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Les Équations aux Dérivées Partielles Stochastiques avec Obstacle

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Résumé

Cette thèse traite des Équations aux Dérivées Partielles Stochastiques Quasilinéaires. Elle est divisée en deux parties. La première partie concerne le problème d'obstacle pour les équations aux dérivées partielles stochastiques quasilinéaires et la deuxième partie est consacrée à l'étude des équations aux dérivées partielles stochastiques quasilinéaires dirigées par un $G-$mouvement brownien.

Dans la première partie, on montre d'abord l'existence et l'unicité d'un problème d'obstacle pour les équations aux dérivées partielles stochastiques quasilinéaires (en bref OSPDE). Notre méthode est basée sur des techniques analytiques venant de la théorie du potentiel parabolique. La solution est exprimée comme une paire (u, ν) où u est un processus prévisible continu qui prend ses valeurs dans un espace de Sobolev et ν est une mesure régulière aléatoire satisfaisant la condition de Skohorod.

Ensuite, on établit un principe du maximum pour la solution locale des équations aux dérivées partielles stochastiques quasilinéaires avec obstacle. La preuve est basée sur une version de la formule d'Itô et les estimations pour la partie positive d'une solution locale qui est négative sur le bord du domaine considéré.

L'objectif de la deuxème partie est d'étudier l'existence et l'unicité de la solution des équations aux dérivées partielles stochastiques dirigées par $G-$mouvement brownien dans le cadre d'un espace muni d'une espérance sous-linéaire. On établit une formule d'Itô pour la solution et un théorème de comparaison.

Abstract

This thesis deals with quasilinear Stochastic Partial Differential Equations (in short SPDE). It is divided into two parts, the first part concerns the obstacle problem for quasilinear SPDE and the second part solves quasilinear SPDE driven by $G-$Brownian motion.

In the first part we begin with the existence and uniqueness result for the obstacle problem of quasilinear stochastic partial differential equations (in short OSPDE). Our method is based on analytical technics coming from the parabolic potential theory. The solution is expressed as a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying minimal Skohorod condition.

Then we prove a maximum principle for a local solution of quasilinear stochastic partial differential equations with obstacle. The proofs are based on a version of Itô's formula and estimates for the positive part of a local solution which is negative on the lateral boundary.

The objective of the second part is to study the well-posedness of stochastic partial differential equations driven by $G-$Brownian motion in the framework of sublinear expectation spaces. One can also establish an Itô formula for the solution and a comparison theorem.

Contents

Introduction

L'objet de cette thèse est l'étude les Équations aux Dérivées Partielles Stochastiques (dans la suite EDPSs). Elle est divisée en deux parties : La première partie est consacrée au problème d'obstacle pour les EDPSs quasilinéaires. On étudie l'existence et l'unicité de la solution et on établit un principe de maximum pour les solutions locales. Dans la deuxième partie, on étudie les EDPSs dirigées par un $G-$mouvement brownien dans le cadre d'un espace muni d'une espérance sous-linéaire.

Le point de départ est l'EDPS suivante :

$$\begin{aligned} du_t(x) &= \partial_i\left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right)dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ &\quad + \sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \end{aligned} \tag{1}$$

où a est une matrice mesurable bornée symétrique qui définit un opérateur du seconde ordre sur $\mathcal{O}\subset\mathbb{R}^d$, avec la condition nulle au bord. La valeur initiale $u_0=\xi$ est une variable aléatoire prenant ses valeurs dans $L^2(\mathcal{O})$ et f, $g=(g_1,...,g_d)$ et $h=(h_1,...h_i,...)$ sont les fonctions aléatoires non linéaires. Pour un obstacle donné par $S:\Omega\times[0,T]\times\mathcal{O}\to\mathbb{R}$, nous étudions le problème d'obstacle pour l'EDPS (1), i.e. on veut trouver une solution de (1) qui satisfait "$u\geq S$" où l'obstacle S est régulier dans un certain sens et contrôlé par la solution d'une autre EDPS.

Nualart et Pardoux [55] ont étudié le problème d'obstacle pour l'équation de la chaleur non-linéaire sur l'intervalle spatial $[0,1]$ avec la condition de Dirichlet, dirigée par un bruit blanc spatio-temporel. Ils ont prouvé l'existence et l'unicité de la solution et leur méthode s'est fortement basée sur l'inégalité variationnelle déterministe. Donati-Martin et Pardoux [26] ont généralisé le modèle de Nualart et Pardoux. La non-linéarité apparaît à la fois dans le coefficient de drift et dans le coefficient de diffusion. Ils ont prouvé l'existence d'une solution par la méthode de pénalisation mais ils n'ont pas obtenu l'unicité. Ensuite en 2009, Xu et Zhang ont résolu le problème de l'unicité, voir [79]. Pourtant, dans tous leurs modèles, il n'y avait pas de terme de divergence c'est à dire qu'ils n'ont pas considéré le cas où les coefficients dépendent de ∇u.

Le travail de El Karoui et al [27] traite le problème d'obstacle pour les EDPs semilinéaires déterministes dans le cadre des équations différentielles stochastiques rétrogrades (en bref EDSR). L'équation (1) est considérée avec f dépendant de u et ∇u, alors que la fonction

g est nulle (aussi h) et l'obstacle S est continu. Ils ont considéré la solution de viscosité du problème d'obstacle pour l'équation (1) , et représenté cette solution comme un processus. Le nouvel objet principal de ce cadre d'EDSR est un processus croissant continu qui contrôle l'ensemble $\{u = v\}$. Bally et al [5] (voir [52]) souligne que la continuité de ce processus permet d'étendre la classe de notion de la solution variationnelle forte. (voir Théorème 2.2 de [7] p.238) et d'exprimer la solution d'un problème d'obstacle comme une paire (u, ν) où ν a son support inclus dans l'ensemble $\{u = v\}$.

Matoussi et Stoica [53] ont prouvé un résultat d'existence et d'unicité pour le problème d'obstacle de l'EDP stochastique quasilinéaire rétrograde sur l'espace $\mathbb{R}^d$ tout entier et dirigée par un mouvement brownien de dimension infinie. Leur méthode est basée sur l'interprétation probabiliste de la solution à l'aide de l'équation différentielle doublement stochastique rétrograde (en bref EDDSR). Ils ont aussi prouvé que la solution est une paire (u, ν) où u est un processus continu prévisible qui prend ses valeurs dans un espace de Sobolev et ν est une mesure régulière aléatoire qui satisfait la condition de Skohorod. En particulier, ils ont donné une interpréstation probabiliste de la mesure régulière ν en fonction d'un processus croissant continu K où (Y, Z, K) est la solution d'une EDSRD réfléchie généralisée.

Notre objectif est de prouver l'existence et l'unicité sous des hypothèses convenables sur ξ, f, g and h de l'EDPS suivante avec l'obstacle S, ce qu'on écrit formellement comme :

$$\begin{cases} du_t(x) = \partial_i \left(a_{i,j}(x) \partial_j u_t(x) + g_i(t, x, u_t(x), \nabla u_t(x)) \right) dt + f(t, x, u_t(x), \nabla u_t(x)) dt \\ \qquad\qquad + \displaystyle\sum_{j=1}^{+\infty} h_j(t, x, u_t(x), \nabla u_t(x)) dB_t^j, \\ u_t \geq S_t \, , \\ u_0 = \xi. \end{cases} \tag{2}$$

Heuristiquement, une paire (u, ν) est une solution d'un problème d'obstacle pour (2) si les conditions suivantes sont satisfaites :

1. $u \in \mathcal{H}_T$ et $u(t,x) \geq S(t,x)$, $dP \otimes dt \otimes dx - p.p.$ et $u_0(x) = \xi$, $dP \otimes dx - p.p.$;
2. ν est une mesure aléatoire définie sur $[0, T) \times \mathcal{O}$;
3. la relation suivante est vraie presque partout, pour tout $t \in [0, T]$ et $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^2(\mathcal{O})$,
$$\begin{aligned} &(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(u_s, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g_s^i(u_s, \nabla u_s), \partial_i \varphi_s) ds \\ &= \int_0^t (f_s(u_s, \nabla u_s), \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j(u_s, \nabla u_s), \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds); \end{aligned}$$
4. $$\int_0^T \int_{\mathcal{O}} (u(s,x) - S(s,x)) \nu(dx, ds) = 0, \quad p.s.$$

Mais la mesure aléatoire qui dans un certain sens oblige la solution à rester au-dessus de la barrière est un temps local donc, en général, elle n'est pas absolument continue par rapport à la mesure de Lebesgue. Par conséquent, par exemple, la condition

$$\int_0^T \int_{\mathcal{O}} (u(s,x) - S(s,x))\nu(dx,ds) = 0$$

n'a pas de sens. On doit donc considérer des version précisées de u et S définies $\nu-$presque surement.

Pour faire face à cette difficulté, nous introduisons la notion de capacité parabolique sur $[0,T] \times \mathcal{O}$ et la notion de version quasi-continue des fonctions introduites par Michel Pierre dans plusieurs articles (voit par exemple [70, 71]) dans lesquels il a étudié l'EDP parabolique avec obstacle à l'aide de potentiel parabolique. Il a prouvé que la solution existe uniquement et est quasi-continue. Remarquons que ces outils sont aussi utlisés par Klimsiak ([41]) pour obtenir une interprétation probabiliste de l'EDP semilinéaire avec obstacle.

Afin de donner une définition rigoureuse de la solution de (2), nous utiliserons les techniques de la théorie du potentiel parabolique développée par M. Pierre dans le cadre stochastique. Le point critique est de construire la solution qui admet une version quasi-continue donc définie en dehors d'un ensemble polaire et les mesures régulières qui en général ne sont pas absolument continues par rapport à la mesure de Lebesgue mais ne chargent pas les ensembles polaires. Nous établissons d'abord un résultat de quasi-continuité pour la solution de l'EDPS (1) avec la condition nulle au bord sur un domaine $\mathcal{O}$ et dirigée par un mouvement brownien de dimension infinie. Ce résultat n'est pas évident et la preuve s'appuie sur un argument de trajectoire et le résultat d'existence de Mignot et Puel [54] pour un problème d'obstacle de l'EDP déterministe. En plus, nous prouvons dans notre cadre que la mesure réfléchie ν est une mesure régulière aléatoire et nous donnons la représentation analytique d'une telle mesure en terme de potentiel parabolique au sens donné par M. Pierre in [70]. Le théorème principal qu'on obtient est :

Theorem 0.1. *Supposons que f, g et h satisfont des hypothèses de continuité de type Lipschitz et d'intégrabilité, $\xi \in L^2(\Omega \times \mathcal{O})$, que S est quasi-continu et $S_t \leq S'_t$ où S' est la solution de l'EDPS linéaire :*

$$\begin{cases} dS'_t &= LS'_t dt + f'_t dt + \sum_{i=1}^d \partial_i g'_{i,t} dt + \sum_{j=1}^{+\infty} h'_{j,t} dB^j_t \\ S'(0) &= S'_0, \end{cases}$$

avec $S'_0 \in L^2(\Omega \times \mathcal{O})$, f', g' et h' des processus adaptés de carré intégrable.
Alors il existe une unique solution (u,ν) du problème d'obstacle pour l'EDPS (2) associée à (ξ, f, g, h, S) i.e. u est un processes continu prévisible qui prend ses valeurs dans un espace de Sobolev, $u \geq S$ et ν est une mesure réguilère aléatoire tels que :

1. *la relation suivante est vraie presque surement, pour tout $t \in [0,T]$ et $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes$*

$\mathcal{C}_c^2(\mathcal{O})$,

$$(u_t,\varphi_t)-(\xi,\varphi_0)-\int_0^t(u_s,\partial_s\varphi_s)ds+\int_0^t\mathcal{E}(u_s,\varphi_s)ds+\sum_{i=1}^d\int_0^t(g_s^i(u_s,\nabla u_s),\partial_i\varphi_s)ds$$
$$=\int_0^t(f_s(u_s,\nabla u_s),\varphi_s)ds+\sum_{j=1}^{+\infty}\int_0^t(h_s^j(u_s,\nabla u_s),\varphi_s)dB_s^j+\int_0^t\int_{\mathcal{O}}\varphi_s(x)\nu(dx,ds).$$

2. u admet une version quasi-continue, $\widetilde{u}$, et on a la condition de Skohorod

$$\int_0^T\int_{\mathcal{O}}(\widetilde{u}(s,x)-S(s,x))\nu(dx,ds)=0\quad p.s.$$

Nous commençons la preuve par le cas linéaire, i.e. f, g et h ne dependent pas en u et ∇u. A l'aide de la méthode de pénalisation on prouve que le problème d'obstacle admet une unique solution. Et puis on établit une formule d'Itô par approximation afin d'utiliser l'itération de Picard pour obtenir le résultat dans le cas non linéaire. Grace à la formule d'Itô on peut déduire un théorème de comparaison pour les solutions des EDPS avec obstacle.

Dans le chapitre suivant nous prouvons une principe de maximum pour les solutions locales de l'EDPS quasilinéaire avec obstacle. On commence par l'EDPS avec obstacle (2).

Dans la théorie des Équations aux Dérivées Partielles déterministes, le principe de maximum joue un rôle important car elle donne la relation entre la borne de la solution sur le bord et la borne de la solution sur tout le domaine. Dans le cas déterministe, le principe du maximum pour les équations paraboliques quasilinéaires a été prouvé par Aronson-Serrin (voit Théorème 1 de [3]). Dans [23], Denis-Matoussi-Stoica ont adapté la méthode de Aronson-Serrin dans le cadre stochastique et prouvé la principe de maximum pour l'EDPS avec l'opérateur du second ordre homogène et dirigée par un mouvement brownien de dimension infinie. Les preuves sont basées sur la notion de semigroupe associée à l'opérateur du second ordre et sur la propriété de régularisation du semigroupe. Dans [24] Denis-Matoussi ont généralisé les résultats dans [23] au cas de l'EDPS avec opérateur du second ordre non-homogène et dirigée par un bruit qui est blanc par rapport au temps et coloré par rapport à l'espace. Les preuves sont basées sur la fonction de Green associée à l'opérateur non-homogène et les résultats de Aronson [1] concernant l'existence et les estimations Gaussiennes de la solution faible fondamentale de l'EDP parabolique.

Il existe une large littérature sur les EDPS paraboliques sans obstacle. L'étude de la norme L^p-de la norme uniforme sur l'espace-temps des trajectoires de l'EDP stochastique est initiée par N. V. Krylov dans [42], pour un aperçu plus complet des travaux existants sur ce sujet voir [23, 24] et les références indiquées. Concernant le problème d'obstacle, il y a deux approches, l'une est probabiliste (voir [53, 41]) basée sur la formule de Feynmann-Kac via les équations différentielles doublement stochastiques rétrogrades et l'autre est analytique (voir [26, 55, 79]) basée sur la fonction de Green.

À notre connaissance, jusqu'à maintenant il n'y a pas de principe de maximum pour l'EDPS quasilinéaire avec obstacle et même très peu de résultats dans le cas déterministe. L'objet de ce chapitre est d'obtenir, sous les hypothèses d'intégrabilité convenables sur les coefficients, des estimées de la norme L^p- de la norme uniforme (par rapport aux temps et espace) de la solution, un principe du maximum pour les solutions locales de l'équation (2) et des théorèmes de comparaison similaires à ceux obtenus dans le cas sans obstacle dans [21, 23]. On obtient, par exemple, le résultat suivant :

Theorem 0.2. *Soit $(M_t)_{t\geq 0}$ un processus d'Itô vérifiant certaines conditions d'intégrabilité, $p \geq 2$ et u une solution faible locale du problème d'obstacle (2). Supposons que $\partial\mathcal{O}$ est Lipschitzien et $u \leq M$ sur $\partial\mathcal{O}$, alors, pour tout $t \in [0,T]$:*

$$E \left\| (u-M)^+ \right\|_{\infty,\infty;t}^p \leq k\,(p,t)\,\mathcal{C}(S,f,g,h,M)$$

où $\mathcal{C}(S,f,g,h,M)$ dépend seulement de l'obstacle S, la condition initiale ξ, les coefficients f,g,h et la condition au bord M et k est une fonction qui dépend seulement des contantes de structure de l'EDPS. $\|\cdot\|_{\infty,\infty;t}$ désignant la norme uniforme sur $[0,t]\times\mathcal{O}$.

Remarquons qu'afin d'obtenir ce résultat, nous définissons la notion de solutions locales au problème d'obstacle, qui sont des solutions faibles sans la conditions au bord. Par exemple la solution obtenue dans une domaine plus grande $\mathcal{D} \supset \mathcal{O}$ avec la condition nulle au bord, quand considérée sur $\mathcal{O}$ devient une solution locale. Concernant le problème d'obstacle, nous avons besoin d'introduire ce que nous appelons les mesures régulières locales.

Ce chapitre est organisé comme suit : dans la deuxième section nous introduisons les notions et les hypothèses et détaillons les conditions d'intégrabilité qui seront utilisées dans ce chapitre. Dans la troisième section, nous prouvons un résultat d'existence et unicité pour l'EDPS avec obstacle (2) avec la condition nulle au bord sous les hypothèses d'intégrabilité plus faibles sur f et aussi donnons une estimation de la partie positive de la solution. Dans la quatrième section, nous établissons les estimées L^p- de la norme uniforme de la solution avec la condition de Dirichlet. La cinquième section est consacrée au résultat principal : la principe de maximum pour les solutions locales dont la preuve est basée sur une formule d'Itô satisfaite par la partie positive de la solution locale avec la condition au bord M. La dernière section est l'appendice dans laquelle nous donnons les preuves des lemmes.

L'objectif de la deuxième partie est d'étudier l'existence et l'unicité de solution des équations aux dérivées partielles stochastiques dirigées par un $G-$mouvement brownien (en bref EDPSG) dans le cadre d'un espace muni d'une espérance sous-linéaire.

Motivé par des problèmes d'incertitude, la notion de mesure de risque et le superhedging en finance, Peng [64, 65, 66] a introduit le $G-$mouvement brownien. L'espérance $\mathbb{E}[\cdot]$ associée au$G-$mouvement brownien est une espérance sous-linéaire qui est appelée $G-$espérance. Le calcul stochastique par rapport au $G-$mouvement brownien a été introduit dans [66]. Il y a eu plusieurs travaux concernant l'existence et l'unicité des équations différentielles stochastiques dirigées par $G-$mouvement brownien, voir [4, 30, 47, 48, 66] et des équations différentielles stochastiques rétrogrades, voir [66].

À notre connaissance, jusqu'à maintenant il n'y a pas eu de résultats sur les équations aux dérivées partielles stochastiques dirigées par un $G-$mouvement brownien. On veut étudier la solvabilité de l'équation aux dérivées partielles stochastique dirigée par un $G-$mouvement brownien suivante :

$$\begin{aligned} du_t(x) &= \partial_i \left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right) dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ &\quad + \sum_{j=1}^{d_1} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \end{aligned} \tag{3}$$

où a est une matrice mesurable bornée symmetrique qui définit un opérateur du second ordre sur $\mathcal{O} \subset \mathbb{R}^d$, avec la condition nulle au bord. La valeur initiale est donnée comme $u_0 = \xi \in L^2(\mathcal{O})$, et f, $g = (g_1, ..., g_d)$ et $h = (h_1, ...h_{d_1})$ sont des fonctions aléatoires nonlinéaires qui satisfont les conditions de Lipschitz avec les coefficients lipschitziens appropriés, B est un $G-$mouvement brownien de d_1-dimension.
À cet effet, nous avons besoin de développer le calcul stochastique pour les processus à valeurs dans un espace de Hilbert par rapport au $G-$mouvement brownien et de prouver l'inégalité de Burkholder-Davis-Gundy. L'existence et l'unicité de l'EDPSG est comme suivant :

Theorem 0.3. *Sous les hypothèses de continuité de type Lipschitz et d'intégrabilité de f, g et h, il existe une unique solution u de* (3) *dans un espace approprié.*

Puis on établit une formula d'Itô pour la solution de (3) ainsi qu'un théorème de comparaison.

Introduction

The objective of this thesis is to study the Quasilinear Stochastic Partial Differential Equations. It is divided into two parts. The first part concerns the obstacle problem for Quasilinear Stochastic Partial Differential Equations. We study the existence and uniqueness of solution and prove a maximum principle for local solutions. In the second part, we study quasilinear stochastic PDEs driven by $G-$Brownian motion in the framework of sublinear expectation spaces.

The starting point of the first part is the following parabolic stochastic partial differential equation (in short SPDE)

$$\begin{aligned} du_t(x) &= \partial_i \left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right)dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ &\quad + \sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \end{aligned} \tag{4}$$

where a is a symmetric bounded measurable matrix which defines a second order operator on $\mathcal{O} \subset R^d$, with null Dirichlet condition. The initial condition is given as $u_0 = \xi$, a $L^2(\mathcal{O})-$valued random variable, and f, $g = (g_1, ..., g_d)$ and $h = (h_1, ...h_i, ...)$ are non-linear random functions. The existence and uniqueness result for SPDE (4) without obstacle has been studied, see for example [17] and [20]. Now given an obstacle $S : \Omega \times [0,T] \times \mathcal{O} \to \mathbb{R}$, we study the obstacle problem for the SPDE (4), i.e. we want to find a solution of (4) which satisfies "$u \geq S$" where the obstacle S is regular in some sense and controlled by the solution of an SPDE.

Nualart and Pardoux [55] have studied the obstacle problem for a nonlinear heat equation on the spatial interval $[0,1]$ with Dirichlet boundary conditions, driven by an additive space-time white noise. They proved the existence and uniqueness of the solution and their method relied heavily on the results for a deterministic variational inequality. Donati-Martin and Pardoux [26] generalized the model of Nualart and Pardoux. The nonlinearity appears both in the drift and in the diffusion coefficients. They proved the existence of the solution by penalization method but they didn't obtain the uniqueness result. And then in 2009, Xu and Zhang solved the problem of the uniqueness, see [79]. However, in all their models, there isn't the term of divergence and they do not consider the case where the coefficients depend on ∇u.

The work of El Karoui and al [27] treats the obstacle problem for deterministic semilinear PDE's within the framework of backward stochastic differential equations (BSDE in short).

Namely the equation (4) is considered with f depending of u and ∇u, while the function g is null (as well h) and the obstacle v is continuous. They considered the viscosity solution of the obstacle problem for the equation (4) , they represented this solution stochastically as a process and the main new object of this BSDE framework is a continuous increasing process that controls the set $\{u = v\}$. Bally et al [5] (see also [52]) point out that the continuity of this process allows one to extend the classical notion of strong variational solution (see Theorem 2.2 of [7] p.238) and express the solution to the obstacle as a pair (u, ν) where ν is supported by the set $\{u = v\}$.

Matoussi and Stoica [53] have proved an existence and uniqueness result for the obstacle problem of backward quasilinear stochastic PDE on the whole space $\mathbb{R}^d$ and driven by a finite dimensional Brownian motion. The method is based on the probabilistic interpretation of the solution by using the backward doubly stochastic differential equation (in short DBSDE). They have also proved that the solution is a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying minimal Skohorod condition. In particular they gave for the regular measure ν a probabilistic interpretation in term of the continuous increasing process K where (Y, Z, K) is the solution of a reflected generalized BDSDE.

Our aim is to prove existence and uniqueness, under suitable assumptions on ξ, f, g and h, for the following SPDE with given obstacle S that we write formally as:

$$\begin{cases} du_t(x) = \partial_i \left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right)dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ \qquad\qquad + \displaystyle\sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \\ u_t \geq S_t\,, \\ u_0 = \xi. \end{cases} \tag{5}$$

Heuristically, a pair (u, ν) is a solution of the obstacle problem for (5) if we have the followings:

1. $u \in \mathcal{H}_T$ and $u(t,x) \geq S(t,x)$, $dP \otimes dt \otimes dx - a.e.$ and $u_0(x) = \xi$, $dP \otimes dx - a.e.$;
2. ν is a random measure defined on $[0,T) \times \mathcal{O}$;
3. the following relation holds almost surely, for all $t \in [0,T]$ and $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^2(\mathcal{O})$,
$$(u_t,\varphi_t) - (\xi,\varphi_0) - \int_0^t (u_s, \partial_s\varphi_s)ds + \int_0^t \mathcal{E}(u_s,\varphi_s)ds + \sum_{i=1}^d \int_0^t (g_s^i(u_s,\nabla u_s), \partial_i\varphi_s)ds$$
$$= \int_0^t (f_s(u_s,\nabla u_s),\varphi_s)ds + \sum_{j=1}^{+\infty}\int_0^t (h_s^j(u_s,\nabla u_s),\varphi_s)dB_s^j + \int_0^t\int_{\mathcal{O}} \varphi_s(x)\nu(dx,ds);$$
4. $$\int_0^T \int_{\mathcal{O}} (u(s,x) - S(s,x))\nu(dx,ds) = 0, \quad a.s..$$

But, the random measure which in some sense obliges the solution to stay above the barrier is a local time so, in general, it is not absolutely continuous w.r.t Lebesgue measure. As a consequence, for example, the condition

$$\int_0^T \int_{\mathcal{O}} (u(s,x) - S(s,x))\nu(dxds) = 0$$

makes no sense. Hence we need to consider precise version of u and S defined $\nu-$almost surely.

In order to tackle this difficulty, we introduce the notions of parabolic capacity on $[0,T]\times\mathcal{O}$ and quasi-continuous version of functions introduced by Michel Pierre in several works (see for example [70, 71]) in which he has studied the parabolic PDE with obstacle using the parabolic potential as a tool. He proved that the solution uniquely exists and is quasi-continuous. Let us remark that these tools were also used by Klimsiak ([41]) to get a probabilistic interpretation to semilinear PDE's with obstacle.

To give a rigorous definition to the notion of solution to (5), we will use the technics of parabolic potential theory developed by M. Pierre in the stochastic framework. The key point is to construct a solution which admits a quasi continuous version hence defined outside a polar set and that regular measures which in general are not absolutely continuous w.r.t. the Lebesgue measure, do not charge polar sets. Hence, we first prove a quasi-continuity result for the solution of the SPDE (4) with null Dirichlet condition on given domain $\mathcal{O}$ and driven by an infinite dimensional Brownian motion. This result is not obvious and its based on a mixing pathwise arguments and Mignot and Puel [54] existence result of the obstacle problem for some deterministic PDEs. Moreover, we prove in our context that the reflected measure ν is a regular random measure and we give the analytical representation of such measure in term of parabolic potential in the sense given by M. Pierre in [70]. The main theorem we obtain is:

Theorem 0.4. *Assume that f, g and h satisfy some Lipschitz continuity and integrability hypotheses, $\xi \in L^2(\Omega\times\mathcal{O})$, S is quasi-continuous and $S_t \le S'_t$ where S' is the solution of the linear SPDE*

$$\begin{cases} dS'_t &= LS'_t dt + f'_t dt + \sum_{i=1}^d \partial_i g'_{i,t} dt + \sum_{j=1}^{+\infty} h'_{j,t} dB^j_t \\ S'(0) &= S'_0, \end{cases}$$

where $S'_0 \in L^2(\Omega\times\mathcal{O})$, f', g' and h' are square integrable adapted processes.
Then there exists a unique solution (u,ν) of the obstacle problem for the SPDE (5) associated to (ξ, f, g, h, S) i.e. u is a predictable continuous process which takes values in a proper Sobolev space, $u \ge S$ and ν is a random regular measure such that:

1. *the following relation holds almost surely, for all $t\in[0,T]$ and $\forall\varphi\in\mathcal{C}_c^\infty(\mathbb{R}^+)\otimes\mathcal{C}_c^2(\mathcal{O})$,*

$$(u_t,\varphi_t) - (\xi,\varphi_0) - \int_0^t (u_s, \partial_s\varphi_s)ds + \int_0^t \mathcal{E}(u_s,\varphi_s)ds + \sum_{i=1}^d \int_0^t (g^i_s(u_s,\nabla u_s), \partial_i\varphi_s)ds$$
$$= \int_0^t (f_s(u_s,\nabla u_s),\varphi_s)ds + \sum_{j=1}^{+\infty}\int_0^t (h^j_s(u_s,\nabla u_s),\varphi_s)dB^j_s + \int_0^t\int_{\mathcal{O}} \varphi_s(x)\nu(dx,ds)$$

2. u admits a quasi-continuous version, $\widetilde{u}$, and we have the mininal Skohorod condition

$$\int_0^T \int_{\mathcal{O}} (\widetilde{u}(s,x) - S(s,x))\nu(dx,ds) = 0 \quad a.s.$$

We begin the proof with the linear case, i.e. f, g and h do not depend on u and ∇u. Using the method of penalization we proved that the obstacle problem admits a unique solution. Then we established an Itô formula by approximation in order to use Picard iteration to get the result in the nonlinear case. With the help of Itô's formula we deduced a comparison theorem for the solution of SPDE with obstacle.

In the following chapter we proved a maximum principle for local solutions of quasilinear stochastic PDEs with obstacle. We begin with the Stochastic PDEs with obstacle (5).

In the theory of deterministic Partial Differential Equations, the maximum principle plays an important role since it gives a relation between the bound of the solution on the boundary and a bound on the whole domain. In the deterministic case, the maximum principle for quasilinear parabolic equations was proved by Aronson-Serrin (see Theorem 1 of [3]). In [23], Denis-Matoussi-Stoica adapted the method of Aronson-Serrin to the stochastic framework and proved maximum principle for SPDE with homogeneous second order operator and driven by a finite dimensional Brownian motion. The proofs are based on the notion of semigroup associated to the second order operator and on the regularizing property of the semigroup. In [24] Denis-Matoussi generalized the results in [23] to the case of SPDE with non-homogeneous second order operator and driven by a noise which is white in time and colored in space. The proofs are based on the Green function associated to the operator and use heavily the results of Aronson [1] on the existence and the Gaussian estimates of the weak fundamental solution of a parabolic PDE.

There is a huge literature on parabolic SPDE's without obstacle. The study of the L^p-norms w.r.t. the randomness of the space-time uniform norm on the trajectories of a stochastic PDE was started by N. V. Krylov in [42], for a more complete overview of existing works on this subject see [23, 24] and the references therein. Concerning the obstacle problem, there are two approaches, a probabilistic one (see [53, 41]) based on the Feynmann-Kac's formula via the backward doubly stochastic differential equations and the analytical one (see [26, 55, 79]) based on the Green function.

To our knowledge, up to now there is no maximum principle result for quasilinear SPDE with obstacle and even very few results in the deterministic case. The aim of this chapter is to obtain, under suitable integrability conditions on the coefficients, L^p-estimates for the uniform norm (in time and space) of the solution, a maximum principle for local solutions of equation (5) and comparison theorems similar to those obtained in the without obstacle case in [21, 23]. This yields for example the following result:

Theorem 0.5. *Let $(M_t)_{t\geq 0}$ be an Itô process satisfying some integrability conditions, $p \geq 2$ and u be a local weak solution of the obstacle problem (5). Assume that $\partial\mathcal{O}$ is Lipschitz and $u \leq M$ on $\partial\mathcal{O}$, then for all $t \in [0,T]$:*

$$E\left\|(u-M)^+\right\|_{\infty,\infty;t}^p \leq k\,(p,t)\,\mathcal{C}(S,f,g,h,M)$$

where $\mathcal{C}(S,f,g,h,M)$ depends only on the barrier S, the initial condition ξ, coefficients f,g,h, the boundary condition M and k is a function which only depends on the structure constants of the SPDE, $\|\cdot\|_{\infty,\infty;t}$ is the uniform norm on $[0,t]\times\mathcal{O}$.

Let us remark that in order to get such a result, we define the notion of local solutions to the obstacle problem (5), which, roughly speaking, are weak solutions without conditions at the boundary. For example a solution obtained in a larger domain $\mathcal{D}\supset\mathcal{O}$ with null conditions on $\partial\mathcal{D}$, when regarded on $\mathcal{O}$ becomes a local solution. Regarding the obstacle problem, we need to introduce what we call *local regular measures*.

This chapter is organized as follows: in section 2 we introduce notations and hypotheses and we take care to detail the integrability conditions which are used all along this chapter. In section 3, we prove an existence and uniqueness result for the SPDE (5) with obstacle with null Dirichlet condition under a weaker integrability hypothesis on f and also give an estimate of the positive part of the solution. In section 4, we establish the L^p-estimate for uniform norm of the solution with null Dirichlet boundary condition. Section 5 is devoted to the main result: the maximum principle for local solutions whose proof is based on an Itô's formula satisfied by the positive part of any local solution with lateral boundary condition, M. The last section is an Appendix in which we give the proofs of several lemmas.

The objective of the second part of this thesis is to study the existence and uniqueness of solutions to stochastic partial differential equations driven by $G-$Brownian motion in the framework of sublinear expectation spaces (GSPDE for short).

Motivated by uncertainty problems, risk measures and the superhedging in finance, Peng [64, 65, 66] introduced $G-$Brownian motion. The expectation $\mathbb{E}[\cdot]$ associated with $G-$Brownian motion is a sublinear expectation which is called $G-$expectation. The stochastic calculus with respect to the $G-$Brownian motion has been established in [66]. There have been several works concerning the well-posedness of Stochastic Differential Equations driven by $G-$Brownian motion, see [4, 30, 47, 48, 66] and also Backward Stochastic Differential Equations, see [66].

To our knowledge, up to now there is no result about Stochastic PDEs driven by $G-$Brownian motion, we want to study the solvability of the following stochastic partial differential equation driven by $G-$Brownian motion:

$$\begin{aligned} du_t(x) &= \partial_i\left(a_{i,j}(x)\partial_j u_t(x)+g_i(t,x,u_t(x),\nabla u_t(x))\right)dt+f(t,x,u_t(x),\nabla u_t(x))dt\\ &\quad+\sum_{j=1}^{d_1}h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \end{aligned} \tag{6}$$

where a is a symmetric bounded measurable matrix which defines a second order operator on $\mathcal{O}\subset\mathbb{R}^d$, with null Dirichlet condition. The initial condition is given as $u_0=\xi\in L^2(\mathcal{O})$, and f, $g=(g_1,...,g_d)$ and $h=(h_1,...h_{d_1})$ are non-linear random functions which satisfy Lipschitz condition with proper Lipschitz coefficients, B is a d_1-dimensional $G-$Brownian motion.

For this purpose, we need to develop the stochastic calculus for Hilbert space valued process with respect to $G-$Brownian motion and to prove the Burkholder-Davis-Gundy's inequality. The existence and uniqueness result of GSPDE is as follows:

Theorem 0.6. *Under the assumptions Lipschitz continuity and integrability of f, g and h, there exists a unique solution u of* (6) *in a proper space.*

We can also establish an Itô formula for the solution of (6) and a comparison theorem.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we will recall some results which will be useful for us. The first one concerns the notion of parabolic potential. The so-called obstacle problem for Stochastic PDEs means that we want to find a solution of a SPDE which always stays above a given barrier. Hence the natural idea is that when the solution touch the barrier we add a force to push it up. In mathematical language, that is, we add a term of measure ν into the equation. This random measure ν is a local time, in general, it is not absolutely continuous with respect to Lebesgue measure. So, for example, the minimal Skorohod condition make no sense. We have to consider a precise version of solution which is defined $\nu-$almost surely. In order to do this, we follow the technics of parabolic potential theory developed by M. Pierre [70, 71] in stochastic framework. The key point is that we construct a solution which admits a quasi-continuous version hence defined outside a polar set and that regular measures do not charge polar sets. With the help of parabolic potential we can give an analytical representation of regular measures. This analytical representation will give us more information.
The second one is the wellposedness of SPDE without obstacle. The existence and uniqueness of solution, Itô's formula and comparison theorem will play a basic role in our work. See, for example, [17], [20] and [42].

1.2 Parabolic Potential Analysis

In this section we recall some definitions and results in [70] and [71]. We will specify some spaces and modify some notations for our convenience.
In [70] and [71], Michel Pierre has studied the obstacle problem for parabolic equation with Dirichlet boundary condition as follows:

$$\begin{cases} u \geq \varsigma, \quad u(0) = u_0, \quad u_t|_{\partial\mathcal{O}} = 0 \\ \dfrac{\partial u}{\partial t} - \triangle u \geq 0, \quad (\dfrac{\partial u}{\partial t} - \triangle u)(u - \varsigma) = 0. \end{cases} \tag{1.1}$$

where $\mathcal{O}$ is an open domain in $\mathbb{R}^d$, the obstacle $\varsigma : [0,T[\times\mathcal{O} \to \mathbb{R}$ and the initial value $u_0 : \mathcal{O} \to \mathbb{R}$.

When the obstacle is regular, the problem is well-posed (see [9], [10] and [11]). When the obstacle is not regular (only measurable), the problem has been studied by Mignot-Puel (see [54]) using the method of parabolic variational inequality. They have proved the existence of the smallest weak solution which can be viewed as a solution to the obstacle problem, but, in general, there isn't the uniqueness of the weak solutions. Michel Pierre used an analytical method to study the existence and uniqueness of solution to this problem. The fundamental difficult is that, unlike the case of elliptic case, the solution of parabolic equation is not quasi-continuous. The elliptic equation with obstacle is well-posed thanks to the fact that all the elements v of $H_0^1(\mathcal{O})$ admit a unique quasi-continuous version $\widetilde{v}$ satisfies that for all u of $H_0^1(\mathcal{O})$ with $-\triangle u \geq 0$,

$$\int_{[0,T[\times\mathcal{O}} \nabla u \nabla v = \int_{[0,T[\times\mathcal{O}} \widetilde{v} d(-\triangle u)$$

where $-\triangle u$ is a finite energy measure. To tackle this difficult, Michel Pierre began the problem with the case of quasi-continuous obstacle, in this case the solution uniquely exists and is quasi-continuous. Moreover, this solution coincide with the smallest solution of the inequality variational that has been founded by Mignot-Puel. Then he dealt with the case of quasi-s.c.s obstacle and the case of any obstacle which, in fact, can be reduced to the case of quasi-s.c.s. In these two cases he needed the so-called 'représentant précis' in order to give a quasi-continuous version of the solution. We will focus on the case of quasi-continuous obstacle.

1.2.1 Parabolic potentials

Let $\mathcal{O}$ be an open domain in $\mathbb{R}^d$. $L^2(\mathcal{O})$ is the space of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$. $H_0^1(\mathcal{O})$ denotes the usual Sobolev space with null Dirichlet conditions on $\mathcal{O}$, we shall denote $H_0^{-1}(\mathcal{O})$ its dual space. It is well-known that

$$H_0^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \cong (L^2(\mathcal{O}))'(= L^2(\mathcal{O})) \hookrightarrow H_0^{-1}(\mathcal{O}).$$

$\mathcal{K}$ denotes $L^\infty([0,T]; L^2(\mathcal{O})) \cap L^2([0,T]; H_0^1(\mathcal{O}))$ equipped with the norm:

$$\begin{aligned} \| v \|_{\mathcal{K}}^2 &= \| v \|_{L^\infty([0,T];L^2(\mathcal{O}))}^2 + \| v \|_{L^2([0,T];H_0^1(\mathcal{O}))}^2 \\ &= \sup_{t\in[0,T[} \| v_t \|^2 + \int_0^T \left(\| v_t \|^2 + \mathcal{E}(v_t) \right) dt. \end{aligned}$$

$\mathcal{C}$ denotes the space of continuous functions with compact support in $[0,T[\times\mathcal{O}$ and finally:

$$\mathcal{W} = \{\varphi \in L^2([0,T]; H_0^1(\mathcal{O}));\ \frac{\partial\varphi}{\partial t} \in L^2([0,T]; H^{-1}(\mathcal{O}))\},$$

endowed with the norm$\| \varphi \|_{\mathcal{W}}^2 = \| \varphi \|_{L^2([0,T];H_0^1(\mathcal{O}))}^2 + \| \dfrac{\partial\varphi}{\partial t} \|_{L^2([0,T];H^{-1}(\mathcal{O}))}^2$.
It is known (see [49]) that $\mathcal{W}$ is continuously embedded in $C([0,T]; L^2(\mathcal{O}))$, the set of $L^2(\mathcal{O})$-valued continuous functions on $[0,T]$. So without ambiguity, we will also consider $\mathcal{W}_T = \{\varphi \in \mathcal{W}; \varphi(T) = 0\}$, $\mathcal{W}^+ = \{\varphi \in \mathcal{W}; \varphi \geq 0\}$, $\mathcal{W}_T^+ = \mathcal{W}_T \cap \mathcal{W}^+$.

Definition 1.1. *An element $v \in \mathcal{K}$ is said to be a* ***parabolic potential*** *if it satisfies:*

$$\forall \varphi \in \mathcal{W}_T^+, \ \int_0^T -(\frac{\partial \varphi_t}{\partial t}, v_t)dt + \int_0^T \mathcal{E}(\varphi_t, v_t)dt \geq 0.$$

We denote by $\mathcal{P}$ the set of all parabolic potentials.

The next representation property will help us to define the so-called regular measure:

Proposition 1.2. *(Proposition 1.1 in [71]) Let $v \in \mathcal{P}$, then there exists a unique positive Radon measure on $[0,T[\times\mathcal{O}$, denoted by ν^v, such that:*

$$\forall \varphi \in \mathcal{W}_T \cap \mathcal{C}, \ \int_0^T (-\frac{\partial \varphi_t}{\partial t}, v_t)dt + \int_0^T \mathcal{E}(\varphi_t, v_t)dt = \int_0^T \int_{\mathcal{O}} \varphi(t,x)d\nu^v.$$

Moreover, v admits a right-continuous (resp. left-continuous) version $\widehat{v}$ (resp. $\bar{v}$) : $[0,T] \mapsto L^2(\mathcal{O})$.
Such a Radon measure, ν^v is called ***a regular measure*** *and we write:*

$$\nu^v = \frac{\partial v}{\partial t} + Av.$$

Remark 1.3. *As a consequence, we can also define for all $v \in \mathcal{P}$:*

$$v_T = \lim_{t \uparrow T} \bar{v}_t \ \in L^2(\mathcal{O}).$$

Proposition 1.2 is a consequence of the following lemma:

Lemma 1.4. *Let L be a positive linear mapping on $\mathcal{W}_T$, then there exists a unique positive Radon measure ν on $[0,T[\times\mathcal{O}$ such that:*

$$\forall v \in \mathcal{W}_T \cap \mathcal{C}, \quad L(v) = \int_{[0,T[\times\mathcal{O}} v(s,x)\nu(dxds).$$

The existence of the measure comes from Hahn-Banach theorem and the uniqueness is a consequence of the fact that $\mathcal{W}_T \cap \mathcal{C}$ is dense in $\mathcal{C}$.

Integral by parts yields the following proposition which describes the regular potential and the measure associated:

Proposition 1.5. *Let $v \in \mathcal{W}$, then*

1. $$(v \in \mathcal{P}) \Leftrightarrow (\frac{\partial v}{\partial t} + Av \geq 0 \ \ in \ \ L^2([0,T]; H_0^{-1}(\mathcal{O})), \quad v(0) \geq 0);$$

2. $\forall u \in \mathcal{C} \cap \mathcal{W}_T$,

$$\int_{[0,T[\times\mathcal{O}} u(t,x)\nu(dxdt) = (v(0), u(0)) + \int_0^T (\frac{\partial v}{\partial t} + Av, u)dt.$$

Each potential can be approximated by the regular potentials (i.e. the ones in $\mathcal{W}$):

Proposition 1.6. *(Proposition 1.3 in [71]) If $v \in \mathcal{P}$ and $v_0 = \lim_{h\to 0^+} \frac{1}{h}\int_0^h v(s)ds$, for all $\lambda > 0$, if v_λ is the solution of*

$$v^\lambda \in \mathcal{W},\ v^\lambda(0) = v_0,\ v^\lambda + \lambda(\frac{\partial v^\lambda}{\partial t} + Av^\lambda) = v.$$

then $v^\lambda \in \mathcal{P}$ and when λ tends to zero, v^λ increasingly converges to v in $L^2(0,T;L^2(\mathcal{O}))$ and weakly in $L^2(0,T;H_0^1(\mathcal{O}))$. Moreover, ν^{v^λ} converges vaguely to ν^v on $[0,T]\times\mathcal{O}$.

1.2.2 Capacity

Firstly, we define the capacity of compact sets. For this purpose, we introduce the following definition:

Definition 1.7. *Let $K \subset [0,T[\times\mathcal{O}$ be compact, $v \in \mathcal{P}$ is said to be $\nu-$superior than 1 on K, if there exists a sequence $v_n \in \mathcal{P}$ with $v_n \geq 1$ a.e. on a neighborhood of K converging to v in $L^2([0,T];H_0^1(\mathcal{O}))$.*

We denote:

$$\mathcal{S}_K = \{v \in \mathcal{P};\ v \text{ is } \nu - \text{superior to } 1 \text{ on } K\}.$$

Then we have:

Proposition 1.8. *(Proposition 2.1 in [71]) Let $K \subset [0,T[\times\mathcal{O}$ compact, then $\mathcal{S}_K$ admits a smallest element $v_K \in \mathcal{P}$ and the measure ν_K^v whose support is in K satisfies*

$$\int_0^T\int_{\mathcal{O}} d\nu_K^v = \inf_{v\in\mathcal{P}}\{\int_0^T\int_{\mathcal{O}} d\nu^v;\ v \in \mathcal{S}_K\}.$$

Now, we can give the definition of capacity of any borelian sets:

Definition 1.9. *(Parabolic Capacity)*

- *Let $K \subset [0,T[\times\mathcal{O}$ be compact, we define $cap(K) = \int_0^T\int_{\mathcal{O}} d\nu_K^v$;*
- *let $O \subset [0,T[\times\mathcal{O}$ be open, we define $cap(O) = \sup\{cap(K);\ K \subset O \text{ compact}\}$;*
- *for any borelian $E \subset [0,T[\times\mathcal{O}$, we define $cap(E) = \inf\{cap(O);\ O \supset E \text{ open}\}$.*

Definition 1.10. *A property is said to hold quasi-everywhere (in short q.e.) if it holds outside a set of null capacity.*

Proposition 1.11. *Let $K \subset \mathcal{O}$ a compact set, then $\forall t \in [0,T[$*

$$cap(\{t\}\times K) = \lambda_d(K),$$

where λ_d is the Lebesgue measure on $\mathcal{O}$.
As a consequence, if $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is a map defined quasi-everywhere then it defines uniquely a map from $[0,T[$ into $L^2(\mathcal{O})$. In other words, for any $t \in [0,T[$, u_t is defined without any ambiguity as an element in $L^2(\mathcal{O})$. Moreover, if $u \in \mathcal{P}$, it admits version $\bar{u}$ which is left continuous on $[0,T]$ with values in $L^2(\mathcal{O})$ so that $u_T = \bar{u}_{T^-}$ is also defined without ambiguity.

