.9) were given in **[31]**, **[32]** and go back to **[6]**. Estimates (6.8) were proved in **[15]**. Estimate (6.10) was obtained in **[31]**, **[34]**.

7. Proof of Theorem 2.2

We suppose that $\widetilde{\psi}_{z_0,1}(\cdot,-\lambda)$, $\psi_{z_0,2}(\cdot,\lambda)$, $\delta h_{z_0}(\lambda)$ are defined as in Section 6 but with $v_j - E$ in place of v_j, $j = 1,2$. Note that functions $\widetilde{\psi}_{z_0,1}(\cdot,-\lambda)$, $\psi_{z_0,2}(\cdot,\lambda)$ satisfy (1.1) in B_r with $v = v_j$, $j = 1,2$, respectively. We also use the notation $N_E = N + E$. Then, using (6.10), we have that

$$\left| v_2(z_0) - v_1(z_0) - \frac{2}{\pi}|\lambda|\delta h_{z_0}(\lambda)\right| \le \frac{c_{12}(N_E,r)\left(\ln(3|\lambda|)\right)^2}{|\lambda|^{3/4}} \quad \text{for } z_0 \in B_r,\ |\lambda| \ge \rho_2(N_E,r). \tag{7.1}$$

Let

$$\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)}. \tag{7.2}$$

Combining (3.2), (3.4) and (6.3), we get that

$$|\delta h_{z_0}(\lambda)| \le \delta \left\| \frac{\partial\phi_1}{\partial\nu_+} - \frac{\partial\phi_1}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)} \left\| \frac{\partial\phi_2}{\partial\nu_+} - \frac{\partial\phi_2}{\partial\nu_-}\right\|_{\mathbb{L}^2(B_r)}, \quad (k,l) \in \Theta_E,\ |\operatorname{Im} k| = |\operatorname{Im} l| \ne 0, \tag{7.3}$$

where ϕ_j, $j = 1,2$, denotes the solution of (3.3) with $v = v_j$, satisfying

$$\phi_1(x) = \widetilde{\psi}_{z_0,1}(x,-\lambda), \quad \phi_2(x) = \psi_{z_0,2}(x,\lambda), \quad \text{for } x \in \overline{B}_r. \tag{7.4}$$

Using (3.6), (6.8) and the fact that $C^1(\partial B_r) \subset \mathbb{H}^1(\partial B_r)$, we find that:

$$\left\|\frac{\partial \phi_j}{\partial \nu_+} - \frac{\partial \phi_j}{\partial \nu_-}\right\|_{\mathbb{L}^2(B_r)} \le \sigma c_{13}(r)(1+E)\exp\Big(|\lambda|(4r^2+4r)\Big), \quad \lambda \in \mathbb{C}, \quad |\lambda| \ge \rho_1(N_E, r, \sigma), \quad j = 1,2. \tag{7.5}$$

Here and bellow in this section the constant σ is the same that in (6.8).

Combining (7.3), (7.5), we obtain that

$$|\delta h_{z_0}(\lambda)| \le c_{14}(E, r, \sigma)\exp\Big(|\lambda|(8r^2+8r)\Big)\delta, \quad \lambda \in \mathbb{C}, \quad |\lambda| \ge \rho_1(N_E, r, \sigma). \tag{7.6}$$

Using (7.1) and (7.6), we get that

$$\begin{aligned} |v_2(z_0) - v_1(z_0)| \le c_{14}(E, r, \sigma)\exp\Big(|\lambda|(8r^2+8r)\Big)\delta+ \\ + \frac{c_{12}(N_E, r)\left(\ln(3|\lambda|)\right)^2}{|\lambda|^{3/4}}, \\ z_0 \in B_r, \quad \lambda \in \mathbb{C}, \quad |\lambda| \ge \rho_3(N_E, r, \sigma) = \max\{\rho_1, \rho_2\}. \end{aligned} \tag{7.7}$$

We fix some $\tau \in (0,1)$ and let

$$\beta = \frac{1-\tau}{8r^2+8r}, \quad \lambda = \beta \ln\left(3+\delta^{-1}\right), \tag{7.8}$$

where δ is so small that $|\lambda| \ge \rho_3(N_E, r, \sigma)$. Then due to (7.7), we have that

$$\begin{aligned} \|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} &\le c_{14}(E, r, \sigma)\left(3+\delta^{-1}\right)^{\beta(8r^2+8r)}\delta+ \\ &+ c_{12}(N_E, r)\frac{\left(\ln\left(3\beta\ln\left(3+\delta^{-1}\right)\right)\right)^2}{\left(\beta\ln\left(3+\delta^{-1}\right)\right)^{\frac{3}{4}}} = \\ &= c_{14}(E, r, \sigma)\left(1+3\delta\right)^{1-\tau}\delta^\tau + \\ &+ c_{12}(N_E, r)\beta^{-\frac{3}{4}}\frac{\left(\ln\left(3\beta\ln\left(3+\delta^{-1}\right)\right)\right)^2}{\left(\ln\left(3+\delta^{-1}\right)\right)^{\frac{3}{4}}}, \end{aligned} \tag{7.9}$$

where τ, β and δ are the same as in (7.8).