Corollary 1.12. *If $u^1,\ u^2 : [0,T[\times\mathcal{O} \to \mathbb{R}$, and $u^1 = u^2$, q.e., then $\forall t \in [0,T[,\ u^1(t) = u^2(t),\ \lambda_d - a.e.$.*

In other words, this corollary says that a function defined quasi-everywhere defines a unique function from $[0,T[$ to $L^2(\mathcal{O})$.

The next proposition, whose proof may be found in [70] or [71] (see Proposition 2.5 and Corollary 2.2 in [71]) shall play an important role in the sequel, for example, it ensures that the minimal Skohorod condition makes a sense (see Definition 2.22 in Chapter 2):

Proposition 1.13. *Regular measures do not charge polar sets (i.e. sets of capacity 0).*

Remark 1.14. *The capacity only depends on the space $H_0^1(\mathcal{O})$ in the sense that, if we define another capacity associated to another bilinear form on $H_0^1(\mathcal{O})\times H_0^1(\mathcal{O})$ (satisfying the same hypotheses as the former one), they will be equivalent.*

1.2.3 Quasi-continuity

Now, we come to another very important definition in our work:

Definition 1.15. *(Quasi-continuous) A function $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is called quasi-continuous, if there exists a decreasing sequence of open subsets O_n of $[0,T[\times\mathcal{O}$ with:*

1. *for all n, the restriction of u_n to the complement of O_n is continuous;*
2. $\lim_{n\to+\infty} cap\,(O_n) = 0$.

We say that u admits a quasi-continuous version, if there exists $\widetilde{u}$ quasi-continuous such that $\widetilde{u} = u$ a.e..

Proposition 1.16. *(Theorem 3.1 in [71]) If $\varphi \in \mathcal{W}$, then it admits a unique quasi-continuous version that we denote by $\widetilde{\varphi}$. Moreover, for all $v \in \mathcal{P}$, the following relation holds:*

$$\int_{[0,T[\times\mathcal{O}} \widetilde{\varphi} d\nu^v = \int_0^T (-\partial_t\varphi, v) + \mathcal{E}(\varphi, v)\, dt + (\varphi_T, v_T)\,.$$

The uniqueness comes from the following lemma:

Lemma 1.17. *Let $v : [0,T[\times\mathcal{O} \to \mathbb{R}$ be quasi-continuous, if $v \geq 0$ almost everywhere on $[0,T[\times\mathcal{O}$, then $v \geq 0$ quasi-everywhere on $[0,T[\times\mathcal{O}$.*

Corollary 1.18. *Let v^1 and v^2 be quasi-continuous, then*

$$(v^1 = v^2 \ almost-everywhere) \Rightarrow (v^1 = v^2 \ quasi-everywhere).$$

The proof of the existence is based on the following two lemmas. Before recalling the lemmas, we introduce a notation: for any function $\psi : [0,T[\times\mathcal{O} \to R$ and $v_0 \in L^2(\mathcal{O})$, we define

$$\kappa(\psi, v_0) = ess \inf\{v \in \mathcal{P};\ v \geq \psi \ a.e.,\ u(0) \geq v_0\}.$$

Lemma 1.19. *There exists $k > 0$ such that, for all $w \in \mathcal{W}$ and all $v_0 \in L^2(\mathcal{O})$,*

$$\| \kappa(w, v_0) \|_{\mathcal{K}} \leq k(\| w \|_{\mathcal{W}} + \| v_0 \|_{L^2(\mathcal{O})}).$$

Lemma 1.20. *There exists $C > 0$ such that, for all open set $\vartheta \subset [0,T[\times\mathcal{O}$ and $v \in \mathcal{P}$ with $v \geq 1$ a.e. on ϑ:*

$$cap\vartheta \leq C \| v \|_{\mathcal{K}}^2 .$$

That is: a set ϑ is of null capacity if there exists a sequence of elements of $\mathcal{W}$ which are superior than 1 on the neighborhood of ϑ converge to 0 in $\mathcal{W}$. This provides an idea to prove the quasi-continuity of a function. See, for example, Theorem 2.18 in Chapter 2 which concerns the quasi-continuity of the solution of SPDEs.

Corollary 1.21. *All the elements $u \in \mathcal{P}$ admits a quasi-s.c.i. version $\bar{u}$ defined by*

$$\bar{u} = \sup_{\lambda > 0} \ quasi \ ess \ \widetilde{u}^\lambda \quad q.e.$$

where $\widetilde{u}^\lambda$ is defined as in Proposition 1.6.
Moreover, $\bar{u} \in L^1(\nu^v)$ for all $v \in \mathcal{P}$ with the estimation

$$\int_0^T \int_{\mathcal{O}} \bar{u} d\nu^v \leq C \| u \|_{\mathcal{K}} \| v \|_{\mathcal{K}} . \tag{1.2}$$

Proof: From Proposition 1.6 we know that $\forall u \in \mathcal{P}$ there exists a sequence of $u^\lambda \in \mathcal{W}$ increasingly converges to u. By Proposition 1.16, we get the following relation

$$\begin{aligned} \int_0^T \int_{\mathcal{O}} \widetilde{u}^\lambda d\nu^v &= \int_0^T (-\frac{\partial u^\lambda}{\partial t}, v) + \int_0^T \mathcal{E}(u^\lambda, v) + (u_T^\lambda, v_T) \\ &\leq \int_0^T \mathcal{E}(u^\lambda, v) + \int_0^T \mathcal{E}(u^\lambda, v) + (u_T^\lambda, v_T) \\ &\leq C \| u^\lambda \|_{\mathcal{K}} \| v \|_{\mathcal{K}}, \end{aligned}$$

the second inequality is obtained from the fact that $u^\lambda \in \mathcal{P}$, then we take the sup and get the estimation (1.2). $\square$

Remark 1.22. *Note that $\bar{u}$ is left-continuous from $[0,T]$ to $L^2(\mathcal{O})$, hence, it is not a 'good' representative of the solution of a parabolic equation.*

We end this subsection by a convergence lemma which plays an important role in our approach (Lemma 3.8 in [71]), see, for example, Theorem 2.25 in Chapter 2 and Theorem 3.22 in Chapter 3:

Lemma 1.23. *If $v^n \in \mathcal{P}$ is a bounded sequence in $\mathcal{K}$ and converges weakly to v in $L^2(0,T;H_0^1(\mathcal{O}))$; if u^n is a sequence of quasi-continuous functions and $|u^n|$ is bounded by a element in $\mathcal{P}$. Suppose that there exists $w_0 \in \mathcal{P}$ and u quasi-continuous with*

$$\lim_{n\to\infty} \parallel u - u^n \parallel_{\bar{w}_0} = 0,$$

where $\parallel u - u^n \parallel_{\bar{w}_0} = \inf\{\alpha \in]0,\infty],\ |u-u^n| \leq \alpha \bar{w}_0 \text{ q.e.}\}$. Then

$$\lim_{n\to+\infty} \int_0^T \int_{\mathcal{O}} u^n d\nu^{v^n} = \int_0^T \int_{\mathcal{O}} u d\nu^v.$$

1.2.4 Applications to PDE's with obstacle

For any function $\psi : [0,T[\times\mathcal{O} \to R$ and $u_0 \in L^2(\mathcal{O})$, following M. Pierre [70, 71], F. Mignot and J.P. Puel [54], we define

$$\kappa(\psi, u_0) = ess\inf\{u \in \mathcal{P};\ u \geq \psi \ a.e.,\ u(0) \geq u_0\}. \tag{1.3}$$

This lower bound exists and is an element in $\mathcal{P}$. Moreover, when ψ is quasi-continuous, this potential is the solution of the following reflected problem:

$$\kappa \in \mathcal{P},\ \kappa \geq \psi,\ \frac{\partial \kappa}{\partial t} + A\kappa = 0 \ on \ \{u > \psi\},\ \kappa(0) = u_0.$$

Mignot and Puel have proved in [54] that $\kappa(\psi, u_0)$ is the limit (increasingly and weakly in $L^2(0,T;H_0^1(\mathcal{O}))$) when ϵ tends to 0 of the solution of the following penalized equation

$$u_\epsilon \in \mathcal{W},\ u_\epsilon(0) = u_0,\ \frac{\partial u_\epsilon}{\partial t} + Au_\epsilon - \frac{(u_\epsilon - \psi)^-}{\epsilon} = 0.$$

Let us point out that they obtain this result in the more general case where ψ is only measurable from $[0,T[$ into $L^2(\mathcal{O})$.
For given $f \in L^2(0,T;H^{-1}(\mathcal{O}))$, we denote by $\kappa_{u_0}^f$ the solution of the following problem:

$$\kappa \in \mathcal{W},\ \kappa(0) = u_0,\ \frac{\partial \kappa}{\partial t} + A\kappa = f.$$

The next theorem ensures existence and uniqueness of the solution of parabolic PDE with quasi-continuous obstacle, it is proved in [70], Theorem 1.1. The proof is based on a regularization argument of the obstacle:

Theorem 1.24. *Let $\psi : [0,T[\times\mathcal{O} \to \mathbb{R}$ be quasi-continuous, suppose that there exists $\zeta \in \mathcal{P}$ with $|\psi| \le \zeta$ a.e., $f \in L^2(0,T;H^{-1}(\mathcal{O}))$, and the initial value $u_0 \in L^2(\mathcal{O})$ with $u_0 \ge \psi(0)$, then there exists a unique $u \in \kappa_{u_0}^f + \mathcal{P}$ quasi-continuous such that:*

$$u(0) = u_0,\ \widetilde{u} \ge \psi,\ q.e.; \quad \int_0^T \int_{\mathcal{O}} (\widetilde{u} - \widetilde{\psi}) d\nu^{u-\kappa_{u_0}^f} = 0.$$

The first basic tool is the well-posedness of the obstacle problem for parabolic equation in the case where the obstacle is 'regular', see Charrier-Troianiello (see [11], [10]):

Theorem 1.25. *(regular obstacle) Let $\psi \in \mathcal{W} \cap \mathcal{P} - \mathcal{W} \cap \mathcal{P}$ and $u_0 \in L^2(\mathcal{O})$, and*

$$u = \inf ess\{v \in \kappa_{u_0}^0 + \mathcal{P}, v \ge \psi \ a.e.\}.$$

Then $u \in \mathcal{W} \cap (\kappa_{u_0}^0 + \mathcal{P})$ and we have

$$\frac{\partial u}{\partial t} + Au \le (\frac{\partial \psi}{\partial t} + A\psi)^+ \ in \ L^2(0,T;H^{-1}(\mathcal{O})).$$

Moreover, u is unique element of $\mathcal{W}$ such that

$$u(0) = u_0 \vee \psi(0), \ \int_0^T (\frac{\partial u}{\partial t} + Au, u - \psi) = 0.$$

The following lemma allows us to approximate quasi-continuous obstacle by 'regular' processes.

Lemma 1.26. *Let $u \in \mathcal{P}$ quasi-continuous, then there exists a sequence of $u^n \in \mathcal{W} \cap \mathcal{C}$ and $w \in \mathcal{P}$ with*

$$\lim_{n\to\infty} \parallel u^n - u \parallel_{L^2(0,T;H_0^1(\mathcal{O}))} = \lim_{n\to\infty} \parallel \widetilde{u}^n - \widetilde{u} \parallel_{\bar{w}} = 0.$$

This lemma is a consequence of the density of $\mathcal{W} \cap \mathcal{C}$ in $\mathcal{C}$.

Remark 1.27. *With the help of the 'représentant précis' (see [71]), the similar existence and uniqueness result of the general case, for example, the obstacle is quasi-s.c.s., can be obtained. But in our work, we only concentrate on the case of quasi-continuous obstacle.*

1.3 Existence and uniqueness for SPDE

In this section, using an analytical method, we prove an existence and uniqueness result for the following stochastic partial differential equation (see, for example, [17], [20] and [42]):

$$\begin{aligned} du_t(x) &= Lu_t(x)dt + f(t,x,u_t(x),\nabla u_t(x))dt + \sum_{i=1}^d \partial_i g_i(t,x,u_t(x),\nabla u_t(x))dt \\ &+ \sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \end{aligned} \tag{1.4}$$

where L is an elliptic second order symmetric differential operator defined on some domain $\mathcal{O} \subset \mathbb{R}^d$, with Dirichlet boundary condition. We also establish an Itô formula and a comparison theorem for the solution. These results play a basic role in our work.

1.3.1 Hypotheses and definitions

We consider a sequence $((B^i(t))_{t\geq 0})_{i\in\mathbb{N}^*}$ of independent Brownian motions defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions.

Let $\mathcal{O} \subset \mathbb{R}^d$ be an open domain in $\mathbb{R}^d$ and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$. Let A be a symmetric second order differential operator given by

$$A := -L = -\sum_{i,j=1}^{d} \partial_i(a^{i,j}\partial_j).$$

We assume that $a = (a^{i,j})_{i,j}$ is a measurable symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{d} a^{i,j}(x)\xi^i\xi^j \leq \Lambda|\xi|^2, \ \forall x \in \mathcal{O}, \ \xi \in \mathbb{R}^d,$$

where λ and Λ are positive constants.

Let $(F, \mathcal{E})$ be the associated Dirichlet form given by $F := \mathcal{D}(A^{1/2}) = H_0^1(\mathcal{O})$and

$$\mathcal{E}(u,v) := (A^{1/2}u, A^{1/2}v) \ and \ \mathcal{E}(u) = \parallel A^{1/2}u \parallel^2, \ \forall u, v \in F$$

where $(\cdot,\cdot)$ and $\parallel \cdot \parallel$ are respectively the inner product and the norm on $L^2(\mathcal{O})$. $H_0^1(\mathcal{O})$ is the first order Sobolev space of functions vanishing at the boundary.

We consider the quasilinear stochastic partial differential equation (1.4) with initial condition $u(0,\cdot) = \xi(\cdot)$ and Dirichlet boundary condition $u(t,x) = 0, \ \forall \ (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O}$.

We assume that we have predictable random functions

$$\begin{aligned}
&f : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \\
&g = (g_1, ..., g_d) : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \\
&h = (h_1, ..., h_i, ...) : \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{\mathbb{N}^*}.
\end{aligned}$$

In the sequel, $|\cdot|$ will always denote the underlying Euclidean or l^2-norm. For example

$$|h(t,\omega,x,y,z)|^2 = \sum_{i=1}^{+\infty} |h_i(t,\omega,x,y,z)|^2.$$

Assumption (H): There exist non negative constants $C, \ \alpha, \ \beta$ such that for almost all ω, the following inequalities hold for all $(x,y,z,t) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$:

1. $|f(t,\omega,x,y,z) - f(t,\omega,x,y',z')| \leq C(|y-y'| + |z-z'|)$,
2. $(\sum_{i=1}^{d} |g_i(t,\omega,x,y,z) - g_i(t,\omega,x,y',z')|^2)^{\frac{1}{2}} \leq C|y-y'| + \alpha|z-z'|$,
3. $(|h(t,\omega,x,y,z) - h(t,\omega,x,y',z')|^2)^{\frac{1}{2}} \leq C|y-y'| + \beta|z-z'|$,
4. the contraction property: $2\alpha + \beta^2 < 2\lambda$.

With the uniform ellipticity condition we have the following equivalent conditions:

$$\| f(u, \nabla u) - f(v, \nabla v) \| \leq C \| u - v \| + C\lambda^{-1/2}\mathcal{E}^{1/2}(u-v),$$

$$\| g(u, \nabla u) - g(v, \nabla v) \|_{L^2(\mathcal{O};\mathbb{R}^d)} \leq C \| u - v \| + \alpha\lambda^{-1/2}\mathcal{E}^{1/2}(u-v),$$

$$\| h(u, \nabla u) - h(v, \nabla v) \|_{L^2(\mathcal{O};\mathbb{R}^{\mathbb{N}^*})} \leq C \| u - v \| + \beta\lambda^{-1/2}\mathcal{E}^{1/2}(u-v).$$

Moreover for simplicity, we fix a terminal time $T > 0$, we assume that

Assumption (I):

$$\xi \in L^2(\Omega \times \mathcal{O}) \textit{ is an } \mathcal{F}_0 - \textit{measurable random variable}$$

$$f(\cdot,\cdot,\cdot,0,0) := f^0 \in L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R})$$
$$g(\cdot,\cdot,\cdot,0,0) := g^0 = (g_1^0, ..., g_d^0) \in L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^d)$$
$$h(\cdot,\cdot,\cdot,0,0) := h^0 = (h_1^0, ..., h_i^0, ...) \in L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^{\mathbb{N}^*}).$$

Now we introduce the notion of weak solution.
We denote by $\mathcal{H}_T$ the space of $H_0^1(\mathcal{O})-$valued predictable $L^2(\mathcal{O})-$continuous processes $(u_t)_{t\in[0,T]}$ which satisfy

$$\left(E \sup_{t\in[0,T]} \| u_t \|^2 + E\int_0^T \mathcal{E}(u_t)dt \right)^{1/2} < +\infty.$$

The space $\mathcal{H}_T$ is the basic space in which we are going to look for solutions.
The space of test functions is denote by $\mathcal{D} = \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^2(\mathcal{O})$, where $\mathcal{C}_c^\infty(\mathbb{R}^+)$ is the space of all real valued infinite differentiable functions with compact support in $\mathbb{R}^+$ and $\mathcal{C}_c^2(\mathcal{O})$ the set of C^2-functions with compact support in $\mathcal{O}$.

Definition 1.28. *(Weak solution) We say that $u \in \mathcal{H}_T$ is a weak solution of the equation (1.4) with initial condition $u_0 = \xi$ the following relation holds almost surely, for all $t \in [0,T]$ and $\forall\varphi \in \mathcal{D}$,*

$$(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s\varphi_s)ds + \int_0^t \mathcal{E}(u_s, \varphi_s)ds + \sum_{i=1}^d \int_0^t (g_s^i(u_s, \nabla u_s), \partial_i\varphi_s)ds$$
$$= \int_0^t (f_s(u_s, \nabla u_s), \varphi_s)ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j(u_s, \nabla u_s), \varphi_s)dB_s^j.$$

Definition 1.29. *(Mild solution) We call $u \in \mathcal{H}_T$ a mild solution of the equation (1.4) with initial condition $u_0 = \xi$ if the following equality is verified almost surely, for each $t \in [0,T]$,*

$$u_t = P_t\xi + \int_0^t P_{t-s}f_s ds + \int_0^t P_{t-s}div g_s ds + \sum_{j=1}^{+\infty} \int_0^t P_{t-s}h_s^j dB_s^j. \tag{1.5}$$

We begin with proving all the terms in the RHS of (1.5) are well defined.

Remark 1.30. *(The spectral theorem for self-adjoint operators) Let A be a self-adjoint operator, we recall that if $x \in H$, then for all $t > 0$, $P_t x \in \mathcal{D}(A)$, and the map $t \to P_t x$ is $H-$continuous on $[0, +\infty[$ and $H-$differentiable on $]0, +\infty[$ and its derivative is*

$$\partial_t P_t x = -AP_t x = -P_t A x.$$

The spectral decomposition of A is:

$$A = \int_0^{+\infty} \lambda dE_\lambda,$$

where E_λ is the resolution of identity associated to A. So

$$P_t x = \int_0^{+\infty} e^{-\lambda t} dE_\lambda x.$$

1.3.2 Mild solution

Lemma 1.31. *Let ξ be in $L^2(\mathcal{O} \times \Omega)$. Then*

1. *the process $\Gamma :\ t \in [0,T] \to P_t \xi$ admits a version in $\mathcal{H}_T$;*
2. *for all $\varphi \in \mathcal{D}$ and for all $t \in [0,T]$, we have*

$$\int_0^t (\Gamma_s, \partial_s \varphi_s) ds = (\Gamma_t, \varphi_t) - (\xi, \varphi_0) + \int_0^t \mathcal{E}(\Gamma_s, \varphi_s) ds \quad P - a.e. \tag{1.6}$$

3. *For all $0 \le s \le t \le T$, $\int_s^t \Gamma_u du$ belongs to $\mathcal{D}(L)$ $P-$a.e. and*

$$\Gamma_t - \Gamma_s = L(\int_s^t \Gamma_u du)\,, \quad P - a.e.$$

Proof. : We fix $\omega \in \Omega$, $\xi(\omega) \in L^2(\mathcal{O})$. It is well known that $\forall\ t \in]0,T]$, $P_t \xi(\omega) \in F$. From now on, we omit ω from the notation.
By Remark 1.30, we have

$$\forall t \in [0,T],\ P_t \xi = \int_0^{+\infty} e^{-\lambda t} dE_\lambda \xi\,.$$

Therefore,

$$\begin{aligned}
\mathcal{E}(P_t\xi, P_t\xi) &= -(LP_t\xi, P_t\xi) = -(\frac{\partial P_t \xi}{\partial t}, P_t\xi) = -(\frac{\partial}{\partial t}\int_0^\infty e^{-\lambda t} dE_\lambda \xi, \int_0^\infty e^{-\lambda t} dE_\lambda \xi) \\
&= (\int_0^\infty \lambda e^{-\lambda t} dE_\lambda \xi, \int_0^\infty e^{-\lambda t} dE_\lambda \xi) = \int_0^\infty \lambda e^{-2\lambda t} d(E_\lambda \xi, \xi)
\end{aligned}$$

and

$$\| P_t \xi \|^2 = (\int_0^\infty e^{-\lambda t} dE_\lambda \xi, \int_0^\infty e^{-\lambda t} dE_\lambda \xi) = \int_0^\infty e^{-2\lambda t} d(E_\lambda \xi, \xi)\,.$$

Hence,

$$\forall t \in [0,T],\ \| P_t\xi \|_F^2 = \mathcal{E}(P_t\xi, P_t\xi) + \| P_t\xi \|^2 = \int_0^{+\infty} (1+\lambda) e^{-2\lambda t} d(E_\lambda \xi, \xi).$$

This yields

$$\begin{aligned}
\int_0^T \| P_t\xi \|_F^2 \, dt &= \int_0^\infty (1+\lambda) \frac{1-e^{-2\lambda T}}{2\lambda} d(E_\lambda\xi,\xi) \\
&= \int_0^\infty \frac{1-e^{-2\lambda T}}{2\lambda} d(E_\lambda\xi,\xi) + \int_0^\infty \frac{1-e^{-2\lambda T}}{2} d(E_\lambda\xi,\xi) \\
&\leq \int_0^\infty T d(E_\lambda\xi,\xi) + \int_0^\infty \frac{1}{2} d(E_\lambda\xi,\xi) \\
&= (T+\frac{1}{2}) \| \xi \|^2 < +\infty,
\end{aligned}$$

which proves 1.
Assume first that $\xi \in \mathcal{D}(L)$. Then, for all $t \in [0,T]$, $\int_0^t P_s\xi ds$ belongs to $\mathcal{D}(L)$, and the map $t \in [0,T] \to P_t\xi$ is $L^2(\mathcal{O})-$differentiable, $\frac{\partial P_t\xi}{\partial t} = L(P_t\xi)$, so

$$\forall t \in [0,T],\quad P_t\xi - \xi = \int_0^t L(P_s\xi)ds = L(\int_0^t P_s\xi ds),$$

which is assertion 3.
Moreover, for all $\varphi \in \mathcal{D}$,

$$\begin{aligned}
\partial_t(P_t\xi, \varphi_t) &= (LP_t\xi, \varphi_t) + (P_t\xi, \partial_t\varphi_t) \\
&= -\mathcal{E}(P_t\xi, \varphi_t) + (P_t\xi, \partial_t\varphi_t)
\end{aligned}$$

Then we get the relation (1.6) by integrating by part. This relation will be used to obtain the equivalence between weak solution and mild solution.
For the general case, $\xi \in L^2(\mathcal{O})$, there exists a sequence ξ^n in $\mathcal{D}(L)$ which converges to ξ in $L^2(\mathcal{O})$. Thanks to the proof of 1, we know that $(P_t\xi^n)$ converges to $P_t\xi$ in $L^2([0,T]; H_0^1(\mathcal{O}))$ which yields 2 by density.
Moreover, for $t \in [0,T]$, by 3, we have, for all $n,\ m \in \mathbb{N}^*$,

$$L(\int_0^t P_u\xi^n du - \int_0^t P_u\xi^m) = P_t(\xi^n - \xi^m) - (\xi^n - \xi^m).$$

As P_t is continuous on $L^2(\mathcal{O})$, it is clear that $(L(\int_0^t P_u\xi^n du))$ is a Cauchy sequence in $L^2(\mathcal{O})$ and so converges. As L is a closed operator, we conclude that $\int_0^t P_u\xi du$ belongs to $\mathcal{D}(L)$ and that

$$L(\int_0^t P_u\xi du) = \lim_{n\to+\infty} L(\int_0^t P_u\xi^n du).$$

∎

Remark 1.32. *We have the following relation*

$$(E_\lambda\xi, E_\lambda\xi) = (E_\lambda^* E_\lambda\xi, \xi) = (E_\lambda^2\xi, \xi) = (E\xi, \xi).$$

Lemma 1.33. *Let $f \in L^2([0,T] \times \Omega \times \mathcal{O})$ and adapted. Then*

1. *the process $\alpha : \ t \in [0,T] \to \int_0^t P_{t-s} f_s ds$ admits a version in $\mathcal{H}_T$;*
2. *for all $\varphi \in \mathcal{D}$ and all $t \in [0,T]$, we have*

$$\int_0^t (\alpha_s, \partial_s \varphi_s) ds = (\alpha_t, \varphi_t) - \int_0^t (f_s, \varphi_s) dt + \int_0^t \mathcal{E}(\alpha_s, \varphi_s) ds \quad P-a.e.$$

3. *for all $0 \le s \le t \le T$, $\int_s^t \alpha_u du$ belongs to $\mathcal{D}(L)$ $P-a.e.$ and*

$$\alpha_t - \alpha_s = L(\int_s^t \alpha_u du) + \int_s^t f_u du \quad P-a.e.$$

Proof. Assume firts that $f \in C^1([0,T]) \otimes L^2(\Omega) \otimes \mathcal{D}(L)$ and is adapted, we fix $\omega \in \Omega$, so for all $t \in [0,T]$, $\alpha_t(\omega) \in \mathcal{D}(L)$ and $t \to \alpha_t(\omega)$ is $L^2(\mathcal{O})-$differentiable and satisfies

$$\forall \ t \in [0,T], \ \frac{d\alpha_t}{dt}(\omega) = f_t(\omega) + L\alpha_t(\omega).$$

From now on we omit ω from the notation.
Integrating by part we get, for all $\varphi \in \mathcal{D}$ and all $t \in [0,T]$,

$$\int_0^t (\alpha_s, \partial_s \varphi_s) ds = (\alpha_t, \varphi_t) - \int_0^t (f_s, \varphi_s) ds + \int_0^t \mathcal{E}(\alpha_s, \varphi_s) ds.$$

Moreover, still integrating by part, we have, for all $t \in [0,T]$,

$$\begin{aligned} \| \alpha_t \|^2 &= 2 \int_0^t (\partial_s \alpha_s, \alpha_s) ds \\ &= 2 \int_0^t (f_s + L\alpha_s, \alpha_s) ds \\ &= 2 \int_0^t (f_s, \alpha_s) ds - 2 \int_0^t \mathcal{E}(\alpha_s) ds. \end{aligned}$$

This yields

$$\| \alpha_t \|^2 + 2 \int_0^t \mathcal{E}(\alpha_s) ds = 2 \int_0^t (f_s, \alpha_s) ds \le \int_0^t (\| f_s \|^2 + \| \alpha_s \|^2) ds.$$

Taking the supreme, we get

$$\sup_{t \in [0,T]} \| \alpha_t \|^2 \le \int_0^T \| f_t \|^2 \, dt + \int_0^T \sup_{t \in [0,T]} \| \alpha_t \|^2 \, dt.$$

Thanks to the Grownall's lemma, we have

$$\sup_{t \in [0,T]} \| \alpha_t \|^2 \le e^T \int_0^T \| f_t \|^2 \, dt,$$

and

$$2\int_0^T \mathcal{E}(\alpha_t)dt \leq \int_0^T \parallel f_t \parallel^2 + \parallel \alpha_t \parallel^2 dt \leq (1+Te^T)\int_0^T \parallel f_t \parallel^2 dt.$$

By density argument, we get 1 and 2.
Consider $0 \leq s \leq t \leq T$, $f \in L^2([0,T]\times\Omega\times\mathcal{O})$ and a sequence $(f^n)_{n\in\mathbb{N}^*}$ of elements in $C^1([0,T])\otimes L^2(\Omega)\otimes\mathcal{D}(L)$ which converges to f in $L^2([0,T]\times\Omega\times\mathcal{O})$. We put

$$\forall n\in\mathbb{N}^*,\ \forall t\in[0,T],\ \alpha_t^n = \int_0^t P_{t-s}f_s^n ds.$$

It is clear that for all $n\in\mathbb{N}^*$ and $P-$almost all $\omega\in\Omega$, $\int_s^t \alpha_u^n du \in \mathcal{D}(L)$ and

$$L(\int_s^t \alpha_u^n du) = \alpha_t^n - \alpha_s^n - \int_s^t f_u^n du.$$

Thanks to the relations we have established at the beginning of this proof, we conclude that $\int_s^t \alpha_u^n du$ converges to $\int_s^t \alpha_u du$ in $L^2(\mathcal{O})$ and that moreover, $L(\int_s^t \alpha_u^n du)$ converges in $L^2(\mathcal{O})$ to $\alpha_t - \alpha_s - \int_s^t f_u du$, for $P-$almost all $\omega\in\Omega$. This ensures that $\int_s^t \alpha_u du$ belongs to $\mathcal{D}(L)$ and that

$$\alpha_t - \alpha_s = L(\int_s^t \alpha_u du) + \int_s^t f_u du.$$

∎

Lemma 1.34. *Let g be in $L^2([0,T]\times\Omega\times\mathcal{O})$ and adapted. Then*

1. *the process $\gamma:\ t\to\int_0^t P_{t-s}divg_s ds$ admits a version in $\mathcal{H}_T$;*
2. *for all $\varphi\in\mathcal{D}$ and for all $t\in[0,T]$, we have*

$$\int_0^t (\gamma_s,\partial_s\varphi_s)ds = (\gamma_t,\varphi_t) + \int_0^t (g_s,\partial\varphi_s)ds + \int_0^t \mathcal{E}(\gamma_s,\varphi_s)ds \quad P-a.e.$$

3. *for all $0\leq s\leq t\leq T$, $\int_s^t \gamma_u du$ belongs to $\mathcal{D}(L)$ $P-a.e.$ and*

$$\gamma_t - \gamma_s = L(\int_s^t \gamma_u du) + \int_s^t divg_u du \quad P-a.e.$$

Proof. Assume first that $g\in C^1([0,T])\otimes L^2(\Omega)\otimes\mathcal{D}(L^{3/2})$ and is adapted. It is clear that $divg\in C^1([0,T])\otimes L^2(\Omega)\otimes\mathcal{D}(L)$ and is adapted.

We fix $\omega\in\Omega$, for all $t\in[0,T]$, $\gamma_t(\omega)\in\mathcal{D}(L)$ and $t\to\gamma_t(\omega)$ is $L^2(\mathcal{O})-$differentiable and satisfies

$$\forall\ t\in[0,T],\ \frac{d\gamma_t}{dt}(\omega) = divg_t(\omega) + L\gamma_t(\omega).$$

From now on, we omit ω from the notation.
Integrating by part, for all $\varphi\in\mathcal{D}$ and for all $t\in[0,T]$, we get:

$$\begin{aligned}\int_0^t (\gamma_s,\partial_s\varphi_s)ds &= (\gamma_t,\varphi_t) - \int_0^t (divg_s,\varphi_s)ds + \int_0^t \mathcal{E}(\gamma_s,\varphi_s)ds\\ &= (\gamma_t,\varphi_t) + \int_0^t (g_s,\partial\varphi_s)ds + \int_0^t \mathcal{E}(\gamma_s,\varphi_s)ds.\end{aligned}$$

Moreover, still integrating by part, we obtain, $\forall t \in [0,T]$,

$$\begin{aligned}
\| \gamma_t \|_{L^2}^2 &= 2\int_0^t (\partial_s \gamma_s, \gamma_s) ds = 2\int_0^t (div g_s + L\gamma_s, \gamma_s) ds \\
&= 2\int_0^t (div g_s, \gamma_s) ds + 2\int_0^t (L\gamma_s, \gamma_s) ds \\
&= -2\int_0^t (g_s, \partial \gamma_s) ds - 2\int_0^t \mathcal{E}(\gamma_s) ds \qquad (1.7)
\end{aligned}$$

Using the inequality $ab \leq c_\epsilon a^2 + \epsilon b^2$ and the uniform elliptic condition, we get

$$\| \gamma_t \|^2 + 2\int_0^t \mathcal{E}(\gamma_s) ds \leq \int_0^t \left(\frac{1}{\epsilon} \| g_s \|^2 + \frac{\epsilon}{\lambda} \mathcal{E}(\gamma_s) \right) ds\,,$$

therefore,

$$\| \gamma_t \|^2 + (2 - \frac{\epsilon}{\lambda}) \int_0^t \mathcal{E}(\gamma_s) ds \leq \frac{1}{\epsilon} \int_0^t \| g_s \|^2 ds.$$

We can take ϵ small enough such that $(2 - \frac{\epsilon}{\lambda}) > 0$, then taking the supreme, we have the following two relations:

$$\sup_{t \in [0,T]} \| \gamma_t \|^2 \leq \frac{1}{\epsilon} \int_0^T \| g_s \|^2 ds$$

and

$$\int_0^T \mathcal{E}(\gamma_s) ds \leq \frac{\lambda}{\epsilon(2\lambda - \epsilon)} \int_0^T \| g_s \|^2 ds.$$

By density argument, we get 1 and 2.
Consider now $0 \leq s \leq t \leq T$, $g \in L^2([0,T] \times \Omega \times \mathcal{O})$ and a sequence $g_n \in C^1([0,T]) \otimes L^2(\Omega) \otimes \mathcal{D}(L^{3/2})$ which converges to g in $L^2([0,T] \times \Omega \times \mathcal{O})$.
We put

$$\forall n \in \mathbb{N}^*,\ \forall t \in [0,T],\ \gamma_t^n = \int_0^t P_{t-s} div g_s^n ds.$$

It is clear that for all $n \in \mathbb{N}^*$ and $P-$almost all $\omega \in \Omega$, $\int_s^t \gamma_u^n du \in \mathcal{D}(L)$ and

$$L(\int_s^t \gamma_u^n du) = \gamma_t^n - \gamma_s^n - \int_s^t div g_u^n du.$$

Thanks to the relations we have established at the beginning of the proof, we conclude that $\int_s^t \gamma_u^n du$ converges to $\int_s^t \gamma_u du$ in $L^2(\mathcal{O})$ and that moreover, $L(\int_s^t \gamma_u^n du)$ converges in $L^2(\mathcal{O})$ to $\gamma_t - \gamma_s - \int_s^t div g_u du$, for $P-$almost all $\omega \in \Omega$. This ensures that $\int_s^t \gamma_u du$ belongs to $\mathcal{D}(L)$ and that

$$\gamma_t - \gamma_s = L(\int_s^t \gamma_u du) + \int_s^t div g_u du.$$

∎

Lemma 1.35. *Let $h \in L^2([0,T] \times \Omega \times \mathcal{O})$ and adapted. Then*

1. *the process $\beta :\ t \in [0,T] \to \int_0^t P_{t-s} h_s dB_s$ admits a version in $\mathcal{P}(F)$;*

2. for all $\varphi \in \mathcal{D}$ and for all $t \in [0,T]$, we have

$$\int_0^t (\beta_s, \partial_s \varphi_s) ds = (\beta_t, \varphi_t) + \int_0^t \mathcal{E}(\beta_s, \varphi_s) ds - \int_0^t (h_s, \varphi_s) dB_s \ P - a.e.$$

3. for all $0 \le s \le t \le T$, $\int_s^t \beta_u du$ belongs to $\mathcal{D}(L)$ and

$$\beta_t - \beta_s = L(\int_s^t \beta_u du) + \int_s^t h_u dB_u \quad P - a.e.$$

Proof. Assume first that $h \in C^1([0,T]) \otimes L^2(\Omega) \otimes \mathcal{D}(L)$ and adapted. It is clear that the process

$$\forall t \in [0,T], \ \beta_t = \int_0^t P_{t-s} h_s dB_s$$

is a square integrable $\mathcal{D}(L)-$valued martingale (see [D] Prop. 2.3). Hence, for all $t \in [0,T]$, we have almost surely (see Protter (1985) theorem 3.3)

$$\begin{aligned} \beta_t &= \int_0^t P_{s-s} h_s dB_s + \int_0^t (\int_0^s L P_{s-u} h_u dB_u) ds \\ &= \int_0^t h_s dB_s + \int_0^t L\beta_s ds \end{aligned} \tag{1.8}$$

Thanks to the Itô's formula, we have almost surely, for all $\varphi \in \mathcal{D}$ and for all $t \in [0,T]$,

$$(\beta_t, \varphi_t) = (\beta_0, \varphi_0) + \int_0^t (\beta_s, \partial_s \varphi_s) ds + \int_0^t (h_s, \varphi_s) dB_s + \int_0^t (L\beta_s, \varphi_s) ds$$

This yields

$$\int_0^t (\beta_s, \partial_s \varphi_s) ds = (\beta_t, \varphi_t) + \int_0^t \mathcal{E}(\beta_s, \varphi_s) ds - \int_0^t (h_s, \varphi_s) dB_s \ P - a.e.$$

We apply Itô's formula to β^2 and obtain:

$$\forall t \in [0,T], \quad \| \beta_t \|^2 + 2\int_0^t \mathcal{E}(\beta_s) ds = 2\int_0^t (\beta_s, h_s) dB_s + \int_0^t \| h_s \|^2 ds, \quad a.s. \tag{1.9}$$

Taking the supreme and the expectation, we have:

$$\mathbb{E}[\sup_{t\in[0,T]} \| \beta_t \|^2] \le 2\mathbb{E}[\sup_{t\in[0,T]} | \int_0^t (\beta_s, h_s) dB_s |] + 2\mathbb{E}\int_0^T \| h_s \|^2 ds.$$

By Burkholder-Davies-Gundy's inequality, we get that there exists a constant C such that

$$\begin{aligned} \mathbb{E}[\sup_{t\in[0,T]} \| \beta_t \|^2] &\le 2C\mathbb{E}[\int_0^T (\beta_s, h_s) ds]^{1/2} + 2\mathbb{E}\int_0^T \| h_s \|^2 ds \\ &\le C\epsilon \mathbb{E}[\sup_{t\in[0,T]} \| \beta_t \|^2] + (\frac{C}{\epsilon} + 2)\mathbb{E}\int_0^T \| h_s \|^2 ds \end{aligned}$$

Therefore

$$(1 - C\epsilon)\mathbb{E}[\sup_{t\in[0,T]} \| \beta_t \|^2] \le (\frac{C}{\epsilon} + 2)\mathbb{E}\int_0^T \| h_s \|^2 ds$$

We can take ϵ small enough such that $1 - C\epsilon > 0$, thus we have

$$\mathbb{E}[\sup_{t\in[0,T]} \parallel \beta_t \parallel^2] \leq C\mathbb{E}\int_0^T \parallel h_s \parallel^2 ds.$$

Again from (1.9), we obtain:

$$\mathbb{E}\int_0^T \mathcal{E}(\beta_s)ds \leq \mathbb{E}\int_0^T (\beta_s, h_s)dB_s + \frac{1}{2}\mathbb{E}\int_0^T \parallel h_s \parallel^2 ds.$$

Hence,

$$\mathbb{E}\int_0^T \mathcal{E}(\beta_s)ds \leq \frac{1}{2}\mathbb{E}\int_0^T \parallel h_s \parallel^2 ds.$$

By density argument, we get assertions 1 and 2.
For the last one, we remark that if $h \in C^1([0,T]) \otimes L^2(\Omega) \otimes \mathcal{D}(L)$, it is given by relation (1.8). The general case is obtained by density argument. ■

1.3.3 Equivalence between weak and mild solutions

We now consider the mild equation

$$\begin{aligned} u_t(x) &= P_t\xi(x) + \int_0^t P_{t-s}f_s(\cdot, u_s(\cdot), \nabla u_s(\cdot))(x)ds + \int_0^t P_{t-s}divg_s(\cdot, u_s(\cdot), \nabla u_s(\cdot))(x)ds \\ &+ \int_0^t P_{t-s}h_s(\cdot, u_s(\cdot), \nabla u_s(\cdot))(x)dB_s \end{aligned} \tag{1.10}$$

Proposition 1.36. *$u \in \mathcal{H}_T$ is a weak solution of* (1.4) *if and only if it satisfies* (1.10).

Proof. Let u be in $\mathcal{H}_T$. We put, for all $t \in [0,T]$, $\alpha_t = \int_0^t P_{t-s}f_s(\cdot, u_s(\cdot), \nabla u_s(\cdot))(x)ds$, $\gamma_t = \int_0^t P_{t-s}divg_s(\cdot, u_s(\cdot), \nabla u_s(\cdot))(x)ds$ and $\beta_t = \int_0^t P_{t-s}h_s(\cdot, u_s(\cdot), \nabla u_s(\cdot))(x)dB_s$. Thanks to the lemmas in the previous section, it is easy to conclude that if $u \in \mathcal{H}_T$ satisfies (1.10) then u satisfies (1.4).
Conversely, if $u \in \mathcal{H}_T$ is a solution of (1.4), then define the process

$$\begin{aligned} u'_t(x) &= P_t\xi(x) + \int_0^t P_{t-s}f_s(\cdot, u_s(\cdot), \nabla u_s(\cdot))(x)ds + \int_0^t P_{t-s}divg_s(\cdot, u_s(\cdot), \nabla u_s(\cdot))(x)ds \\ &+ \int_0^t P_{t-s}h_s(\cdot, u_s(\cdot), \nabla u_s(\cdot))(x)dB_s. \end{aligned}$$

Using the previous calculus, we have, for all $\varphi \in \mathcal{D}$,

$$\begin{aligned} \int_0^T (u'_t, \partial_t\varphi_t)dt &= -(\xi, \varphi_0) + \int_0^T \mathcal{E}(u'_t, \varphi_t)dt - \int_0^T (f_t(u_t, \nabla u_t), \varphi_t)dt \\ &+ \int_0^T (g_t(u_t, \nabla u_t), \nabla\varphi_t)dt - \int_0^T (h_t(u_t, \nabla u_t), \varphi_t)dB_t \end{aligned}$$

On the other side, as u is a solution of (1.4), for all $\varphi \in \mathcal{D}$,

$$\begin{aligned} \int_0^T (u_t, \partial_t\varphi_t)dt &= -(\xi, \varphi_0) + \int_0^T \mathcal{E}(u_t, \varphi_t) - \int_0^T (f_t(u_t, \nabla u_t), \varphi_t)dt \\ &+ \int_0^T (g_t(u_t, \nabla u_t), \nabla\varphi_t)dt - \int_0^T (h_t(u_t, \nabla u_t), \varphi_t)dB_t \end{aligned}$$

If we put $v_t(x) = u_t(x) - u'_t(x)$, it is clear that v belongs to $\mathcal{H}_T$ and that for all $\varphi \in \mathcal{D}$,

$$\int_0^T (v_t, \partial_t \varphi_t) dt = \int_0^T \mathcal{E}(v_t, \varphi_t) dt \quad a.s.$$

Hence, v is the weak solution of the equation $\partial_t v_t - L v_t = 0$ with initial condition $v_0 = 0$. Then we can conclude thanks to Lemma 4.10 in [17]. ∎

1.3.4 Itô's formula

Lemma 1.37. *Assume that f, g, h belong to $L^2([0,T] \times \Omega \times \mathcal{O})$ and adapted and $\xi \in L^2(\Omega \times \mathcal{O})$ and consider $u := \mathcal{U}(\xi, f, g, h)$. Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^{1,2}$. We denote by Φ' and Φ'' the derivatives of Φ with respect to the space variables and by $\frac{\partial \Phi}{\partial t}$ the partial derivative with respect to times. We assume that these derivatives are bounded and $\Phi'(t, 0) = 0$ for all $t \in [0,T]$. Then we have the following relation $P-$almost surely, for all $t \in [0,T]$,*

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi(t, u_t(x)) dx + \int_0^t \mathcal{E}(\Phi'(s, u_s), u_s) ds = \int_{\mathcal{O}} \Phi(0, \xi(x)) dx + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, u_s(x)) dx ds \\
&+ \int_0^t (\Phi'(s, u_s), f_s) ds - \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s) \partial_i u_s(x) g_s^i(x) dx ds + \sum_{j=1}^{\infty} \int_0^t (\Phi'(s, u_s), h_s^j) dB_s^j \\
&+ \frac{1}{2} \sum_{j=1}^{\infty} \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s(x)) (h_s^j(x))^2 dx ds.
\end{aligned}$$

Proof. Assume first that $f,\ h \in C^1([0,T]) \otimes L^2(\Omega) \otimes \mathcal{D}(A)$, $g \in C^1([0,T]) \otimes L^2(\Omega) \otimes \mathcal{D}(A^{3/2})$ and adapted and $\xi \in L^2(\Omega) \otimes \mathcal{D}(A)$, then u is a semi-martingale and it posses the following form:

$$\forall t \in [0,T],\ u_t = \xi - \int_0^t A u_s ds + \int_0^t f_s ds + \int_0^t div g_s ds + \int_0^t h_s dB_s.$$

Thanks to Itô's formula for Hilbert-valued semi-martingale we have almost surely for all $t \in [0,T]$:

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi(t, u_t(x)) dx = \int_{\mathcal{O}} \Phi(0, \xi(x)) dx + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, u_s(x)) dx ds - \int_0^t (\Phi'(s, u_s), A u_s) ds \\
&+ \int_0^t (\Phi'(s, u_s), f_s) ds + \int_0^t \int_{\mathcal{O}} \Phi'(s, u_s(x)) div g_s(x) dx ds + \int_0^t (\Phi'(s, u_s), h_s) dB_s \\
&+ \frac{1}{2} \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s(x)) h_s^2(x) dx ds
\end{aligned}$$

Then, as

$$(\Phi'(s, u_s), A u_s) = \mathcal{E}(\Phi'(s, u_s), u_s)$$

and

$$\int_{\mathcal{O}} \Phi'(s, u_s) div g_s dx = - \int_{\mathcal{O}} \Phi''(s, u_s) \partial u_s g_s dx$$

we get the desired equality.

For the martingale part, by the Burkholder-Davis-Gundy inequality

$$\begin{aligned}&\mathbb{E}(\sup_{t\in[0,T]}\int_0^t(\Phi'(s,u_s),h_s)dB_s)\le C\mathbb{E}(\int_0^T(\Phi'(s,u_s),h_s)^2ds)^{1/2}\\&\le C\mathbb{E}(\int_0^T\|\Phi'(s,u_s)\|^2\|h_s\|^2ds)^{1/2}\\&\le C\mathbb{E}[\sup_{s\in[0,T]}\|\Phi'(s,u_s)\|^2+\int_0^T\|h_s\|^2ds]<\infty\end{aligned}$$

we deduce that $\int_0^t(\Phi'(s,u_s),h_s)dB_s$ is a martingale.

The general case is obtained by approximation: if f, g et $h\in L^2([0,T]\times\Omega\times\mathcal{O})$ et $\xi\in L^2$, $\exists f^k,\ h^k\in C^1([0,T])\otimes L^2(\Omega)\otimes\mathcal{D}(A)$, $g^k\in C^1([0,T])\otimes L^2(\Omega)\otimes\mathcal{D}(A^{3/2})$ and $\xi^k\in L^2(\Omega)\otimes\mathcal{D}(A)$, and they converge strongly respectively. $u^k:=\mathcal{U}(\xi^k,f^k,g^k,h^k)$ is a Hilbert-valued semi-martingale and it satisfies $\lim_{k\to\infty}\mathbb{E}\|u^k-u\|_T^2=0$, where $u:=\mathcal{U}(\xi,f,g,h)$. For such u_k Itô's formula is valid, then, thanks to the dominated convergence theorem, we take the limit and get the desired result. ■

1.3.5 Existence and uniqueness result

Theorem 1.38. *Under the hypotheses in subsection 1.3.1, the equation (1.4) admits a unique solution in $\mathcal{H}_T$.*

Let γ and δ be 2 positive constants. On $\mathcal{H}_T$, we introduce the norm

$$\|u\|_{\gamma,\delta}=E(\int_0^Te^{-\gamma s}(\delta\|u_s\|^2+\|\nabla u_s\|^2)ds),$$

which clearly defines an equivalent norm on $\mathcal{H}_T$.

Proof. We define the application $\Lambda:\ \mathcal{H}_T\to\ \mathcal{H}_T$ as following:

$$(\Lambda u)_t=P_t\xi+\int_0^tP_{t-s}f(s,u_s,\nabla u_s)ds+\int_0^tP_{t-s}divg_s(s,u_s,\nabla u_s)ds+\int_0^tP_{t-s}h(s,u_s,\nabla u_s)dB_s$$

we will prove that Λ is a contraction with respect to the norm $\|\cdot\|_{\gamma,\delta}$.
Denoting $\bar{u}_t=\Lambda u_t-\Lambda v_t$ with u and v are in $\mathcal{H}_T$, $\bar{f}=f(u,\nabla u)-f(v,\nabla v)$, $\bar{g}=g(u,\nabla u)-g(v,\nabla v)$ and $\bar{h}=h(u,\nabla u)-h(v,\nabla v)$. Applying Itô's formula to $e^{-\gamma T}\bar{u}_T^2$ we have almost surely:

$$\begin{aligned}&e^{-\gamma T}\|\bar{u}_T\|^2+2\int_0^Te^{-\gamma s}\mathcal{E}(\bar{u}_s)ds=-\gamma\int_0^Te^{-\gamma s}\|\bar{u}_s\|^2ds+2\int_0^Te^{-\gamma s}(\bar{u}_s,\bar{f}_s)ds\\&-2\sum_{i=1}^d\int_0^Te^{-\gamma s}(\partial_i\bar{u}_s,\bar{g}_s^i)ds+2\sum_{j=1}^\infty\int_0^Te^{-\gamma s}(\bar{u}_s,\bar{h}_s^j)dB_s^j+\int_0^Te^{-\gamma s}\||\bar{h}_s|\|^2ds\end{aligned}$$

The following calculus are based on the Lipschitz conditions and Cauchy-Schwarz's inequality:

$$\begin{aligned}2\int_0^T e^{-\gamma s}(\bar{u}_s,\bar{f}_s)ds &\leq \frac{1}{\epsilon}\int_0^T e^{-\gamma s}\parallel \bar{u}_s\parallel^2 ds+\epsilon\int_0^T e^{-\gamma s}\parallel \bar{f}_s\parallel^2 ds\\ &\leq \frac{1}{\epsilon}\int_0^T e^{-\gamma s}\parallel \bar{u}_s\parallel^2 ds+C\epsilon\int_0^T e^{-\gamma s}\parallel u_s-v_s\parallel^2 ds\\ &+ C\epsilon\int_0^T e^{-\gamma s}\parallel \nabla(u_s-v_s)\parallel^2 ds\end{aligned}$$

and

$$\begin{aligned}2\sum_{i=1}^d\int_0^T e^{-\gamma s}(\bar{g}^i_s,\partial_i\bar{u}_s)ds &\leq 2\int_0^T e^{-\gamma s}\parallel \nabla\bar{u}_s\parallel (C\parallel u_s-v_s\parallel+\alpha\parallel\nabla(u_s-v_s)\parallel)ds\\ &\leq C\epsilon\int_0^T e^{-\gamma s}\parallel\nabla\bar{u}_s\parallel^2 ds+\frac{C}{\epsilon}\int_0^T e^{-\gamma s}\parallel u_s-v_s\parallel^2 ds\\ &\quad +\alpha\int_0^T e^{-\gamma s}\parallel\nabla\bar{u}_s\parallel^2 ds+\alpha\int_0^T e^{-\gamma s}\parallel\nabla(u_s-v_s)\parallel^2 ds\end{aligned}$$

and

$$\int_0^T e^{-\gamma s}\parallel |\bar{h}_s|\parallel^2 ds\leq C(1+\frac{1}{\epsilon})\int_0^T e^{-\gamma s}\parallel u_s-v_s\parallel^2 ds+\beta^2(1+\epsilon)\int_0^T e^{-\gamma s}\parallel\nabla(u_s-v_s)\parallel^2 ds$$

where C, α and β are the constants in the Lipschitz conditions. Using the elliptic condition and taking expectation, we get:

$$\begin{aligned}&(\gamma-\frac{1}{\epsilon})E\int_0^T e^{-\gamma s}\parallel\bar{u}_s\parallel^2 ds+(2\lambda-\alpha-C\epsilon)E\int_0^T e^{-\gamma s}\parallel\nabla\bar{u}_s\parallel^2 ds\leq\\ &C(1+\epsilon+\frac{2}{\epsilon})\int_0^T e^{-\gamma s}\parallel u_s-v_s\parallel^2 ds+(C\epsilon+\alpha+\beta^2(1+\epsilon))E\int_0^T e^{-\gamma s}\parallel\nabla(u_s-v_s)\parallel^2 ds\end{aligned}$$

We choose ϵ small enough and then γ such that

$$C\epsilon+\alpha+\beta^2(1+\epsilon)<2\lambda-\alpha-C\epsilon \textit{ and } \frac{\gamma-1/\epsilon}{2\lambda-\alpha-C\epsilon}=\frac{C(1+\epsilon+2/\epsilon)}{C\epsilon+\alpha+\beta^2(1+\epsilon)}$$

If we set $\delta=\frac{\gamma-1/\epsilon}{2\lambda-\alpha-C\epsilon}$, we have the following inequality:

$$\parallel\bar{u}\parallel_{\gamma,\delta}\leq\frac{C\epsilon+\alpha+\beta^2(1+\epsilon)}{2\lambda-\alpha-C\epsilon}\parallel u-v\parallel_{\gamma,\delta}.$$

We conclude thanks to the fixed point theorem. ∎

1.3.6 Comparison theorem

In this subsection we will establish a comparison theorem for the solution of SPDE (1.4) as following:

Theorem 1.39. *Let f' be another coefficient which satisfies the same hypotheses as f and $\xi' \in L^2(\mathcal{O})$. Let u' be the solution of*

$$\begin{aligned} du'_t(x) &= Lu'_t(x)dt + f'(t,x,u'_t(x),\nabla u'_t(x))dt + \sum_{i=1}^{d} \partial_i g_i(t,x,u'_t(x),\nabla u'_t(x))dt \\ &+ \sum_{j=1}^{+\infty} h_j(t,x,u'_t(x),\nabla u'_t(x))dB^j_t, \end{aligned}$$

with initial condition $u'_0 = \xi'$.
Assume that $\xi \leq \xi'$ $dx \otimes dP - a.e.$ and

$$f(t,x,u_t(x),\nabla u_t(x)) \leq f'(t,x,u'_t(x),\nabla u'_t(x)) \quad dt \otimes dx \otimes dP - a.e.$$

then

$$\forall t \in [0,T], \quad u_t \leq u'_t \quad dx \otimes dP - a.e.$$

We put $\widehat{u} = u - u'$, $\widehat{\xi} = \xi - \xi'$, $\widehat{f}_t = f(t,u_t,\nabla u_t) - f'(t,u'_t,\nabla u'_t)$, $\widehat{g}_t = g(t,u_t,\nabla u_t) - g(t,u'_t,\nabla u'_t)$ and $\widehat{h}_t = h(t,u_t,\nabla u_t) - h(t,u'_t,\nabla u'_t)$. The main idea is to evaluate $E \parallel \widehat{u}^+_t \parallel^2$, thanks to Itô's formula and then apply Gronwall's lemma. Hence, we begin with the following lemma:

Lemma 1.40. *Let u be the solution of* (1.4)*, the following relation holds $P-$almost surely, for all $t \in [0,T]$,*

$$\begin{aligned} &\int_{\mathcal{O}} \left(u^+_t(x)\right)^2 dx + 2\int_0^t \mathcal{E}\left(u^+_s, u^+_s\right) ds = \int_{\mathcal{O}} \left(\xi^+(x)\right)^2 dx + 2\int_0^t \left(u^+_s, f_s(u_s,\nabla u_s)\right) ds \\ &-2\int_0^t \sum_{i=1}^{d} \left(1_{\{u_s>0\}}\partial_i u_s, g_{i,s}(u_s,\nabla u_s)\right) ds + \int_0^t \left(1_{\{u_s>0\}}, |h_s(u_s,\nabla u_s)|^2\right) ds \\ &+2\sum_{j=1}^{\infty}\int_0^t \left(u^+_s, h_{j,s}(u_s,\nabla u_s)\right) dB^j_s. \end{aligned} \tag{1.11}$$

Proof. We approximate $\psi(y) = (y^+)^2$ by a sequence of regular functions: Let φ be an increasing $\mathcal{C}^\infty$ function such that $\varphi(y) = 0$ for any $y \in]-\infty,1]$ and $\varphi(y) = 1$ for any $y \in [2,\infty[$. We set $\psi_n(y) = y^2\varphi(ny)$, for each $y \in \mathbb{R}$ and all $n \in \mathbb{N}^*$. It is easy to verify that $(\psi_n)_{n\in\mathbb{N}^*}$ converges uniformly to the function ψ and that

$$\lim_{n\to\infty} \psi'_n(y) = 2y^+, \ \lim_{n\to\infty} \psi''_n(y) = 2 \cdot I_{\{y>0\}},$$

for any $y \in \mathbb{R}$. Moreover we have the estimates

$$0 \leq \psi_n(y) \leq \psi(y), \ 0 \leq \psi'_n(y) \leq Cy, \ \left|\psi''_n(y)\right| \leq C,$$

for any $y \geq 0$ and all $n \in \mathbb{N}^*$, where C is a constant. We have for all $n \in \mathbb{N}^*$ and each

$t \in [0,T]$, a.s.,

$$\int_{\mathcal{O}} \psi_n \left(u_t\left(x\right)\right) dx + \int_0^t \mathcal{E}\left(\psi_n'\left(u_s\right), u_s\right) ds = \int_{\mathcal{O}} \psi_n\left(\xi\left(x\right)\right) dx + \int_0^t \left(\psi_n'\left(u_s\right), f_s\left(u_s, \nabla u_s\right)\right) ds$$
$$- \int_0^t \sum_{i=1}^d \left(\psi_n''\left(u_s\right) \partial_i u_s, g_{i,s}\left(u_s, \nabla u_s\right)\right) ds + \frac{1}{2}\int_0^t \left(\psi_n''\left(u_s\right), \left|h_s\left(u_s, \nabla u_s\right)\right|^2\right) ds$$
$$+ \sum_{j=1}^{\infty} \int_0^t \left(\psi_n'\left(u_s\right), h_{j,s}\left(u_s, \nabla u_s\right)\right) dB_s^j.$$

Taking the limit, thanks to the dominated convergence theorem, we get (1.11). ∎

Proof. **(Proof of Comparison Theorem)**
As $u = \mathcal{U}(\xi, f, g, h)$ and $u' = \mathcal{U}(\xi', f', g, h)$, $\widehat{u}$ is the solution of SPDE associated to $(\widehat{\xi}, \widehat{f}, \widehat{g}, \widehat{h})$. So that we can apply (1.11) to $(\widehat{u}^+)^2$ and get for all $t \in [0,T]$,

$$\int_{\mathcal{O}} \left(\widehat{u}_t^+\left(x\right)\right)^2 dx + 2\int_0^t \mathcal{E}\left(\widehat{u}_s^+, \widehat{u}_s^+\right) ds = \int_{\mathcal{O}} \left(\widehat{\xi}^+\left(x\right)\right)^2 dx + 2\int_0^t \left(\widehat{u}_s^+, \widehat{f}_s\right) ds$$
$$-2\int_0^t \sum_{i=1}^d \left(1_{\{\widehat{u}_s>0\}}\partial_i \widehat{u}_s, \widehat{g}_{i,s}\right) ds + \int_0^t \left(1_{\{\widehat{u}_s>0\}}, \left|\widehat{h}_s\right|^2\right) ds + 2\sum_{j=1}^{\infty}\int_0^t \left(\widehat{u}_s^+, \widehat{h}_{j,s}\right) dB_s^j, \quad a.s.$$

As we assume that $f(u, \nabla u) \le f'(u', \nabla u')$,

$$\begin{aligned} \widehat{u}_s^+ \widehat{f}_s &= \widehat{u}_s^+\{f(s, u_s, \nabla u_s) - f'(s, u_s, \nabla u_s)\} + \widehat{u}_s^+\{f'(s, u_s, \nabla u_s) - f'(s, u_s', \nabla u_s')\} \\ &\le \widehat{u}_s^+\{f'(s, u_s, \nabla u_s) - f'(s, u_s', \nabla u_s')\}. \end{aligned}$$

then with the Lipschitz condition, using Cauchy-Schwartz's inequality, we have the following relations:

$$\int_0^t (\widehat{u}_s^+, \widehat{f}_s) ds \le (C + \frac{C}{\epsilon})\int_0^t \| \widehat{u}_s^+ \|^2 ds + \frac{C\epsilon}{\lambda}\int_0^t \mathcal{E}(\widehat{u}_s^+) ds.$$
$$\int_0^t (\nabla \widehat{u}_s^+, \widehat{g}_s) ds \le \frac{\epsilon + \alpha}{\lambda}\int_0^t \mathcal{E}(\widehat{u}_s^+) ds + \frac{C}{\epsilon}\int_0^t \| \widehat{u}_s^+ \|^2 ds$$
$$\int_0^t \| I_{\{\widehat{u}_s>0\}}|\widehat{h}_s| \|^2 ds \le C\int_0^t \| \widehat{u}_s^+ \|^2 ds + \frac{\beta^2 + \epsilon}{\lambda}\int_0^t \mathcal{E}(\widehat{u}_s^+) ds.$$

Taking expectation we obtain the following inequality:

$$E \| \widehat{u}_t^+ \|^2 + (2 - \frac{2\alpha + 2\epsilon}{\lambda} - \frac{2C\epsilon}{\lambda} - \frac{\beta^2 + \epsilon}{\lambda}) E\int_0^t \mathcal{E}(\widehat{u}_s^+) ds \le CE\int_0^t \| \widehat{u}_s^+ \|^2 ds.$$

We can take ϵ small enough such that $2 - \frac{2\alpha+2\epsilon}{\lambda} - \frac{2C\epsilon}{\lambda} - \frac{\beta^2+\epsilon}{\lambda} > 0$, we have

$$E \| \widehat{u}_s^+ \|^2 \le CE\int_0^t \| \widehat{u}_s^+ \|^2 ds,$$

then we deduce the result from Gronwall's lemma. ∎

Chapter 2

The Obstacle Problem for Quasilinear Stochastic PDEs: Analytical approach

2.1 Introduction

The starting point of this chapter is the following parabolic stochastic partial differential equation (in short SPDE)

$$\begin{aligned} du_t(x) &= \partial_i\left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right)dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ &\quad + \sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \end{aligned} \tag{2.1}$$

where a is a symmetric bounded measurable matrix which defines a second order operator on $\mathcal{O} \subset R^d$, with null Dirichlet condition. The initial condition is given as $u_0 = \xi$, a $L^2(\mathcal{O})-$valued random variable, and f, $g = (g_1, ..., g_d)$ and $h = (h_1, ...h_i, ...)$ are non-linear random functions. Given an obstacle $S : \Omega \times [0,T] \times \mathcal{O} \to \mathbb{R}$, we study the obstacle problem for the SPDE (2.1), i.e. we want to find a solution of (2.1) which satisfies "$u \geq S$" where the obstacle S is regular in some sense and controlled by the solution of a SPDE.

Nualart and Pardoux [55] have studied the obstacle problem for a nonlinear heat equation on the spatial interval $[0,1]$ with Dirichlet boundary conditions, driven by an additive space-time white noise. They proved the existence and uniqueness of the solution and their method relied heavily on the results for a deterministic variational inequality. Donati-Martin and Pardoux [26] generalized the model of Nualart and Pardoux. The nonlinearity appears both in the drift and the diffusion coefficients. They proved the existence of the solution by penalization method but they didn't obtain the uniqueness result. And then in 2009, Xu and Zhang solved the problem of the uniqueness, see [79]. However, in all their models, there isn't the term of divergence and they do not consider the case where the coefficients depend on ∇u.

The work of El Karoui et al [27] treats the obstacle problem for deterministic semi linear PDE's within the framework of backward stochastic differential equations (BSDE in short). Namely the equation (2.1) is considered with f depending of u and ∇u, while the function g is null (as well h) and the obstacle v is continuous. They considered the viscosity solution of the obstacle problem for the equation (2.1) , they represented this solution stochastically as a process and the main new object of this BSDE framework is a continuous increasing process that controls the set $\{u = v\}$. Bally et al [5] (see also [52]) point out that the continuity of this process allows one to extend the classical notion of strong variational solution (see Theorem 2.2 of [7] p.238) and express the solution to the obstacle as a pair (u, ν) where ν is supported by the set $\{u = v\}$.

Matoussi and Stoica [53] have proved an existence and uniqueness result for the obstacle problem of backward quasilinear stochastic PDE on the whole space $\mathbb{R}^d$ and driven by a finite dimentionnal Brownian motion. The method is based on the probabilistic interpretation of the solution by using the backward doubly stochastic differential equation (DBSDE). They have also proved that the solution is a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying minimal Skohorod condition. In particular they gave for the regular measure ν a probabilistic interpretation in term of the continuous increasing process K where (Y, Z, K) is the solution of a reflected generalized BDSDE.

Michel Pierre [70, 71] has studied the parabolic PDE with obstacle using the parabolic potential as a tool. He proved that the solution uniquely exists and is quasi-continuous. With the help of Pierre's result, under suitable assumptions on f, g and h, our aim is to prove existence and uniqueness for the following SPDE with given obstacle S that we write formally as:

$$\begin{cases} du_t(x) = \partial_i \left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right) dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ \qquad\qquad + \displaystyle\sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \\ u_t(x) \geq S_t(x)\,, \forall (t,x) \in \mathbb{R}^+ \times \mathcal{O}, \\ u_0(x) = \xi(x)\,,\ \forall x \in \mathcal{O}, \\ u_t(x) = 0,\ \forall (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O}\,. \end{cases} \tag{2.2}$$

To give a rigorous definition to the notion of solution to this equation, we will use the technics of parabolic potential theory developed by M. Pierre in the stochastic framework. We first prove a quasi-continuity result for the solution of the SPDE (2.1) with null Dirichlet condition on given domain $\mathcal{O}$ and driven by an infinite dimensional Brownian motion. This result is not obvious and is based on a mixing pathwise arguments and Mignot and Puel [54] existence result of the obstacle problem for some deterministic PDEs. Moreover, we prove in our context that the reflected measure ν is a regular random measure and we give the analytical representation of such measure in term of parabolic potential in the sense given by M. Pierre in [70]. The main theorem we obtain is:

Theorem 2.1. *Assume that f, g and h satisfy some Lipschitz continuity and integrability* *hypotheses,* $\xi \in L^2(\Omega \times \mathcal{O})$*,* S *is quasi-continuous and* $S_t \leq S'_t$ *where* S' *is the solution of the linear SPDE with null boundary condition:*

$$\begin{cases} dS'_t &= LS'_t dt + f'_t dt + \sum_{i=1}^d \partial_i g'_{i,t} dt + \sum_{j=1}^{+\infty} h'_{j,t} dB^j_t \\ S'(0) &= S'_0, \end{cases}$$

where $S'_0 \in L^2(\Omega \times \mathcal{O})$*,* f'*,* g' *and* h' *are square integrable adapted processes.*
Then there exists a unique solution (u, ν) *of the obstacle problem for the SPDE (3.1) associated to* (ξ, f, g, h, S) *i.e.* u *is a predictable continuous process which takes values in a proper Sobolev space,* $u \geq S$ *and* ν *is a random regular measure such that:*

1. *the following relation holds almost surely, for all* $t \in [0, T]$ *and* $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^2(\mathcal{O})$*,*

$$(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(u_s, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g^i_s(u_s, \nabla u_s), \partial_i \varphi_s) ds$$
$$= \int_0^t (f_s(u_s, \nabla u_s), \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h^j_s(u_s, \nabla u_s), \varphi_s) dB^j_s + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds)$$

2. u *admits a quasi-continuous version,* $\widetilde{u}$*, and we have the mininal Skohorod condition*

$$\int_0^T \int_{\mathcal{O}} (\widetilde{u}(s, x) - S(s, x)) \nu(dx, ds) = 0 \quad a.s.$$

This chapter is divided as follows: in the second section, we set the assumptions then we introduce in the third section the notion of regular measure associated to parabolic potentials. The fourth section is devoted to prove the quasi-continuity of the solution of SPDE without obstacle. The fifth section is the main part of the chapter in which we prove existence and uniqueness of the solution, to do that we begin with the linear case, and then by Picard iteration we get the result in the nonlinear case, we also establish the Ito's formula. Finally, in the sixth section, we prove a comparison theorem for the solution of SPDE with obstacle.

2.2 Preliminaries

We consider a sequence $((B^i(t))_{t \geq 0})_{i \in \mathbb{N}^*}$ of independent Brownian motions defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions.

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded open domain and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$, it is an Hilbert space equipped with the usual scalar product and norm as follows

$$(u, v) = \int_{\mathcal{O}} u(x) v(x) dx, \qquad \| u \| = (\int_{\mathcal{O}} u^2(x) dx)^{1/2}.$$

Let A be a symmetric second order differential operator, with domain $\mathcal{D}(A)$, given by

$$A := -\sum_{i,j=1}^d \partial_i (a^{i,j} \partial_j).$$

We assume that $a = (a^{i,j})_{i,j}$ is a measurable symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{d} a^{i,j}(x)\xi^i\xi^j \leq \Lambda|\xi|^2,\ \forall x \in \mathcal{O},\ \xi \in \mathbb{R}^d,$$

where λ and Λ are positive constants.

Let $(F, \mathcal{E})$ be the associated Dirichlet form given by $F := \mathcal{D}(A^{1/2}) = H_0^1(\mathcal{O})$ and

$$\mathcal{E}(u,v) := (A^{1/2}u, A^{1/2}v)\ and\ \mathcal{E}(u) = \| A^{1/2}u \|^2,\ \forall u, v \in F,$$

where $H_0^1(\mathcal{O})$ is the first order Sobolev space of functions vanishing at the boundary. We shall denote $H^{-1}(\mathcal{O})$ its dual space.

We consider the quasilinear stochastic partial differential equation (1) with initial condition $u(0,\cdot) = \xi(\cdot)$ and Dirichlet boundary condition $u(t,x) = 0,\ \forall\ (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O}$.

We assume that we have predictable random functions

$$\begin{aligned}
&f : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R},\\
&g = (g_1, ..., g_d) : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d,\\
&h = (h_1, ..., h_i, ...) : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{\mathbb{N}^*},
\end{aligned}$$

In the sequel, $|\cdot|$ will always denote the underlying Euclidean or l^2-norm. For example

$$|h(t,\omega,x,y,z)|^2 = \sum_{i=1}^{+\infty} |h_i(t,\omega,x,y,z)|^2.$$

Assumption (H): There exist non negative constants $C,\ \alpha,\ \beta$ such that for almost all ω, the following inequalities hold for all $(t,x,y,z) \in \mathbb{R}^+ \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d$:

1. $|f(t,\omega,x,y,z) - f(t,\omega,x,y',z')| \leq C(|y-y'| + |z-z'|)$,
2. $(\sum_{i=1}^{d} |g_i(t,\omega,x,y,z) - g_i(t,\omega,x,y',z')|^2)^{\frac{1}{2}} \leq C|y-y'| + \alpha|z-z'|$,
3. $(|h(t,\omega,x,y,z) - h(t,\omega,x,y',z')|^2)^{\frac{1}{2}} \leq C|y-y'| + \beta|z-z'|$,
4. the contraction property: $2\alpha + \beta^2 < 2\lambda$.

Remark 2.2. *This last contraction property ensures existence and uniqueness for the solution of the SPDE without obstacle (see [20]).*

With the uniform ellipticity condition we have the following equivalent conditions:

$$\| f(u,\nabla u) - f(v,\nabla v) \| \leq C \| u - v \| + C\lambda^{-1/2}\mathcal{E}^{1/2}(u-v)$$

$$\| g(u,\nabla u) - g(v,\nabla v) \|_{L^2(\mathcal{O};\mathbb{R}^d)} \leq C \| u - v \| + \alpha\lambda^{-1/2}\mathcal{E}^{1/2}(u-v)$$

$$\| h(u,\nabla u) - h(v,\nabla v) \|_{L^2(\mathcal{O};\mathbb{R}^{\mathbb{N}^*})} \leq C \| u - v \| + \beta\lambda^{-1/2}\mathcal{E}^{1/2}(u-v)$$

Assumption (I): Moreover we assume that for any $T > 0$,

$$\xi \in L^2(\Omega \times \mathcal{O}) \textit{ is an } \mathcal{F}_0 - \textit{measurable random variable}$$

$$f(\cdot,\cdot,\cdot,0,0) := f^0 \in L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R})$$
$$g(\cdot,\cdot,\cdot,0,0) := g^0 = (g_1^0, ..., g_d^0) \in L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^d)$$
$$h(\cdot,\cdot,\cdot,0,0) := h^0 = (h_1^0, ..., h_i^0, ...) \in L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^{\mathbb{N}^*}).$$

Now we introduce the notion of weak solution.
For simplicity, we fix the terminal time $T > 0$. We denote by $\mathcal{H}_T$ the space of $H_0^1(\mathcal{O})$-valued predictable continuous processes $(u_t)_{t\in[0,T]}$ which satisfy

$$E \sup_{t\in[0,T]} \| u_t \|^2 + E \int_0^T \mathcal{E}(u_t)dt < +\infty.$$

It is the natural space for solutions.
The space of test functions is denote by $\mathcal{D} = \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^2(\mathcal{O})$, where $\mathcal{C}_c^\infty(\mathbb{R}^+)$ is the space of all real valued infinite differentiable functions with compact support in $\mathbb{R}^+$ and $\mathcal{C}_c^2(\mathcal{O})$ the set of C^2-functions with compact support in $\mathcal{O}$.
Heuristquely, a pair (u, ν) is a solution of the obstacle problem for (2.1) with null boundary condition if we have the followings:

1. $u \in \mathcal{H}_T$ and $u(t,x) \geq S(t,x),\ dP \otimes dt \otimes dx - a.e.$ and $u_0(x) = \xi,\ dP \otimes dx - a.e.$;
2. ν is a random measure defined on $(0,T) \times \mathcal{O}$;
3. the following relation holds almost surely, for all $t \in [0,T]$ and $\forall \varphi \in \mathcal{D}$,
$$(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s)ds + \int_0^t \mathcal{E}(u_s, \varphi_s)ds + \sum_{i=1}^d \int_0^t (g_s^i(u_s, \nabla u_s), \partial_i \varphi_s)ds$$
$$= \int_0^t (f_s(u_s, \nabla u_s), \varphi_s)ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j(u_s, \nabla u_s), \varphi_s)dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x)\nu(dx, ds);$$
4. $$\int_0^T \int_{\mathcal{O}} (u(s,x) - S(s,x))\nu(dx, ds) = 0, \quad a.s..$$

But, the random measure which in some sense obliges the solution to stay above the barrier is a local time so, in general, it is not absolutely continuous w.r.t Lebesgue measure. As a consequence, for example, the condition

$$\int_0^T \int_{\mathcal{O}} (u(s,x) - S(s,x))\nu(dxds) = 0$$

makes no sense. Hence we need to consider precise version of u and S defined $\nu-$almost surely.
In order to tackle this difficulty, we introduce in the next section the notions of parabolic capacity on $[0,T] \times \mathcal{O}$ and quasi-continuous version of functions introduced by Michel Pierre

in several works (see for example [70, 71]). Let us remark that these tools were also used by Klimsiak ([41]) to get a probabilistic interpretation to semilinear PDE's with obstacle.

Finally and to end this section, we give an important example of stochastic noise which is covered by our framework:

Example 2.3. *Let W be a noise white in time and colored in space, defined on a standard filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\big)$ whose covariance function is given by:*

$$\forall s,t \in \mathbb{R}^+,\ \forall x,y \in \mathcal{O}, \quad E[\dot{W}(x,s)\dot{W}(y,t)] = \delta(t-s)k(x,y),$$

where $k : \mathcal{O}\times\mathcal{O} \mapsto \mathbb{R}^+$ is a symmetric and measurable function.
Consider the following SPDE driven by W:

$$\begin{aligned} du_t(x) = \big(\sum_{i,j=1}^{d} \partial_i a_{i,j}(x)\partial_j u_t(x) + f(t,x,u_t(x),\nabla u_t(x)) + \sum_{i=1}^{d} \partial_i g_i(t,x,u_t(x),\nabla u_t(x))\big)dt \\ + \widetilde{h}(t,x,u_t(x),\nabla u_t(x))\, W(dt,x), \end{aligned} \tag{2.3}$$

where f and g are as above and $\widetilde{h}$ is a random real valued function.
We assume that the covariance function k defines a trace class operator denoted by K in $L^2(\mathcal{O})$. It is well known (see [74]) that there exists an orthogonal basis $(e_i)_{i\in\mathbb{N}^*}$ of $L^2(\mathcal{O})$ consisting of eigenfunctions of K with corresponding eigenvalues $(\lambda_i)_{i\in\mathbb{N}^*}$ such that

$$\sum_{i=1}^{+\infty} \lambda_i < +\infty,$$

and

$$k(x,y) = \sum_{i=1}^{+\infty} \lambda_i e_i(x)e_i(y).$$

It is also well known that there exists a sequence $((B^i(t))_{t\geq 0})_{i\in\mathbb{N}^*}$ of independent standard Brownian motions such that

$$W(dt,\cdot) = \sum_{i=1}^{+\infty} \lambda_i^{1/2} e_i B^i(dt).$$

So that equation (2.3) is equivalent to (2.1) with $h = (h_i)_{i\in\mathbb{N}^*}$ where

$$\forall i \in \mathbb{N}^*,\ h_i(s,x,y,z) = \sqrt{\lambda_i}\widetilde{h}(s,x,y,z)e_i(x).$$

Assume as in [76] that for all $i\in\mathbb{N}^*$, $\|e_i\|_\infty < +\infty$ and

$$\sum_{i=1}^{+\infty} \lambda_i \|e_i\|_\infty^2 < +\infty.$$

Since

$$\Big(|h(t,\omega,x,y,z) - h(t,\omega,x,y\partial,z\partial)|^2\Big)^{\frac{1}{2}} = \left(\sum_{i=1}^{+\infty} \lambda_i\|e_i\|_\infty^2\right) \Big|\widetilde{h}(t,x,y,z) - \widetilde{h}(t,x,y\partial,z\partial)\Big|^2,$$

*h satisfies the Lipschitz hypothesis **(H)-(ii)** if and only if $\widetilde{h}$ satisfies a similar Lipschitz hypothesis.*

2.3 Parabolic potential analysis

2.3.1 Parabolic capacity and potentials

In this section we will recall some important definitions and results concerning the obstacle problem for parabolic PDE in [70] and [71].
$\mathcal{K}$ denotes $L^\infty(0,T;L^2(\mathcal{O}))\cap L^2(0,T;H_0^1(\mathcal{O}))$ equipped with the norm:

$$\begin{aligned}\| v \|_{\mathcal{K}}^2 &= \| v \|^2_{L^\infty(0,T;L^2(\mathcal{O}))} + \| v \|^2_{L^2(0,T;H_0^1(\mathcal{O}))}\\ &= \sup_{t\in[0,T[} \| v_t \|^2 + \int_0^T \big(\| v_t \|^2 + \mathcal{E}(v_t)\big)\, dt.\end{aligned}$$

$\mathcal{C}$ denotes the space of continuous functions on compact support in $[0,T[\times\mathcal{O}$ and finally:

$$\mathcal{W}=\{\varphi\in L^2(0,T;H_0^1(\mathcal{O}));\ \frac{\partial\varphi}{\partial t}\in L^2(0,T;H^{-1}(\mathcal{O}))\},$$

endowed with the norm$\| \varphi \|^2_{\mathcal{W}}=\| \varphi \|^2_{L^2(0,T;H_0^1(\mathcal{O}))}+\| \frac{\partial\varphi}{\partial t} \|^2_{L^2(0,T;H^{-1}(\mathcal{O}))}$.
It is known (see [49]) that $\mathcal{W}$ is continuously embedded in $C([0,T];L^2(\mathcal{O}))$, the set of $L^2(\mathcal{O})$-valued continuous functions on $[0,T]$. So without ambiguity, we will also consider $\mathcal{W}_T=\{\varphi\in\mathcal{W};\varphi(T)=0\}$, $\mathcal{W}^+=\{\varphi\in\mathcal{W};\varphi\geq 0\}$, $\mathcal{W}_T^+=\mathcal{W}_T\cap\mathcal{W}^+$.
We now introduce the notion of parabolic potentials and regular measures which permit to define the parabolic capacity.

Definition 2.4. *An element $v\in\mathcal{K}$ is said to be a* ***parabolic potential*** *if it satisfies:*

$$\forall\varphi\in\mathcal{W}_T^+,\ \int_0^T -(\frac{\partial\varphi_t}{\partial t},v_t)dt+\int_0^T\mathcal{E}(\varphi_t,v_t)dt\geq 0.$$

We denote by $\mathcal{P}$ the set of all parabolic potentials.

The next representation property is crucial:

Proposition 2.5. *(Proposition 1.1 in [71]) Let $v\in\mathcal{P}$, then there exists a unique positive Radon measure on $[0,T[\times\mathcal{O}$, denoted by ν^v, such that:*

$$\forall\varphi\in\mathcal{W}_T\cap\mathcal{C},\ \int_0^T(-\frac{\partial\varphi_t}{\partial t},v_t)dt+\int_0^T\mathcal{E}(\varphi_t,v_t)dt=\int_0^T\int_{\mathcal{O}}\varphi(t,x)d\nu^v.$$

Moreover, v admits a right-continuous (resp. left-continuous) version $\widehat{v}$ (resp. $\bar{v}$) : $[0,T]\mapsto L^2(\mathcal{O})$.
Such a Radon measure, ν^v is called ***a regular measure*** *and we write:*

$$\nu^v=\frac{\partial v}{\partial t}+Av.$$

Remark 2.6. *As a consequence, we can also define for all $v\in\mathcal{P}$:*

$$v_T=\lim_{t\uparrow T}\bar{v}_t\ \in L^2(\mathcal{O}).$$

Definition 2.7. *Let $K \subset [0,T[\times\mathcal{O}$ be compact, $v \in \mathcal{P}$ is said to be $\nu-$superior than 1 on K, if there exists a sequence $v_n \in \mathcal{P}$ with $v_n \geq 1$ a.e. on a neighborhood of K converging to v in $L^2(0,T;H_0^1(\mathcal{O}))$.*

We denote:

$$\mathcal{S}_K = \{v \in \mathcal{P};\ v \text{ is } \nu - \text{superior to } 1 \text{ on } K\}.$$

Proposition 2.8. *(Proposition 2.1 in [71]) Let $K \subset [0,T[\times\mathcal{O}$ compact, then $\mathcal{S}_K$ admits a smallest $v_K \in \mathcal{P}$ and the measure ν_K^v whose support is in K satisfies*

$$\int_0^T \int_{\mathcal{O}} d\nu_K^v = \inf_{v\in\mathcal{P}}\{\int_0^T \int_{\mathcal{O}} d\nu^v;\ v \in \mathcal{S}_K\}.$$

Definition 2.9. *(Parabolic Capacity)*

- *Let $K \subset [0,T[\times\mathcal{O}$ be compact, we define cap$(K) = \int_0^T \int_{\mathcal{O}} d\nu_K^v$;*
- *let $O \subset [0,T[\times\mathcal{O}$ be open, we define cap$(O) = \sup\{cap(K);\ K \subset O$ compact$\}$;*
- *for any borelian $E \subset [0,T[\times\mathcal{O}$, we define cap$(E) = \inf\{cap(O);\ O \supset E$ open$\}$.*

Definition 2.10. *A property is said to hold quasi-everywhere (in short q.e.) if it holds outside a set of null capacity.*

Definition 2.11. *(Quasi-continuous)*

A function $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is called quasi-continuous, if there exists a decreasing sequence of open subsets O_n of $[0,T[\times\mathcal{O}$ with:

1. *for all n, the restriction of u_n to the complement of O_n is continuous;*
2. *$\lim_{n\to+\infty} cap\ (O_n) = 0$.*

We say that u admits a quasi-continuous version, if there exists $\widetilde{u}$ quasi-continuous such that $\widetilde{u} = u$ a.e..

The next proposition, whose proof may be found in [70] or [71] shall play an important role in the sequel:

Proposition 2.12. *Let $K \subset \mathcal{O}$ a compact set, then $\forall t \in [0,T[$*

$$cap(\{t\} \times K) = \lambda_d(K),$$

where λ_d is the Lebesgue measure on $\mathcal{O}$.
As a consequence, if $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is a map defined quasi-everywhere then it defines uniquely a map from $[0,T[$ into $L^2(\mathcal{O})$. In other words, for any $t \in [0,T[$, u_t is defined without any ambiguity as an element in $L^2(\mathcal{O})$. Moreover, if $u \in \mathcal{P}$, it admits version $\bar{u}$ which is left continuous on $[0,T]$ with values in $L^2(\mathcal{O})$ so that $u_T = \bar{u}_{T-}$ is also defined without ambiguity.

Remark 2.13. *The previous proposition applies if for example u is quasi-continuous.*

Proposition 2.14. *(Theorem III.1 in [71]) If $\varphi \in \mathcal{W}$, then it admits a unique quasi-continuous version that we denote by $\widetilde{\varphi}$. Moreover, for all $v \in \mathcal{P}$, the following relation holds:*

$$\int_{[0,T[\times\mathcal{O}} \widetilde{\varphi} d\nu^v = \int_0^T (-\partial_t \varphi, v) + \mathcal{E}(\varphi, v)\, dt + (\varphi_T, v_T)\,.$$

2.3.2 Applications to PDE's with obstacle

For any function $\psi : [0,T[\times\mathcal{O} \to R$ and $u_0 \in L^2(\mathcal{O})$, following M. Pierre [70, 71], F. Mignot and J.P. Puel [54], we define

$$\kappa(\psi, u_0) = ess \inf\{u \in \mathcal{P};\ u \geq \psi\ a.e.,\ u(0) \geq u_0\}. \tag{2.4}$$

This lower bound exists and is an element in $\mathcal{P}$. Moreover, when ψ is quasi-continuous, this potential is the solution of the following reflected problem:

$$\kappa \in \mathcal{P},\ \kappa \geq \psi,\ \frac{\partial \kappa}{\partial t} + A\kappa = 0\ on\ \{u > \psi\},\ \kappa(0) = u_0.$$

Mignot and Puel have proved in [54] that $\kappa(\psi, u_0)$ is the limit (increasingly and weakly in $L^2(0,T;H_0^1(\mathcal{O}))$) when ϵ tends to 0 of the solution of the following penalized equation

$$u_\epsilon \in \mathcal{W},\ u_\epsilon(0) = u_0,\ \frac{\partial u_\epsilon}{\partial t} + Au_\epsilon - \frac{(u_\epsilon - \psi)^-}{\epsilon} = 0.$$

Let us point out that they obtain this result in the more general case where ψ is only measurable from $[0,T[$ into $L^2(\mathcal{O})$.
For given $f \in L^2(0,T;H^{-1}(\mathcal{O}))$, we denote by $\kappa_{u_0}^f$ the solution of the following problem:

$$\kappa \in \mathcal{W},\ \kappa(0) = u_0,\ \frac{\partial \kappa}{\partial t} + A\kappa = f.$$

The next theorem ensures existence and uniqueness of the solution of parabolic PDE with obstacle, it is proved in [70], Theorem 1.1. The proof is based on a regularization argument of the obstacle, using the results of [11].

Theorem 2.15. *Let $\psi : [0,T[\times\mathcal{O} \to \mathbb{R}$ be quasi-continuous, suppose that there exists $\zeta \in \mathcal{P}$ with $|\psi| \leq \zeta$ a.e., $f \in L^2(0,T;H^{-1}(\mathcal{O}))$, and the initial value $u_0 \in L^2(\mathcal{O})$ with $u_0 \geq \psi(0)$, then there exists a unique $u \in \kappa_{u_0}^f + \mathcal{P}$ quasi-continuous such that:*

$$u(0) = u_0,\ \widetilde{u} \geq \psi,\ q.e.;\quad \int_0^T \int_{\mathcal{O}} (\widetilde{u} - \widetilde{\psi}) d\nu^{u-\kappa_{u_0}^f} = 0$$

We end this section by a convergence lemma which plays an important role in our approach (Lemma 3.8 in [71]):

Lemma 2.16. *If $(v^n)_n \in \mathcal{P}$ is a bounded sequence in $\mathcal{K}$ and converges weakly to v in $L^2(0,T;H_0^1(\mathcal{O}))$; if u is a quasi-continuous function and $|u|$ is bounded by a element in $\mathcal{P}$. Then*

$$\lim_{n\to+\infty}\int_0^T\int_{\mathcal{O}} u d\nu^{v^n} = \int_0^T\int_{\mathcal{O}} u d\nu^v.$$

Remark 2.17. *For the more general case one can see [71] Lemma 3.8.*

2.4 Quasi-continuity of the solution of SPDE without obstacle

As a consequence of well-known results (see for example [20], Theorem 8), we know that under assumptions **(H)** and **(I)**, SPDE (2.1) with zero Dirichlet boundary condition, admits a unique solution in $\mathcal{H}_T$, we denote it by $\mathcal{U}(\xi, f, g, h)$.
The main theorem of this section is the following:

Theorem 2.18. *Under assumptions **(H)** and **(I)**, $u = \mathcal{U}(\xi, f, g, h)$ the solution of SPDE (2.1) admits a quasi-continuous version denoted by $\widetilde{u}$ i.e. $u = \widetilde{u}$ $dP \times dt \times dx$ a.e. and for almost all $w \in \Omega$, $(t,x) \to \widetilde{u}_t(w,x)$ is quasi-continuous.*

Before giving the proof of this theorem, we need the following lemmas. The first one is proved in [71], Lemma 3.3:

Lemma 2.19. *There exists $C > 0$ such that, for all open set $\vartheta \subset [0,T[\times\mathcal{O}$ and $v \in \mathcal{P}$ with $v \geq 1$ a.e. on ϑ:*

$$cap\vartheta \leq C \parallel v \parallel_{\mathcal{K}}^2 .$$

Let $\kappa = \kappa(u, u^+(0))$ be defined by relation (2.4). One has to note that κ is a random function. From now on, we always take for κ the following measurable version

$$\kappa = \sup_n v^n,$$

where $(v^n)_n$ is the non-decreasing sequence of random functions given by

$$\begin{cases} \dfrac{\partial v_t^n}{\partial t} = Lv_t^n + n(v_t^n - u_t)^- \\ v_0^n = u^+(0). \end{cases} \tag{2.5}$$

Using the results recalled in Subsection 2.3, we know that for almost all $\omega \in \Omega$, $(v^n(\omega))_n$ converges weakly to $v(\omega) = \kappa(u(\omega), u^+(0)(\omega))$ in $L^2(0,T;H_0^1(\mathcal{O}))$ and that $v \geq u$.