Using (7.9), we obtain that

$$\|v_1 - v_2\|_{\mathbb{L}^\infty(B_r)} \le c_{15}(N, E, r, \sigma)\left(\ln\left(3+\delta^{-1}\right)\right)^{-\frac{3}{4}}\left(\ln\left(3\ln\left(3+\delta^{-1}\right)\right)\right)^2 \tag{7.10}$$

for $\delta = ||S_1^+(E) - S_2^+(E)||_{\mathbb{L}^2(\partial B_r \times \partial B_r)} \le \delta_2(N_E, r, \sigma)$, where δ_2 is a sufficiently small positive constant. Estimate (7.10) in the general case (with modified c_{15}) follows from (7.10) for $\delta \le \delta_2(N_E, r, \sigma)$ and the property that $\|v_j\|_{\mathbb{L}^\infty(B_r)} \le N$.

This completes the proof of (2.8).

8. Proof of Lemma 3.1

In this section we assume for simplicity that $r = 1$ and therefore $\partial B_r = \mathbb{S}^{d-1}$.
We fix an orthonormal basis in $\mathbb{L}^2(\partial B_r)$:

$$\begin{aligned}&\{f_{jp} : j \geq 0;\ 1 \leq p \leq p_j\},\\ &f_{jp} \text{ is a spherical harmonic of degree } j,\end{aligned} \tag{8.1}$$

where p_j is the dimension of the space of spherical harmonics of order j,

$$p_j = \binom{j+d-1}{d-1} - \binom{j+d-3}{d-1}, \tag{8.2}$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad \text{for } n \geq 0 \tag{8.3}$$

and

$$\binom{n}{k} = 0 \quad \text{for } n < 0. \tag{8.4}$$

The precise choice of f_{jp} is irrelevant for our purposes. Besides orthonormality, we only need f_{jp} to be the restriction of a homogeneous harmonic polynomial of degree j to the sphere ∂B_r and so $|x|^j f_{jp}(x/|x|)$ is harmonic pn $\mathbb{R}^d$. In the Sobolev spaces $\mathbb{H}^s(\partial B_r)$ the norm is defined by

$$\Bigg\|\sum_{j,p} c_{jp} f_{jp}\Bigg\|^2_{\mathbb{H}^s(\partial B_r)} = \sum_{j,p}(1+j)^{2s}|c_{jp}|^2. \tag{8.5}$$

The solution ϕ of the exterior Dirichlet problem

$$\begin{aligned} -\Delta\phi = E\phi, \quad &\text{in } \mathbb{R}^d \setminus \overline{B}_r,\\ \lim_{|x|\to+\infty} |x|^{\frac{d-1}{2}}\left(\frac{\partial}{\partial|x|}\phi - i\sqrt{E}\phi\right) &= 0,\\ \phi|_{\partial B_r} = u \in \mathbb{H}^1(\partial B_r)&, \end{aligned} \tag{8.6}$$

can be expressed in the following form (see, for example, [**4**], [**8**]):

$$\phi = \sum_{j,p} c_{jp}\phi_{jp}, \tag{8.7}$$

where c_{jp} are expansion coefficients of u in the basis $\{f_{jp} : j \geq 0;\ 1 \leq p \leq p_j\}$, and

$$\begin{aligned} &\phi_{jp} \text{ denotes the solution of (8.6) with } u = f_{jp},\\ &\phi_{jp}(x) = h_{jp}(|x|) f_{jp}(x/|x|),\\ &h_{jp}(|x|) = |x|^{-\frac{d-2}{2}} \frac{H^{(1)}_{j+\frac{d-2}{2}}(\sqrt{E}|x|)}{H^{(1)}_{j+\frac{d-2}{2}}(\sqrt{E})}, \end{aligned} \tag{8.8}$$

where $H_\mu^{(1)}$ is the Hankel function of the first kind. Let

$$\phi_{jp}^0(x) = |x|^{-j-d+2} f_{jp}(x/|x|). \tag{8.9}$$

Note that ϕ_{jp}^0 is harmonic in $\mathbb{R}^d \setminus \{0\}$ and

$$\lim_{|x|\to+\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial}{\partial |x|} \phi_{jp}^0 - i\sqrt{E}\phi_{jp}^0 \right) = 0 \quad \text{for } j + \frac{d-3}{2} > 0. \tag{8.10}$$

Using the Green formula and the radiation condition for ϕ_{jp}, ϕ_{jp}^0, we get that

$$\begin{aligned} \int\limits_{\mathbb{R}^d\setminus B_r} E\phi_{jp}\phi_{jp}^0 dx &= \int\limits_{\mathbb{R}^d\setminus B_r} \left(\Delta\phi_{jp}^0\phi_{jp} - \Delta\phi_{jp}\phi_{jp}^0\right) dx = \\ &= \int\limits_{\partial B_r} \left(\frac{\partial \phi_{jp}^0}{\partial \nu_+}\phi_{jp} - \frac{\partial \phi_{jp}}{\partial \nu_+}\phi_{jp}^0 \right) dx \quad \text{for } j + \frac{d-3}{2} > 0. \end{aligned} \tag{8.11}$$