Lemma 2.20. *We have the following estimate:*

$$E \parallel \kappa \parallel_{\mathcal{K}}^2 \leq C\left(E \parallel u_0^+ \parallel^2 + E \parallel u_0 \parallel^2 + E\int_0^T \parallel f_t^0 \parallel^2 + \parallel |g_t^0| \parallel^2 + \parallel |h_t^0| \parallel^2 dt\right),$$

where C is a constant depending only on the structure constants of the equation.

Proof. All along this proof, we shall denote by C or C_ϵ some constant which may change from line to line.
The following estimate for the solution of the SPDE we consider is well-known:

$$E\sup_{t\in[0,T]}\parallel u_t\parallel^2+E\int_0^T\mathcal{E}(u_t)dt\le CE(\parallel u_0\parallel^2+\int_0^T(\parallel f_t^0\parallel^2+\parallel|g_t^0|\parallel^2+\parallel|h_t^0|\parallel^2)dt)\quad(2.6)$$

where C is a constant depending only on the structure constants of the equation.
Consider the approximation $(v^n)_n$ defined by (3.47), P-almost surely, it converges weakly to $v=\kappa(u,u^+(0))$ in $L^2(0,T;H_0^1(\mathcal{O}))$.
We remark that v^n-u satisfies the following equation:

$$d(v_t^n-u_t)+A(v_t^n-u_t)dt=-f_t(u_t,\nabla u_t)dt-\sum_{i=1}^d\partial_i g_t^i(u_t,\nabla u_t)dt-\sum_{j=1}^{+\infty}h_t^j(u_t,\nabla u_t)dB_t^j+n(v_t^n-u_t)^-dt,$$

applying the Itô's formula to $(v^n-u)^2$, see Lemma 7 in [21], we have

$$\begin{aligned}&\parallel v_t^n-u_t\parallel^2+2\int_0^t\mathcal{E}(v_s^n-u_s)ds=\parallel u_0^+-u_0\parallel^2-2\int_0^t(v_s^n-u_s,f_s(u_s,\nabla u_s))ds\\&+2\sum_{i=1}^d\int_0^t(\partial_i(v_s^n-u_s),g_s^i(u_s,\nabla u_s))ds+\int_0^t\parallel|h_s(u_s,\nabla u_s)|\parallel^2ds\\&-2\sum_{j=1}^{+\infty}\int_0^t(v_s^n-u_s,h_s^j(u_s,\nabla u_s))dB_s^j+2\int_0^t(n(v_s^n-u_s)^-,v_s^n-u_s)ds.\end{aligned}\quad(2.7)$$

The last term in the right member of (2.7) is obviously non-positive so

$$\begin{aligned}&\parallel v_t^n-u_t\parallel^2+2\int_0^t\mathcal{E}(v_s^n-u_s)ds\le\parallel u_0^+-u_0\parallel^2-2\int_0^t(v_s^n-u_s,f_s(u_s,\nabla u_s))ds\\&+\int_0^t\parallel|h_s(u_s,\nabla u_s)|\parallel^2ds+2\sum_{i=1}^d\int_0^t(\partial_i(v_s^n-u_s),g_s^i(u_s,\nabla u_s))ds\\&-2\sum_{j=1}^{+\infty}\int_0^t(v_s^n-u_s,h_s^j(u_s,\nabla u_s))dB_s^j.\end{aligned}\quad(2.8)$$

Then taking expectation and using Cauchy-Schwarz's inequality, we get

$$\begin{aligned}&E\parallel v_t^n-u_t\parallel^2+(2-\frac{\epsilon}{\lambda})E\int_0^t\mathcal{E}(v_s^n-u_s)ds\le E\parallel u_0^+-u_0\parallel^2+E\int_0^t\parallel v_s^n-u_s\parallel^2ds\\&+E\int_0^t\parallel f_s(u_s,\nabla u_s)\parallel^2ds+C_\epsilon E\int_0^t\parallel|g_s(u_s,\nabla u_s)|\parallel^2ds+E\int_0^t\parallel|h_s(u_s,\nabla u_s)|\parallel^2ds.\end{aligned}$$

Therefore, by using the Lipschitz conditions on the coefficients we have:

$$\begin{aligned}&E\parallel v_t^n-u_t\parallel^2+(2-\frac{\epsilon}{\lambda})E\int_0^t\mathcal{E}(v_s^n-u_s)ds\le E\parallel u_0^+-u_0\parallel^2+E\int_0^t\parallel v_s^n-u_s\parallel^2ds\\&+CE\int_0^t(\parallel f_s^0\parallel^2+\parallel|g_s^0|\parallel^2+\parallel|h_s^0|\parallel^2)ds+CE\int_0^t\parallel u_s\parallel^2ds+(\frac{C}{\lambda}+\frac{\alpha}{\lambda}+\frac{\beta^2}{\lambda})E\int_0^t\mathcal{E}(u_s)ds.\end{aligned}$$

Combining with (2.6), this yields

$$E \parallel v_t^n - u_t \parallel^2 + (2 - \frac{\epsilon}{\lambda}) E \int_0^t \mathcal{E}(v_s^n - u_s) ds \leq E \parallel u_0^+ - u_0 \parallel^2 + E \int_0^t \parallel v_s^n - u_s \parallel^2 ds$$
$$+ CE(\parallel u_0 \parallel^2 + \int_0^T (\parallel f_t^0 \parallel^2 + \parallel |g_t^0| \parallel^2 + \parallel |h_t^0| \parallel^2) dt).$$

We take now ϵ small enough such that $(2 - \frac{\epsilon}{\lambda}) > 0$, then, with Gronwall's lemma, we obtain for each $t \in [0, T]$

$$E \parallel v_t^n - u_t \parallel^2 \leq C e^{c'T} (E \parallel u_0^+ - u_0 \parallel^2 + E \parallel u_0 \parallel^2 + E \int_0^T \parallel f_t^0 \parallel^2 + \parallel |g_t^0| \parallel^2 + \parallel |h_t^0| \parallel^2 dt).$$

As we a priori know that P-almost surely, $(v^n)_n$ tends to κ strongly in $L^2([0,T] \times \mathcal{O})$, the previous estimate yields, thanks to the dominated convergence theorem, that $(v_n)_n$ converges to κ strongly in $L^2(\Omega \times [0,T] \times \mathcal{O})$ and

$$\sup_{t \in [0,T]} E \parallel \kappa_t - u_t \parallel^2 \leq C e^{c'T} (E \parallel u_0^+ - u_0 \parallel^2 + E \parallel u_0 \parallel^2 + E \int_0^T \parallel f_t^0 \parallel^2 + \parallel |g_t^0| \parallel^2 + \parallel |h_t^0| \parallel^2 dt).$$

Moreover, as $(v^n)_n$ tends to κ weakly in $L^2([0,T]; H_0^1(\mathcal{O}))$ P-almost-surely, we have for all $t \in [0,T]$:

$$E \int_0^T \mathcal{E}(\kappa_s - u_s) ds \leq \liminf_n E \int_0^T \mathcal{E}(v_s^n - u_s) ds$$
$$\leq TCe^{c'T} (E \parallel u_0^+ - u_0 \parallel^2 + E \parallel u_0 \parallel^2 + E \int_0^T \parallel f_t^0 \parallel^2 + \parallel |g_t^0| \parallel^2 + \parallel |h_t^0| \parallel^2 dt).$$

Let us now study the stochastic term in (2.8). Let define the martingales

$$M_t^n = \sum_{j=1}^{+\infty} \int_0^t (v_s^n - u_s, h_s^j) dB_s^j \quad \text{and} \quad M_t = \sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s^j) dB_s^j.$$

Then

$$E[|M_T^n - M_T|^2] = E \int_0^T \sum_{j=1}^{+\infty} (\kappa_s - v_s^n, h_s)^2 \, ds \leq E \int_0^T \parallel \kappa_s - v_s^n \parallel^2 \parallel |h_s| \parallel^2 \, ds.$$

Using the strong convergence of $(v^n)_n$ to κ we conclude that $(M^n)_n$ tends to M in L^2 sense. Passing to the limit in (2.8), we get:

$$\begin{aligned}
\parallel \kappa_t - u_t \parallel^2 + 2 \int_0^t \mathcal{E}(\kappa_s - u_s) ds &\leq \parallel u_0^+ - u_0 \parallel^2 - 2 \int_0^t (\kappa_s - u_s, f_s(u_s, \nabla u_s)) ds \\
&+ 2 \sum_{i=1}^d \int_0^t (\partial_i(\kappa_s - u_s), g_s^i(u_s, \nabla u_s)) ds \\
&- 2 \sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s^j(u_s, \nabla u_s)) dB_s^j + \int_0^t \parallel |h_s(u_s, \nabla u_s)| \parallel^2 ds.
\end{aligned}$$

As a consequence of the Burkholder-Davies-Gundy's inequalities, we get

$$\begin{aligned}
& E\sup_{t\in[0,T]}|\sum_{j=1}^{+\infty}\int_0^t(\kappa_s-u_s,h_s^j(u_s,\nabla u_s))dB_s^j| \\
\leq\; & CE[\int_0^T\sum_{j=1}^{+\infty}(\kappa_s-u_s,h_s^j(u_s,\nabla u_s))^2ds]^{1/2} \\
\leq\; & CE[\int_0^T\sum_{j=1}^{+\infty}\sup_{t\in[0,T]}\parallel\kappa_t-u_t\parallel^2\parallel h_s^j(u_s,\nabla u_s)\parallel^2 ds]^{1/2} \\
\leq\; & CE[\sup_{t\in[0,T]}\parallel\kappa_t-u_t\parallel(\int_0^T\parallel|h_t(u_t,\nabla u_t)|\parallel^2 dt)^{1/2}] \\
\leq\; & \epsilon E\sup_{t\in[0,T]}\parallel\kappa_t-u_t\parallel^2+C_\epsilon E\int_0^T\parallel|h_t(u_t,\nabla u_t)|\parallel^2 dt.
\end{aligned}$$

By Lipschitz conditions on h and (2.6) this yields

$$\begin{aligned}
E\sup_{t\in[0,T]}|\sum_{j=1}^{+\infty}\int_0^t(\kappa_s-u_s,h_s(u_s,\nabla u_s))dB_s| &\leq \epsilon E\sup_{t\in[0,T]}\parallel\kappa_t-u_t\parallel^2+C(E\parallel u_0\parallel^2 \\
&+ E\int_0^T(\parallel f_t^0\parallel^2+\parallel|g_t^0|\parallel^2+\parallel|h_t^0|\parallel^2)dt)
\end{aligned}$$

Hence,

$$\begin{aligned}
(1-\epsilon)E\sup_{t\in[0,T]}\parallel\kappa_t-u_t\parallel^2 &+ (2-\frac{\epsilon}{\lambda})E\int_0^T\mathcal{E}(\kappa_t-u_t)dt\leq C(E\parallel u_0^+-u_0\parallel^2+E\parallel u_0\parallel^2 \\
&+ E\int_0^T\parallel f_t^0\parallel^2+\parallel|g_t^0|\parallel^2+\parallel|h_t^0|\parallel^2 dt).
\end{aligned}$$

We can take ϵ small enough such that $1-\epsilon>0$ and $2-\frac{\epsilon}{\lambda}>0$, hence,

$$\begin{aligned}
E\sup_{t\in[0,T]}\parallel\kappa_t-u_t\parallel^2 &+ E\int_0^T\mathcal{E}(\kappa_t-u_t)dt\leq C(E\parallel u_0^+-u_0\parallel^2+E\parallel u_0\parallel^2 \\
&+ E\int_0^T\parallel f_t^0\parallel^2+\parallel|g_t^0|\parallel^2+\parallel|h_t^0|\parallel^2 dt).
\end{aligned}$$

Then, combining with (2.6), we get the desired estimate:

$$\begin{aligned}
E\sup_{t\in[0,T]}\parallel\kappa_t\parallel^2+E\int_0^T\mathcal{E}(\kappa_t)dt &\leq C(E\parallel u_0^+\parallel^2+E\parallel u_0\parallel^2 \\
&+ E\int_0^T\parallel f_t^0\parallel^2+\parallel|g_t^0|\parallel^2+\parallel|h_t^0|\parallel^2 dt).
\end{aligned}$$

∎

Proof of Theorem 2.18: For simplicity, we put

$f_t(x)=f(t,x,u_t(x),\nabla u_t(x))$, $g_t(x)=g(t,x,u_t(x),\nabla u_t(x))$ and $h_t(x)=h(t,x,u_t(x),\nabla u_t(x))$.

We introduce (P_t) the semi-group associated to operator A and put for each $n \in \mathbb{N}^*$, $i \in \{1, \cdots, d\}$ and each $j \in \mathbb{N}^*$:

$$u_0^n = P_{\frac{1}{n}} u_0, \quad f^n = P_{\frac{1}{n}} f, \quad g_i^n = P_{\frac{1}{n}} g_i, \quad and\ h_j^n = P_{\frac{1}{n}} h_j.$$

Then $(u_0^n)_n$ converges to u_0 in $L^2(\Omega; L^2(\mathcal{O}))$, $(f^n)_n$, $(g^n)_n$ and $(h^n)_n$ are sequences of elements in $L^2(\Omega \times [0,T]; \mathcal{D}(A))$ which converge respectively to f, g and h in $L^2(\Omega \times [0,T]; L^2(\mathcal{O}))$. For all $n \in \mathbb{N}$ we define

$$\begin{aligned} u_t^n &= P_t u_0^n + \int_0^t P_{t-s} f_s^n ds + \sum_{i=1}^d \int_0^t P_{t-s} \partial_i g_{i,s}^n ds + \sum_{j=1}^{+\infty} \int_0^t P_{t-s} h_{j,s}^n dB_s^j \\ &= P_{t+\frac{1}{n}} u_0 + \int_0^t P_{t+\frac{1}{n}-s} f_s ds + \sum_{i=1}^d \int_0^t P_{t+\frac{1}{n}-s} \partial_i g_{i,s} ds + \sum_{j=1}^{+\infty} \int_0^t P_{t+\frac{1}{n}-s} h_{j,s} dB_s^j. \end{aligned}$$

We denote by $G(t,x,s,y)$ the kernel associated to P_t, then

$$\begin{aligned} u^n(t,x) &= \int_{\mathcal{O}} G(t+\frac{1}{n}, x, 0, y) u_0(y) dy + \int_0^t \int_{\mathcal{O}} G(t+\frac{1}{n}, x, s, y) f(s,y) dy ds \\ &+ \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} G(t+\frac{1}{n}, x, s, y) \partial_i g_s^i(y) dy ds + \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} G(t+\frac{1}{n}, x, s, y) h_s^j(y) dy dB_s^i. \end{aligned}$$

But, as A is strictly elliptic, G is uniformly continuous in space-time variables on any compact away from the diagonal in time (see Theorem 6 in [1]) and satisfies Gaussian estimates (see Aronson [2]), this ensures that for all n, u^n is P-almost surely continuous in (t,x).

We consider a sequence of random open sets

$$\vartheta_n = \{|u^{n+1} - u^n| > \epsilon_n\}, \quad \Theta_p = \bigcup_{n=p}^{+\infty} \vartheta_n.$$

Let $\kappa_n = \kappa(\frac{1}{\epsilon_n}(u^{n+1}-u^n), \frac{1}{\epsilon_n}(u^{n+1}-u^n)^+(0)) + \kappa(-\frac{1}{\epsilon_n}(u^{n+1}-u^n), \frac{1}{\epsilon_n}(u^{n+1}-u^n)^-(0))$, from the definition of κ and the relation (see [71])

$$\kappa(|v|) \leq \kappa(v, v^+(0)) + \kappa(-v, v^-(0))$$

we know that κ_n satisfy the conditions of Lemma 2.19, i.e. $\kappa_n \in \mathcal{P}$ et $\kappa_n \geq 1$ $a.e.$ on ϑ_n, thus we get the following relation

$$cap\,(\Theta_p) \leq \sum_{n=p}^{+\infty} cap\,(\vartheta_n) \leq \sum_{n=p}^{+\infty} \| \kappa_n \|_{\mathcal{K}}^2 .$$

Thus, remarking that $u^{n+1} - u^n = \mathcal{U}(u_0^{n+1} - u_0^n, f^{n+1} - f^n, g^{n+1} - g^n, h^{n+1} - h^n)$, we apply Lemma 2.20 to $\kappa(\frac{1}{\epsilon_n}(u^{n+1}-u^n), \frac{1}{\epsilon_n}(u^{n+1}-u^n)^+(0))$ and $\kappa(-\frac{1}{\epsilon_n}(u^{n+1}-u^n), \frac{1}{\epsilon_n}(u^{n+1}-u^n)^-(0))$ and obtain:

$$\begin{aligned} E[cap\,(\Theta_p)] \leq \sum_{n=p}^{+\infty} E \| \kappa_n \|_{\mathcal{K}}^2 &\leq 2C \sum_{n=p}^{+\infty} \frac{1}{\epsilon_n^2} (E \| u_0^{n+1} - u_0^n \|^2 + E \int_0^T \| f_t^{n+1} - f_t^n \|^2 \\ &+ \| |g_t^{n+1} - g_t^n| \|^2 + \| |h_t^{n+1} - h_t^n| \|^2 \, dt) \end{aligned}$$

Then, by extracting a subsequence, we can consider that

$$E \parallel u_0^{n+1} - u_0^n \parallel^2 + E\int_0^T \parallel f_t^{n+1} - f_t^n \parallel^2 + \parallel |g_t^{n+1} - g_t^n| \parallel^2 + \parallel |h_t^{n+1} - h_t^n| \parallel^2 dt \leq \frac{1}{2^n}$$

Then we take $\epsilon_n = \frac{1}{n^2}$ to get

$$E[cap\ (\Theta_p)] \leq \sum_{n=p}^{+\infty} \frac{2Cn^4}{2^n}$$

Therefore

$$\lim_{p\to+\infty} E[cap\ (\Theta_p)] = 0.$$

For almost all $\omega \in \Omega$, $u^n(\omega)$ is continuous in (t,x) on $(\Theta_p(w))^c$ and $(u^n(\omega))_n$ converges uniformly to u on $(\Theta_p(w))^c$ for all p, hence, $u(\omega)$ is continuous in (t,x) on $(\Theta_p(w))^c$, then from the definition of quasi-continuous, we know that $u(\omega)$ admits a quasi-continuous version since *cap* (Θ_p) tends to 0 almost surely as p tends to $+\infty$. □

2.5 Existence and uniqueness of the solution of the obstacle problem

2.5.1 Weak solution

Assumption (O): The obstacle S is assumed to be an adapted process, quasi-continuous, such that $S_0 \leq \xi$ P-almost surely and controlled by the solution of an SPDE, i.e. $\forall t \in [0,T]$,

$$S_t \leq S'_t \tag{2.9}$$

where S' is the solution of the linear SPDE with null boundary condition:

$$\begin{cases} dS'_t &= LS'_t dt + f'_t dt + \sum_{i=1}^d \partial_i g'_{i,t} dt + \sum_{j=1}^{+\infty} h'_{j,t} dB_t^j \\ S'(0) &= S'_0, \end{cases} \tag{2.10}$$

where $S'_0 \in L^2(\Omega \times \mathcal{O})$ is $\mathcal{F}_0$-measurable, f', g' and h' are adapted processes respectively in $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R})$, $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^d)$ and $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^{\mathbb{N}})$.

Remark 2.21. *Here again, we know that S' uniquely exists and satisfies the following estimate:*

$$E \sup_{t\in[0,T]} \parallel S'_t \parallel^2 + E\int_0^T \mathcal{E}(S'_t)dt \leq CE\left[\parallel S'_0 \parallel^2 + \int_0^T (\parallel f'_t \parallel^2 + \parallel |g'_t| \parallel^2 + \parallel |h'_t| \parallel^2)dt\right] \tag{2.11}$$

Moreover, from Theorem 2.18, S' admits a quasi-continuous version.
Let us also remark that even if this assumption seems restrictive since S' is driven by the same operator and Brownian motions as u, it encompasses a large class of examples.

We now are able to define rigorously the notion of solution to the problem with obstacle we consider.

Definition 2.22. *A pair (u,ν) is said to be a solution of the obstacle problem for (2.1) with Dirichlet boundary condition if*

1. $u \in \mathcal{H}_T$ *and* $u(t,x) \geq S(t,x),\ dP\otimes dt\otimes dx-a.e.$ *and* $u_0(x)=\xi,\ dP\otimes dx-a.e.$;
2. ν *is a random regular measure defined on* $[0,T)\times\mathcal{O}$;
3. *the following relation holds almost surely, for all* $t\in[0,T]$ *and* $\forall\varphi\in\mathcal{D}$,

$$\begin{aligned}&(u_t,\varphi_t)-(\xi,\varphi_0)-\int_0^t(u_s,\partial_s\varphi_s)ds+\int_0^t\mathcal{E}(u_s,\varphi_s)ds+\sum_{i=1}^d\int_0^t(g_s^i(u_s,\nabla u_s),\partial_i\varphi_s)ds\\&=\int_0^t(f_s(u_s,\nabla u_s),\varphi_s)ds+\sum_{j=1}^{+\infty}\int_0^t(h_s^j(u_s,\nabla u_s),\varphi_s)dB_s^j+\int_0^t\int_{\mathcal{O}}\varphi_s(x)\nu(dx,ds)\end{aligned}\tag{2.12}$$

4. u *admits a quasi-continuous version,* $\widetilde{u}$, *and we have*

$$\int_0^T\int_{\mathcal{O}}(\widetilde{u}(s,x)-S(s,x))\nu(dx,ds)=0,\quad a.s..$$

The main result of this paper is the following:

Theorem 2.23. *Under assumptions **(H)**, **(I)** and **(O)**, there exists a unique weak solution of the obstacle problem for the SPDE (2.1) associated to* $(\xi,\ f,\ g,\ h,\ S)$.
We denote by $\mathcal{R}(\xi,f,g,h,S)$ *the solution of SPDE (2.1) with obstacle when it exists and is unique.*

As the proof of this theorem is quite long, we split it in several steps: first we prove existence and uniqueness in the linear case then establish an Itô formula and finally prove the Theorem thanks to a fixed point argument.

2.5.2 Proof of Theorem 2.23 in the linear case

All along this subsection, we assume that f, g and h do not depend on u and ∇u, so we consider that f, g and h are adapted processes respectively in $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R})$, $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^d)$ and $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^{\mathbb{N}^*})$.
For $n\in\mathbb{N}$, let u^n be the solution of the following SPDE

$$du_t^n=Lu_t^ndt+f_tdt+\sum_{i=1}^d\partial_ig_{i,t}dt+\sum_{j=1}^{+\infty}h_{j,t}dB_t^j+n(u_t^n-S_t)^-dt\tag{2.13}$$

with initial condition $u_0^n=\xi$ and null Dirichlet boundary condition. We know from Theorem 8 in [DS04] that this equation admits a unique solution in $\mathcal{H}_T$ and that the solution admits $L^2(\mathcal{O})-$continuous trajectories.

Lemma 2.24. *u^n satisfies the following estimate:*

$$E\sup_{t\in[0,T]}\parallel u_t^n\parallel^2+E\int_0^T\mathcal{E}(u_t^n)dt+E\int_0^T n\parallel(u_t^n-S_t)^-\parallel^2 dt\le C,$$

where C is a constant depending only on the structure constants of the SPDE.

Proof. From (2.13) and (2.10), we know that u^n-S' satisfies the following equation:

$$d(u_t^n-S_t')=L(u_t^n-S_t')dt+\widetilde{f}_t dt+\sum_{i=1}^d\partial_i\widetilde{g}_t^i dt+\sum_{j=1}^{+\infty}\widetilde{h}_t^j dB_t^j+n(u_t^n-S_t)^-dt$$

where $\widetilde{f}=f-f'$, $\widetilde{g}=g-g'$and $\widetilde{h}=h-h'$. Applying Itô's formula to $(u^n-S')^2$, we have:

$$\parallel u_t^n-S_t'\parallel^2+2\int_0^t\mathcal{E}(u_s^n-S_s')ds=2\int_0^t((u_s^n-S_s'),\widetilde{f}_s)ds+2\sum_{j=1}^{+\infty}\int_0^t((u_s^n-S_s'),\widetilde{h}_s^j)dB_s^j$$
$$-2\sum_{i=1}^d\int_0^t(\partial_i(u_s^n-\widetilde{S}_s),\widetilde{g}_s^i)ds+2\int_0^t\int_{\mathcal{O}}(u_s^n-S_s')n(u_s^n-S_s)^-ds+\int_0^t\parallel|\widetilde{h}_s|\parallel^2 ds.$$

We remark first:

$$\begin{aligned}\int_0^t\int_{\mathcal{O}}(u_s^n-S_s')n(u_s^n-S_s)^-ds &= \int_0^t\int_{\mathcal{O}}(u_s^n-S_s+S_s-S_s')n(u_s^n-S_s)^-ds\\ &= -\int_0^t\int_{\mathcal{O}}n((u_s^n-S_s)^-)^2ds+\int_0^t\int_{\mathcal{O}}(S_s-S_s')n(u_s^n-S_s)^-dxds\end{aligned}$$

the last term in the right member is non-positive because $S_t\le S_t'$, thus,

$$\parallel u_t^n-S_t'\parallel^2+2\int_0^t\mathcal{E}(u_s^n-S_s')ds+2\int_0^t n\parallel(u_s^n-S)^-\parallel^2 ds\le 2\int_0^t(u_s^n-S_s',\widetilde{f}_s)ds$$
$$-2\sum_{i=1}^d\int_0^t(\partial_i(u_s^n-S_s'),\widetilde{g}_s^i)ds+2\sum_{j=1}^{+\infty}\int_0^t(u_s^n-S_s',\widetilde{h}_s^j)dB_s^j+\int_0^t\parallel|\widetilde{h}_s|\parallel^2 ds.$$

Then using Cauchy-Schwarz's inequality, we have $\forall t\in[0,T]$,

$$2|\int_0^t(u_s^n-S_s',\widetilde{f}_s)ds|\le\epsilon\int_0^T\parallel u_s^n-S_s'\parallel^2 ds+\frac{1}{\epsilon}\int_0^T\parallel\widetilde{f}_s\parallel^2 ds$$

and

$$2|\sum_{i=1}^d\int_0^t(\partial_i(u_s^n-S_s'),\widetilde{g}_s^i)ds|\le\epsilon\int_0^T\parallel\nabla(u_s^n-S_s')\parallel^2 ds+\frac{1}{\epsilon}\int_0^T\parallel|\widetilde{g}|\parallel^2 ds.$$

Moreover, thanks to the Burkholder-Davies-Gundy inequality, we get

$$\begin{aligned}
E\sup_{t\in[0,T]}|\sum_{j=1}^{+\infty}\int_0^t(u_s^n-S'_s,\widetilde{h}_s^j)dB_s^j| &\le c_1E[\int_0^T\sum_{j=1}^{+\infty}(u_s^n-S'_s,\widetilde{h}_s^j)^2ds]^{1/2}\\
&\le c_1E[\int_0^T\sum_{j=1}^{+\infty}\sup_{s\in[0,T]}\|u_s^n-S'_s\|^2\|\widetilde{h}_s^j\|^2ds]^{1/2}\\
&\le c_1E[\sup_{s\in[0,T]}\|u_s^n-S'_s\|(\int_0^T\||\widetilde{h}_s|\|^2ds)^{1/2}]\\
&\le \epsilon E\sup_{s\in[0,T]}\|u_s^n-S'_s\|^2+\frac{c_1}{4\epsilon}E\int_0^T\||\widetilde{h}_s|\|^2ds.
\end{aligned}$$

Then using the strict ellipticity assumption and the inequalities above, we get

$$\begin{aligned}
&(1-2\epsilon(T+1))E\sup_{t\in[0,T]}\|u_t^n-S'_t\|^2+(2\lambda-\epsilon)E\int_0^T\mathcal{E}(u_s^n-S'_s)ds+2E\int_0^T n\|(u_s^n-S_s)^-\|^2ds\\
&\le C(E\|\xi\|^2+\frac{2}{\epsilon}E\int_0^T\|\widetilde{f}_s\|^2+\frac{2}{\epsilon}\||\widetilde{g}_s|\|^2+(\frac{c_1}{2\epsilon}+1)\||\widetilde{h}_s|\|^2ds).
\end{aligned}$$

We take ϵ small enough such that $(1-2\epsilon(T+1))>0$, this yields $(2\lambda-\epsilon)>0$

$$E\sup_{t\in[0,T]}\|u_t^n-S'_t\|^2+E\int_0^T\mathcal{E}(u_t^n-S'_t)dt+E\int_0^T n\|(u_t^n-S_t)^-\|^2dt\le C.$$

Then with (2.11), we obtain the desired estimate. ■

Proof. [End of the proof of Theorem 2.23] We now introduce z, the solution of the corresponding SPDE without obstacle:

$$dz_t+Az_tdt=f_tdt+\sum_{i=1}^d\partial_ig_{i,t}dt+\sum_{j=1}^{+\infty}h_{j,t}dB_t^j,$$

starting from $z_0=\xi$, with null Dirichlet condition on the boundary. As a consequence of Theorem 2.18, we can take for z a quasi-continuous version.
For each $n\in\mathbb{N}$, we put $v^n=u^n-z$. Clearly, v^n satisfies

$$dv_t^n+Av_t^ndt=n(v_t^n-(S_t-z_t))^-dt=n(u_t^n-S_t)^-dt.$$

Since $S-z$ is quasi-continuous almost-surely, by the results established by Mignot and Puel in [54], we know that P-almost surely, the sequence $(v^n)_n$ is increasing and converges in $L^2([0,T]\times\mathcal{O})$ P-almost surely to v and that the sequence of random measures $\nu^{v^n}=n(u_t^n-S_t)^-dtdx$ converges vaguely to a measure associated to v: $\nu=\nu^v$. As a consequence of the previous lemma, $(u^n)_n$ and $(v^n)_n$ are bounded sequences in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$ which is an Hilbert space (equipped the norm $(E\int_0^T\|u_t\|^2_{H_0^1(\mathcal{O})}dt)^{1/2}$), by a double extraction argument, we can construct subsequences $(u^{n_k})_k$ and $(v^{n_k})_k$ such that the first one converges weakly in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$ to an element that we denote u and the second one to an element which necessary is equal to v since $(v^n)_n$ is increasing. Moreover, we can construct sequences $(\widehat{u}^n)_n$ and $(\widehat{v}^n)_n$ of convex combinations of elements of the form

$$\widehat{u}^n = \sum_{k=1}^{N_n} \alpha_k^n u^{n_k} \text{ and } \widehat{v}^n = \sum_{k=1}^{N_n} \alpha_k^n v^{n_k}$$

converging strongly to u an v respectively in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$.

From the fact that u^n is the weak solution of (2.13), we get

$$(u_t^n, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s^n, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(u_s^n, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g_s^i, \partial_i \varphi_s) ds$$
$$= \int_0^t (f_s, \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j, \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) n(u_s^n - S_s)^- dx ds \ a.s. \quad (2.14)$$

Hence

$$(\widehat{u}_t^n, \varphi_t) - (\xi, \varphi_0) - \int_0^t (\widehat{u}_s^n, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(\widehat{u}_s^n, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g_s^i, \partial_i \varphi_s) ds$$
$$= \int_0^t (f_s, \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j, \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \left(\sum_{k=1}^{N_n} n_k (u_s^{n_k} - S_s)^- \right) dx ds \quad (2.15)$$

We have

$$\int_0^t \int_{\mathcal{O}} \varphi_s(x) \left(\sum_{k=1}^{N_n} n_k (u_s^{n_k} - S_s)^- \right) dx ds \ = \ \int_0^T -(\frac{\partial \varphi_t}{\partial t}, \widehat{v}_t^n) dt + \int_0^T \mathcal{E}(\varphi_t, \widehat{v}_t^n) dt$$

so that we have almost-surely, at least for a subsequence:

$$\begin{aligned} \lim_{n \to +\infty} \int_0^t \int_{\mathcal{O}} \varphi_s(x) \left(\sum_{k=1}^{N_n} n_k (u_s^{n_k} - S_s)^- \right) dx ds &= \int_0^T -(\frac{\partial \varphi_t}{\partial t}, v_t) dt + \int_0^T \mathcal{E}(\varphi_t, v_t) dt \\ &= \int_0^T \int_{\mathcal{O}} \varphi_t(x) \nu(dx, dt). \end{aligned}$$

As $(\widehat{u}^n)_n$ converges to u in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$, by making n tend to $+\infty$ in (2.15), we obtain:

$$(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s \varphi_s) ds + \int_0^t \mathcal{E}(u_s, \varphi_s) ds + \sum_{i=1}^d \int_0^t (g_s^i, \partial_i \varphi_s) ds$$
$$= \int_0^t (f_s, \varphi_s) ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j, \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds), \quad a.s..$$

In the next subsection, we'll show that u satisfies an Itô's formula, as a consequence by applying it to u_t^2, using standard arguments we get that $u \in \mathcal{H}_T$ so for almost all $\omega \in \Omega$,

$u(\omega) \in \mathcal{K}$. And from Theorem 9 in [20], we know that for almost all $\omega \in \Omega$, $z(\omega) \in \mathcal{K}$. Therefore, for almost all $\omega \in \Omega$, $v(\omega) = u(w) - z(w) \in \mathcal{K}$. Hence, $\nu = \partial_t v + Av$ is a regular measure by definition. Moreover, by [70, 71] we know that v admits a quasi continuous version $\widetilde{v}$ which satisfies the minimality condition

$$\int\int (\widetilde{v} - S + \widetilde{z})\nu(dxdt) = 0. \tag{2.16}$$

z is quasi-continuous version hence $\widetilde{u} = z + \widetilde{v}$ is a quasi-continuous version of u and we can write (2.16) as

$$\int\int (\widetilde{u} - S)\nu(dxdt) = 0.$$

The fact that $u \geq S$ comes from the fact that $v \geq u - z$, so at this stage we have proved that (u, ν) is a solution to the obstacle problem we consider.
Uniqueness comes from the fact that both z and v are unique, which ends the proof of Theorem 2.23.
■

2.5.3 Itô's formula

The following Itô's formula for the solution of the obstacle problem is fundamental to get all the results in the non linear case. Let us also remark, that any solution of the non-linear equation (2.1) may be viewed as the solution of a linear one so that it satisfies also the Itô's formula.

Theorem 2.25. *Under assumptions of the previous subsection 2.5.2, let u be the solution of SPDE(2.1) with obstacle and $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^{1,2}$. We denote by Φ' and Φ'' the derivatives of Φ with respect to the space variables and by $\frac{\partial \Phi}{\partial t}$ the partial derivative with respect to time. We assume that these derivatives are bounded and $\Phi'(t,0) = 0$ for all $t \geq 0$. Then $P - a.s.$ for all $t \in [0,T]$,*

$$\begin{aligned}\int_{\mathcal{O}} \Phi(t, u_t(x))dx + \int_0^t \mathcal{E}(\Phi'(s,u_s), u_s)ds &= \int_{\mathcal{O}} \Phi(0, \xi(x))dx + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, u_s(x))dxds \\ &+ \int_0^t (\Phi'(s,u_s), f_s)ds - \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s(x))\partial_i u_s(x) g_i(x)dxds + \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s,u_s), h_j)dB_s^j \\ &+ \frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s(x))(h_{j,s}(x))^2 dxds + \int_0^t \int_{\mathcal{O}} \Phi'(s, \widetilde{u}_s(x))\nu(dxds).\end{aligned}$$

Proof. We keep the same notations as in the previous subsection and so consider the sequence $(u^n)_n$ approximating u and also $(\widehat{u}^n)_n$ the sequence of convex combinations $\widehat{u}^n = \sum_{k=1}^{N_n} \alpha_k^n u^{n_k}$ converging strongly to u in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$.
Moreover, by standard arguments such as the Banach-Saks theorem, since $(u^n)_n$ is non-decreasing, we can choose the convex combinations such that $(\widehat{u}^n)_n$ is also a non-decreasing sequence. We start by a key lemma:

Lemma 2.26. *Let $t \in [0,T]$, then*

$$\lim_{n\to+\infty} E\int_0^t \int_{\mathcal{O}} (\widehat{u}_s^n - S_s)^- \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds = 0.$$

Proof. We write as above $u^n = v^n + z$ and we denote $\widehat{\nu}^n = \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^-$ so that

$$\int_0^t \int_{\mathcal{O}} (\widehat{u}_s^n - S_s)\widehat{\nu}^n(dxds) = \int_0^t \int_{\mathcal{O}} \widehat{v}_s^n \widehat{\nu}^n(dxds) + \int_0^t \int_{\mathcal{O}} (z_s - S_s)\widehat{\nu}^n(dxds)$$

From Lemma 2.16, we know that

$$\int_0^t \int_{\mathcal{O}} (z_s - S_s)\widehat{\nu}^n(dxds) \to \int_0^t \int_{\mathcal{O}} (z_s - S_s)\nu(dxds).$$

Moreover, by Lemma II.6 in [70] we have for all n:

$$\frac{1}{2} \parallel \widehat{v}_T^n \parallel^2 + \int_0^T \mathcal{E}(\widehat{v}_s^n)ds = \int_0^T \int_{\mathcal{O}} \widehat{v}_s^n \widehat{\nu}^n(dxds),$$

and

$$\frac{1}{2} \parallel v_T \parallel^2 + \int_0^T \mathcal{E}(v_s)ds = \int_0^T \int_{\mathcal{O}} \widetilde{v}_s \nu(dxds).$$

As $(\widehat{v}_n)_n$ tends to v in $L^2([0,T], H_0^1(\mathcal{O}))$,

$$\lim_{n\to+\infty} \int_0^T \mathcal{E}(\widehat{v}_s^n)ds = \int_0^T \mathcal{E}(v_s)ds.$$

Let us prove that $(\parallel \widehat{v}_T^n \parallel)_n$ tends to $\parallel v_T \parallel$.
Since, $(\widehat{v}_T^n)_n$ is non-decreasing and bounded in $L^2(\mathcal{O})$ it converges in $L^2(\mathcal{O})$ to $m = \sup_n \widehat{v}_T^n$.
Let $\rho \in H_0^1(\mathcal{O})$ then the map defined by $\varphi(t,x) = \rho(x)$ belongs to $\mathcal{W}$ hence as a consequence of Proposition 3.39

$$\int_{[0,T[\times\mathcal{O}} \rho d\widehat{\nu}^n = \int_0^T \mathcal{E}(\rho, \widehat{v}_s^n)\, ds + (\rho, \widehat{v}_T^n),$$

and

$$\int_{[0,T[\times\mathcal{O}} \widetilde{\rho} d\nu = \int_0^T \mathcal{E}(\rho, v_s)\, ds + (\rho, v_T),$$

making n tend to $+\infty$ and using one more time Lemma 2.16, we get

$$\lim_{n\to+\infty} (\rho, \widehat{v}_T^n) = (\rho, m) = (\rho, v_T).$$

Since ρ is arbitrary, we have $v_T = m$ and so $\lim_{n\to+\infty} \parallel \widehat{v}_T^n \parallel = \parallel v_T \parallel$ and this yields

$$\lim_{n\to+\infty} \int_0^T \int_{\mathcal{O}} \widehat{v}_s^n \widehat{\nu}^n(dxds) = \int_0^T \int_{\mathcal{O}} \widetilde{v}_s \nu(dxds) = \int_0^T \int_{\mathcal{O}} (S_s - z_s)\nu(dxds).$$

This proves that

$$\lim_{n\to+\infty} \int_0^t \int_{\mathcal{O}} (\widehat{u}_s^n - S_s)\widehat{\nu}^n(dxds) = 0,$$

we conclude by remarking that

$$\lim_{n\to+\infty}\int_0^t\int_{\mathcal{O}}(\widehat{u}_s^n-S_s)^+\widehat{\nu}^n(dxds)\leq\lim_{n\to+\infty}\int_0^t\int_{\mathcal{O}}(u_s-S_s)\widehat{\nu}^n(dxds)=\int_0^t\int_{\mathcal{O}}(\widetilde{u}_s-S_s)\nu(dxds)=0.$$

■

Proof of Theorem 2.25: We consider the penalized solution $(u^n)_n$, we know that its convex combination $(\widehat{u}^n)_n$ converges strongly to u in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$. And $\widehat{u}^n$ satisfies the following SPDE

$$d\widehat{u}_t^n+A\widehat{u}_t^n dt=f_t dt+\sum_{i=1}^d\partial_i g_t^i dt+\sum_{j=1}^{+\infty}h_t^j dB_t^j+\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dt$$

From the Itô formula for the solution of SPDE without obstacle (see Lemma 7 in [21]), we have, almost surely, for all $t\in[0,T]$,

$$\begin{aligned}&\int_{\mathcal{O}}\Phi(t,\widehat{u}_t^n(x))dx+\int_0^t\mathcal{E}(\Phi'(s,\widehat{u}_s^n),\widehat{u}_s^n)ds=\int_{\mathcal{O}}\Phi(0,\xi(x))dx+\int_0^t\int_{\mathcal{O}}\frac{\partial\Phi}{\partial s}(s,\widehat{u}_s^n)dxds\\&+\int_0^t(\Phi'(s,\widehat{u}_s^n),f_s)ds-\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\Phi''(s,\widehat{u}_s^n(x))\partial_i\widehat{u}_s^n(x)g_i(x)dxds+\sum_{j=1}^{+\infty}\int_0^t(\Phi'(s,\widehat{u}_s^n),h_j)dB_s^j\\&+\frac{1}{2}\sum_{j=1}^{+\infty}\int_0^t\int_{\mathcal{O}}\Phi''(s,\widehat{u}_s^n(x))(h_j(x))^2dxds+\int_0^t\int_{\mathcal{O}}\Phi'(s,\widehat{u}_s^n)\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds.\end{aligned}$$

Because of the strong convergence of $(\widehat{u}^n)_n$, the convergence of all the terms except the last one are clear. To obtain the convergence of the last term, we do as follows:

$$\begin{aligned}\int_0^t\int_{\mathcal{O}}\Phi'(s,\widehat{u}_s^n)\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds&=\int_0^t\int_{\mathcal{O}}(\Phi'(s,\widehat{u}_s^n)-\Phi'(s,S_s))\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds\\&+\int_0^t\int_{\mathcal{O}}\Phi'(s,S_s)\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds.\end{aligned}$$

For the first term in the right member, we have:

$$\begin{aligned}&|\int_0^t\int_{\mathcal{O}}(\Phi'(s,\widehat{u}_s^n)-\Phi'(s,S_s))\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds|\\&\leq C\int_0^t\int_{\mathcal{O}}|\widehat{u}_s^n-S_s|\cdot\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds\\&=C\int_0^t\int_{\mathcal{O}}((\widehat{u}_s^n-S_s)^++(\widehat{u}_s^n-S_s)^-)\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds\\&=C\int_0^t\int_{\mathcal{O}}(\widehat{u}_s^n-S_s)^+\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds+C\int_0^t\int_{\mathcal{O}}(\widehat{u}_s^n-S_s)^-\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^-dxds\end{aligned}$$

We have the following inequality because $(\widehat{u}^n)_n$ converges to u increasingly:

$$\begin{aligned}\int_0^t\int_{\mathcal{O}}(\widehat{u}_s^n - S_s)^- \sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^- dxds &\leq \int_0^t\int_{\mathcal{O}}(u_s - S_s)^+ \sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^- dxds\\ &= \int_0^t\int_{\mathcal{O}}(u_s - S_s) \sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^- dxds\end{aligned}$$

With Lemma 2.16, we know that

$$\lim_{n\to\infty}\int_0^t\int_{\mathcal{O}}(u_s - S_s)\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^- dxds \to \int_0^t\int_{\mathcal{O}}(\widetilde{u}_s - \widetilde{S}_s)\nu(dxds) = 0.$$

And from Lemma 2.26, we have

$$\int_0^t\int_{\mathcal{O}}(\widehat{u}_s^n - S_s)^- \sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^- dxds \to 0.$$

Therefore,

$$\int_0^t\int_{\mathcal{O}}(\Phi'(s,\widehat{u}_s^n) - \Phi'(s,S_s))\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^- dxds \to 0.$$

Moreover, with Lemma 2.16, we have

$$\int_0^t\int_{\mathcal{O}}\Phi'(s,S_s)\sum_{k=1}^{N_n}\alpha_k^n n_k(u_s^{n_k}-S_s)^- dxds \to \int_0^t\int_{\mathcal{O}}\Phi'(s,S_s)\nu(dxds)$$

and

$$\begin{aligned}\left|\int_0^t\int_{\mathcal{O}}\Phi'(s,u_s)\nu(dxds) - \int_0^t\int_{\mathcal{O}}\Phi'(s,S_s)\nu(dxds)\right| &\leq C\int_0^t\int_{\mathcal{O}}|\widetilde{u}_s - S_s|\nu(dxds)\\ &= C\int_0^t\int_{\mathcal{O}}(\widetilde{u}_s - S_s)\nu(dxds) = 0\end{aligned}$$

Therefore, taking limit, we get the desired Itô formula. ∎

2.5.4 Itô's formula for the difference of the solutions of two RSPDEs

We still consider (u,ν) solution of the linear equation as in Subsection 2.5.2

$$\begin{cases} du_t + Au_t dt = f_t dt + \sum_{i=1}^d \partial_i g_t^i dt + \sum_{j=1}^{+\infty} h_t^j dB_t^j + \nu(dt,x)\\ u \geq S\end{cases}$$

and consider another linear equation with adapted coefficients $\bar{f}$, $\bar{g}$, $\bar{h}$ respectively in $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R})$, $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^d)$ and $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^{\mathbb{N}})$ and obstacle $\bar{S}$ which satisfies the same hypotheses (O) as S i.e; $\bar{S}_0 \leq \xi$ and $\bar{S}$ is dominated by the solution of an SPDE (not necessarily the same as S). We denote by $(y,\bar{\nu})$ the unique solution to the associated SPDE with obstacle with initial condition $y_0 = u_0 = \xi$.

$$\begin{cases} dy_t + Ay_t dt = \bar{f}_t dt + \sum_{i=1}^d \partial_i \bar{g}_t^i dt + \sum_{j=1}^{+\infty} \bar{h}_t^j dB_t^j + \bar{\nu}(dt,x)\\ y \geq \bar{S}\end{cases}$$

Theorem 2.27. *Let Φ as in Theorem 2.25, then the difference of the two solutions satisfy the following Itô's formula for all $t \in [0,T]$:*

$$\int_{\mathcal{O}} \Phi(t, u_t(x) - y_t(x))dx + \int_0^t \mathcal{E}(\Phi'(s, u_s - y_s), u_s - y_s)ds = \int_0^t (\Phi'(s, u_s - y_s), f_s - \bar{f}_s)ds$$
$$- \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \Phi"(s, u_s - y_s)\partial_i(u_s - y_s)(g_s^i - \bar{g}_s^i)dxds + \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s, u_s - y_s), h_s^j - \bar{h}_s^j)dB_s^j$$
$$+ \frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi"(s, u_s - y_s)(h_s^j - \bar{h}_s^j)^2 dxds + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, u_s - y_s)dxds$$
$$+ \int_0^t \int_{\mathcal{O}} \Phi'(s, \widetilde{u}_s - \widetilde{y}_s)(\nu - \bar{\nu})(dx, ds) \quad a.s. \tag{2.17}$$

Proof. We begin with the penalized solutions. The corresponding penalization equations are

$$du_t^n + Au_t^n dt = f_t dt + \sum_{i=1}^d \partial_i g_t^i dt + \sum_{j=1}^{+\infty} h_t^j dB_t^j + n(u_t^n - S_t)^- dt$$

and

$$dy_t^m + Ay_t^m dt = \bar{f}_t dt + \sum_{i=1}^d \partial_i \bar{g}_t^i dt + \sum_{j=1}^{+\infty} \bar{h}_t^j dB_t^j + m(y_t^m - \bar{S}_t)^- dt$$

from the proofs above, we know that the penalized solution converges weakly to the solution and we can take convex combinations $\widehat{u}^n = \sum_{i=1}^{N_n} \alpha_i^n u^{n_i}$ and $\widehat{y}^n = \sum_{i=1}^{N'_n} \beta_i^n y^{n'_i}$ such that $(\widehat{u}^n)_n$ and $(\widehat{y}^n)_n$ are non-decreasing and converge strongly to u and y respectively in $L^2([0,T], H_0^1(\mathcal{O}))$.
As in the proof of Theorem 2.25, we first establish a key lemma:

Lemma 2.28. *For all $t \in [0,T]$,*

$$\lim_{n \to +\infty} E \int_0^t \int_{\mathcal{O}} \widehat{u}_s^n \sum_{k=1}^{N'_n} \beta_k^n n'_k (y_s^{n'_k} - \bar{S}_s)^- dxds = E \int_0^t \int_{\mathcal{O}} \widetilde{u}\bar{\nu}(ds, dx),$$

and

$$\lim_{n \to +\infty} E \int_0^t \int_{\mathcal{O}} \widehat{y}_s^n \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds = E \int_0^t \int_{\mathcal{O}} \widetilde{y}\nu(ds, dx).$$

Proof. We put for all n:

$$\nu^n(ds, dx) = \sum_{k=1}^{N_n} \alpha_k^n n_k (u_s^{n_k} - S_s)^- dxds \text{ and } \bar{\nu}^n(ds, dx) = \sum_{k=1}^{N'_n} \beta_k^n n'_k (y_s^{n'_k} - \bar{S}_s)^- dxds.$$

As in the proof of Lemma 2.26, we write for all n: $u^n = z + v^n$.
In the same spirit, we introduce $\bar{z}$ the solution of the linear spde:

$$d\bar{z}_t + A\bar{z}_t = \bar{f}_t dt + \sum_{i=1}^d \partial_i \bar{g}_t^i dt + \sum_{j=1}^{+\infty} \bar{h}_t^j dB_t^j,$$

with initial condition $\bar{z}_0 = \xi$ and put $\forall n \in \mathbb{N},\ \bar{v}^n = y^n - \bar{z}$, $\widehat{\bar{v}}^n = \widehat{y}^n - \bar{z}$ and $\bar{v} = y - \bar{z}$. As a consequence of Lemma II.6 in [71], we have for all n, P-almost surely:

$$\frac{1}{2} \parallel \widehat{v}_t^n - \widehat{\bar{v}}_t^n \parallel^2 + \int_0^t \mathcal{E}(\widehat{v}_s^n - \widehat{\bar{v}}_s^n) ds = \int_0^t \int_{\mathcal{O}} (\widehat{v}_s^n - \widehat{\bar{v}}_s^n)(\nu^n - \bar{\nu}^n)(dx, ds),$$

and

$$\frac{1}{2} \parallel v_t - \bar{v}_t \parallel^2 + \int_0^t \mathcal{E}(v_s - \bar{v}_s) ds = \int_0^t \int_{\mathcal{O}} (\widetilde{v}_s - \widetilde{\bar{v}}_s)(\nu - \bar{\nu})(dx, ds).$$

But, as in the proof of Lemma 2.26, we get that $\widehat{v}_t^n - \widehat{\bar{v}}_t^n$ tends to $v_t - \bar{v}_t$ in $L^2(\mathcal{O})$ almost surely and

$$\lim_n \int_0^t \int_{\mathcal{O}} \widehat{v}_s^n \nu^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \widetilde{v}_s \nu(dx, ds),$$

$$\lim_n \int_0^t \int_{\mathcal{O}} \widehat{\bar{v}}_s^n \bar{\nu}^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \widetilde{\bar{v}}_s \bar{\nu}(dx, ds).$$

This yields:

$$\lim_n \Big(\int_0^t \int_{\mathcal{O}} \widehat{v}_s^n \bar{\nu}^n(dx, ds) + \int_0^t \int_{\mathcal{O}} \widehat{\bar{v}}_s^n \nu^n(dx, ds) \Big) = \int_0^t \int_{\mathcal{O}} \widetilde{v}_s \bar{\nu}(dx, ds) + \int_0^t \int_{\mathcal{O}} \widetilde{\bar{v}}_s \nu(dx, ds).$$

But, we have

$$\limsup_n \int_0^t \int_{\mathcal{O}} \widehat{v}_s^n \bar{\nu}^n(dx, ds) \quad \leq \quad \limsup_n \int_0^t \int_{\mathcal{O}} v_s \bar{\nu}^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \widetilde{v}_s \bar{\nu}(dx, ds),$$

and in the same way:

$$\limsup_n \int_0^t \int_{\mathcal{O}} \widehat{\bar{v}}_s^n \nu^n(dx, ds) \leq \int_0^t \int_{\mathcal{O}} \widetilde{\bar{v}}_s \nu(dx, ds).$$

Let us remark that these inequalities also hold for any subsequence. From this, it is easy to deduce that necessarily:

$$\lim_n \int_0^t \int_{\mathcal{O}} \widehat{v}_s^n \bar{\nu}^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \widetilde{v}_s \bar{\nu}(dx, ds),$$

and

$$\lim_n \int_0^t \int_{\mathcal{O}} \widehat{\bar{v}}_s^n \nu^n(dx, ds) = \int_0^t \int_{\mathcal{O}} \widetilde{\bar{v}}_s \nu(dx, ds).$$

We end the proof of this lemma by using similar arguments as in the proof of Lemma 2.26. ■

End of the proof of Theorem 2.27: We begin with the equation which $\widehat{u}^n - \widehat{y}^n$ satisfies:

$$\begin{aligned} d(\widehat{u}_t^n - \widehat{y}_t^n) + A(\widehat{u}_t^n - \widehat{y}_t^n) dt &= (f_t - \bar{f}_t) dt + \sum_{i=1}^d \partial_i (g_t^i - \bar{g}_t^i) dt + \sum_{j=1}^{+\infty} (h_t^j - \bar{h}_t^j) dB_t^j \\ &+ (\nu^n - \bar{\nu}^n)(x, dt) \end{aligned}$$

Applying Itô's formula to $\Phi(\widehat{u}^n - \widehat{y}^n)$, we have

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi(t, \widehat{u}_t^n(x) - \widehat{y}_t^n(x))dx + \int_0^t \mathcal{E}(\Phi'(s, \widehat{u}_s^n - \widehat{y}_s^n), \widehat{u}_s^n - \widehat{y}_s^n)ds = \int_0^t (\Phi'(s, \widehat{u}_s^n - \widehat{y}_s^n), f_s - \bar{f}_s)ds \\
&-\sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \Phi\text{"}(s, \widehat{u}_s^n - \widehat{y}_s^n)\partial_i(\widehat{u}_s^n - \widehat{y}_s^n)(g_s^i - \bar{g}_s^i)dxds + \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s, \widehat{u}_s^n - \widehat{y}_s^n), h_s^j - \bar{h}_s^j)dB_s^j \\
&+\frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi\text{"}(s, \widehat{u}_s^n - \widehat{y}_s^n)(h_s^j - \bar{h}_s^j)^2 dxds + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, \widehat{u}_s^n - \widehat{y}_s^n)dxds \\
&+\int_0^t \int_{\mathcal{O}} \Phi'(s, \widehat{u}_s^n - \widehat{y}_s^n)(\nu^n - \bar{\nu}^n)(dx, dt)
\end{aligned}$$

Because that $(\widehat{u}^n)_n$ and $(\widehat{y}^n)_n$ converge strongly to u and y respectively, the convergence of all the terms except the last term are clear. For the convergence of the last term, we do as follows:

$$\begin{aligned}
&\left| \int_0^t \int_{\mathcal{O}} [\Phi'(s, \widehat{u}_s^n - \widehat{y}_s^n) - \Phi'(s, u_s - \widehat{y}_s^n)]\nu^n(dxds) + \int_0^t \int_{\mathcal{O}} [\Phi'(s, u_s - \widehat{y}_s^n) - \Phi'(s, u_s - y_s)]\nu^n(dxds) \right| \\
&\leq C \int_0^t \int_{\mathcal{O}} |\widehat{u}_s^n - u_s|\nu^n(dxds) + \int_0^t \int_{\mathcal{O}} |\widehat{y}_s^n - y_s|\nu^n(dx, ds)
\end{aligned}$$

As a consequence of Lemma 2.26 and using the fact that $\widehat{u}^n \leq u$:

$$\lim_n \int_0^t \int_{\mathcal{O}} |\widehat{u}_s^n - u_s|\nu^n(dxds) = \lim_n \int_0^t \int_{\mathcal{O}} (u_s - \widehat{u}_s^n)\nu^n(dxds) = 0.$$

By Lemma 2.28 and the fact that $\widehat{y}^n \leq y$:

$$\lim_n \int_0^t \int_{\mathcal{O}} |\widehat{y}_s^n - y_s|\nu^n(dx, ds) = \lim_n \int_0^t \int_{\mathcal{O}} (y_s - \widehat{y}_s^n)\nu^n(dx, ds) = 0,$$

this yields:

$$\lim_n \int_0^t \int_{\mathcal{O}} (\Phi'(s, \widehat{u}_s^n - \widehat{y}_s^n) - \Phi'(s, u_s - y_s)\nu^n(dx, dt) = 0,$$

but by Lemma 2.16, we know that

$$\lim_n \int_0^t \int_{\mathcal{O}} \Phi'(s, u_s - y_s)\nu^n(dx, dt) = \int_0^t \int_{\mathcal{O}} \Phi'(s, \widetilde{u}_s - \widetilde{y}_s)\bar{\nu}(dx, dt),$$

so

$$\lim_n \int_0^t \int_{\mathcal{O}} \Phi'(s, \widehat{u}_s^n - \widehat{y}_s^n)\nu^n(dx, dt) = \int_0^t \int_{\mathcal{O}} \Phi'(s, \widetilde{u}_s - \widetilde{y}_s)\nu(dx, dt).$$

In the same way, we prove:

$$\lim_n \int_0^t \int_{\mathcal{O}} \Phi'(s, \widehat{u}_s^n - \widehat{y}_s^n)\bar{\nu}^n(dx, dt) = \int_0^t \int_{\mathcal{O}} \Phi'(s, \widetilde{u}_s - \widetilde{y}_s)\bar{\nu}(dx, dt).$$

The proof is now complete. ∎

2.5.5 Proof of Theorem 2.23 in the nonlinear case

Let γ and δ be 2 positive constants. On $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$, we introduce the norm

$$\| u \|_{\gamma,\delta} = E(\int_0^T e^{-\gamma s}(\delta \| u_s \|^2 + \| \nabla u_s \|^2)ds),$$

which clearly defines an equivalent norm on $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$.
Let us consider the Picard sequence (u^n) defined by $u^0 = \xi$ and for all $n \in \mathbb{N}$ we denote by (u^{n+1}, ν^{n+1}) the solution of the linear SPDE with obstacle

$$(u^{n+1}, \nu^{n+1}) = \mathcal{R}(\xi, f(u^n, \nabla u^n), g(u^n, \nabla u^n), h(u^n, \nabla u^n), S).$$

Then, by Itô's formula (2.17), we have

$$\begin{aligned}
&e^{-\gamma T} \| u_T^{n+1} - u_T^n \|^2 + 2\int_0^T e^{-\gamma s}\mathcal{E}(u_s^{n+1} - u_s^n)ds = -\gamma \int_0^T e^{-\gamma s} \| u_s^{n+1} - u_s^n \|^2 \, ds \\
&+2\int_0^T e^{-\gamma s}(\widehat{f}_s, u_s^{n+1} - u_s^n)ds - 2\sum_{i=1}^d \int_0^T e^{-\gamma s}(\widehat{g}_s^i, \partial_i(u_s^{n+1} - u_s^n))ds \\
&+2\sum_{j=1}^{+\infty} \int_0^T e^{-\gamma s}(\widehat{h}_s^j, u_s^{n+1} - u_s^n)dB_s^j + \int_0^T e^{-\gamma s} \| |\widehat{h}_s| \|^2 \, ds \\
&+2\int_0^T \int_{\mathcal{O}} e^{-\gamma s}(u_s^{n+1} - u_s^n)(\nu^{n+1} - \nu^n)(dxds)
\end{aligned}$$

where $\widehat{f} = f(u^n, \nabla u^n) - f(u^{n-1}, \nabla u^{n-1})$, $\widehat{g} = g(u^n, \nabla u^n) - g(u^{n-1}, \nabla u^{n-1})$ and $\widehat{h} = h(u^n, \nabla u^n) - h(u^{n-1}, \nabla u^{n-1})$.Clearly, the last term is non-positive so using Cauchy-Schwarz's inequality and the Lipschitz conditions on f, g and h, we have

$$\begin{aligned}
2\int_0^T e^{-\gamma s}(u_s^{n+1} - u_s^n, \widehat{f}_s)ds &\leq \frac{1}{\epsilon}\int_0^T e^{-\gamma s} \| u_s^{n+1} - u_s^n \|^2 \, ds + \epsilon \int_0^T e^{-\gamma s} \| \widehat{f}_s \|^2 \, ds \\
&\leq \frac{1}{\epsilon}\int_0^T e^{-\gamma s} \| u_s^{n+1} - u_s^n \|^2 \, ds + C\epsilon \int_0^T e^{-\gamma s} \| u_s^n - u_s^{n-1} \|^2 \, ds \\
&+ C\epsilon \int_0^T e^{-\gamma s} \| \nabla(u_s^n - u_s^{n-1}) \|^2 \, ds
\end{aligned}$$

and

$$\begin{aligned}
&2\sum_{i=1}^d \int_0^T e^{-\gamma s}(\widehat{g}_s^i, \partial_i(u_s^{n+1} - u_s^n))ds \leq 2\int_0^T e^{-\gamma s} \| \nabla(u_s^{n+1} - u_s^n) \| (C \| u_s^n - u_s^{n-1} \| \\
&+\alpha \| \nabla(u_s^n - u_s^{n-1}) \|)ds \leq C\epsilon \int_0^T e^{-\gamma s} \| \nabla(u_s^{n+1} - u_s^n) \|^2 \, ds + \frac{C}{\epsilon}\int_0^T e^{-\gamma s} \| u_s^n - u_s^{n-1} \|^2 \, ds \\
&+\alpha \int_0^T e^{-\gamma s} \| \nabla(u_s^{n+1} - u_s^n) \|^2 \, ds + \alpha \int_0^T e^{-\gamma s} \| \nabla(u_s^n - u_s^{n-1}) \|^2 \, ds
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T e^{-\gamma s} \| |\widehat{h}_s| \|^2 \, ds &\leq C(1 + \frac{1}{\epsilon})\int_0^T e^{-\gamma s} \| u_s^n - u_s^{n-1} \|^2 \, ds \\
&+ \beta^2(1+\epsilon)\int_0^T e^{-\gamma s} \| \nabla(u_s^n - u_s^{n-1}) \|^2 \, ds
\end{aligned}$$

where C, α and β are the constants in the Lipschitz conditions. Using the elliptic condition and taking expectation, we get:

$$(\gamma-\frac{1}{\epsilon})E\int_0^T e^{-\gamma s}\parallel u_s^{n+1}-u_s^n\parallel^2 ds+(2\lambda-\alpha-C\epsilon)E\int_0^T e^{-\gamma s}\parallel\nabla(u_s^{n+1}-u_s^n)\parallel^2 ds\leq$$
$$C(1+\epsilon+\frac{2}{\epsilon})\int_0^T e^{-\gamma s}\parallel u_s^n-u_s^{n-1}\parallel^2 ds+(C\epsilon+\alpha+\beta^2(1+\epsilon))E\int_0^T e^{-\gamma s}\parallel\nabla(u_s^n-u_s^{n-1})\parallel^2 ds$$

We choose ϵ small enough and then γ such that

$$C\epsilon+\alpha+\beta^2(1+\epsilon)<2\lambda-\alpha-C\epsilon\ and\ \frac{\gamma-1/\epsilon}{2\lambda-\alpha-C\epsilon}=\frac{C(1+\epsilon+2/\epsilon)}{C\epsilon+\alpha+\beta^2(1+\epsilon)}$$

If we set $\delta=\frac{\gamma-1/\epsilon}{2\lambda-\alpha-C\epsilon}$, we have the following inequality:

$$\parallel u^{n+1}-u^n\parallel_{\gamma,\delta}\leq\frac{C\epsilon+\alpha+\beta^2(1+\epsilon)}{2\lambda-\alpha-C\epsilon}\parallel u^n-u^{n-1}\parallel_{\gamma,\delta}\leq...\leq(\frac{C\epsilon+\alpha+\beta^2(1+\epsilon)}{2\lambda-\alpha-C\epsilon})^n\parallel u^1\parallel_{\gamma,\delta}$$

when $n\to\infty$, $(\frac{C\epsilon+\alpha+\beta^2(1+\epsilon)}{2\lambda-\alpha-C\epsilon})^n\to 0$, we deduce that u^n converges strongly to u in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$.
Moreover, as $(u^{n+1},\nu^{n+1})=\mathcal{R}(\xi,f(u^n,\nabla u^n),g(u^n,\nabla u^n),h(u^n,\nabla u^n),S)$, we have for any $\varphi\in\mathcal{D}$:

$$(u_t^{n+1},\varphi_t)-(\xi,\varphi_0)-\int_0^t(u_s^n,\partial_s\varphi_s)ds+\int_0^t\mathcal{E}(u_s^{n+1},\varphi_s)ds+\sum_{i=1}^d\int_0^t(g_s^i(u_s^n,\nabla u_s^n),\partial_i\varphi_s)ds$$
$$=\int_0^t(f_s(u_s^n,\nabla u_s^n),\varphi_s)ds+\sum_{j=1}^{+\infty}\int_0^t(h_s^j(u_s^n,\nabla u_s^n),\varphi_s)dB_s^j+\int_0^t\int_{\mathcal{O}}\varphi_s(x)\nu^{n+1}(dxds).\ \ a.s.$$

Let v^{n+1} the random parabolic potential associated to ν^{n+1}:

$$\nu^{n+1}=\partial_t v^{n+1}+Av^{n+1}.$$

We denote $z^{n+1}=u^{n+1}-v^{n+1}$, so

$$z^{n+1}=\mathcal{U}(\xi,f(u^n,\nabla u^n),g(u^n,\nabla u^n),h(u^n,\nabla u^n))$$

converges strongly to z in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$. As a consequence of the strong convergence of $(u^{n+1})_n$, we deduce that $(v^{n+1})_n$ converges strongly to v in $L^2(\Omega\times[0,T];H_0^1(\mathcal{O}))$. Therefore, for fixed ω,

$$\int_0^t(-\frac{\partial_s\varphi_s}{\partial s},v_s)ds+\int_0^t\mathcal{E}(\varphi_s,v_s)ds=\lim\int_0^t(-\frac{\partial_s\varphi_s}{\partial s},v_s^{n+1})ds+\int_0^t\mathcal{E}(\varphi_s,v_s^{n+1})ds\geq 0$$

i.e. $v(\omega)\in\mathcal{P}$. Then from Proposition 2.5, we obtain a regular measure associated with v, and $(\nu^{n+1})_n$ converges vaguely to ν.
Taking the limit, we obtain

$$(u_t,\varphi_t)-(\xi,\varphi_0)-\int_0^t(u_s,\partial_s\varphi_s)ds+\int_0^t\mathcal{E}(u_s,\varphi_s)ds+\sum_{i=1}^d\int_0^t(g_s^i(u_s,\nabla u_s),\partial_i\varphi_s)ds$$
$$=\int_0^t(f_s(u_s,\nabla u_s),\varphi_s)ds+\sum_{j=1}^{+\infty}\int_0^t(h_s^j(u_s,\nabla u_s),\varphi_s)dB_s^j+\int_0^t\int_{\mathcal{O}}\varphi_s(x)\nu(dx,ds),\ \ a.s..$$

From the fact that u and z are in $\mathcal{H}_T$, we know that v is also in $\mathcal{H}_T$, by definition, ν is a random regular measure. □

2.6 Comparison theorem

2.6.1 A comparison Theorem in the linear case

We first establish a comparison theorem for the solutions of linear SPDE with obstacle in the case where the obstacles are the same, this gives a comparison between the regular measures.
So, for this part only, we consider the same hypotheses as in the Subsection 2.5.2. So we consider adapted processes f, g, h respectively in $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R})$, $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^d)$ and $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^{\mathbb{N}})$, an obstacle S which satisfies assumption **(O)** and $\xi\in L^2(\Omega\times\mathcal{O})$ is an $\mathcal{F}_0$-measurable random variable such that $\xi\leq S_0$. We denote by (u,ν) be the solution of $\mathcal{R}(\xi,\ f,\ g,\ h,\ S)$.
We are given another $\xi'\in L^2(\Omega\times\mathcal{O})$ is $\mathcal{F}_0$-measurable and such that $\xi'\leq S_0$ and another adapted process f' in $L^2([0,T]\times\Omega\times\mathcal{O};\mathbb{R})$. We denote by (u',ν') the solution of $\mathcal{R}(\xi',\ f',\ g,\ h,\ S)$. We have the followingg comparison theorem:

Theorem 2.29. *Assume that the following conditions hold*

1. $\xi\leq\xi',\ dx\otimes d\mathbb{P}-a.e.;$
2. $f\leq f',\ dt\otimes dx\otimes d\mathbb{P}-a.e..$

Then for almost all $\omega\in\Omega$, $u\leq u',\ q.e.$ and $\nu\geq\nu'$ in the sense of distribution.

Proof. We consider the following two penalized equations:

$$du_t^n=Au_t^ndt+f_tdt+\sum_{i=1}^d\partial_ig_t^idt+\sum_{j=1}^{+\infty}h_t^jdB_t^j+n(u_t^n-S_t)^-dt$$

$$du_t^{'n}=Au_t^{'n}dt+f_t'dt+\sum_{i=1}^d\partial_ig_t^idt+\sum_{j=1}^{+\infty}h_t^jdB_t^j+n(u_t^{'n}-S_t)^-dt$$

we denote

$$F_t(x,u_t^n)=f_t(x)+n(u_t^n-S_t)^-$$

$$F_t'(x,u_t^n)=f_t'(x)+n(u_t^n-S_t)^-$$

with assumption 2 we have that $F_t(x,u_t^n)\leq F_t'(x,u_t^n),\ dt\otimes dx\otimes dP-a.e.$, therefore, from the comparison theorem for SPDE (without obstacle, see [D]), we know that $\forall t\in[0,T]$, $u_t^n\leq u_t^{'n},\ dx\otimes dP-a.e.$, thus $n(u_t^n-S_t)^-\geq n(u_t^{'n}-S_t)^-$.
The results are immediate consequence of the construction of (u,ν) and (u',ν') given in Subsection 2.5.2. ∎

2.6.2 A comparison theorem in the general case

We now come back to the general setting and still consider $(u,\nu)=\mathcal{R}(\xi,f,g,h,S)$ the solution of the SPDE with obstacle

$$\begin{cases} du_t(x)=Lu_t(x)dt+f(t,x,u_t(x),\nabla u_t(x))dt+\sum_{i=1}^{d}\partial_i g_i(t,x,u_t(x),\nabla u_t(x))dt \\ \qquad\qquad +\sum_{j=1}^{+\infty}h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j+\nu(x,dt) \\ u\geq S\ ,\ u_0=\xi, \end{cases}$$

where we assume hypotheses **(H)**, **(I)** and **(O)**.
We consider another coefficients f' which satisfies the same assumptions as f, another obstacle S' which satisfies **(O)** and another initial condition ξ' belonging to $L^2(\Omega\times\mathcal{O})$ and $\mathcal{F}_0$ adapted such that $\xi'\geq S_0'$. We denote by $(u',\nu')=\mathcal{R}(\xi',f',g,h,S')$.

Theorem 2.30. *Assume that the following conditions hold*

1. $\xi\leq\xi',\ dx\otimes d\mathbb{P}-a.e.$
2. $f(u,\nabla u)\leq f'(u,\nabla u),\ dt\otimes dx\otimes\mathbb{P}-a.e.$
3. $S\leq S',\ dt\otimes dx\otimes\mathbb{P}-a.e.$

Then for almost all $\omega\in\Omega$, $u(t,x)\leq u'(t,x)$, q.e..

We put $\widehat{u}=u-u'$, $\widehat{\xi}=\xi-\xi'$, $\widehat{f}_t=f(t,u_t,\nabla u_t)-f'(t,u'_t,\nabla u'_t)$, $\widehat{g}_t=g(t,u_t,\nabla u_t)-g(t,u'_t,\nabla u'_t)$ and $\widehat{h}_t=h(t,u_t,\nabla u_t)-h(t,u'_t,\nabla u'_t)$. The main idea is to evaluate $E\parallel\widehat{u}_t^+\parallel^2$, thanks to Itô's formula and then apply Gronwall's inequality. Therefore, we start by the following lemma

Lemma 2.31. *For all $t\in[0,T]$, we have*

$$\begin{aligned} &E\parallel\widehat{u}_t^+\parallel^2+2E\int_0^t\mathcal{E}(\widehat{u}_s^+)ds=E\parallel\widehat{\xi}^+\parallel^2+2E\int_0^t(\widehat{u}_s^+,\widehat{f}_s)ds-2E\int_0^t(\nabla\widehat{u}_s^+,\widehat{g}_s)ds \\ &+2E\int_0^t\int_{\mathcal{O}}\widehat{u}_s^+(x)(\nu-\nu')(dxds)+E\int_0^t\parallel I_{\{\widehat{u}_s>0\}}|\widehat{h}_s|\parallel^2 ds\quad a.s.. \end{aligned}\tag{2.18}$$

Proof. We approximate the function $\psi:y\in R\to(y^+)^2$ by a sequence (ψ_n) of regular functions: let φ be a C^∞ increasing function such that

$$\forall y\in]-\infty,1],\ \varphi(y)=0\ \textit{and}\ \forall y\in[2,+\infty[,\ \varphi(y)=1.$$

We set for all n:

$$\forall y\in R,\ \psi_n(y)=y^2\varphi(ny).$$

It is easy to verify that (ψ_n) converges uniformly to the function ψ and that moreover we have the estimates:

$$\forall y \in R^+,\ \forall n,\ 0 \leq \psi_n(y) \leq \psi(y),\ 0 \leq \psi_n'(y) \leq Cy,\ |\psi_n''(y)| \leq C.$$

Thanks to Theorem 2.27, for all n and $t \in [0,T]$ we have

$$\begin{aligned} & E\int_{\mathcal{O}} \psi_n(\widehat{u}_s)dx + E\int_0^t \mathcal{E}(\psi_n'(\widehat{u}_s),\widehat{u}_s)ds = E\int_{\mathcal{O}} \psi_n(\widehat{\xi})dx + E\int_0^t (\psi_n'(\widehat{u}_s),\widehat{f}_s)ds \\ & -E\int_0^t (\nabla\psi_n'(\widehat{u}_s),\widehat{g}_s)ds + E\int_0^t\int_{\mathcal{O}} \psi_n'(\widehat{u}_s)\widehat{\nu}(dxds) + \frac{1}{2}E\int_0^t\int_{\mathcal{O}} \psi_n''(\widehat{u}_s)\widehat{h}_s^2 dxds \end{aligned} \tag{2.19}$$

Taking the limit, thanks to the dominated convergence theorem, we obtain the convergences of all the terms except $E\int_0^t\int_{\mathcal{O}} \psi_n'(\widehat{u}_s(x))\widehat{\nu}(dxds)$.
From (2.19), we know that

$$-E\int_0^t\int_{\mathcal{O}} \psi_n'(\widehat{u}_s(x))\widehat{\nu}(dxds) \leq C.$$

Moreover, we have the following relation:

$$\begin{aligned} & -E\int_0^t\int_{\mathcal{O}} \psi_n'(\widehat{u}_s(x))\widehat{\nu}(dxds) \\ = & -E\int_0^t\int_{\mathcal{O}} \psi_n'(S_s^1(x)-u_s^2(x))\nu^1(dxds) + E\int_0^t\int_{\mathcal{O}} \psi_n'(u_s^1(x)-S_s^2(x))\nu^2(dxds) \\ = & \ E\int_0^t\int_{\mathcal{O}} \psi_n'(u_s^2(x)-S_s^1(x))\nu^1(dxds) + E\int_0^t\int_{\mathcal{O}} \psi_n'(u_s^1(x)-S_s^2(x))\nu^2(dxds). \end{aligned}$$

By Fatou's lemma, we obtain

$$2E\int_0^t\int_{\mathcal{O}} (u_s^2(x)-S_s^1(x))^+\nu^1(dxds) + 2E\int_0^t\int_{\mathcal{O}} (u_s^1(x)-S_s^2(x))^+\nu^2(dxds) < +\infty.$$

Hence, the convergence of the term $E\int_0^t\int_{\mathcal{O}} \psi_n'(\widehat{u}_s(x))\widehat{\nu}(dxds)$ comes from the dominated convergence theorem. ∎

Proof of Theorem 2.30: Applying (2.18) to $(\widehat{u}_t^+)^2$, we have

$$\begin{aligned} & E\parallel \widehat{u}_t^+ \parallel^2 + 2E\int_0^t I_{\{\widehat{u}_s>0\}}\mathcal{E}(\widehat{u}_s)ds = 2E\int_0^t (\widehat{u}_s^+,\widehat{f}_s)ds + 2E\int_0^t (\widehat{u}_s^+,\widehat{g}_s)ds \\ & +E\int_0^t \parallel I_{\{\widehat{u}_s>0\}}|\widehat{h}_s| \parallel^2 ds + 2E\int_0^t\int_{\mathcal{O}} (u_s-u_s')^+(x)(\nu-\nu')(dx,ds). \end{aligned}$$

As we assume that $f(u,\nabla u) \leq f'(u,\nabla u)$,

$$\begin{aligned} \widehat{u}_s^+\widehat{f}_s &= \widehat{u}_s^+\{f(s,u_s,\nabla u_s)-f'(s,u_s,\nabla u_s)\} + \widehat{u}_s^+\{f'(s,u_s,\nabla u_s)-f'(s,u_s',\nabla u_s')\} \\ &\leq \widehat{u}_s^+\{f'(s,u_s,\nabla u_s)-f'(s,u_s',\nabla u_s')\}. \end{aligned}$$

then with the Lipschitz condition, using Cauchy-Schwartz's inequality, we have the following relations:

$$E\int_0^t (\widehat{u}_s^+,\widehat{f}_s)ds \leq (C+\frac{C}{\epsilon})E\int_0^t \parallel \widehat{u}_s^+ \parallel^2 ds + \frac{C\epsilon}{\lambda}E\int_0^t \mathcal{E}(\widehat{u}_s^+)ds.$$

$$E\int_0^t (\nabla \widehat{u}_s^+, \widehat{g}_s)ds \le \frac{\epsilon+\alpha}{\lambda} E\int_0^t \mathcal{E}(\widehat{u}_s^+)ds + \frac{C}{\epsilon} E\int_0^t \parallel \widehat{u}_s^+ \parallel^2 ds$$

$$E\int_0^t \parallel I_{\{\widehat{u}_s>0\}} |\widehat{h}_s| \parallel^2 ds \le CE\int_0^t \parallel \widehat{u}_s^+ \parallel^2 ds + \frac{\beta^2+\epsilon}{\lambda} E\int_0^t \mathcal{E}(\widehat{u}_s^+)ds.$$

The last term is equal to $-2E\int_0^t\int_{\mathcal{O}}(u_s-u'_s)^+(x)\nu'(dx,ds) \le 0$, because that on $\{u \le u'\}$, $(u-u')^+ = 0$ and on $\{u > u'\}$, $\nu(dx,ds) = 0$. Thus we have the following inequality

$$E \parallel \widehat{u}_t^+ \parallel^2 + (2 - \frac{2\alpha+2\epsilon}{\lambda} - \frac{2C\epsilon}{\lambda} - \frac{\beta^2+\epsilon}{\lambda}) E\int_0^t \mathcal{E}(\widehat{u}_s^+)ds \le CE\int_0^t \parallel \widehat{u}_s^+ \parallel^2 ds.$$

we can take ϵ small enough such that $2 - \frac{2\alpha+2\epsilon}{\lambda} - \frac{2C\epsilon}{\lambda} - \frac{\beta^2+\epsilon}{\lambda} > 0$, we have

$$E \parallel \widehat{u}_t^+ \parallel^2 \le CE\int_0^t \parallel \widehat{u}_s^+ \parallel^2 ds,$$

then we deduce the result from the Gronwall's lemma. □

Remark 2.32. *Applying the comparison theorem to the same obstacle gives another proof of the uniqueness of the solution.*

Chapter 3

Maximum Principle for Quasilinear Stochastic PDEs with Obstacle

3.1 Introduction

In this chapter, we consider an obstacle problem for the following parabolic Stochastic PDE (SPDE for short)

$$\begin{cases} du_t(x) = \partial_i \left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right) dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ \qquad\qquad + \displaystyle\sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j + \nu(t,dx)\,, \\ u_t \geq S_t\,, \\ u_0 = \xi\,. \end{cases} \tag{3.1}$$

Here, S is the given obstacle, a is a matrix defining a symmetric operator on a bounded open domain $\mathcal{O}$, f, g, h are random coefficients.
In a recent work [25] we have proved existence and uniqueness of the solution of equation (3.1) under standard Lipschitz hypotheses and L^2-type integrability conditions on the coefficients. Let us recall that the solution is a couple (u, ν), where u is a process with values in the first order Sobolev space and ν is a random regular measure forcing u to stay above S and satisfying a minimal Skohorod condition.

In order to give a rigorous meaning to the notion of solution, inspired by the works of M. Pierre in the deterministic case (see [70, 71]), we introduce the notion of parabolic capacity. The key point is that in [25], we construct a solution which admits a quasi continuous version hence defined outside a polar set and that regular measures which in general are not absolutely continuous w.r.t. the Lebesgue measure, do not charge polar sets.

There is a huge literature on parabolic SPDE's without obstacle. The study of the L^p-norms w.r.t. the randomness of the space-time uniform norm on the trajectories of a stochastic PDE was started by N. V. Krylov in [42], for a more complete overview of existing works on this subject see [23, 24] and the references therein. Concerning the obstacle problem, there are two approaches, a probabilistic one (see [53, 41]) based on the Feynmann-Kac's formula via the backward doubly stochastic differential equations and the analytical one (see [26, 55, 79]) based on the Green function.

To our knowledge, up to now there is no maximum principle result for quasilinear SPDE with obstacle and even very few results in the deterministic case. The aim of this work is to obtain, under suitable integrability conditions on the coefficients, L^p-estimates for the uniform norm (in time and space) of the solution, a maximum principle for local solutions of equation (3.1) and comparison theorems similar to those obtained in the without obstacle case in [21, 23]. This yields for example the following result:

Theorem 3.1. *Let $(M_t)_{t\geq 0}$ be an Itô process satisfying some integrability conditions, $p \geq 2$ and u be a local weak solution of the obstacle problem (3.1). Assume that $\partial\mathcal{O}$ is Lipschitz and $u \leq M$ on $\partial\mathcal{O}$, then for all $t \in [0,T]$:*

$$E\left\|(u-M)^+\right\|^p_{\infty,\infty;t} \leq k\,(p,t)\,\mathcal{C}(S,f,g,h,M)$$

where $\mathcal{C}(S,f,g,h,M)$ depends only on the barrier S, the initial condition ξ, coefficients f,g,h, the boundary condition M and k is a function which only depends on the structure constants of the SPDE, $\|\cdot\|_{\infty,\infty;t}$ is the uniform norm on $[0,t]\times\mathcal{O}$.

Let us remark that in order to get such a result, we define the notion of local solutions to the obstacle problem (3.1) and so introduce what we call *local regular measures*.

This chapter is organized as follows: in section 2 we introduce notations and hypotheses and we take care to detail the integrability conditions which are used all along this chapter. In section 3, we prove an existence and uniqueness result for the SPDE (3.1) without obstacle with null Dirichlet condition under a weaker integrability hypothesis on f and also give an estimate of the positive part of the solution. In section 4, we establish the L^p-estimate for uniform norm of the solution with null Dirichlet boundary condition. Section 5 is devoted to the main result: the maximum principle for local solutions whose proof is based on an Itô's formula satisfied by the positive part of any local solution with lateral boundary condition, M. The last section is an Appendix in which we give the proofs of several lemmas.

3.2 Preliminaries

3.2.1 $L^{p,q}-$space

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded open domain and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$, it is an Hilbert space equipped with the usual scalar product and norm as follows

$$(u,v) = \int_{\mathcal{O}} u(x)v(x)dx, \qquad \| u \|= (\int_{\mathcal{O}} u^2(x)dx)^{1/2}.$$

In general, we shall extend the notation

$$(u,v)=\int_{\mathcal{O}}u(x)v(x)dx,$$

where $u,\ v$ are measurable functions defined on $\mathcal{O}$ such that $uv\in L^1(\mathcal{O})$.
The first order Sobolev space of functions vanishing at the boundary will be denoted by $H_0^1(\mathcal{O})$, its natural scalar product and norm are

$$(u,v)_{H_0^1(\mathcal{O})}=(u,v)+\int_{\mathcal{O}}\sum_{i=1}^{d}\left(\partial_i u\left(x\right)\right)\left(\partial_i v\left(x\right)\right)dx,\ \|u\|_{H_0^1(\mathcal{O})}=\left(\|u\|_2^2+\|\nabla u\|_2^2\right)^{\frac{1}{2}}.$$

As usual we shall denote $H^{-1}(\mathcal{O})$ its dual space.
We shall denote by $H^1_{loc}(\mathcal{O})$ the space of functions which are locally square integrable in $\mathcal{O}$ and which admit first order derivatives that are also locally square integrable.

For each $t>0$ and for all real numbers $p,\ q\geq 1$, we denote by $L^{p,q}([0,t]\times\mathcal{O})$ the space of (classes of) measurable functions $u:[0,t]\times\mathcal{O}\longrightarrow\mathbb{R}$ such that

$$\|u\|_{p,q;\,t}:=\left(\int_0^t\left(\int_{\mathcal{O}}|u(s,x)|^p\,dx\right)^{q/p}ds\right)^{1/q}$$

is finite. The limiting cases with p or q taking the value ∞ are also considered with the use of the essential sup norm.
Now we introduce some other spaces of functions and discuss a certain duality between them. Like in [21] and [23], for self-containeness, we recall the following definitions:
Let $(p_1,q_1),\ (p_2,q_2)\in[1,\infty]^2$ be fixed and set

$$I=I\left(p_1,q_1,p_2,q_2\right):=\Big\{(p,q)\in[1,\infty]^2\ /\ \exists\ \rho\in[0,1]\ s.t.$$
$$\frac{1}{p}=\rho\frac{1}{p_1}+(1-\rho)\frac{1}{p_2},\frac{1}{q}=\rho\frac{1}{q_1}+(1-\rho)\frac{1}{q_2}\Big\}.$$

This means that the set of inverse pairs $\left(\frac{1}{p},\frac{1}{q}\right)$, (p,q) belonging to I, is a segment contained in the square $[0,1]^2$, with the extremities $\left(\frac{1}{p_1},\frac{1}{q_1}\right)$ and $\left(\frac{1}{p_2},\frac{1}{q_2}\right)$.
We introduce:

$$L_{I;t}=\bigcap_{(p,q)\in I}L^{p,q}\left([0,t]\times\mathcal{O}\right).$$

We know that this space coincides with the intersection of the extreme spaces,

$$L_{I;t}=L^{p_1,q_1}\left([0,t]\times\mathcal{O}\right)\cap L^{p_2,q_2}\left([0,t]\times\mathcal{O}\right)$$

and that it is a Banach space with the following norm

$$\|u\|_{I;t}:=\|u\|_{p_1,q_1;t}\vee\|u\|_{p_2,q_2;t}\,.$$

The other space of interest is the algebraic sum

$$L^{I;t}:=\sum_{(p,q)\in I}L^{p,q}\left([0,t]\times\mathcal{O}\right),$$

which represents the vector space generated by the same family of spaces. This is a normed vector space with the norm

$$\|u\|^{I;t} := \inf\left\{\sum_{i=1}^{n}\|u_i\|_{k_i,l_i;t} \;/\; u=\sum_{i=1}^{n}u_i, u_i\in L^{k_i,l_i}\left([0,t]\times\mathcal{O}\right), (k_i,l_i)\in I,\ i=1,...n;\ n\in\mathbb{N}^*\right\}.$$

Clearly one has $L^{I;t}\subset L^{1,1}\left([0,t]\times\mathcal{O}\right)$ and $\|u\|_{1,1;t}\le c\|u\|^{I;t}$, for each $u\in L^{I;t}$, with a certain constant $c>0$.

We also remark that if $(p,q)\in I$, then the conjugate pair (p',q'), with $\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$, belongs to another set, I', of the same type. This set may be described by

$$I'=I'\left(p_1,q_1,p_2,q_2\right) := \left\{\left(p',q'\right) / \;\exists\,(p,q)\in I\ s.t.\ \frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1\right\}$$

and it is not difficult to check that $I'\left(p_1,q_1,p_2,q_2\right)=I\left(p_1',q_1',p_2',q_2'\right)$, where p_1',q_1',p_2' and q_2' are defined by $\frac{1}{p_1}+\frac{1}{p_1'}=\frac{1}{q_1}+\frac{1}{q_1'}=\frac{1}{p_2}+\frac{1}{p_2'}=\frac{1}{q_2}+\frac{1}{q_2'}=1$.

Moreover, by Hölder's inequality, it follows that one has

$$\int_0^t\int_{\mathcal{O}}u\left(s,x\right)v\left(s,x\right)dxds\le\|u\|_{I;t}\|v\|^{I';t}, \tag{3.2}$$

for any $u\in L_{I;t}$ and $v\in L^{I';t}$. This inequality shows that the scalar product of $L^2\left([0,t]\times\mathcal{O}\right)$ extends to a duality relation for the spaces $L_{I;t}$ and $L^{I';t}$.