Due to (8.8) and (8.9), we have that

$$\left| \int\limits_{\partial B_r} \frac{\partial \phi_{jp}^0}{\partial \nu_+} \phi_{jp} dx \right| = (j+d-2) \int\limits_{\partial B_r} f_{jp}^2 dx = j+d-2. \tag{8.12}$$

Using also the following property of the Hankel function of the first kind (see, for example, [**38**]):

$$|H_\mu^{(1)}(x)| \text{ is a decreasing function of } x \text{ for } x \in \mathbb{R}_+, \mu \in \mathbb{R}, \tag{8.13}$$

we get that

$$\begin{aligned} \left| \int\limits_{\mathbb{R}^d\setminus B_r} \phi_{jp}\phi_{jp}^0 dx \right| &= \left| \int\limits_1^{+\infty} t^{-j-d+2} h_{jp}(t) t^{d-1} dt \right| = \\ = \left| \int\limits_1^{+\infty} t^{-j-\frac{d}{2}} \frac{H^{(1)}_{j+\frac{d-2}{2}}(\sqrt{E}t)}{H^{(1)}_{j+\frac{d-2}{2}}(\sqrt{E})} dt \right| &\le \int\limits_1^{+\infty} t^{-j-\frac{d}{2}} dt = \frac{1}{j+\frac{d}{2}-1} \le 2 \\ &\text{for } j + \frac{d-3}{2} > 0. \end{aligned} \tag{8.14}$$

Combining (8.8), (8.9), (8.11), (8.12) and (8.14), we obtain that

$$\left| \int\limits_{\partial B_r} \frac{\partial \phi_{jp}^0}{\partial \nu_+} \phi_{jp} dx \right| = \left| \frac{h'_{jp}(r)}{h_{jp}(r)} \right| \le j+d-2+2E \quad \text{for } j + \frac{d-3}{2} > 0. \tag{8.15}$$

Let consider the cases when $j + \frac{d-3}{2} \le 0$.

Case 1. $j = 0$, $d = 2$. Using the property $dH_0^{(1)}(t)/dt = -H_1^{(1)}(t)$, we get that

$$\frac{h'_{jp}(r)}{h_{jp}(r)} = \sqrt{E}\frac{H_1^{(1)}(\sqrt{E})}{H_0^{(1)}(\sqrt{E})}. \tag{8.16}$$

We recall that functions $H_0^{(1)}$ and $H_1^{(1)}$ have the following asymptotic forms (see, for example [**38**]):

$$\begin{aligned} H_0^{(1)}(t) &\sim \frac{2i}{\pi}\ln(t/2) \quad \text{as} \quad t \to +0, \\ H_0^{(1)}(t) &\sim \sqrt{\frac{2}{\pi t}}e^{i(t-\pi/4)} \quad \text{as} \quad t \to +\infty, \\ H_1^{(1)}(t) &\sim -\frac{i}{\pi}(2/t) \quad \text{as} \quad t \to +0, \\ H_1^{(1)}(t) &\sim \sqrt{\frac{2}{\pi t}}e^{i(t-3\pi/4)} \quad \text{as} \quad t \to +\infty. \end{aligned} \tag{8.17}$$

Using (8.13) and (8.17), we get that for some $c > 0$

$$\frac{H_1^{(1)}(t)}{H_0^{(1)}(t)} \le c\left(1 + 1/t\right). \tag{8.18}$$

Combining (8.16) and (8.18), we obtain that for $j = 0$, $d = 2$

$$\left|\frac{h'_{jp}(r)}{h_{jp}(r)}\right| \le c(1 + \sqrt{E}). \tag{8.19}$$

Case 2. $j = 0$, $d = 3$. We have that

$$H^{(1)}_{j+\frac{d-2}{2}}(t) = \sqrt{\frac{2}{\pi t}}e^{i(t-\pi/2)}. \tag{8.20}$$

Using (8.8) and (8.20), we get that for $j = 0$, $d = 3$

$$\frac{h'_{jp}(r)}{h_{jp}(r)} = -1 + i\sqrt{E}. \tag{8.21}$$

Combining (8.5)-(8.8), (8.15), (8.19) and (8.21), we get that for some constant $c' = c'(d) > 0$

$$\left\|\left.\frac{\partial\phi}{\partial\nu_+}\right|_{\partial B_r}\right\|^2_{\mathbb{L}^2(\partial B_r)} = \sum_{j,p} c_{jp}^2\left|\frac{h'_{jp}(r)}{h_{jp}(r)}\right|^2 \le c'(1+E)^2\sum_{j,p}(1+j)^2 c_{jp}^2. \tag{8.22}$$

Using (8.5) and (8.22), we obtain (3.6)

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