Now let us recall that the Sobolev inequality states that

$$\|u\|_{2^*}\le c_S\|\nabla u\|_2, \tag{3.3}$$

for each $u\in H_0^1\left(\mathcal{O}\right)$, where $c_S>0$ is a constant that depends on the dimension and $2^*=\frac{2d}{d-2}$ if $d>2$, while 2^* may be any number in $]2,\infty[$ if $d=2$ and $2^*=\infty$ if $d=1$. Therefore one has

$$\|u\|_{2^*,2;t}\le c_S\|\nabla u\|_{2,2;t},$$

for each $t\ge 0$ and each $u\in L^2_{loc}\left(\mathbb{R}_+;H_0^1\left(\mathcal{O}\right)\right)$. If $u\in L^\infty_{loc}\left(\mathbb{R}_+;L^2\left(\mathcal{O}\right)\right)\bigcap L^2_{loc}\left(\mathbb{R}_+;H_0^1\left(\mathcal{O}\right)\right)$, one has

$$\|u\|_{2,\infty;t}\vee\|u\|_{2^*,2;t}\le c_1\left(\|u\|^2_{2,\infty;t}+\|\nabla u\|^2_{2,2;t}\right)^{\frac{1}{2}},$$

with $c_1=c_S\vee 1$.

One particular case of interest for us in relation with this inequality is when $p_1=2,q_1=\infty$ and $p_2=2^*,q_2=2$. If $I=I\left(2,\infty,2^*,2\right)$, then the corresponding set of associated conjugate numbers is $I'=I'\left(2,\infty,2^*,2\right)=I\left(2,1,\frac{2^*}{2^*-1},2\right)$, where for $d=1$ we make the convention that $\frac{2^*}{2^*-1}=1$. In this particular case we shall use the notation $L_{\#;t}:=L_{I;t}$ and $L^*_{\#;t}:=L^{I';t}$ and the respective norms will be denoted by

$$\|u\|_{\#;t}:=\|u\|_{I;t}=\|u\|_{2,\infty;t}\vee\|u\|_{2^*,2;t},\ \|u\|^*_{\#;t}:=\|u\|^{I';t}.$$

Thus we may write

$$\|u\|_{\#;t}\le c_1\left(\|u\|^2_{2,\infty;t}+\|\nabla u\|^2_{2,2;t}\right)^{\frac{1}{2}}, \tag{3.4}$$

for any $u \in L^{\infty}_{loc}\left(\mathbb{R}_+; L^2(\mathcal{O})\right) \bigcap L^2_{loc}\left(\mathbb{R}_+; H^1_0(\mathcal{O})\right)$ and $t \geq 0$ and the duality inequality becomes

$$\int_0^t \int_{\mathcal{O}} u(s,x)\, v(s,x)\, dxds \leq \|u\|_{\#;t} \|v\|^*_{\#;t}, \tag{3.5}$$

for any $u \in L_{\#;t}$ and $v \in L^*_{\#;t}$.

For $d \geq 3$ and some parameter $\theta \in [0,1[$ we set:

$$\Gamma^*_\theta = \left\{ (p,q) \in [1,\infty]^2 \,/\, \frac{d}{2p} + \frac{1}{q} = 1 - \theta \right\},$$

$$L^*_\theta = \sum_{(p,q)\in\Gamma^*_\theta} L^{p,q}([0,t]\times\mathcal{O})$$

$$\|u\|^*_{\theta;t} := \inf \left\{ \sum_{i=1}^n \|u_i\|_{k_i,l_i;t} \,/\, u = \sum_{i=1}^n u_i, u_i \in L^{k_i,l_i}([0,t]\times\mathcal{O}), \right.$$
$$\left. (k_i,l_i) \in \Gamma^*_\theta,\ i = 1,...n;\ n \in \mathbb{N}^* \right\}.$$

If $d = 1, 2.$ we put

$$\Gamma_\theta = \left\{ (p,q) \in [1,\infty]^2 \,/\, \frac{2^*}{2^*-2}\frac{1}{p} + \frac{1}{q} = \frac{2^*}{2^*-2} + \theta \right\},$$

$$\Gamma^*_\theta = \left\{ (p,q) \in [1,\infty]^2 \,/\, \frac{2^*}{2^*-2}\frac{1}{p} + \frac{1}{q} = 1 - \theta \right\}$$

and by using similar calculations with the convention $\frac{2^*}{2^*-2} = 1$ if $d = 1$.

We remark that $\Gamma^*_\theta = I\left(\infty, \frac{1}{1-\theta}, \frac{d}{2(1-\theta)}, \infty\right)$ and that the norm $\|u\|^*_{\theta;t}$ coincides with $\|u\|^{\Gamma^*_\theta;t} = \|u\|^{I\left(\infty,\frac{1}{1-\theta},\frac{d}{2(1-\theta)},\infty\right);t}$. We recall that the norm $\|u\|^*_{\#;t}$ is associated to the set $I\left(2, 1, \frac{2^*}{2^*-1}, 2\right)$, i.e. $\|u\|^*_{\#;t}$ coincides with $\|u\|^{I\left(2,1,\frac{2^*}{2^*-1},2\right);t}$.

3.2.2 Hypotheses

We consider a sequence $((B^i(t))_{t\geq 0})_{i\in\mathbb{N}^*}$ of independent Brownian motions defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions.

Let A be a symmetric second order differential operator defined on the bounded open subset $\mathcal{O} \subset \mathbb{R}^d$, with domain $\mathcal{D}(A)$, given by

$$A := -L = -\sum_{i,j=1}^d \partial_i (a^{i,j}\partial_j).$$

We assume that $a = (a^{i,j})_{i,j}$ is a measurable symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^d a^{i,j}(x)\xi^i\xi^j \leq \Lambda|\xi|^2,\ \forall x \in \mathcal{O},\ \xi \in \mathbb{R}^d,$$

where λ and Λ are positive constants. The energy associated with the matrix a will be denoted by

$$\mathcal{E}\left(w,v\right)=\sum_{i,j=1}^{d}\int_{\mathcal{O}}a^{i,j}(x)\partial_{i}w(x)\partial_{j}v(x)\,dx. \tag{3.6}$$

It's defined for functions w, $v\in H_0^1(\mathcal{O})$, or for $w\in H_{loc}^1(\mathcal{O})$ and $v\in H_0^1(\mathcal{O})$ with compact support.

We assume that we have predictable random functions

$$\begin{aligned}
&f:\mathbb{R}^+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R},\\
&g=(g_1,...,g_d):\mathbb{R}^+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d,\\
&h=(h_1,...,h_i,...):\mathbb{R}^+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^{\mathbb{N}^*}.
\end{aligned}$$

We define

$$f(\cdot,\cdot,\cdot,0,0):=f^0,\ g(\cdot,\cdot,\cdot,0,0):=g^0=(g_1^0,...,g_d^0)\text{ and }h(\cdot,\cdot,\cdot,0,0):=h^0=(h_1^0,...,h_i^0,...).$$

In the sequel, $|\cdot|$ will always denote the underlying Euclidean or l^2-norm. For example

$$|h(t,\omega,x,y,z)|^2=\sum_{i=1}^{+\infty}|h_i(t,\omega,x,y,z)|^2.$$

Remark 3.2. *Let us note that this general setting of the SPDE* (3.1) *we consider, encompasses the case of an SPDE driven by a space-time noise, colored in space and white in time as in [76] for example (see also Example 1 in [25]).*

Assumption (H): There exist non negative constants $C,\ \alpha,\ \beta$ such that for almost all ω, the following inequalities hold for all $(x,y,z,t)\in\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^+$:

1. $|f(t,\omega,x,y,z)-f(t,\omega,x,y',z')|\le C(|y-y'|+|z-z'|)$,
2. $|g(t,\omega,x,y,z)-g(t,\omega,x,y',z')|\le C|y-y'|+\alpha|z-z'|$,
3. $|h(t,\omega,x,y,z)-h(t,\omega,x,y',z')|\le C|y-y'|+\beta|z-z'|$,
4. the contraction property: $2\alpha+\beta^2<2\lambda$.

Moreover we introduce some integrability conditions on the coefficients $f^0,\ g^0,\ h^0$ and the initial data ξ : **(HI2)** and the weaker one **(HI#)** concerns the case of global solutions and the one denoted **(HIL)** concerns the case of local solutions.
Along this article, we fix a terminal time $T>0$.

Assumption (HI2)

$$E\left(\|\xi\|_2^2+\|f^0\|_{2,2;T}^2+\||g^0|\|_{2,2;T}^2+\||h^0|\|_{2,2;T}^2\right)<\infty.$$

Assumption (HI#)

$$E\left(\|\xi\|_2^2+\left(\|f^0\|_{\#;T}^*\right)^2+\||g^0\||_{2,2;T}^2+\||h^0\||_{2,2;T}^2\right)<\infty.$$

Assumption (HIL)

$$E\int_K|\xi(x)|^2dx+E\int_0^T\int_K\left(|f_s^0(x)|^2+|g_s^0(x)|^2+|h_s^0(x)|^2\right)dxdt<\infty,$$

for any compact set $K\subset\mathcal{O}$.

Remark 3.3. *Note that* $(2,1)$ *is the pair of conjugates of the pair* $(2,\infty)$ *and so* $(2,1)$ *belongs to the set* I' *which defines the space* $L_{\#;t}^*$. *Since* $\|v\|_{2,1;t}\leq\sqrt{t}\,\|v\|_{2,2;t}$ *for each* $v\in L^{2,2}\left([0,t]\times\mathcal{O}\right)$, *it follows that*

$$L^{2,2}\left([0,t]\times\mathcal{O}\right)\subset L^{2,1;t}\subset L_{\#;t}^*,$$

and $\|v\|_{\#;t}^*\leq\sqrt{t}\,\|v\|_{2,2;t}$, *for each* $v\in L^{2,2}\left([0,t]\times\mathcal{O}\right)$. *This shows that the condition* ***(HI#)*** *is weaker than* ***(HI2)***.

3.2.3 Weak solutions

We now introduce $\mathcal{H}_T$, the space of $H_0^1(\mathcal{O})$-valued predictable processes $(u_t)_{t\in[0,T]}$ such that

$$\left(E\sup_{0\leq s\leq T}\|u_s\|_2^2+\int_0^T E\,\mathcal{E}\left(u_s\right)ds\right)^{1/2}\;<\;\infty\;.$$

We define $\mathcal{H}_{loc}=\mathcal{H}_{loc}(\mathcal{O})$ to be the set of $H_{loc}^1(\mathcal{O})$-valued predictable processes defined on $[0,T]$ such that for any compact subset K in $\mathcal{O}$:

$$\left(E\sup_{0\leq s\leq T}\int_K u_s(x)^2\,dx+E\int_0^T\int_K|\nabla u_s(x)|^2\,dxds\right)^{1/2}\;<\;\infty.$$

The space of test functions is the algebraic tensor product $\mathcal{D}=\mathcal{C}_c^\infty(\mathbb{R}^+)\otimes\mathcal{C}_c^2(\mathcal{O})$, where $\mathcal{C}_c^\infty(\mathbb{R}^+)$ denotes the space of all real infinite differentiable functions with compact support in $\mathbb{R}^+$ and $\mathcal{C}_c^2(\mathcal{O})$ the set of C^2-functions with compact support in $\mathcal{O}$.

Now we recall the definition of the regular measure which has been defined in [25].
$\mathcal{K}$ denotes $L^\infty([0,T];L^2(\mathcal{O}))\cap L^2([0,T];H_0^1(\mathcal{O}))$ equipped with the norm:

$$\begin{aligned}\|\,v\,\|_{\mathcal{K}}^2 &= \|\,v\,\|_{L^\infty([0,T];L^2(\mathcal{O}))}^2+\|\,v\,\|_{L^2([0,T];H_0^1(\mathcal{O}))}^2\\ &= \sup_{t\in[0,T[}\|\,v_t\,\|^2+\int_0^T\left(\|\,v_t\,\|^2+\mathcal{E}(v_t)\right)\,dt.\end{aligned}$$

$\mathcal{C}$ denotes the space of continuous functions with compact support in $[0,T[\times\mathcal{O}$ and finally:

$$\mathcal{W} = \{\varphi \in L^2([0,T];H_0^1(\mathcal{O}));\ \frac{\partial\varphi}{\partial t} \in L^2([0,T];H^{-1}(\mathcal{O}))\},$$

endowed with the norm$\parallel \varphi \parallel^2_{\mathcal{W}} = \parallel \varphi \parallel^2_{L^2([0,T];H_0^1(\mathcal{O}))} + \parallel \frac{\partial\varphi}{\partial t} \parallel^2_{L^2([0,T];H^{-1}(\mathcal{O}))}$.
It is known (see [49]) that $\mathcal{W}$ is continuously embedded in $C([0,T];L^2(\mathcal{O}))$, the set of $L^2(\mathcal{O})$-valued continuous functions on $[0,T]$. So without ambiguity, we will also consider $\mathcal{W}_T = \{\varphi \in \mathcal{W}; \varphi(T) = 0\}$, $\mathcal{W}^+ = \{\varphi \in \mathcal{W}; \varphi \geq 0\}$, $\mathcal{W}_T^+ = \mathcal{W}_T \cap \mathcal{W}^+$.

Definition 3.4. *An element $v \in \mathcal{K}$ is said to be a **parabolic potential** if it satisfies:*

$$\forall\varphi \in \mathcal{W}_T^+,\ \int_0^T -(\frac{\partial\varphi_t}{\partial t}, v_t)dt + \int_0^T \mathcal{E}(\varphi_t, v_t)dt \geq 0.$$

We denote by $\mathcal{P}$ the set of all parabolic potentials.

The next representation property is crucial:

Proposition 3.5. *(Proposition 1.1 in [71]) Let $v \in \mathcal{P}$, then there exists a unique positive Radon measure on $[0,T[\times\mathcal{O}$, denoted by ν^v, such that:*

$$\forall\varphi \in \mathcal{W}_T \cap \mathcal{C},\ \int_0^T (-\frac{\partial\varphi_t}{\partial t}, v_t)dt + \int_0^T \mathcal{E}(\varphi_t, v_t)dt = \int_0^T \int_{\mathcal{O}} \varphi(t,x)d\nu^v.$$

Moreover, v admits a right-continuous (resp. left-continuous) version $\widehat{v}$ (resp. $\bar{v}$) : $[0,T] \mapsto L^2(\mathcal{O})$.
*Such a Radon measure, ν^v is called **a regular measure** and we write:*

$$\nu^v = \frac{\partial v}{\partial t} + Av.$$

Definition 3.6. *Let $K \subset [0,T[\times\mathcal{O}$ be compact, $v \in \mathcal{P}$ is said to be $\nu-$superior than 1 on K, if there exists a sequence $v_n \in \mathcal{P}$ with $v_n \geq 1$ a.e. on a neighborhood of K converging to v in $L^2([0,T];H_0^1(\mathcal{O}))$.*

We denote:

$$\mathcal{S}_K = \{v \in \mathcal{P};\ v \text{ is } \nu - \text{superior to } 1 \text{ on } K\}.$$

Proposition 3.7. *(Proposition 2.1 in [71]) Let $K \subset [0,T[\times\mathcal{O}$ compact, then $\mathcal{S}_K$ admits a smallest $v_K \in \mathcal{P}$ and the measure ν_K^v whose support is in K satisfies*

$$\int_0^T \int_{\mathcal{O}} d\nu_K^v = \inf_{v\in\mathcal{P}}\{\int_0^T \int_{\mathcal{O}} d\nu^v;\ v \in \mathcal{S}_K\}.$$

Definition 3.8. *(Parabolic Capacity)*

- *Let $K \subset [0,T[\times\mathcal{O}$ be compact, we define $cap(K) = \int_0^T \int_{\mathcal{O}} d\nu_K^v$;*
- *let $O \subset [0,T[\times\mathcal{O}$ be open, we define $cap(O) = \sup\{cap(K);\ K \subset O \text{ compact}\}$;*

- *for any borelian* $E \subset [0, T[\times\mathcal{O}$*, we define* $cap(E) = \inf\{cap(O);\ O \supset E \text{ open}\}$.

Definition 3.9. *A property is said to hold quasi-everywhere (in short q.e.) if it holds outside a set of null capacity.*

Definition 3.10. *(Quasi-continuous)*

A function $u : [0, T[\times\mathcal{O} \to \mathbb{R}$ *is called quasi-continuous, if there exists a decreasing sequence of open subsets* O_n *of* $[0, T[\times\mathcal{O}$ *with:*

1. *for all* n*, the restriction of* u_n *to the complement of* O_n *is continuous;*
2. $\lim_{n\to+\infty} cap\,(O_n) = 0$.

We say that u *admits a quasi-continuous version, if there exists* $\bar{u}$ *quasi-continuous such that* $\bar{u} = u$ *a.e.*

The next proposition, whose proof may be found in [70] or [71] shall play an important role in the sequel:

Proposition 3.11. *Let* $K \subset \mathcal{O}$ *a compact set, then* $\forall t \in [0, T[$

$$cap(\{t\} \times K) = \lambda_d(K),$$

where λ_d *is the Lebesgue measure on* $\mathcal{O}$.
As a consequence, if $u : [0, T[\times\mathcal{O} \to \mathbb{R}$ *is a map defined quasi-everywhere then it defines uniquely a map from* $[0, T[$ *into* $L^2(\mathcal{O})$*. In other words, for any* $t \in [0, T[$*,* u_t *is defined without any ambiguity as an element in* $L^2(\mathcal{O})$*. Moreover, if* $u \in \mathcal{P}$*, it admits version* $\bar{u}$ *which is left continuous on* $[0, T]$ *with values in* $L^2(\mathcal{O})$ *so that* $u_T = \bar{u}_{T^-}$ *is also defined without ambiguity.*

Remark 3.12. *The previous proposition applies if for example* u *is quasi-continuous.*

To establish a maximum principle for local solutions we need to define the notion of *local regular measures*:

Definition 3.13. *We say that a Radon measure* ν *on* $[0, T[\times\mathcal{O}$ *is a local regular measure if for any non-negative* ϕ *in* $\mathcal{C}_c^\infty(\mathcal{O})$*,* $\phi\nu$ *is a regular measure.*

Proposition 3.14. *Local regular measures do not charge polar sets (i.e. sets of capacity 0).*

Proof. Let A be a polar set and consider a sequence (ϕ_n) in $\mathcal{C}_c^\infty(\mathcal{O})$, $0 \le \phi_n \le 1$, converging to 1 everywhere on $\mathcal{O}$. By Fatou's lemma,

$$\int_{[0,T[\times\mathcal{O}} I_A d\nu(x, t) \le \liminf_{n\to\infty} \int_{[0,T[\times\mathcal{O}} I_A \phi_n d\nu(x, t) = 0.$$

∎

We end this part by a convergence lemma which plays an important role in our approach (Lemma 3.8 in [71]):

Lemma 3.15. *If $v^n \in \mathcal{P}$ is a bounded sequence in $\mathcal{K}$ and converges weakly to v in $L^2([0,T]; H_0^1(\mathcal{O}))$; if u is a quasi-continuous function and $|u|$ is bounded by a element in $\mathcal{P}$. Then*

$$\lim_{n\to+\infty} \int_0^T \int_{\mathcal{O}} u d\nu^{v^n} = \int_0^T \int_{\mathcal{O}} u d\nu^v.$$

We now give the assumptions on the obstacle that we shall need in the different cases that we shall consider.

Assumption (O): The obstacle $S : [0,T] \times \Omega \times \mathcal{O} \to \mathbb{R}$ is an adapted random field almost surely quasi-continuous, in the sense that for P-almost all $\omega \in \Omega$, the map $(t,x) \to S_t(\omega, x)$ is quasi-continuous. Moreover, $S_0 \leq \xi$ P-almost surely and S is controlled by the solution of an SPDE, i.e. $\forall t \in [0,T]$,

$$S_t \leq S'_t, \quad dP \otimes dt \otimes dx - a.e. \tag{3.7}$$

where S' is the solution of the linear SPDE

$$\left\{ \begin{array}{rcl} dS'_t & = & LS'_t dt + f'_t dt + \sum_{i=1}^d \partial_i g'_{i,t} dt + \sum_{j=1}^{+\infty} h'_{j,t} dB^j_t \\ S'(0) & = & S'_0, \end{array} \right. \tag{3.8}$$

with null boundary Dirichlet conditions.

Assumption (OL): The obstacle $S : [0,T] \times \Omega \times \mathcal{O} \to \mathbb{R}$ is an adapted random field, almost surely quasi-continuous, such that $S_0 \leq \xi$ P-almost surely and controlled by a **local** solution of an SPDE, i.e. $\forall t \in [0,T]$,

$$S_t \leq S'_t, \quad dP \otimes dt \otimes dx - a.e.$$

where S' is a **local** solution of the linear SPDE (for the definition of local solution see for example Definition 1 in [23])

$$\left\{ \begin{array}{rcl} dS'_t & = & LS'_t dt + f'_t dt + \sum_{i=1}^d \partial_i g'_{i,t} dt + \sum_{j=1}^{+\infty} h'_{j,t} dB^j_t \\ S'(0) & = & S'_0. \end{array} \right.$$

Assumption (HO2)

$$E\left(\|\xi\|_2^2 + \|f'\|_{2,2;T}^2 + \||g'|\|_{2,2;T}^2 + \||h'|\|_{2,2;T}^2 \right) < \infty.$$

Assumption (HO#)

$$E\left(\|S'_0\|_2^2 + \left(\|f'\|_{\#;T}^* \right)^2 + \||g'|\|_{2,2;T}^2 + \||h'|\|_{2,2;T}^2 \right) < \infty.$$

Assumption (HOL)

$$E \int_K |S'_0|^2 dx + E \int_0^T \int_K \left(|f'_t(x)|^2 + |g'_t(x)|^2 + |h'_t(x)|^2 \right) dx dt < \infty$$

for any compact set $K \subset \mathcal{O}$.

Remark 3.16. *It is well-known that under **(HO2)** S' belongs to $\mathcal{H}_T$, is unique and satisfies the following estimate:*

$$E\sup_{t\in[0,T]}\parallel S'_t\parallel^2+E\int_0^T\mathcal{E}(S'_t)dt\leq CE\left[\parallel S'_0\parallel^2+\int_0^T(\parallel f'_t\parallel^2+\parallel|g'_t|\parallel^2+\parallel|h'_t|\parallel^2)dt\right],\tag{3.9}$$

see for example Theorem 8 in [20]. Moreover, as a consequence of Theorem 3 in [25], we know that S' admits a quasi-continuous version.
*Under the weaker condition **(HO#)**, S' also exists, is unique and satisfies the following estimate (see Theorem 3 in [23]):*

$$E\sup_{t\in[0,T]}\parallel S'_t\parallel^2+E\int_0^T\mathcal{E}(S'_t)dt\leq CE\left[\parallel S'_0\parallel^2+\int_0^T\left((\parallel f'_t\parallel^*_{\#})^2+\parallel|g'_t|\parallel^2+\parallel|h'_t|\parallel^2\right)dt\right].\tag{3.10}$$

Definition 3.17. *A pair (u,ν) is said to be a solution of the problem (3.1) if*

1. *$u\in\mathcal{H}_T$, $u(t,x)\geq S(t,x)$, $dP\otimes dt\otimes dx-a.e.$ and $u_0(x)=\xi$, $dP\otimes dx-a.e.$;*
2. *ν is a random regular measure defined on $[0,T[\times\mathcal{O}$;*
3. *the following relation holds almost surely, for all $t\in[0,T]$ and all $\varphi\in\mathcal{D}$,*

$$\begin{aligned}(u_t,\varphi_t)=&(\xi,\varphi_0)+\int_0^t(u_s,\partial_s\varphi_s)ds-\int_0^t\mathcal{E}(u_s,\varphi_s)ds\\&-\sum_{i=1}^d\int_0^t(g^i_s(u_s,\nabla u_s),\partial_i\varphi_s)ds+\int_0^t(f_s(u_s,\nabla u_s),\varphi_s)ds\\&+\sum_{j=1}^{+\infty}\int_0^t(h^j_s(u_s,\nabla u_s),\varphi_s)dB^j_s+\int_0^t\int_{\mathcal{O}}\varphi_s(x)\nu(dx,ds);\end{aligned}\tag{3.11}$$

4. *u admits a quasi-continuous version, $\widetilde{u}$, and we have*

$$\int_0^T\int_{\mathcal{O}}(\widetilde{u}(s,x)-S(s,x))\nu(dx,ds)=0,\quad P-a.s.$$

We denote by $\mathcal{R}(\xi,f,g,h,S)$ the solution of the obstacle problem when it exists and it is unique.

Definition 3.18. *A pair (u,ν) is said to be a local solution of the problem (3.1) if*

1. *$u\in\mathcal{H}_{loc}$, $u(t,x)\geq S(t,x)$, $dP\otimes dt\otimes dx-a.e.$ and $u_0(x)=\xi$, $dP\otimes dx-a.e.$;*
2. *ν is a local random regular measure defined on $[0,T[\times\mathcal{O}$;*

3. *the following relation holds almost surely, for all $t \in [0,T]$ and all $\varphi \in \mathcal{D}$,*

$$\begin{aligned}(u_t, \varphi_t) =&(\xi, \varphi_0) + \int_0^t (u_s, \partial_s \varphi_s) ds - \int_0^t \mathcal{E}(u_s, \varphi_s) ds \\ &- \sum_{i=1}^d \int_0^t (g_s^i(u_s, \nabla u_s), \partial_i \varphi_s) ds + \int_0^t (f_s(u_s, \nabla u_s), \varphi_s) ds \\ &+ \sum_{j=1}^{+\infty} \int_0^t (h_s^j(u_s, \nabla u_s), \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds);\end{aligned} \tag{3.12}$$

4. *u admits a quasi-continuous version, $\widetilde{u}$, and we have*

$$\int_0^T \int_{\mathcal{O}} (\widetilde{u}(s,x) - S(s,x)) \nu(dx, ds) = 0, \quad P - a.s.$$

We denote by $\mathcal{R}_{loc}(\xi, f, g, h, S)$ the set of all the local solutions (u, ν).
Finally, in the sequel, we introduce some constants ϵ, $\delta > 0$, we shall denote by C_ϵ, C_δ some constants depending only on ϵ, δ, typically those appearing in the kind of inequality

$$|ab| \leq \epsilon a^2 + C_\epsilon b^2.$$

3.3 Existence, uniqueness and estimates for the solutions with null Dirichlet condition under a weaker integrability condition

In this section, we prove existence and uniqueness under a weaker integrability on f^0 and the obstacle S, improving results obtained in [25], Theorem 4 and then give an Itô's formula and estimate for the positive part of the solution, which is a crucial step leading to the maximum principle. Let us note that these results have been established in the case of SPDE without obstacle (see Section 3 in [23] and [24]).
All along this section, we suppose that **(H)**, **(O)**, **(HI#)** and **(HO#)** hold.

3.3.1 Existence, uniqueness and estimates for the solutions

To get the estimates we need, we apply Itô's formula to $u - S'$, in order to take advantage of the fact that $S - S'$ is non-positive and that as u is solution of (3.1) and S' satisfies (3.8), $u - S'$ satisfies

$$\begin{cases} d(u_t - S'_t) = \partial_i (a_{i,j}(x) \partial_j (u_t(x) - S'_t(x))) dt + (f(t,x,u_t(x), \nabla u_t(x)) - f'(t,x)) dt \\ \; + \partial_i (g_i(t,x,u_t(x), \nabla u_t(x)) - g'_i(t,x)) dt + (h_j(t,x,u_t(x), \nabla u_t(x)) - h'_j(t,x)) dB_t^j \\ \quad + \nu(x, dt), \\ (u - S')_0 = \xi - S'_0, \\ u - S' \geq S - S' \end{cases} \tag{3.13}$$

that is why we introduce the following functions:

$$\bar{f}(t,\omega,x,y,z) = f(t,\omega,x,y+S'_t,z+\nabla S'_t) - f'(t,\omega,x)$$

$$\bar{g}(t,\omega,x,y,z) = g(t,\omega,x,y+S'_t,z+\nabla S'_t) - g'(t,\omega,x)$$

$$\bar{h}(t,\omega,x,y,z) = h(t,\omega,x,y+S'_t,z+\nabla S'_t) - h'(t,\omega,x).$$

Let us remark that the Skohorod condition for $u - S'$ is satisfied since

$$\int_0^T \int_{\mathcal{O}} (u_s(x) - S'_s(x)) - (S_s(x) - S'_s(x))\nu(ds,dx) = \int_0^T \int_{\mathcal{O}} (u_s(x) - S_s(x))\nu(ds,dx) = 0.$$

It is obvious that $\bar{f}$, $\bar{g}$ and $\bar{h}$ satisfy the Lipschitz conditions with the same Lipschitz coefficients as f, g and h. Then, using Remark 3.3, we check the integrability conditions for $\bar{f}^0$, $\bar{g}^0$ and $\bar{h}^0$:

$$\begin{aligned}
\left\|\bar{f}^0\right\|^*_{\#;T} &= \left\|f(S',\nabla S') - f'\right\|^*_{\#;T} \leq \left\|f(S',\nabla S')\right\|^*_{\#;T} + \left\|f'\right\|^*_{\#;T} \\
&\leq \left\|f^0\right\|^*_{\#;T} + C\left\|S'\right\|^*_{\#;T} + C\left\|\nabla S'\right\|^*_{\#;T} + \left\|f'\right\|^*_{\#;T} \\
&\leq \left\|f^0\right\|^*_{\#;T} + C\sqrt{t}\left\|S'\right\|_{2,2;T} + C\sqrt{T}\left\|\nabla S'\right\|_{2,2;T} + \left\|f'\right\|^*_{\#;T}.
\end{aligned}$$

We know that (see Remark 3.16):

$$E\left(\left\|S'\right\|^2_{2,2;T} + \left\|\nabla S'\right\|^2_{2,2;T}\right) < \infty.$$

Hence, for each t, we have

$$E\left(\left\|\bar{f}^0\right\|^*_{\#;T}\right)^2 < \infty.$$

We also have:

$$\begin{aligned}
\left\|\bar{g}^0\right\|_{2,2;T} &= \left\|\bar{g}(S',\nabla S') - g'\right\|_{2,2;T} \leq \left\|\bar{g}(S',\nabla S')\right\|_{2,2;T} + \left\|g'\right\|_{2,2;T} \\
&\leq \left\|g^0\right\|_{2,2;T} + C\left\|S'\right\|_{2,2;T} + \alpha\left\|\nabla S'\right\|_{2,2;T} + \left\|g'\right\|_{2,2;T} < \infty.
\end{aligned}$$

And the same thing for $\bar{h}$. Hence,

$$E\left(\left(\left\|\bar{f}\right\|^*_{\#;T}\right)^2 + \|\bar{g}\|^2_{2,2;T} + \left\|\bar{h}\right\|^2_{2,2;T}\right) < \infty. \tag{3.14}$$

We now state the main Theorem of this subsection:

Theorem 3.19. *Under conditions **(H)**, **(O)**, **(HI#)** and **(HO#)**, the obstacle problem (3.1) admits a unique solution (u,ν), where u is in $\mathcal{H}_T$ and ν is a random regular measure.*

For the proof of this theorem, we need the following two lemmas whose proofs are given in the appendix. The first lemma concerns the Itô's formula for the solution of SPDE (3.1) without obstacle under **(H)** and **(HI#)**. Let us remark that in [23], the existence and uniqueness result has been established but not the Itô's formula.

Lemma 3.20. *Under the assumptions **(H)** and **(HI#)**, the SPDE (3.1) without obstacle admits a unique solution $u \in \mathcal{H}_T$. Moreover, it satisfies the Itô's formula i.e. if $\varphi : \mathbb{R} \to \mathbb{R}$ is a function of class C^2 such that φ'' is bounded and $\varphi'(0) = 0$, then the following relation holds almost surely, for all $t \in [0,T]$,*

$$\int_{\mathcal{O}} \varphi(u_t(x))\,dx + \int_0^t \mathcal{E}\left(\varphi'(u_s), u_s\right) ds = \int_{\mathcal{O}} \varphi(\xi(x))\,dx + \int_0^t \left(\varphi'(u_s), f_s(u_s, \nabla u_s)\right) ds$$
$$- \int_0^t \sum_{i=1}^d \left(\partial_i\left(\varphi'(u_s)\right), g_{i,s}(u_s, \nabla u_s\right) ds + \frac{1}{2}\int_0^t \left(\varphi''(u_s), |h_s(u_s, \nabla u_s)|^2\right) ds$$
$$+ \sum_{j=1}^\infty \int_0^t \left(\varphi'(u_s), h_{j,s}(u_s, \nabla u_s)\right) dB_s^j. \tag{3.15}$$

The following lemma will be helpful in showing that the solution to problem (3.1) is quasi-continuous.

Lemma 3.21. *The following PDE with random coefficient f^0 and zero Dirichlet boundary condition*

$$\begin{cases} dw_t + Aw_t dt = f_t^0 dt \\ w_0 = 0 \end{cases} \tag{3.16}$$

has a unique solution $w \in \mathcal{H}_T$. Moreover, w admits a quasi-continuous version.

Proof. [Proof of Theorem 3.19.] We split the proof in 2 steps:

Step 1. We prove an existence and uniqueness result for the problem (3.1) under the stronger conditions **(H)**, **(O)**, **(HI2)** and **(HO#)**. The idea of the proof is the same as Theorem 4 in [25].

We begin with the linear case i.e. we assume that f, g and h do not depend on $(u, \nabla u)$, this implies that $f = f^0$, $g = g^0$ and $h = h^0$. We consider the following penalized equation:

$$d(u_t^n - S_t') = L(u_t^n - S_t')dt + \bar{f}_t dt + \sum_{i=1}^d \partial_i \bar{g}_t^i dt + \sum_{j=1}^{+\infty} \bar{h}_t^j dB_t^j + n(u_t^n - S_t)^- dt$$

where $\bar{f} = f - f'$, $\bar{g} = g - g'$and $\bar{h} = h - h'$. Applying Itô's formula (3.15) to $(u^n - S')^2$, we have:

$$\begin{aligned}
\| u_t^n - S_t' \|^2 + 2\int_0^t \mathcal{E}(u_s^n - S_s')ds &= \| \xi - S_0' \|^2 + 2\int_0^t ((u_s^n - S_s'), \bar{f}_s)ds \\
&- 2\sum_{i=1}^d \int_0^t (\partial_i(u_s^n - S_s'), \bar{g}_s^i)ds + 2\sum_{j=1}^{+\infty} \int_0^t ((u_s^n - S_s'), \bar{h}_s^j)dB_s^j \\
&+ 2\int_0^t \int_{\mathcal{O}} (u_s^n - S_s')n(u_s^n - S_s)^- ds + \int_0^t \| |\bar{h}_s| \|^2 \, ds.
\end{aligned}$$

We remark first that:

$$\int_0^t \int_{\mathcal{O}} (u_s^n - S_s')n(u_s^n - S_s)^- ds = \int_0^t \int_{\mathcal{O}} (u_s^n - S_s + S_s - S_s')n(u_s^n - S_s)^- ds$$
$$- \int_0^t \int_{\mathcal{O}} n((u_s^n - S_s)^-)^2 ds + \int_0^t \int_{\mathcal{O}} (S_s - S_s')n(u_s^n - S_s)^- dxds.$$

The last term in the right member is non-positive because $S_t \leq S'_t$, thus,

$$\begin{aligned}\| u_t^n - S'_t \|^2 &+ 2\int_0^t \mathcal{E}(u_s^n - S'_s)ds + 2\int_0^t n \parallel (u_s^n - S_s)^- \parallel^2 ds \leq \parallel \xi - S'_0 \parallel^2 \\ &+ 2\int_0^t (u_s^n - S'_s, \bar{f}_s)ds - 2\sum_{i=1}^d \int_0^t (\partial_i(u_s^n - S'_s), \bar{g}_s^i)ds \\ &+ 2\sum_{j=1}^{+\infty} \int_0^t (u_s^n - S'_s, \bar{h}_s^j)dB_s^j + \int_0^t \parallel |\bar{h}_s| \parallel^2 ds.\end{aligned}$$

Then, the Hölder duality inequality (3.5) and the relation (3.4) lead to the following estimates, for all t in $[0, T]$, for any δ, $\epsilon > 0$,

$$\begin{aligned}2|\int_0^t (u_s^n - S'_s, \bar{f}_s)ds| &\leq \delta \left\|u^n - S'\right\|_{\#;T}^2 + C_\delta \left(\left\|\bar{f}\right\|_{\#;T}^*\right)^2 \\ &\leq C\delta \left(\left\|u^n - S'\right\|_{2,\infty;T}^2 + \left\|\nabla(u^n - S')\right\|_{2,2;T}^2\right) + C_\delta \left(\left\|\bar{f}\right\|_{\#;T}^*\right)^2,\end{aligned}$$

and

$$2|\sum_{i=1}^d \int_0^t (\partial_i(u_s^n - S'_s), \bar{g}_s^i)ds| \leq \epsilon \left\|\nabla(u^n - S')\right\|_{2,2;T}^2 + C_\epsilon \left\||\bar{g}|\right\|_{2,2;T}^2.$$

Moreover, thanks to the Burkholder-Davies-Gundy inequality, we get

$$\begin{aligned}E \sup_{t\in[0,T]} |\sum_{j=1}^{+\infty} \int_0^t (u_s^n - S'_s, \bar{h}_s^j)dB_s^j| &\leq c_1 E[\int_0^T \sum_{j=1}^{+\infty} (u_s^n - S'_s, \bar{h}_s^j)^2 ds]^{1/2} \\ &\leq c_1 E[\int_0^T \sum_{j=1}^{+\infty} \sup_{s\in[0,T]} \parallel u_s^n - S'_s \parallel^2 \parallel \bar{h}_s^j \parallel^2 ds]^{1/2} \\ &\leq c_1 E[\sup_{s\in[0,T]} \parallel u_s^n - S'_s \parallel (\int_0^T \parallel |\bar{h}_s| \parallel^2 ds)^{1/2}] \\ &\leq \epsilon E \sup_{s\in[0,T]} \parallel u_s^n - S'_s \parallel^2 + \frac{c_1}{4\epsilon} E \int_0^T \parallel |\bar{h}_s| \parallel^2 ds.\end{aligned}$$

Then using the strict ellipticity assumption and the inequalities above, we get

$$\begin{aligned}&(1 - 2\epsilon - C\delta)E \sup_{t\in[0,T]} \parallel u_t^n - S'_t \parallel^2 + (2\lambda - \epsilon - C\delta)E \int_0^T \parallel \nabla(u_s^n - S'_s) \parallel^2 ds \\ &\leq C(E \parallel \xi - S'_0 \parallel^2 + E(\parallel \bar{f} \parallel_{\#;T}^*)^2 + E \parallel |\bar{g}| \parallel_{2,2;T}^2 + E \parallel |\bar{h}| \parallel_{2,2;T}^2).\end{aligned}$$

We take ϵ and δ small enough such that $(1 - 2\epsilon - C\delta) > 0$ and $(2\lambda - \epsilon - C\delta) > 0$,

$$E \sup_{t\in[0,T]} \parallel u_t^n - S'_t \parallel^2 + E \int_0^T \mathcal{E}(u_t^n - S'_t)dt \leq C.$$

Then, to prove existence of uniqueness in this case, we can follow line by line the proof based on a weak convergence argument given in [25], Theorem 4. The only difference is that now the estimates depend on $\|\bar{f}^0\|_{\#;t}$ instead of $\|\bar{f}^0\|_{2,2;t}$.

Step 2. Now we turn to the general case, i.e. assume **(H)**, **(O)**, **(HI#)** and **(HO#)**. We consider the following SPDE:

$$dw_t + Aw_t dt = f_t^0 dt \tag{3.17}$$

Thus $u - w$ satisfies the following OSPDE:

$$\begin{aligned} d(u_t - w_t) + A(u_t - w_t)dt &= F_t(u_t - w_t, \nabla(u_t - w_t))dt + divG_t(u_t - w_t, \nabla(u_t - w_t))dt \\ &+ H_t(u_t - w_t, \nabla(u_t - w_t))dB_t + \nu(x, dt), \end{aligned}$$

where

$$F_t(x, y, z) = f_t(x, y + w, z + \nabla w) - f_t^0(x)$$

$$G_t(x, y, z) = g_t(x, y + w, z + \nabla w)$$

$$H_t(x, y, z) = h_t(x, y + w, z + \nabla w).$$

We can easily check that F, G and H satisfy the same Lipschitz conditions as f, g and h and also $F^0 \in L^2(\Omega\times[0,T]\times\mathcal{O}; R)$, $G^0 \in L^2(\Omega\times[0,T]\times\mathcal{O}; R^d)$ and $H^0 \in L^2(\Omega\times[0,T]\times\mathcal{O}; R^{\mathbb{N}^*})$. Moreover, $u - w \geq S - w$ and $S - w \leq S' - w$ where $S' - w$ satisfies the following SPDE:

$$d(S'_t - w_t) + A(S'_t - w_t)dt = (f'_t - f_t^0)dt + divg'_t dt + h'_t dB_t.$$

It is easy to see that $f' - f$, g' and h' satisfy **(HO#)**. Therefore, from Step 1, we know that $(u - w, \nu)$ uniquely exists.
Combining with the existence and uniqueness of w, we deduce that the solution of the problem (3.1) uniquely exists under the weaker assumptions **(HI#)** and **(HO#)**.
And the quasi-continuity of u comes from the quasi-continuity of w and $u - w$. ∎

3.3.2 Estimates of the positive part of the solution with null boundary condition

We recall that we assume that **(H)**, **(HI#)**, **(O)** and **(HO#)** are fulfilled. By Theorem 3.19, we know that the problem (3.1) admits a unique solution with null Dirichlet boundary conditions that we still denote by u. Now we establish an Itô's formula for (u, ν).

Theorem 3.22. *Let (u, ν) be the solution of OSPDE (3.1) and $\varphi : \mathbb{R} \to \mathbb{R}$ be a function of class C^2 and assume that φ'' is bounded and $\varphi'(0) = 0$. Then the following relation holds a.s. for all $t \in [0, T]$:*

$$\begin{aligned} &\int_{\mathcal{O}} \varphi(u_t(x))\, dx + \int_0^t \mathcal{E}(\varphi'(u_s), u_s)\, ds = \int_{\mathcal{O}} \varphi(\xi(x))\, dx + \int_0^t (\varphi'(u_s), f_s(u_s, \nabla u_s))\, ds \\ &- \int_0^t \sum_{i=1}^d (\partial_i(\varphi'(u_s)), g_{i,s}(u_s, \nabla u_s))\, ds + \frac{1}{2}\int_0^t \left(\varphi''(u_s), |h_s(u_s, \nabla u_s)|^2\right) ds \\ &+ \sum_{j=1}^\infty \int_0^t (\varphi'(u_s), h_{j,s}(u_s, \nabla u_s))\, dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi'(u_s)\nu(dxds). \end{aligned}$$

Proof. We begin with the stronger case, where **(H)**, **(O)**, **(HI2)** and **(HO#)** hold. In this case we have the Itô's formula, see step 1 of the proof of Theorem 3.19. Then using an approximation argument we can obtain the Itô's formula in the general case, more precisely: We take the function $f_n(\omega,t,x) := f(\omega,t,x,u,\nabla u) - f^0 + f_n^0$, where f_n^0, $n \in \mathbb{N}$, is a sequence of bounded functions such that $E\left(\|f^0 - f_n^0\|_{\#;t}^*\right)^2 \to 0,\ as\, n \to 0$. We consider the following equation

$$du_t^n(x) + Au_t^n(x)dt = f_t^n(x)dt + div\breve{g}_t(x)dt + \breve{h}_t(x)dB_t + \nu^n(x,dt)$$

where $\breve{g}(\omega,t,x) = g(\omega,t,x,u,\nabla u)$ and $\breve{h}(\omega,t,x) = h(\omega,t,x,u,\nabla u)$. This is a linear equation in u^n so from Theorem 3.19, we know that (u^n,ν^n) uniquely exists.
Applying Itô's formula for the difference of two solutions to $(u^n - u^m)^2$ (see Theorem 6 in [25]),

$$\begin{aligned}\|u_t^n - u_t^m\|^2 + 2\int_0^t \mathcal{E}(u_s^n - u_s^m)ds &= 2\int_0^t (u_s^n - u_s^m, f_s^n - f_s^m)ds \\ &+ 2\int_0^t\int_{\mathcal{O}} (u_s^n - u_s^m)(\nu^n - \nu^m)(dxds)\end{aligned}$$

Remarking that

$$\int\int (u_n - u_m)(\nu_n - \nu_m)(dxds) = \int\int (S - u_m)\nu_n(dxds) - \int\int (u_n - S)\nu_m(dxds) \le 0$$

and for $\delta > 0$, we have

$$2\left|\int_0^t (u_s^n - u_s^m, f_s^n - f_s^m)ds\right| \le \delta\,\|u^n - u^m\|_{\#;t}^2 + C_\delta\left(\|f^n - f^m\|_{\#;t}^*\right)^2.$$

Since $\mathcal{E}(u^n - u^m) \ge \lambda\,\|\nabla(u^n - u^m)\|_2^2$, we deduce that, for all $t \ge 0$, almost surely,

$$\|u_t^n - u_t^m\|^2 + 2\lambda\,\|\nabla(u^n - u^m)\|_{2,2;t}^2 \le \delta\,\|u^n - u^m\|_{\#;t}^2 + C_\delta\left(\|f^n - f^m\|_{\#;t}^*\right)^2 \tag{3.18}$$

Taking the supremum and the expectation, we get

$$E\left(\|u^n - u^m\|_{2,\infty;t}^2 + \|\nabla(u^n - u^m)\|_{2,2;t}^2\right) \le \delta E\,\|u^n - u^m\|_{\#;t}^2 + C_\delta E\left(\|f^n - f^m\|_{\#;t}^*\right)^2.$$

Dominating the term $E\,\|u^n - u^m\|_{\#;t}^2$ by using the estimate (3.4) and taking δ small enough, we obtain the following estimate:

$$E\left(\|u_n - u_m\|_{2,\infty;t}^2 + \|\nabla(u_n - u_m)\|_{2,2;t}^2\right) \le 2C_\delta E\left(\|f^n - f^m\|_{\#;t}^*\right)^2 \to 0,\ when\ n,\, m \to \infty$$

Therefore, u^n has a limit u in $\mathcal{H}_T$. Now we want to find the limit of ν^n: we denote by v^n the parabolic potential associated to ν^n, and $z^n = u^n - v^n$, so z^n satisfies the following SPDE

$$dz_t^n(x) + Az_t^n(x)dt = f_t^n(x)dt - \sum_{i=1}^d \partial_i\breve{g}_t^i(x)dt + \sum_{j=1}^\infty \breve{h}_t^j(x)\,dB_t^j.$$

Applying Itô's formula to $(z^n - z^m)^2$, doing the same calculus as before, we obtain the following relation:

$$E\left(\|z_n - z_m\|^2_{2,\infty;t} + \|\nabla(z_n - z_m)\|^2_{2,2;t}\right) \le 2C_\delta E\left(\|f^n - f^m\|^*_{\#;t}\right)^2 \longrightarrow 0, \quad as\ n,\ m \to \infty.$$

As a consequence:

$$E\left(\|v_n - v_m\|^2_{2,\infty;t} + \|\nabla(v_n - v_m)\|^2_{2,2;t}\right) \longrightarrow 0, \quad as\ n,\ m \to \infty.$$

Therefore, (v^n) has a limit v in $\mathcal{H}_T$. So, by extracting a subsequence, we can assume that (v^n) converges to v in $\mathcal{K}$, $P-$almost-surely. Then, it's clear that $v \in \mathcal{P}$, and we denote by ν the regular random measure associated to the potential v. Moreover, we have $P-a.s.$ $\forall \varphi \in \mathcal{W}_t^+$,

$$\begin{aligned}\int_0^t\int_{\mathcal{O}} \varphi(x,s)\nu(dxds) &= \lim_{n\to\infty}\int_0^t\int_{\mathcal{O}} \varphi(x,s)\nu^n(dxds)\\ &= \lim_{n\to\infty}\int_0^t -(v^n_s, \frac{\partial\varphi_s}{\partial s})ds + \int_0^t \mathcal{E}(v^n_s,\varphi_s)ds\\ &= \int_0^t -(v_s, \frac{\partial\varphi_s}{\partial s})ds + \int_0^t \mathcal{E}(v_s,\varphi_s)ds.\end{aligned}$$

Hence, (u^n,ν^n) converges to (u,ν). Moreover, by Theorem 3.19, we know that the solution of problem (3.1) uniquely exists and we apply the Itô's formula for (u^n,ν^n).

$$\begin{aligned}&\int_{\mathcal{O}} \varphi\left(u^n_t(x)\right)dx + \int_0^t \mathcal{E}\left(\varphi'\left(u^n_s\right), u^n_s\right)ds = \int_{\mathcal{O}} \varphi\left(\xi(x)\right)dx + \int_0^t \left(\varphi'\left(u^n_s\right), f^n_s\right)ds\\ &- \int_0^t \sum_{i=1}^d \left(\partial_i\left(\varphi'\left(u^n_s\right)\right), \breve{g}^i_s\right)ds + \frac{1}{2}\int_0^t \left(\varphi''\left(u^n_s\right), \left|\breve{h}_s\right|^2\right)ds + \sum_{j=1}^\infty \int_0^t \left(\varphi'\left(u^n_s\right), \breve{h}^j_s\right)dB^j_s\\ &+ \int_0^t\int_{\mathcal{O}} \varphi'(u^n_s)\nu^n(dxds) \qquad a.s. \end{aligned}\tag{3.19}$$

Now, we pass to the limit as n tend to $+\infty$. First, by using Lemma 3.15:

$$\int_0^t\int_{\mathcal{O}} \varphi'(u^n_s)\nu^n(dxds) = \int_0^t\int_{\mathcal{O}} \varphi'(S_s)\nu^n(dxds) \to \int_0^t\int_{\mathcal{O}} \varphi'(S_s)\nu(dxds) = \int_0^t\int_{\mathcal{O}} \varphi'(u_s)\nu(dxds).$$

Moreover,

$$\begin{aligned}&\left|\int_0^t \left(\varphi'\left(u^n_s\right), f^n_s\right)ds - \int_0^t \left(\varphi'\left(u_s\right), f_s\right)ds\right|\\ \le\ &\left|\int_0^t \left(\varphi'\left(u^n_s\right) - \varphi'\left(u_s\right), f^n_s\right)ds\right| + \left|\int_0^t \left(\varphi'\left(u_s\right), f^n_s - f_s\right)ds\right|\\ \le\ &C\left\|u^n - u\right\|_{\#;t}\left\|f^n\right\|^*_{\#;t} + C\left\|u\right\|_{\#;t}\left\|f^n - f\right\|^*_{\#;t}.\end{aligned}$$

The relation (3.4) and the strong convergence of u^n yield that $E\left\|u^n - u\right\|_{\#;t} \to 0$, as $n \to \infty$. So, by extracting a subsequence, we can assume that the right member in the previous inequality tends to 0 $P-$almost surely. So we have

$$\lim_{n\to+\infty}\int_0^t \left(\varphi'\left(u^n_s\right), f^n_s\right)ds = \int_0^t \left(\varphi'\left(u_s\right), f_s\right)ds.$$

The convergence of the other terms in (3.19) are easily deduced from the strong convergence of (u^n) to u in $\mathcal{H}_T$ and yield the desired formula. ■

This yields the estimate of the $\mathcal{H}_T$-norm of u under **(HI#)**:

Proposition 3.23. *Under the same hypotheses and notations as in the previous theorem, we have:*

$$\begin{aligned} E\left(\|u\|_{2,\infty;t}^2+\|\nabla u\|_{2,2;t}^2\right) &\leq k(t)E(\|\xi-S'_0\|_2^2+\left(\|\bar f^0\|_{\#;t}^*\right)^2+\|\bar g^0\|_{2,2;t}^2+\|\bar h^0\|_{2,2;t}^2 \\ &+ \|S'_0\|_2^2+\left(\|f'\|_{\#;t}^*\right)^2+\|g'\|_{2,2;t}^2+\|h'\|_{2,2;t}^2) \end{aligned}$$

for each $t\geq 0$, where $k(t)$ is a constant that only depends on the structure constants and t.

Proof. Applying the above Itô's formula to $(u-S')^2$, since $(u-S',\nu)=\mathcal{R}(\xi-S'_0,\bar f,\bar g,\bar h,S-S')$, we have, almost surely, for all $t\in[0,T]$:

$$\begin{aligned} &\left\|u_t-S'_t\right\|^2+2\int_0^t\mathcal{E}(u_s-S'_s)ds=\left\|\xi-S'_0\right\|^2+2\int_0^t(u_s-S'_s,\bar f_s(u_s-S'_s,\nabla(u_s-S'_s)))ds \\ &-2\int_0^t(\nabla(u_s-S'_s),\bar g_s(u_s-S'_s,\nabla(u_s-S'_s))ds+2\int_0^t(u_s-S'_s,\bar h_s(u_s-S'_s,\nabla(u_s-S'_s))dB_s \\ &+\int_0^t\left\|\bar h_s(u_s-S'_s,\nabla(u_s-S'_s))\right\|^2ds+2\int_0^t\int_{\mathcal{O}}(u_s-S'_s)(x)\nu(dxds). \end{aligned}$$

Remarking the following relation

$$\int_0^t\int_{\mathcal{O}}(u_s-S'_s)\nu(dxds)\leq\int_0^t\int_{\mathcal{O}}(u_s-S_s)\nu(dxds)=0.$$

The Lipschitz conditions in $\bar f$, $\bar g$ and $\bar h$ and Cauchy-Schwarz's inequality lead the following relations: for δ, $\epsilon>0$, we have

$$\begin{aligned} \int_0^t\left(u_s-S'_s,\bar f_s\left(u_s-S'_s,\nabla(u_s-S'_s)\right)\right)ds &\leq \epsilon\left\|\nabla(u-S')\right\|_{2,2;t}^2+c_\epsilon\left\|u-S'\right\|_{2,2;t}^2 \\ &+\delta\left\|u-S'\right\|_{\#;t}^2+c_\delta\left(\left\|\bar f^0\right\|_{\#;t}^*\right)^2, \end{aligned}$$

and

$$\begin{aligned} \int_0^t(\nabla(u_s-S'_s),\bar g_s(u_s-S'_s,\nabla(u_s-S'_s))ds &\leq (\alpha+\epsilon)\left\|\nabla(u-S')\right\|_{2,2;t}^2 \\ &+c_\epsilon\left\|u-S'\right\|_{2,2;t}^2+c_\epsilon\left\|\bar g^0\right\|_{2,2;t}^2, \end{aligned}$$

and

$$\int_0^t\left\|\bar h_s(u_s-S'_s,\nabla(u_s-S'_s))\right\|^2ds\leq(\beta^2+\epsilon)\left\|\nabla(u-S')\right\|_{2,2;t}^2+c_\epsilon\left\|u-S'\right\|_{2,2;t}^2+c_\epsilon\left\|\bar h^0\right\|_{2,2;t}^2.$$

Since $\mathcal{E}(u-S')\geq\lambda\left\|\nabla(u-S')\right\|_2^2$, we deduce that for all $t\in[0,T]$, almost surely,

$$\begin{aligned} &\left\|u_t-S'_t\right\|_2^2+2\left(\lambda-\alpha-\frac{\beta^2}{2}-\frac{5}{2}\epsilon\right)\left\|\nabla(u-S')\right\|_{2,2;t}^2\leq\left\|\xi-S'_0\right\|_2^2+\delta\left\|u-S'\right\|_{\#;t}^2 \\ &+2c_\delta\left(\left\|\bar f^0\right\|_{\#;t}^*\right)^2+2c_\epsilon\left\|\bar g^0\right\|_{2,2;t}^2+c_\epsilon\left\|\bar h^0\right\|_{2,2;t}^2+5c_\epsilon\left\|u-S'\right\|_{2,2;t}^2+2M_t, \end{aligned}$$

where $M_t := \sum_{j=1}^{\infty} \int_0^t \left(u_s - S'_s, \bar{h}_s^j (u_s - S'_s, \nabla (u_s - S'_s)) \right) dB_s^j$ represents the martingale part. Further, using a stopping procedure while taking the expectation, the martingale part vanishes, so that we get

$$E \left\| u_t - S'_t \right\|_2^2 + 2 \left(\lambda - \alpha - \frac{\beta^2}{2} - \frac{5}{2} \epsilon \right) E \left\| \nabla (u - S') \right\|_{2,2;t}^2 \leq E \left\| \xi - S'_0 \right\|_2^2 + \delta E \left\| u - S' \right\|_{\#;t}^2$$

$$+2 c_\delta E \left(\left\| \bar{f}^0 \right\|_{\#;t}^* \right)^2 + 2 c_\epsilon E \left\| \bar{g}^0 \right\|_{2,2;t}^2 + c_\epsilon E \left\| \bar{h}^0 \right\|_{2,2;t}^2 + 5 c_\epsilon \int_0^t E \left\| u_s - S'_s \right\|_2^2 ds.$$

Then we choose $\epsilon = \frac{1}{5} \left(\lambda - \alpha - \frac{\beta^2}{2} \right)$, set $\gamma = \lambda - \alpha - \frac{\beta^2}{2}$ and apply Gronwall's lemma obtaining

$$E \left\| u_t - S'_t \right\|_2^2 + \gamma E \left\| \nabla (u - S') \right\|_{2,2;t}^2 \leq \left(\delta E \left\| u - S' \right\|_{\#;t}^2 + E \left[F \left(\delta, \xi - S'_0, \bar{f}^0, \bar{g}^0, \bar{h}^0, t \right) \right] \right) e^{5 c_\epsilon t} \qquad (*)$$

where $F \left(\delta, \xi - S'_0, \bar{f}^0, \bar{g}^0, \bar{h}^0, t \right) = \left(\left\| \xi - S'_0 \right\|^2 + 2 c_\delta \left(\left\| \bar{f}^0 \right\|_{\#;t}^* \right)^2 + 2 c_\epsilon \left\| \bar{g}^0 \right\|_{2,2;t}^2 + c_\epsilon \left\| \bar{h}^0 \right\|_{2,2;t}^2 \right)$.
As a consequence one gets

$$E \left\| u - S' \right\|_{2,2;t}^2 \leq \frac{1}{5 c_\epsilon} \left(\delta E \left\| u - S' \right\|_{\#;t}^2 + E \left[F \left(\delta, \xi - S'_0, \bar{f}^0, \bar{g}^0, \bar{h}^0, t \right) \right] \right) \left(e^{5 c_\epsilon t} - 1 \right). \qquad (**)$$

Now we return to the inequality (3.46) and take the supremum, getting

$$\left\| u - S' \right\|_{2,\infty;t}^2 \leq \delta \left\| u - S' \right\|_{\#;t}^2 + F \left(\delta, \xi - S'_0, \bar{f}^0, \bar{g}^0, \bar{h}^0, t \right) + 5 c_\epsilon \left\| u - S' \right\|_{2,2;t}^2 + 2 \sup_{s \leq t} M_s \tag{3.20}$$

We would like to take the expectation in this relation and for that reason we need to estimate the bracket of the martingale part,

$$\begin{aligned} \langle M \rangle_t^{\frac{1}{2}} &\leq \left\| u - S' \right\|_{2,\infty;t} \left\| \bar{h}(u - S', \nabla (u - S')) \right\|_{2,2;t} \\ &\leq \eta \left\| u - S' \right\|_{2,\infty;t}^2 + c_\eta \left(\left\| u - S' \right\|_{2,2;t}^2 + \left\| \nabla (u - S') \right\|_{2,2;t}^2 + \left\| \bar{h}^0 \right\|_{2,2;t}^2 \right) \end{aligned}$$

with η another small parameter to be properly chosen. Using this estimate and the inequality of Burkholder-Davis-Gundy we deduce from the inequality (3.20):

$$(1 - 2 C_{BDG} \eta) E \left\| u - S' \right\|_{2,\infty;t}^2 \leq \delta E \left\| u - S' \right\|_{\#;t}^2 + E \left[F \left(\delta, \xi - S'_0, \bar{f}^0, \bar{g}^0, \bar{h}^0, t \right) \right]$$

$$+ (5 c_\varepsilon + 2 C_{BDG} c_\eta) E \left\| u - S' \right\|_{2,2;t}^2 + 2 C_{BDG} c_\eta E \left\| \nabla (u - S') \right\|_{2,2;t}^2 + 2 C_{BDG} c_\eta E \left\| \bar{h}^0 \right\|_{2,2;t}^2$$

where C_{BDG} is the constant corresponding to the Burkholder-Davis-Gundy inequality. Further we choose the parameter $\eta = \frac{1}{4 C_{BDG}}$ and combine this estimate with (*) and (**) to deduce an estimate of the form

$$\begin{aligned} E \left(\left\| u - S' \right\|_{2,\infty;t}^2 + \left\| \nabla (u - S') \right\|_{2,2;t}^2 \right) &\leq \delta c_2 (t) E \left\| u - S' \right\|_{\#;t}^2 \\ &+ c_3 (\delta, t) E \left[R \left(\delta, \xi - S'_0, \bar{f}^0, \bar{g}^0, \bar{h}^0, t \right) \right] \end{aligned}$$

where $R \left(\delta, \xi - S'_0, \bar{f}^0, \bar{g}^0, \bar{h}^0, t \right) = \left(\left\| \xi - S'_0 \right\|^2 + \left(\left\| \bar{f}^0 \right\|_{\#;t}^* \right)^2 + \left\| \bar{g}^0 \right\|_{2,2;t}^2 + \left\| \bar{h}^0 \right\|_{2,2;t}^2 \right)$ and $c_3(\delta, t)$ is a constant that depends of δ and t, while $c_2 (t)$ is independent of δ. Dominating the term

$E\left\|u-S'\right\|_{\#;t}^2$ by using the estimate (3.4) and then choosing $\delta=\frac{1}{2c_1^2c_2(t)}$, we get the following estimate:

$$E\left(\left\|u-S'\right\|_{2,\infty;t}^2+\left\|\nabla(u-S')\right\|_{2,2;t}^2\right)\leq k(t)E\left(\left\|\xi-S'_0\right\|_2^2+\left(\left\|\bar{f}^0\right\|_{\#;t}^*\right)^2+\left\|\bar{g}^0\right\|_{2,2;t}^2+\left\|\bar{h}^0\right\|_{2,2;t}^2\right).$$

Combining with the estimate for S' (see Remark 3.16), we obtain the estimate asserted by our corollary. ■

In the following Proposition, we establish a crucial relation for the positive part of u:

Proposition 3.24. *Under the hypotheses of the above Proposition with same notations, the following relation holds a.s. for all $t\in[0,T]$:*

$$\begin{aligned}\int_{\mathcal{O}}(u_t^+(x))^2dx+2\int_0^t\mathcal{E}(u_s^+)ds &= \int_{\mathcal{O}}(\xi^+(x))^2dx+2\int_0^t(u_s^+,f_s(u_s,\nabla u_s))ds\\ &- 2\int_0^t(\nabla u_s^+,g_s(u_s,\nabla u_s))ds+2\int_0^t(u_s^+,h_s(u_s,\nabla u_s))dB_s\\ &+ \int_0^t\left\|I_{\{u_s>0\}}h_s(u_s,\nabla u_s)\right\|^2ds+2\int_0^t\int_{\mathcal{O}}u_s^+(x)\nu(dxds).\end{aligned}$$

Proof. We approximate $\psi(y)=(y^+)^2$ by a sequence of regular functions: Let φ be an increasing $\mathcal{C}^\infty$ function such that $\varphi(y)=0$ for any $y\in]-\infty,1]$ and $\varphi(y)=1$ for any $y\in[2,\infty[$. We set $\psi_n(y)=y^2\varphi(ny)$, for each $y\in\mathbb{R}$ and all $n\in\mathbb{N}^*$. It is easy to verify that $(\psi_n)_{n\in\mathbb{N}^*}$ converges uniformly to the function ψ and that

$$\lim_{n\to\infty}\psi_n'(y)=2y^+,\ \lim_{n\to\infty}\psi_n''(y)=2\cdot I_{\{y>0\}},$$

for any $y\in\mathbb{R}$. Moreover we have the estimates

$$0\leq\psi_n(y)\leq\psi(y),\ 0\leq\psi_n'(y)\leq Cy,\ \left|\psi_n''(y)\right|\leq C, \tag{3.21}$$

for any $y\geq0$ and all $n\in\mathbb{N}^*$, where C is a constant. We have for all $n\in\mathbb{N}^*$ and each $t\geq0$, a.s.,

$$\begin{aligned}&\int_{\mathcal{O}}\psi_n(u_t(x))\,dx+\int_0^t\mathcal{E}\left(\psi_n{}'(u_s),u_s\right)ds=\int_{\mathcal{O}}\psi_n(\xi(x))\,dx+\int_0^t\left(\psi_n{}'(u_s),f_s(u_s,\nabla u_s)\right)ds\\ &-\int_0^t\sum_{i=1}^d\left(\psi_n{}''(u_s)\,\partial_iu_s,g_{i,s}(u_s,\nabla u_s)\right)ds+\frac12\int_0^t\left(\psi_n{}''(u_s),\left|h_s(u_s,\nabla u_s)\right|^2\right)ds\\ &\quad+\sum_{j=1}^\infty\int_0^t\left(\psi_n{}'(u_s),h_{j,s}(u_s,\nabla u_s)\right)dB_s^j+\int_0^t\int_{\mathcal{O}}\psi_n{}'(u_s)\,\nu(dxds).\end{aligned} \tag{3.22}$$

Taking the limit, thanks to the dominated convergence theorem, we know that all the terms except $\int_0^t\int_{\mathcal{O}}\psi_n{}'(u_s)\,\nu(dxds)$ converge. From (3.21) and (3.22), it is easy to verify

$$\sup_n\int_0^t\int_{\mathcal{O}}\phi_n'(u_s)\nu(dxds)\leq C.$$

Then, by Fatou's lemma, we have

$$\int_0^t \int_{\mathcal{O}} u_s^+(x)\nu(dxds) = \liminf_{n\to\infty} \int_0^t \int_{\mathcal{O}} \phi_n'(u_s)\nu(dxds) < +\infty, \qquad a.s.$$

Hence, the convergence of the last term comes from the dominated convergence theorem. ■

Now we prove an estimate for the positive part u^+ of the solution. For this we need the following notation:

$$\begin{aligned}
&\bar{f}^{u-S',0} = I_{\{u>S'\}}\bar{f}^0,\ \bar{g}^{u-S',0} = I_{\{u>S'\}}\bar{g}^0,\ \bar{h}^{u-S',0} = I_{\{u>S'\}}\bar{h}^0,\\
&\bar{f}^{u-S'} = \bar{f} - \bar{f}^0 + \bar{f}^{u-S',0},\ \bar{g}^{u-S'} = \bar{g} - \bar{g}^0 + \bar{g}^{u-S',0},\ \bar{h}^{u-S'} = \bar{h} - \bar{h}^0 + \bar{h}^{u-S',0},\\
&\bar{f}^{u-S',0+} = I_{\{u>S'\}}\left(\bar{f}^0 \vee 0\right),\ (\xi - S_0')^+ = (\xi - S_0') \vee 0. \qquad (3.23)
\end{aligned}$$

Proposition 3.25. *Under the hypotheses of the above Proposition with same notations, one has the following estimate:*

$$\begin{aligned}
E\left(\|u^+\|_{2,\infty;t}^2\right) \leq\ & 2k(t)E\big(\|(\xi - S_0')^+\|_2^2 + \left(\left\|\bar{f}^{u-S',0+}\right\|_{\#;t}^*\right)^2 + \left\|\bar{g}^{u-S',0}\right\|_{2,2;t}^2 + \left\|\bar{h}^{u-S',0}\right\|_{2,2;t}^2\\
+\ & \left\|S_0'^+\right\|_2^2 + \left(\left\|f'^{,0+}\right\|_{\#;t}^*\right)^2 + \left\|g'^{,0}\right\|_{2,2;t}^2 + \left\|h'^{,0}\right\|_{2,2;t}^2\big).
\end{aligned}$$

Proof. Since $(u - S', \nu) = \mathcal{R}(\xi - S_0', \bar{f}, \bar{g}, \bar{h}, S - S')$, by Proposition 3.24, we have almost surely $\forall t \in [0,T]$:

$$\begin{aligned}
&\int_{\mathcal{O}} ((u_t - S_t')^+(x))^2 dx + 2\int_0^t \mathcal{E}((u_s - S_s')^+)ds\\
&= \int_{\mathcal{O}} ((\xi - S_0')^+(x))^2 dx + 2\int_0^t ((u_s - S_s')^+, \bar{f}_s(u_s - S_s', \nabla(u_s - S_s')))ds\\
&-2\int_0^t (\nabla(u_s - S_s')^+, \bar{g}_s(u_s - S_s', \nabla(u_s - S_s'))ds + 2\int_0^t ((u_s - S_s')^+, \bar{h}_s(u_s - S_s', \nabla(u_s - S_s'))dB_s\\
&+ \int_0^t \left\|I_{\{u_s - S_s' > 0\}}(\bar{h}_s(u_s - S_s', \nabla(u_s - S_s')))\right\|^2 ds + 2\int_0^t \int_{\mathcal{O}} (u_s - S_s')^+(x)\nu(dxds).
\end{aligned}$$

As the support of ν is $\{u = S\}$, we have the following relation

$$\int_0^t \int_{\mathcal{O}} (u_s - S_s')^+ \nu(dxds) = \int_0^t \int_{\mathcal{O}} (S_s - S_s')^+ \nu(dxds) = 0.$$

Then we repeat word by word the proof of Proposition 3.23, replacing $u - S'$, $\bar{f}$, $\bar{g}$, $\bar{h}$ and $\xi - S_0'$ by $(u - S')^+$, $\bar{f}^{u-S',0+}$, $\bar{g}^{u-S',0}$, $\bar{h}^{u-S',0}$ and $(\xi - S_0')^+$ respectively. Hence, we get the following estimate:

$$\begin{aligned}
E(\|(u - S')^+\|_{2,\infty;t}^2 \ +\ & \|\nabla(u - S')^+\|_{2,2;t}^2) \leq k(t)E\big(\|(\xi - S_0')^+\|_2^2\\
&+ \left(\left\|\bar{f}^{u-S',0+}\right\|_{\#;t}^*\right)^2 + \left\|\bar{g}^{u-S',0}\right\|_{2,2;t}^2 + \left\|\bar{h}^{u-S',0}\right\|_{2,2;t}^2\big).
\end{aligned}$$

Moreover, from Theorem 4 in [23], we know that

$$E\left(\left\|S'^{+}\right\|_{2,\infty;t}^{2}+\left\|\nabla S'^{+}\right\|_{2,2;t}^{2}\right)\leq k\left(t\right)E\left(\left\|S_{0}'^{+}\right\|_{2}^{2}+\left(\left\|f'^{,0+}\right\|_{\#;t}^{*}\right)^{2}+\left\|g'^{,0}\right\|_{2,2;t}^{2}+\left\|h'^{,0}\right\|_{2,2;t}^{2}\right).$$

where $S_0'^{+}=S_0'\vee 0$, $f'^{,0+}=I_{\{S'>0\}}(f'\vee 0)$, $g'^{,0}=I_{\{S'>0\}}g'$ and $h'^{,0}=I_{\{S'>0\}}h'$. Then with the relation:

$$E\left\|(u)^{+}\right\|_{2,\infty;t}^{2}\leq 2E\left(\left\|(u-S')^{+}\right\|_{2,\infty;t}^{2}+\left\|(S')^{+}\right\|_{2,\infty;t}^{2}\right),$$

we get the desired estimate. ∎

3.4 L^p estimate for the uniform norm of solutions with null Dirichlet boundary condition

In this section, we want to study, for some $p\geq 2$, the L^p- estimate of the uniform norm of the solution of (3.1). To get such estimates, we need stronger integrability conditions on the coefficients and the initial condition. To this end, we consider the following assumptions: for $\theta\in[0,1[$ and $p\geq 2$:

Assumption (HI2p)

$$E\left(\|\xi\|_{\infty}^{p}+\left\|f^{0}\right\|_{2,2;T}^{2}+\left\||g^{0}|\right\|_{2,2;T}^{2}+\left\||h^{0}|\right\|_{2,2;T}^{2}\right)<\infty.$$

Assumption (HO∞p)

$$S_0'\in L^{\infty}(\Omega\times\mathcal{O})\ and\ E\left((\left\|f'\right\|_{\infty,\infty;T})^{p}+(\left\||g'|^{2}\right\|_{\infty,\infty;T})^{p/2}+(\left\||h'|^{2}\right\|_{\infty,\infty;T})^{p/2}\right)<\infty.$$

We still consider $\bar f$, $\bar g$ and $\bar h$ which have been introduced at the beginning of Subsection 3.3.1. It is clear that $\bar f$, $\bar g$ and $\bar h$ satisfy condition **(H)** and $\|\xi-S_0'\|_{\infty}\in L^p(\Omega,P)$. Nevertheless, we need a supplementary hypothesis:
Assumption (HDθp)

$$E((\left\|\bar f^{0}\right\|_{\theta;T}^{*})^{p}+(\left\||\bar g^{0}|^{2}\right\|_{\theta;T}^{*})^{p/2}+(\left\||\bar h^{0}|^{2}\right\|_{\theta;T}^{*})^{p/2})<\infty.$$

This assumption is fulfilled in the following case:

Example 3.26. *If $\|\nabla S'\|_{\theta;T}^{*}$, $\left\|f^{0}\right\|_{\theta;T}^{*}$, $\left\|g^{0}\right\|_{\theta;T}^{*}$ and $\left\|h^{0}\right\|_{\theta;T}^{*}$ belong to $L^p(\Omega,P)$, and assumptions **(H)** and **(HO∞p)** hold, then:*

$\bar f$ satisfies the Lipschitz condition with the same Lipschitz coefficients:

$$\begin{aligned}\left|\bar f(t,\omega,x,y,z)-\bar f(t,\omega,x,y',z')\right| &= \left|f(t,\omega,x,y+S_t'(x),z+\nabla S_t'(x))+f'(t,\omega,x)\right.\\ &- \left.f(t,\omega,x,y'+S_t'(x),z'+\nabla S_t'(x))-f'(t,\omega,x)\right|\\ &\leq C\left|y-y'\right|+C\left|z-z'\right|.\end{aligned}$$

$\bar f$ satisfies the integrability condition:

$$\begin{aligned}\|\bar f^0\|^*_{\theta;T} &= \|f(S',\nabla S')-f'\|^*_{\theta;T}\leq \|f(S',\nabla S')\|^*_{\theta;T}+\|f'\|^*_{\theta;T}\\ &\leq \|f^0\|^*_{\theta;T}+C\|S'\|^*_{\theta;T}+C\|\nabla S'\|^*_{\theta;T}+\|f'\|_{\infty,\infty;T}\,.\end{aligned}$$

*And the same for $\bar g$ and $\bar h$, which proves that **(HDθp)** holds.*

We now give the main result of this Section, which is a version of the maximum principle in the case of a solution vanishing on the boundary of $\mathcal{O}$:

Theorem 3.27. *Suppose that assumptions **(H)**, **(O)**, **(HI2p)**, **(HO∞p)** and **(HDθp)** hold, for some $\theta\in[0,1[$ and $p\geq 2$ and that the constants of Lipschitz conditions satisfy*

$$\alpha+\frac{\beta^2}{2}+72\beta^2<\lambda.$$

Let $(u,\ \nu)$ be the solution of OSPDE (3.1) with null boundary condition, then for all $t\in[0,T]$,

$$\begin{aligned}E\|u\|^p_{\infty,\infty;t}\leq c(p)k(t)E\big(&\|\xi\|^p_\infty+\|S'_0\|^p_\infty+\|f'\|^{*p}_{\theta;t}+\||g'|^2\|^{*p/2}_{\theta;t}+\||h'|^2\|^{*p/2}_{\theta;t}\\ &+\|\bar f^0\|^{*p}_{\theta;t}+\||\bar g^0|^2\|^{*p/2}_{\theta;t}+\||\bar h^0|^2\|^{*p/2}_{\theta;t}\big),\end{aligned}$$

where $c(p)$ is a constant which depends on p and $k(t)$ is a constant which depends on the structure constants and $t\in[0,T]$.

Remark 3.28. *The relations $\|f'\|^{*p}_{\theta;t}\leq(\|f'\|_{\infty,\infty;t})^p$, $\||g'|^2\|^{*p/2}_{\theta;t}\leq(\||g'|^2\|_{\infty,\infty;t})^{p/2}$ and $\||h'|^2\|^{*p/2}_{\theta;t}\leq(\||h'|^2\|_{\infty,\infty;t})^{p/2}$ and assumption **(HO∞p))** yield*

$$E\left(\|f'\|^{*p}_{\theta;t}+\||g'|^2\|^{*p/2}_{\theta;t}+\||h'|^2\|^{*p/2}_{\theta;t}\right)<+\infty.$$

As the proof of this theorem is quite long, we split it into several steps.

3.4.1 The case where ξ, $\bar f^0$, $\bar g^0$ and $\bar h^0$ are uniformly bounded

In this subsection, we assume that the hypotheses **(H)**, **(O)**, **(HI2p)**, **(HO∞p)** hold and we add the following stronger one:

$$\xi\in L^\infty(\Omega\times\mathcal{O}),$$

and

$$\bar f^0,\ \bar g^0,\ \bar h^0\in L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O}).$$

Then it is obviously that $\xi-S'_0\in L^\infty(\Omega\times\mathcal{O})$.
Under these hypotheses, we know that the SPDE with obstacle (3.1) admits a unique weak solution $(u,\nu)=\mathcal{R}(\xi,f,g,h,S)$ and that $(u-S',\nu)=\mathcal{R}(\xi-S'_0,\bar f,\bar g,\bar h,S-S')$. We start by proving the following L^l estimate:

Lemma 3.29. *The solution u of the problem (3.1) belongs to $\cap_{l\geq 2}L^l([0,T]\times\mathcal{O}\times\Omega)$. Moreover there exist constants c, $c'>0$ which only depend on C, α, β and on the quantity*

$$K=\|\xi-S'_0\|_{L^\infty(\Omega\times\mathcal{O})}\vee\|\bar{f}^0\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})}\vee\|\bar{g}^0\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})}\vee\|\bar{h}^0\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})}$$

such that, for all real $l\geq 2$,

$$E\int_{\mathcal{O}}|u_t(x)-S'_t(x)|^l dx\leq cK^2 l(l-1)e^{cl(l-1)t} \tag{3.24}$$

$$E\int_0^t\int_{\mathcal{O}}|u_s(x)-S'_s(x)|^{l-2}|\nabla(u_s(x)-S'_s(x))|^2dxds\leq c'K^2l(l-1)e^{cl(l-1)t} \tag{3.25}$$

and

$$E\int_0^t\int_{\mathcal{O}}|u_s(x)-S'_s(x)|^{l-1}\nu(dxds)<+\infty. \tag{3.26}$$

Proof. Notice first that if $(u-S',\nu)=\mathcal{R}(\xi-S'_0,\bar{f},\bar{g},\bar{h},S-S')$, then

$$\bar{f}\left(u-S',\nabla(u-S')\right),\bar{g}_i\left(u-S',\nabla(u-S')\right),\bar{h}_i\left(u-S',\nabla(u-S')\right)\in L^2\left([0,T];L^2\left(\Omega\times\mathcal{O}\right)\right)$$

and consequently we can apply the Itô's formula to $(u-S',\nu)$ (See Theorem 5 in [25]).
We fix a real $l\geq 2$, $T>0$ and introduce the sequence $(\varphi_n)_{n\in\mathbb{N}^*}$ of functions such that for all $n\in\mathbb{N}^*$:

$$\forall x\in\mathbb{R},\ \varphi_n(x)=\begin{cases} |\,x\,|^l & \text{if } |\,x\,|\leq n\\ n^{l-2}\left[\frac{l(l-1)}{2}(|x|-n)^2+l\,n(|x|-n)+n^2\right] & \text{if } |\,x\,|>n\end{cases}$$

One can easily verify that for fixed n, φ_n is twice differentiable with bounded second derivative, $\varphi''_n(x)\geq 0$, and as $n\to\infty$ one has $\varphi_n(x)\longrightarrow|x|^l$, $\varphi'_n(x)\longrightarrow lsgn(x)|x|^{l-1}$, $\varphi''_n(x)\longrightarrow l(l-1)|x|^{l-2}$. Moreover, the following relations hold, for all $x\in\mathbb{R}$ and $n\geq l$:

1. $|\,x\varphi'_n(x)\,|\leq l\varphi_n(x)$.
2. $|\,\varphi'_n(x)\,|\leq|\,x\varphi''_n(x)\,|$.
3. $|\,x^2\varphi''_n(x)\,|\leq l(l-1)\varphi_n(x)$.
4. $|\varphi'_n(x)|\leq l(\varphi_n(x)+1)$.
5. $|\varphi''_n(x)|\leq l(l-1)(\varphi_n(x)+1)$.

Applying Itô's formula to $\varphi_n(u-S')$, we have P-a.s. for all $t\in[0,T]$,

$$\begin{aligned}
&\int_{\mathcal{O}}\varphi_n(u_t(x)-S'_t(x))\,dx \;+\; \int_0^t \mathcal{E}\big(\varphi'_n(u_s-S'_s),\,u_s-S'_s\big)\,ds = \int_{\mathcal{O}}\varphi_n(\xi(x)-S'_0(x))\,dx\\
&+\int_0^t\int_{\mathcal{O}}\varphi'_n(u_s(x)-S'_s(x))\bar f(s,x,u_s-S'_s,\nabla(u_s-S'_s))\,dxds\\
&-\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\varphi''_n(u_s(x)-S'_s(x))\partial_i(u_s(x)-S'_s(x))\,\bar g_i(s,x,u_s-S'_s,\nabla(u_s-S'_s))\,dx\,ds \quad (3.27)\\
&+\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}}\varphi'_n(u_s(x)-S'_s(x))\,\bar h_j(s,x,u_s-S'_s,\nabla(u_s-S'_s))\,dxdB^j_s\\
&+\frac12\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}}\varphi''_n(u_s(x)-S'_s(x))\bar h_j^2(s,x,u_s-S'_s,\nabla(u_s-S'_s))\,dx\,ds\\
&+\int_0^t\int_{\mathcal{O}}\varphi'_n(u_s(x)-S'_s(x))\nu(dxds)\,. \quad (3.28)
\end{aligned}$$

Since the support of ν is $\{u=S\}$, the last term is equal to

$$\int_0^t\int_{\mathcal{O}}\varphi'_n(S_s(x)-S'_s(x))\nu(dxds)$$

and it is negative, because

$$\int_0^t\int_{\mathcal{O}}\varphi'_n(S_s(x)-S'_s(x))I_{\{|S-S'|\le n\}}\nu(dxds)=l\int_0^t\int_{\mathcal{O}}sgn(S-S')\left|S_s(x)-S'_s(x)\right|^{l-1}\nu(dxds)\le 0$$

and

$$\begin{aligned}
\int_0^t\int_{\mathcal{O}}\varphi'_n(S_s(x) \;&-\; S'_s(x))I_{\{|S-S'|>n\}}\nu(dxds)\\
&=\int_0^t\int_{\mathcal{O}}n^{l-2}[l(l-1)(\left|S-S'\right|-n)sgn(S-S')+sgn(S-S')ln]\nu(dxds)<0
\end{aligned}$$

By the uniform ellipticity of the operator A we get

$$\mathcal{E}\big(\varphi'_n(u_s-S'_s),\,u_s-S'_s\big)\;\ge\;\lambda\int_{\mathcal{O}}\varphi''_n(u_s-S'_s)|\nabla(u_s-S'_s)|^2\,dx.$$

Let $\epsilon>0$ be fixed. Using the Lipschitz condition on $\bar f$ and the properties of the functions $(\varphi_n)_n$ we get

$$\begin{aligned}
&|\varphi'_n(u_s-S'_s)|\,|\bar f(s,x,u_s-S'_s,\nabla u_s-S'_s)|\\
&\le\;|\varphi'_n(u_s-S'_s)|\left(|\bar f^0(s,x)|+C\left(|u_s-S'_s|+|\nabla(u_s-S'_s)|\right)\right)\\
&\le\;|\varphi'_n(u_s-S'_s)||\bar f^0(s,x)|+|u_s-S'_s||\varphi''_n(u_s-S'_s)|\left(C|u_s-S'_s|+C|\nabla(u_s-S'_s)|\right)\\
&\le l(\varphi_n(u_s-S'_s)+1)\,|\bar f^0(s,x)|+C|u_s-S'_s|^2|\varphi''_n(u_s-S'_s)|+C|u_s-S'_s||\nabla(u_s-S'_s)||\varphi''_n(u_s-S'_s)|\\
&\le l(\varphi_n(u_s-S'_s)+1)\,|\bar f^0(s,x)|+(C+c_\epsilon)\,|u_s-S'_s|^2\varphi''_n(u_s-S'_s)+\,\epsilon\varphi''_n(u_s-S'_s)|\nabla u_s-S'_s|^2.
\end{aligned}$$

Now using Cauchy-Schwarz inequality and the Lipschitz condition on $\bar{g}$ we get

$$\sum_{i=1}^{d}\varphi_n''(u_s-S_s')\partial_i(u_s-S_s')\,\bar{g}_i(s,x,u_s-S_s',\nabla(u_s-S_s'))$$
$$\leq \varphi_n''(u_s-S_s')\,|\nabla(u_s-S_s')|\,\big(|\bar{g}^0(s,x)|+C|u_s-S_s'|+\alpha|\nabla(u_s-S_s')|\,\big)$$
$$\leq \epsilon\,\varphi_n''(u_s-S_s')|\nabla(u_s-S_s')|^2+2c_\epsilon\varphi_n''(u_s-S_s')\,\big(K^2+C^2|u_s-S_s'|^2\,\big)+\alpha\,\varphi_n''(u_s-S_s')|\nabla(u_s-S_s')|^2$$
$$\leq l(l-1)c_\epsilon K^2+2c_\epsilon(K^2+C^2)l(l-1)|\varphi_n(u_s-S_s')|+(\alpha+\epsilon)\,\varphi_n''(u_s-S_s')|\nabla(u_s-S_s')|^2.$$

In the same way as before

$$\sum_{j=1}^{\infty}\varphi_n''(u_s-S_s')\bar{h}_j^2(s,u_s-S_s',\nabla(u_s-S_s'))$$
$$\leq \varphi_n''(u_s-S_s')\,\big(\,c_\epsilon'(|\bar{h}^0(s,x)|+C|u_s-S_s'|)^2\,+\,(1+\epsilon)\beta^2\,|\nabla(u_s-S_s')|^2\,\big)$$
$$\leq \varphi_n''(u_s-S_s')\,\big(2c_\epsilon'K^2\,+\,2c_\epsilon'C^2|u_s-S_s'|^2+\,(1+\epsilon)\beta^2\,|\nabla(u_s-S_s')|^2\,\big)$$
$$\leq\,2c_\epsilon'l(l-1)K^2+2c_\epsilon'(K^2+C^2)l(l-1)\varphi_n(u_s-S_s')+(1+\epsilon)\,\beta^2\,\varphi_n''(u_s-S_s')|\nabla(u_s-S_s')|^2.$$

Thus taking the expectation, we deduce

$$E\int_{\mathcal{O}}\varphi_n(u_t(x)-S_t'(x))\,dx+(\lambda-\frac{1}{2}(1+\epsilon)\beta^2-(\alpha+2\epsilon)\,)\,E\int_0^t\int_{\mathcal{O}}\varphi_n''(u_s-S_s')\,|\nabla(u_s-S_s')|^2\,dx\,ds$$
$$\leq\,l(l-1)c_\epsilon''K^2\,+\,c_\epsilon''l(l-1)\big(K^2+C^2+C+c_\epsilon\big)E\int_0^t\int_{\mathcal{O}}\varphi_n(u_s(x)-S_s'(x))\,dx\,ds. \tag{3.29}$$

On account of the contraction condition, one can choose $\epsilon>0$ small enough such that

$$\lambda-\frac{1}{2}(1+\epsilon)\beta^2-(\alpha+2\epsilon)>0$$

and then

$$E\int_{\mathcal{O}}\varphi_n(u_t(x)-S_t'(x))\,dx\leq\,cK^2l(l-1)\,+\,cl(l-1)E\int_0^t\int_{\mathcal{O}}\varphi_n(u_s(x)-S_s'(x))\,dx\,ds\,.$$

We obtain by Gronwall's Lemma, that

$$E\int_{\mathcal{O}}\varphi_n(u_t(x)-S_t'(x))\,dx\leq\,c\,K^2l(l-1)\,\exp\big(c\,l(l-1)\,t\big) \tag{3.30}$$

and so it is now easy from (3.29) to get

$$E\int_0^t\int_{\mathcal{O}}\varphi_n''(u_s(x)-S_s'(x))\,|\nabla(u_s-S_s')|^2\,dx\,ds\,\leq\,c'\,K^2l\,(l-1)\,\exp\big(cl(l-1)\,t\big). \tag{3.31}$$

Finally, letting $n\to\infty$ by Fatou's lemma we deduce (3.24) and (3.25).
Then with (3.27), we know that

$$-\int_0^t\int_{\mathcal{O}}\varphi_n'(u_s-S_s')\nu(dxds)=-\int_0^t\int_{\mathcal{O}}\varphi_n'(S_s-S_s')\nu(dxds)\leq C.$$

This yields (3.26) by Fatou's lemma. ■

With the help of Lemma 3.29, we are able to prove the following Itô's formula:

Proposition 3.30. *Assume the hypotheses of the previous lemma. Let (u,ν) be the solution of the problem (3.1). Then for $l \geq 2$, we get the following Itô's formula, P-almost surely, for all $t \in [0,T]$*

$$
\begin{aligned}
&\int_{\mathcal{O}} \left|u_t(x) - S'_t(x)\right|^l dx + \int_0^t \mathcal{E}\left(l\,(u_s - S'_s)^{l-1} sgn(u_s - S'_s),\, u_s - S'_s\right) ds = \int_{\mathcal{O}} \left|\xi(x) - S'_0(x)\right|^l dx \\
&+ l \int_0^t \int_{\mathcal{O}} sgn(u_s - S'_s)\left|u_s(x) - S'_s(x)\right|^{l-1} \bar{f}(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dxds \\
&- l(l-1)\sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \left|u_s(x) - S'_s(x)\right|^{l-2} \partial_i(u_s(x) - S'_s(x))\, \bar{g}_i(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dx\, ds \\
&+ l \sum_{j=1}^\infty \int_0^t \int_{\mathcal{O}} sgn(u_s - S'_s)\, \left|u_s(x) - S'_s(x)\right|^{l-1} \bar{h}_j(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dxdB^j_s \\
&+ \frac{l(l-1)}{2}\sum_{j=1}^\infty \int_0^t \int_{\mathcal{O}} \left|u_s(x) - S'_s(x)\right|^{l-2} \bar{h}^2_j(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dx\, ds \\
&+ l \int_0^t \int_{\mathcal{O}} sgn(u_s - S'_s)\left|u_s(x) - S'_s(x)\right|^{l-1} \nu(dx\, ds)\,.
\end{aligned}
\tag{3.32}
$$

Proof. From the Itô's formula (see Theorem 5 in [25]), with the same notations as in the previous lemma, we have P-almost surely, and for all $t \in [0,T]$ and $n \in \mathbb{N}$

$$
\begin{aligned}
&\int_{\mathcal{O}} \varphi_n(u_t(x) - S'_t(x))\, dx \; + \; \int_0^t \mathcal{E}\big(\varphi'_n(u_s - S'_s),\, u_s - S'_s\big)\, ds = \int_{\mathcal{O}} \varphi_n(\xi(x) - S'_0(x))\, dx \\
&+ \int_0^t \int_{\mathcal{O}} \varphi'_n(u_s(x) - S'_s(x)) \bar{f}(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dxds \\
&- \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \varphi''_n(u_s(x) - S'_s(x))\partial_i(u_s(x) - S'_s(x))\, \bar{g}_i(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dx\, ds \\
&+ \sum_{j=1}^\infty \int_0^t \int_{\mathcal{O}} \varphi'_n(u_s(x) - S'_s(x))\, \bar{h}_j(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dxdB^j_s \\
&+ \frac{1}{2}\sum_{j=1}^\infty \int_0^t \int_{\mathcal{O}} \varphi''_n(u_s(x) - S'_s(x))\bar{h}^2_j(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dx\, ds \\
&+ \int_0^t \int_{\mathcal{O}} \varphi'_n(u_s(x) - S'_s(x))\nu(dxds)\,.
\end{aligned}
$$

Therefore, passing to the limit as $n \to \infty$, the convergences come from the Lemma 3.29 and the dominated convergence theorem. ∎

From now on, we assume the following stronger hypothesis:

$$\alpha + \frac{1}{2}\beta^2 + 72\beta^2 < \lambda. \tag{3.33}$$

At this stage, the idea is to adapt the Moser iteration technics to our setting. To this end, in order to control uniformly the L^l norms and make l tend to $+\infty$, we introduce for each

$l \geq 2$, the processes v and v' given by

$$v_t := \sup_{s\leq t}\left(\int_{\mathcal{O}}\left|u_s - S'_s\right|^l dx + \gamma l\,(l-1)\int_0^s\int_{\mathcal{O}}\left|u_r - S'_r\right|^{l-2}\left|\nabla(u_r - S'_r)\right|^2 dx\,dr\right),$$

$$v'_t := \int_{\mathcal{O}}\left|\xi - S'_0\right|^l dx + l^2 c_1 \left\|\left|u - S'\right|^l\right\|_{1,1;t} + l\,\|\bar{f}^0\|^*_{\theta,t}\left\|\left|u - S'\right|^{l-1}\right\|_{\theta;t}$$
$$+\, l^2\left(c_2\,\||\bar{g}^0|^2\|^*_{\theta;t} + c_3\,\||\bar{h}^0|^2\|^*_{\theta;t}\right)\left\|\left|u - S'\right|^{l-2}\right\|_{\theta;t},$$

where the constants are given by

$$\begin{aligned}\gamma &= \lambda - \alpha - \frac{\epsilon l}{l-1} - \frac{1+\epsilon}{2}\beta^2\\ c_1 &= \frac{C}{2}\left(1 + \frac{C}{4\epsilon}\right) + \frac{3+2\epsilon}{2\epsilon}C^2 + 3\frac{1+\epsilon}{\epsilon^2}C^2\\ c_2 &= \frac{1}{2\epsilon} \quad \text{and} \quad c_3 = \frac{(3+\epsilon)\,(1+\epsilon)}{\epsilon}\end{aligned} \tag{3.34}$$

The main difficulty in the stochastic case is to control the martingale part. We start by estimating the bracket of the local martingale in (3.32)

$$M_t := l\sum_{j=1}^{\infty}\int_0^t\int_{\mathcal{O}} sgn(u_s - S'_s)\,\left|u_s(x) - S'_s(x)\right|^{l-1}\bar{h}_j(s,x,u_s - S'_s,\nabla(u_s - S'_s))\,dxdB^j_s$$

Lemma 3.31. *For arbitrary $\varepsilon > 0$, one has*

$$\begin{aligned}\langle M\rangle_t^{\frac{1}{2}} \leq \varepsilon v_t &+ \frac{l^2}{2\varepsilon}\left(\frac{1+\varepsilon}{\varepsilon}\,\||\bar{h}^0|^2\|^*_{\theta;t}\left\|\left|u - S'\right|^{l-2}\right\|_{\theta;t} + \frac{1+\varepsilon}{\varepsilon}C^2\left\|\left|u - S'\right|^l\right\|_{1,1;t}\right)\\ &+ \sqrt{1+\varepsilon}\sqrt{\frac{l}{l-1}}\frac{\beta}{\sqrt{\gamma}}v_t\,.\end{aligned} \tag{3.35}$$

The proof is the same as Lemma 12 in [21] replacing u by $u - S'$ and also h by $\bar{h}$.

In what follows we will use the notion of domination, which is essential to handle the martingale part. We recall the definition from Revuz and Yor [73].

Definition 3.32. *A non-negative, adapted right continuous process X is dominated by an increasing process A, if*

$$E\left[X_\rho\right] \leq E\left[A_\rho\right]$$

for any bounded stopping time, ρ.

One important result related to this notion is the following domination inequality (see Proposition IV.4.7 in Revuz-Yor, p. 163), for any $k \in]0,1[$,

$$E\big[(X^*_\infty)^k\big] \leq C_k\,E\big[(A_\infty)^k\big] \tag{3.36}$$

where C_k is a positive constant and $X^*_t := \sup_{s\leq t}|X_s|$.

We will also use the fact that if A, A' are increasing processes, then the domination of a process X by A is equivalent to the domination of $X + A'$ by $A + A'$.

Lemma 3.33. *The Process τv is dominated by the process v' where*

$$\tau = 1 - 6\epsilon - 6\sqrt{1+\epsilon}\sqrt{\frac{l}{l-1}}\frac{\beta}{\sqrt{\gamma}}.$$

In other words, we have

$$\begin{aligned} &\tau\, E \sup_{0\le s\le t}\left(\int_{\mathcal{O}} \left|u_s - S'_s\right|^l\, dx + \gamma l\,(l-1)\int_0^s\int_{\mathcal{O}} \left|u_r - S'_r\right|^{l-2}\left|\nabla(u_r - S'_r)\right|^2 dxdr\right)\\ &\le E\int_{\mathcal{O}}\left|\xi - S'_0\right|^l dx \,+\, l^2 c_1 E\,\left\|\left|u-S'\right|^l\right\|_{1,1;t} + lE\,\|\bar{f}^0\|^*_{\theta,t}\left\|\left|u-S'\right|^{l-1}\right\|_{\theta;t} \\ &\quad + l^2 E\left(c_2\,\||\bar{g}^0|^2\|^*_{\theta;t} + c_3\,\||\bar{h}^0|^2\|^*_{\theta;t}\right)\left\|\left|u-S'\right|^{l-2}\right\|_{\theta;t}, \end{aligned} \tag{3.37}$$

where γ, c_1, c_2 and c_3 are the constants given above.

Proof. Starting from the relation (3.32):

$$\begin{aligned} &\int_{\mathcal{O}}\left|u_t(x) - S'_t(x)\right|^l dx + \int_0^t \mathcal{E}\left(l\,(u_s - S'_s)^{l-1} sgn(u_s - S'_s),\, u_s - S'_s\right) ds = \int_{\mathcal{O}}\left|\xi(x) - S'_0(x)\right|^l dx\\ &+ l\int_0^t\int_{\mathcal{O}} sgn(u_s - S'_s)\left|u_s(x) - S'_s(x)\right|^{l-1}\bar{f}(s,x,u_s - S'_s, \nabla(u_s - S'_s))\,dxds\\ &- l(l-1)\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\left|u_s(x) - S'_s(x)\right|^{l-2}\partial_i(u_s(x) - S'_s(x))\,\bar{g}_i(s,x,u_s - S'_s,\nabla(u_s - S'_s))\,dx\,ds\\ &+ l\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}} sgn(u_s - S'_s)\,\left|u_s(x) - S'_s(x)\right|^{l-1}\bar{h}_j(s,x,u_s - S'_s,\nabla(u_s - S'_s))\,dxdB^j_s\\ &+ \frac{l(l-1)}{2}\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}}\left|u_s(x) - S'_s(x)\right|^{l-2}\bar{h}_j^2(s,x,u_s - S'_s,\nabla(u_s - S'_s))\,dx\,ds\\ &+ l\int_0^t\int_{\mathcal{O}} sgn(u_s - S'_s)\left|u_s(x) - S'_s(x)\right|^{l-1}\nu(dx\,ds)\,. \end{aligned}$$

The last term is negative: from the condition of minimality, we have the following relation,

$$\begin{aligned} &\int_0^t\int_{\mathcal{O}} sgn(u_s - S'_s)\left|u_s(x) - S'_s(x)\right|^{l-1}\nu(dxds)\\ =\;&\int_0^t\int_{\mathcal{O}} sgn(S_s - S'_s)\left|S_s(x) - S'_s(x)\right|^{l-1}\nu(dxds) \le 0. \end{aligned}$$

Then we can do the same calculus as in the proof of Lemma 14 in [21], replacing u by $u - S'$ and f, g, h by $\bar{f}$, $\bar{g}$, $\bar{h}$ respectively. ∎

The proofs of the next 3 lemmas are similar to the proofs of Lemmas 15, 16 and 17 in [21], just replacing u by $u - S'$ and replacing f, g and h by $\bar{f}$, $\bar{g}$ and $\bar{h}$ respectively.

Lemma 3.34. *The process v satisfies the estimate*

$$v_t \ge \delta\left\|\left|u - S'\right|^l\right\|_{0;t}$$

with $\delta = 1\wedge\left(2c_S^{-1}\gamma\right)$, where c_S is the constant in the Sobolev inequality (3.3).

Lemma 3.35. *The process*

$$w_t := \left[\left\| \left| u - S' \right|^{\sigma l} \right\|_{\theta;t}^{\frac{1}{\sigma}} \vee \left\| \xi - S'_0 \right\|_\infty^l \vee \left\| \bar{f}^0 \right\|_{\theta;t}^l \vee \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right]$$

is dominated by the process

$$w'_t := 6k\left(t\right) l^2 \left[\left\| \left| u - S' \right|^{l} \right\|_{\theta;t} \vee \left\| \xi - S'_0 \right\|_\infty^l \vee \left\| \bar{f}^0 \right\|_{\theta;t}^l \vee \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right],$$

where $\sigma = \frac{d+2\theta}{d}$ and $k : \mathbb{R}_+ \to \mathbb{R}_+$ is a function independent of l, depending only on the structure constants.

Lemma 3.36. *There exists a function $k_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which involves only the structure constants of our problem and such that the following estimate holds*

$$Ev_t \leq k_1\left(l,t\right) E\left(\int_{\mathcal{O}} \left| \xi - S'_0 \right|^l dx + \left\| \bar{f}^0 \right\|_{\theta;t}^{*l} + \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} + \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right).$$

We now prove Theorem 3.27 in the case where ξ, $\bar{f}^0$, $\bar{g}^0$ and $\bar{h}^0$ are uniformly bounded:

We set $l = p\sigma^n$, with some $n \in \mathbb{N}$. By Lemma 3.35 and the domination inequality (3.36) we deduce, for $n \geq 1$,

$$\begin{aligned} &E\left(\left\| \left| u - S' \right|^{\sigma l} \right\|_{\theta;t}^{\frac{1}{\sigma}} \vee \left\| \xi - S'_0 \right\|_\infty^l \vee \left\| \bar{f}^0 \right\|_{\theta;t}^{*l} \vee \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right)^{\frac{1}{\sigma^n}} \\ &\leq C_{\sigma^{-n}} \left(6k\left(t\right) l^2\right)^{\frac{1}{\sigma^n}} E\left(\left\| \left| u - S' \right|^{l} \right\|_{\theta;t} \vee \left\| \xi - S'_0 \right\|_\infty^l \vee \left\| \bar{f}^0 \right\|_{\theta;t}^{*l} \vee \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right)^{\frac{1}{\sigma^n}}, \end{aligned}$$

where $C_{\sigma^{-n}}$ is the constant in the domination inequality. This constant is estimated by

$$C_{\sigma^{-n}} \leq \sigma^{\frac{n}{\sigma^n}} \left(1 - \frac{1}{\sigma^n}\right)^{-1}.$$

(See the exercise IV.4.30 in Revuz -Yor, p. 171). So let us denote by

$$a_n := \left\| \left| u - S' \right|^{p\sigma^n} \right\|_{\theta;t}^{\frac{1}{\sigma^n}} \vee \left\| \xi - S'_0 \right\|_\infty^p \vee \left\| \bar{f}^0 \right\|_{\theta;t}^{*p} \vee \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}} \vee \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}}$$

and deduce from the above inequality the following one

$$Ea_{n+1} \leq \sigma^{\frac{n}{\sigma^n}} \left(1 - \frac{1}{\sigma^n}\right)^{-1} \left(6k\left(t\right) \left(p\sigma^n\right)^2\right)^{\frac{1}{\sigma^n}} Ea_n.$$

Iterating this relation n times we get

$$Ea_{n+1} \leq \sigma^{3\sum_{m=1}^n \frac{m}{\sigma^m}} \prod_{m=1}^n \left(1 - \frac{1}{\sigma^m}\right)^{-1} \left(6k\left(t\right) p^2\right)^{\sum_{m=1}^n \frac{1}{\sigma^m}} Ea_1.$$

Now we shall let n tend to infinity in this relation. Since in general one has

$$\lim_{q,q'\to\infty} \|F\|_{q,q';t} = \|F\|_{\infty,\infty;t} \,,$$

for any function $F : \mathbb{R}_+ \times \mathcal{O} \to \mathbb{R}$, it is easy to see that $\lim_{n\to\infty} \left\| |u - S'|^{p\sigma^n} \right\|_{\theta;t}^{\frac{1}{\sigma^n}} = \|u - S'\|^p_{\infty,\infty;t}\,.$
Therefore we have

$$\lim_{n\to\infty} a_n = \left\|u - S'\right\|^p_{\infty,\infty;t} \vee \left\|\xi - S'_0\right\|^p_\infty \vee \left\|\bar f^0\right\|^{*p}_{\theta;t} \vee \left\||\bar g^0|^2\right\|^{*\frac{p}{2}}_{\theta;t} \vee \left\||\bar h^0|^2\right\|^{*\frac{p}{2}}_{\theta;t} \,,$$

which implies

$$E\left\|u - S'\right\|^p_{\infty,\infty;t} \le \rho\,(t)\, Ea_1,$$

with

$$\rho\,(t) = \sigma^{3\sum_{m=1}^\infty \frac{m}{\sigma^m}} \prod_{m=1}^{\infty} \left(1 - \frac{1}{\sigma^m}\right)^{-1} \left(5k\,(t)\,p^2\right)^{\sum_{m=1}^\infty \frac{1}{\sigma^m}} .$$

Now we estimate Ea_1 by using the fact that $\delta \left\||u - S'|^{p\sigma}\right\|^{\frac{1}{\sigma}}_{\theta;t} \le v_t$, with p replacing l in the expression of v. So we have

$$\begin{aligned} Ea_1 &= E\left(\left\||u - S'|^{p\sigma}\right\|^{\frac{1}{\sigma}}_{\theta;t} \vee \left\|\xi - S'_0\right\|^p_\infty \vee \left\|\bar f^0\right\|^{*p}_{\theta;t} \vee \left\||\bar g^0|^2\right\|^{*\frac{p}{2}}_{\theta;t} \vee \left\||\bar h^0|^2\right\|^{*\frac{p}{2}}_{\theta;t}\right) \\ &\le E\left(\delta^{-1} v_t + \left\|\xi - S'_0\right\|^p_\infty + \left\|\bar f^0\right\|^{*p}_{\theta;t} + \left\||\bar g^0|^2\right\|^{*\frac{p}{2}}_{\theta;t} \vee \left\||\bar h^0|^2\right\|^{*\frac{p}{2}}_{\theta;t}\right) . \end{aligned}$$

Finally one deduces the following estimate by applying Lemma 3.36 with $l = p$:

$$E\left\|u - S'\right\|^p_{\infty,\infty;t} \le k_2\,(t)\, E\left(\left\|\xi - S'_0\right\|^p_\infty + \left\|\bar f^0\right\|^{*p}_{\theta,t} + \left\||\bar g^0|^2\right\|^{*p/2}_{\theta;t} + \left\||\bar h^0|^2\right\|^{*p/2}_{\theta;t}\right) . \tag{3.38}$$

Moreover (see Theorem 11 [21]), we have

$$E\left\|S'\right\|^p_{\infty,\infty;t} \le k(t) E\left(\left\|S'_0\right\|^p_\infty + \left\|f'\right\|^{*p}_{\theta;t} + \left\||g'|^2\right\|^{*p/2}_{\theta;t} + \left\||h'|^2\right\|^{*p/2}_{\theta;t}\right)$$

Hence,

$$\begin{aligned} E\left\|u\right\|^p_{\infty,\infty;t} &\le c(p)(E\left\|u - S'\right\|^p_{\infty,\infty;t} + E\left\|S'\right\|^p_{\infty,\infty;t}) \\ &\le c(p)k(t)E\big(\left\|\xi\right\|^p_\infty + \left\|S'_0\right\|^p_\infty + \left\|f'\right\|^{*p}_{\theta;t} + \left\||g'|^2\right\|^{*p/2}_{\theta;t} + \left\||h'|^2\right\|^{*p/2}_{\theta;t} \\ &\quad + \left\|\bar f^0\right\|^{*p}_{\theta;t} + \left\||\bar g^0|^2\right\|^{*p/2}_{\theta;t} + \left\||\bar h^0|^2\right\|^{*p/2}_{\theta;t}\big). \end{aligned}$$

This ends the proof of Theorem 3.27 in this particular case where ξ, $\bar f^0$, $\bar g^0$ and $\bar h^0$ are uniformly bounded. We now turn out to the general case.

3.4.2 Proof of Theorem 3.27 in the general case

We now assume that **(H)**, **(O)**, **(HI2p)**, **(HO∞p)** and **(HDθp)** hold. We are going to prove Theorem 3.27 by using an approximation argument. For this, for all $n \in \mathbb{N}^*$,

$1 \le i \le d, 1 \le j \le \infty$ and all (t,w,x,y,z) in $\mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d$, we set

$$\begin{aligned} \bar{f}_n(t,w,x,y,z) &= \bar{f}(t,w,x,y,z) - \bar{f}^0(t,w,x) + \bar{f}^0(t,w,x)\cdot \mathbf{1}_{\{|\bar{f}^0(t,w,x)|\le n\}} \\ \bar{g}_{i,n}(t,w,x,y,z) &= \bar{g}_i(t,w,x,y,z) - \bar{g}_i^0(t,w,x) + \bar{g}_i^0(t,w,x)\cdot \mathbf{1}_{\{|\bar{g}_i^0(t,w,x)|\le n\}} \\ \bar{h}_{j,n}(t,w,x,y,z) &= \bar{h}_j(t,w,x,y,z) - \bar{h}_j^0(t,w,x) + \bar{h}_j^0(t,w,x)\cdot \mathbf{1}_{\{|\bar{h}_j^0(t,w,x)|\le n\}} \\ \xi_n(w,x) &= \xi(w,x)\cdot \mathbf{1}_{\{|\xi(\omega,x)|\le n\}} \end{aligned} \tag{3.39}$$

One can check that for all n, $\bar{f}_n$, $\bar{g}_n$, $\bar{h}_n$ and $\xi^n - S'_0$ satisfy all the assumptions of the Step 1 of the proof, and that Lipschitz constants do not depend on n. And the obstacle $S - S'$ is controlled by 0, which obviously satisfies **(HO2)**. For each $n \in \mathbb{N}^*$, we put $(\bar{u}^n, \nu^n) = \mathcal{R}(\xi^n - S'_0, \bar{f}^n, \bar{g}^n, \bar{h}^n, S - S')$ and we know that $\bar{u}^n$ satisfies the estimate of Step 1. We are now going to prove that $(\bar{u}^n,\, \nu^n)$ converges to $(\bar{u}, \nu) = \mathcal{R}(\xi - S'_0, \bar{f}, \bar{g}, \bar{h}, S - S')$. Let us fix $n \le m$ in $\mathbb{N}^*$ and put $\bar{u}^{n,m} := \bar{u}^n - \bar{u}^m$ and $\nu^{n,m} := \nu^n - \nu^m$ We first note that $\bar{u}^{n,m}$ satisfies the equation

$$\begin{aligned} d\bar{u}_t^{n,m}(x) + A\bar{u}_t^{n,m}(x)\,dt = {}& \bar{f}_{n,m}\left(t,x,\bar{u}_t^{n,m}(x), \nabla \bar{u}_t^{n,m}(x)\right)dt \\ & - \sum_{i=1}^d \partial_i \bar{g}_{i,n,m}\left(t,x,\bar{u}_t^{n,m}(x), \nabla \bar{u}_t^{n,m}(x)\right)dt \\ & + \sum_{j=1}^\infty \bar{h}_{j,n,m}\left(t,x,\bar{u}_t^{n,m}(x), \nabla \bar{u}_t^{n,m}(x)\right)dB_t^j + \nu^{n,m}(x,\, dt) \end{aligned}$$

where

$$\begin{aligned} \bar{f}_{n,m}(t,w,x,y,z) = {}& \bar{f}\left(t,w,x,y+\bar{u}_t^m(x), z + \nabla \bar{u}_t^m(x)\right) - \bar{f}\left(t,w,x,\bar{u}_t^m(x), \nabla \bar{u}_t^m(x)\right) \\ & + \bar{f}_n^0(t,w,x) - \bar{f}_m^0(t,w,x) \end{aligned}$$

and $\bar{g}_{i,n,m}, \bar{h}_{j,n,m}$ have similar expressions. Clearly one has

$$\bar{f}_{n,m}(t,w,x,0,0) = \bar{f}_n^0(t,w,x) - \bar{f}_m^0(t,w,x) := \bar{f}_{n,m}^0(t,w,x)$$

and some similar relations for $\bar{g}_{i,n,m}(t,w,x,0,0)$ and $\bar{h}_{j,n,m}(t,w,x,0,0)$. On the other hand, one easily verifies that

$$\begin{aligned} &E\left\|\xi_n - \xi\right\|_\infty^p \longrightarrow 0, && E\left\|\bar{f}_n^0 - \bar{f}^0\right\|_{\theta;T}^{*p} \longrightarrow 0, \\ &E\left\|\bar{g}_n^0 - \bar{g}^0\right\|_{\theta;T}^{*p} \longrightarrow 0, && E\left\|\bar{h}_n^0 - \bar{h}^0\right\|_{\theta;T}^{*p} \longrightarrow 0. \end{aligned}$$

By Lemma 3.46 with $l = 2$ (see Appendix) we deduce that

$$E\left\|\bar{u}^n - \bar{u}^m\right\|_T^2 \longrightarrow 0, \quad \textit{as } n,\, m \to \infty. \tag{3.40}$$

Therefore, $\bar{u}^n$ has a limit $\bar{u}$ in $\mathcal{H}_T$.

We now study the convergence of (ν^n). Denote by v^n the parabolic potential associated to ν^n, and $z^n = \bar{u}^n - v^n$, so z^n satisfies the following SPDE

$$\begin{aligned} dz_t^n(x) + Az_t^n(x)dt = {}& \bar{f}_n(t,x,\bar{u}_t^n(x), \nabla \bar{u}_t^n(x))dt - \sum_{i=1}^d \partial_i \bar{g}_{i,n}(t,x,\bar{u}_t^n(x), \nabla \bar{u}_t^n(x))dt \\ & + \sum_{j=1}^\infty \bar{h}_{j,n}(t,x,\bar{u}_t^n(x), \nabla \bar{u}_t^n(x))\, dB_t^j. \end{aligned}$$

We define $z^{1,n}$ to be the solution of the following SPDE with initial value $\xi^n - S'_0$ and zero boundary condition:

$$dz_t^{1,n}(x) + Az_t^{1,n}(x)dt = (\bar{f}_t(x, \bar{u}_t^n(x), \nabla\bar{u}_t^n(x)) - \bar{f}_t^0(x))dt - \sum_{i=1}^{d} \partial_i(\bar{g}_{i,t}(x, \bar{u}_t^n(x), \nabla\bar{u}_t^n(x)) - \bar{g}_t^0(x))dt + \sum_{j=1}^{\infty}(\bar{h}_{j,t}(x, \bar{u}_t^n(x), \nabla\bar{u}_t^n(x)) - \bar{h}_t^0(x))\, dB_t^j.$$

This is a linear SPDE in $z^{1,n}$, its solution uniquely exists and belongs to $\mathcal{H}_T$. Applying Itô's formula to $(z^{1,n})^2$ and doing a classical calculation, we get:

$$E\left\|z^{1,n} - z^{1,m}\right\|_T^2 \leq CE\big(\, \|\xi^n - \xi^m\|_2^2 + \|\bar{u}^n - \bar{u}^m\|_T^2\,\big) \to 0, \quad as\ n,\ m \to \infty.$$

Then, we define $z^{2,n}$ to be the solution of the following SPDE with initial value 0 and zero boundary condition:

$$dz_t^{2,n}(x) + Az_t^{2,n}(x)dt = \bar{f}_n^0(t,x)dt - \sum_{i=1}^{d} \partial_i \bar{g}_{i,n}^0(t,x))dt + \sum_{j=1}^{\infty} \bar{h}_{j,n}^0(x)\, dB_t^j.$$

This is still a linear SPDE in $z^{2,n}$, its solution uniquely exists and from the proof of Theorem 11 in [21], we know that

$$E\left\|z^{2,n} - z^{2,m}\right\|_T^2 \leq CE\left(\|\bar{f}_{n,m}^0\|_{\theta;T}^{*2} + \left\||\bar{g}_{n,m}^0|^2\right\|_{\theta;T}^{*} + \left\||\bar{h}_{n,m}^0|^2\right\|_{\theta;T}^{*}\right) \to 0, \quad as\ n,\ m \to \infty.$$

This yields:

$$E\left\|z^n - z^m\right\|_T^2 \longrightarrow 0, \quad as \quad n, m \to \infty.$$

Hence, using (3.40) and the fact that $\bar{u}^n = z^n + v^n$, we get:

$$E\left\|v^n - v^m\right\|_T^2 \longrightarrow 0, \quad as\ n, m \to \infty.$$

Therefore, v^n has a limit v in $\mathcal{H}_T$. So, by extracting a subsequence, we can assume that v^n converges to v in $\mathcal{K}$ almost-surely. Then, it's clear that $v \in \mathcal{P}$, and we denote by ν the regular random measure associated to the potential v. Moreover, we have P-a.s., $\forall \varphi \in \mathcal{W}_t^+$:

$$\begin{aligned}
\int_0^t \int_{\mathcal{O}} \varphi(x,s)\nu(dxds) &= \lim_{n\to\infty} \int_0^t \int_{\mathcal{O}} \varphi(x,s)\nu^n(dxds) \\
&= \lim_{n\to\infty} \int_0^t -(v_s^n, \frac{\partial\varphi_s}{\partial s})ds + \int_0^t \mathcal{E}(v_s^n, \varphi_s)ds \\
&= \int_0^t -(v_s, \frac{\partial\varphi_s}{\partial s})ds + \int_0^t \mathcal{E}(v_s, \varphi_s)ds.
\end{aligned}$$

As a consequence of Lemma 3.45 in the Appendix, we know that

$$E\left\|\bar{u}^n - \bar{u}^m\right\|_{\infty,\infty;T}^p \longrightarrow 0.$$

Therefore, we can apply Proposition 3.30 to $\bar{u}^n$ and pass to the limit and so we obtain that this proposition remains valid in this case. Then, one can end the proof by repeating the first part of Step 1 starting from Proposition 3.30.

We conclude thanks to the uniqueness of the solution of the obstacle problem ensuring that $\bar{u}$ is exactly equals to $u - S'$.

□

3.5 Maximum Principle for the local solution

We now introduce the lateral conditions on the boundary that we consider:

Definition 3.37. *If u belongs to $\mathcal{H}_{loc}$, we say that u is non-negative on the boundary of $\mathcal{O}$ if u^+ belongs to $\mathcal{H}_T$ and we denote it simply: $u \le 0$ on $\partial\mathcal{O}$. More generally, if M is a random field defined on $[0,T] \times \mathcal{O}$, we note $u \le M$ on $\partial\mathcal{O}$ if $u - M \le 0$ on $\partial\mathcal{O}$.*

3.5.1 Itô's formula for the positive part of a local solution

In this section we are in the general framework with **(H)**, **(HIL)**, **(OL)** and **(HOL)** are assumed to be fulfilled. The following proposition represents a key technical result which leads to a generalization of the estimates of the positive part of a local solution. Let $(u,\nu) \in \mathcal{R}_{loc}(\xi, f, g, h, S)$, denote by u^+ its positive part.

Proposition 3.38. *Assume that $\partial\mathcal{O}$ is Lipschitz and that u^+ belongs to $\mathcal{H}_T$, i.e. u is non-positive on the boundary of $\mathcal{O}$.*
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$, which admits a bounded second order derivative and such that $\varphi'(0) = 0$. Then the following relation holds, a.s., for each $t \in [0,T]$,

$$\int_{\mathcal{O}} \varphi(u_t^+(x))dx + \int_0^t \mathcal{E}(\varphi'(u_s^+), u_s^+)ds = \int_{\mathcal{O}} \varphi(\xi^+(x))dx + \int_0^t \int_{\mathcal{O}} \varphi'(u_s^+(x)) f_s(x)dxds$$
$$- \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \varphi"(u_s^+(x))\partial_i u_s^+(x) g_s^i(x)dxds + \frac{1}{2}\int_0^t \int_{\mathcal{O}} \varphi"(u_s^+(x)) I_{\{u_s>0\}} |h_s(x)|^2 dxds$$
$$+ \sum_{i=1}^{\infty} \int_0^t \int_{\mathcal{O}} \varphi'(u_s^+(x)) h_s^j(x)dxdB_s^j + \int_0^t \int_{\mathcal{O}} \varphi'(u_s^+(x))\nu(dxds). \tag{3.41}$$

Proof. We consider $\phi \in \mathcal{C}_c^\infty(\mathcal{O})$, $0 \le \phi \le 1$, and put

$$\forall t \in [0,T], \quad w_t = \phi u_t.$$

A direct calculation yields the following relation:

$$dw_t = Lw_t dt + \bar{f}_t dt + \sum_{i=1}^d \partial_i \widetilde{g_{i,t}} dt + \sum_{j=1}^\infty \widetilde{h_{j,t}} dB_t^j + \phi\nu(x,dt)$$

where

$$\bar{f}_t = \phi f_t - \sum a_{i,j}(\partial_i\phi)(\partial_j u_t) - \sum (\partial_i \phi) g_{i,t}\,,$$
$$\widetilde{g_{i,t}} = \phi g_{i,t} - u_t \sum a_{i,j}\partial_j\phi\,, \quad \widetilde{h_{j,t}} = \phi h_{j,t}\,.$$

Now we prove that $\phi\nu$ is a regular measure:
We know that:

$$\forall \varphi \in \mathcal{W}_T^+, \quad \int (-\frac{\partial \varphi_s}{\partial s}, v_s)ds + \int \mathcal{E}(\varphi_s, v_s)ds = \int\int \varphi(s,x)d\nu. \tag{3.42}$$

We replace φ by $\phi\varphi$ in (3.42), where ϕ is the same as before, and we obtain the following relation:

$$\int(-\frac{\partial\phi\varphi_s}{\partial s}, v_s)ds + \int \mathcal{E}(\phi\varphi_s, v_s)ds = \int\int \phi\varphi(s,x)d\nu$$

note that ϕ does not depend on t and by a similar calculation as before, we get

$$\int(-\frac{\partial\varphi_s}{\partial s}, \phi v_s)ds + \int \mathcal{E}(\varphi_s, \phi v_s)ds + \int(K_s, \varphi_s)ds - \int(k_s, \nabla\varphi_s)ds = \int\int \varphi(s,x)d\phi\nu$$

where

$$K_t = \sum a_{i,j}(\partial_i\phi)(\partial_j v_t), \quad k_t = v_t \sum a_{i,j}\partial_j\phi.$$

We denote by $\bar{z}$ the solution of the following PDE with Dirichlet boundary condition and the initial value 0:

$$d\bar{z}_t + A\bar{z}_t dt = K_t dt + div k_t dt.$$

If we set $\bar{v} = \phi v + \bar{z}$, then $\bar{v}$ satisfies the following relation:

$$\int_0^t(-\frac{\partial\varphi_s}{\partial s}, \bar{v}_s)ds + \int_0^t \mathcal{E}(\varphi_s, \bar{v}_s)ds = \int_0^t\int_{\mathcal{O}} \varphi(x,s)d\phi\nu.$$

It is easy to verify that $\bar{v} \in \mathcal{P}$. Thus $\phi\nu$ is a regular measure associated to $\bar{v}$.
Hence, we deduce that $(\phi u, \phi\nu)$ satisfies an OSPDE with $\phi\xi$ as initial data and zero Dirichlet boundary conditions.
Now, we approximate the function $\psi : \ y \in \mathbb{R} \to \varphi(y^+)$ by a sequence (ψ_n) of regular functions. Let ζ be a $\mathcal{C}^\infty$ increasing function such that

$$\forall y \in]-\infty, 1], \ \zeta(y) = 0 \ and \ \forall y \in [2, +\infty[, \ \zeta(y) = 1.$$

We set for all n:

$$\forall y \in \mathbb{R}, \quad \psi_n(y) = \varphi(y)\zeta(ny).$$

It is easy to verify that (ψ_n) converges uniformly to the function ψ, (ψ_n') converges everywhere to the function $(y \to \varphi'(y^+))$ and (ψ_n'') converges everywhere to the function $(y \to I_{\{y>0\}}\varphi''(y^+))$. Moreover we have the estimates:

$$\forall y \in \mathbb{R}^+, \ n \in \mathbb{N}, \quad 0 \le \psi_n(y) \le \psi(y), \quad 0 \le \psi_n'(y) \le Cy, \quad |\psi_n''(y)| \le C,$$

where C is a constant. Thanks to the Itô's formula for the solution of OSPDE (3.1) (see Theorem 5 [25]), we have almost surely, for $t \in [0,T]$,

$$\int_{\mathcal{O}} \psi_n(w_t(x))dx + \int_0^t \mathcal{E}(\psi_n'(w_s), w_s)ds = \int_{\mathcal{O}} \psi_n(\phi(x)\xi(x))dx + \int_0^t\int_{\mathcal{O}} \psi_n'(w_s(x))\bar{f}_s(x)dxds$$
$$-\sum\int_0^t\int_{\mathcal{O}} \psi_n''(w_s(x))\partial_i w_s(x)\widetilde{g_{i,s}}(x)dxds + \sum\int_0^t\int_{\mathcal{O}} \psi_n'(w_s(x))\widetilde{h_{j,s}}(x)dxdB_s^j$$
$$+\frac{1}{2}\int_0^t\int_{\mathcal{O}} \psi_n''(w_s(x))|\widetilde{h_{j,s}}(x)|^2 dxds + \int_0^t\int_{\mathcal{O}} \psi_n'(w_s(x))d\phi\nu(x,s).$$

Making n tends to $+\infty$ and using the fact that $I_{w_s>0}\partial_i w_s = \partial_i w_s^+$, we get by the dominated convergence theorem:

$$\int_{\mathcal{O}} \varphi(w_t^+(x))dx + \int_0^t \mathcal{E}(\varphi'(w_s^+), w_s^+)ds = \int_{\mathcal{O}} \varphi(\phi(x)\xi^+(x))dx + \int_0^t\int_{\mathcal{O}} \varphi'(w_s^+(x))\bar{f}_s(x)dxds$$
$$-\sum\int_0^t\int_{\mathcal{O}} \varphi''(w_s^+(x))\partial_i w_s^+(x)\widetilde{g_{i,s}}(x)dxds + \sum\int_0^t\int_{\mathcal{O}} \varphi'(w_s^+(x))\widetilde{h_{j,s}}(x)dxdB_s^j$$
$$+\frac{1}{2}\int_0^t\int_{\mathcal{O}} \varphi''(w_s^+(x))I_{\{w_s>0\}}|\widetilde{h_{j,s}}(x)|^2 dxds + \int_0^t\int_{\mathcal{O}} \phi\varphi'(w_s^+(x))d\nu(x,s), \quad a.s.$$

Then we consider a sequence (ϕ_n) in $\mathcal{C}_c^\infty(\mathcal{O})$, $0 \leq \phi_n \leq 1$, converging to 1 everywhere on $\mathcal{O}$ and such that for any $y \in H_0^1(\mathcal{O})$ the sequence $(\phi_n y)$ tends to y in $H_0^1(\mathcal{O})$ and

$$\sup_n \|\phi_n y\|_{H_0^1(\mathcal{O})} \leq C \|y\|_{H_0^1(\mathcal{O})},$$

where C is a constant which does not depend on y. Such a sequence (ϕ_n) exists because $\partial\mathcal{O}$ is assumed to be Lipschitz (see Lemma 19 in [24]).
One has to remark that if $i \in \{1, ...d\}$ and $y \in H_0^1(\mathcal{O})$, then $(y\partial_i\phi_n)$ tends to 0 in $L^2(\mathcal{O})$.
Now, we set $w_n = \phi_n u$ and

$$\widetilde{f_t^n} = \phi_n f_t - \sum a_{i,j}(\partial_i\phi_n)(\partial_j u_t) - \sum(\partial_i\phi_n)g_{i,t}$$

$$\widetilde{g_{i,t}^n} = \phi_n g_{i,t} - u_t \sum a_{i,j}\partial_j\phi_n, \quad \widetilde{h_{j,t}^n} = \phi_n h_{j,t}$$

Applying the Itô's formula above to $\varphi(w_n^+)$, we get

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi(w_{n,t}^+(x))dx + \int_0^t \mathcal{E}(\varphi'(w_{n,s}^+), w_{n,s}^+)ds = \int_{\mathcal{O}} \varphi(\phi_n(x)\xi^+(x))dx + \int_0^t \int_{\mathcal{O}} \varphi'(w_{n,s}^+(x))\bar{f}_s(x)dxds \\
&-\sum \int_0^t \int_{\mathcal{O}} \varphi''(w_{n,s}^+(x))\partial_i w_{n,s}^+(x)\widetilde{g_{i,s}}(x)dxds + \sum \int_0^t \int_{\mathcal{O}} \varphi'(w_{n,s}^+(x))\widetilde{h_{j,s}}(x)dxdB_s^j \\
&+\frac{1}{2}\int_0^t \int_{\mathcal{O}} \varphi''(w_{n,s}^+(x))I_{\{w_{n,s}>0\}}|\widetilde{h_{j,s}}(x)|^2dxds + \int_0^t \int_{\mathcal{O}} \phi_n\varphi'(w_{n,s}^+(x))d\nu(x,s), \quad a.s. \qquad (3.43)
\end{aligned}$$

We have

$$\begin{aligned}
\varphi'(w_{n,s}^+)\bar{f}_s^n \quad &- \quad \sum \varphi''(w_{s,n}^+)\partial_i w_{n,s}^+ \widetilde{g_{i,s}^n} = \varphi'(w_{n,s}^+)\phi_n f_s - \sum a_{i,j}\varphi'(w_{n,s}^+)\partial_j\phi_n\partial_i u_s^+ \\
&+ \quad \sum a_{i,j}\varphi''(w_{n,s}^+)u_s^+\partial_i w_{n,s}^+\partial_j\phi_n - \sum(\varphi'(w_{n,s}^+))g_{i,s}\partial_i\phi_n + \varphi''(w_{n,s}^+)\phi_n g_{i,s}\partial_i w_{n,s}^+.
\end{aligned}$$

Remarking that for all $s \in (0,T]$, $(\phi_n\varphi'(w_{n,s}^+))$ (resp. $(\partial_i\phi_n\varphi'(w_{n,s}^+))$) tends to $\varphi'(u_s^+)$ (resp. 0) in $H_0^1(\mathcal{O})$ (resp. $L^2(\mathcal{O})$) we get by the dominated convergence theorem the convergence of all the terms in equality (3.43) excepted the one involving the measure ν. For this last term, we know that w_n is quasi-continuous and we can use the same argument as in the proof of Proposition 3.24 since we have $\int_0^t \int_{\mathcal{O}} \varphi'(u_s^+)\nu(dxds) < +\infty$. ■

3.5.2 The comparison theorem for the local solutions

Firstly, we prove the Itô's formula for the difference of two local solutions $(u^1, \nu^1) \in \mathcal{R}_{loc}(\xi^1, f^1, g, h, S^1)$ and $(u^2, \nu^2) \in \mathcal{R}_{loc}(\xi^2, f^2, g, h, S^2)$, where (ξ^i, f^i, g, h, S^i) satisfy assumptions **(H)**, **(HIL)**, **(OL)** and **(HOL)**. We denote by $\widehat{u} = u^1 - u^2$, $\widehat{\nu} = \nu^1 - \nu^2$, $\widehat{\xi} = \xi^1 - \xi^2$, and

$$\widehat{f}(t,\omega,x,y,z) = f^1\left(t,\omega,x,y+u_t^2(x), z+\nabla u_t^2(x)\right) - f^2\left(t,\omega,x,u_t^2(x),\nabla u_t^2(x)\right),$$

$$\widehat{g}(t,\omega,x,y,z) = g\left(t,\omega,x,y+u_t^2(x), z+\nabla u_t^2(x)\right) - g\left(t,\omega,x,u_t^2(x),\nabla u_t^2(x)\right),$$

$$\widehat{h}(t,\omega,x,y,z) = h\left(t,\omega,x,y+u_t^2(x), z+\nabla u_t^2(x)\right) - h\left(t,\omega,x,u_t^2(x),\nabla u_t^2(x)\right).$$

Proposition 3.39. *Assume that $\partial\mathcal{O}$ is Lipschitz and that $\widehat{u}^+$ belongs to $\mathcal{H}_T$. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a function of class $\mathcal{C}^2$, which admits a bounded second order derivative and such that $\varphi'(0)=0$. Then the following relation holds for each $t\in[0,T]$,*

$$\int_{\mathcal{O}}\varphi(\widehat{u}_t^+(x))dx+\int_0^t\mathcal{E}(\varphi'(\widehat{u}_s^+),\widehat{u}_s^+)ds=\int_{\mathcal{O}}\varphi(\widehat{\xi}^+(x))dx+\int_0^t\int_{\mathcal{O}}\varphi'(\widehat{u}_s^+(x))\widehat{f}_s(x)dxds$$
$$-\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\varphi\text{''}(\widehat{u}_s^+(x))\partial_i\widehat{u}_s^+(x)\widehat{g}_s^i(x)dxds+\frac{1}{2}\int_0^t\int_{\mathcal{O}}\varphi\text{''}(\widehat{u}_s^+(x))I_{\{\widehat{u}_s>0\}}|\widehat{h}_s(x)|^2dxds$$
$$+\sum_{i=1}^{\infty}\int_0^t\int_{\mathcal{O}}\varphi'(\widehat{u}_s^+(x))\widehat{h}_s^j(x)dxdB_s^j+\int_0^t\int_{\mathcal{O}}\varphi'(\widehat{u}_s^+(x))\widehat{\nu}(dxds)\qquad a.s. \tag{3.44}$$

Proof. We consider $\phi\in\mathcal{C}_c^\infty(\mathcal{O}),\ 0\le\phi\le1$, and put

$$\forall t\in[0,T],\quad \widehat{w}_t=\phi\widehat{u}_t.$$

From the proof of Proposition 3.38, we know that $(\phi u^1,\phi\nu^1)$ and $(\phi u^2,\phi\nu^2)$ are the solutions of problem (3.1) with null Dirichlet boundary conditions. We have the Itô's formula for $\widehat{w}$, see Theorem 6 in [25]. Then we do the same approximations as in the proof of Proposition 3.38, we can get the desired formula. ∎

We have the following comparison theorem:

Theorem 3.40. *Assume that (ξ^i,f^i,g,h,S^i), $i=1,2$, satisfy assumptions **(H)**, **(HIL)**, **(OL)** and **(HOL)**. Let $(u^i,\nu^i)\in\mathcal{R}_{loc}\left(\xi^i,f^i,g,h,S^i\right),i=1,2$ and suppose that the process $\left(u^1-u^2\right)^+$ belongs to $\mathcal{H}_T$ and that one has*

$$E\left(\left\|f^1\left(.,.,u^2,\nabla u^2\right)-f^2\left(.,.,u^2,\nabla u^2\right)\right\|_{\#;t}^*\right)^2<\infty,\quad \textit{for all}\quad t\in[0,T].$$

If $\xi^1\le\xi^2$ a.s., $f^1\left(t,\omega,u^2,\nabla u^2\right)\le f^2\left(t,\omega,u^2,\nabla u^2\right)$, $dt\otimes dx\otimes dP$-a.e. and $S^1\le S^2$, $dt\otimes dx\otimes dP$-a.s., then one has $u^1(t,x)\le u^2(t,x)$, $dt\otimes dx\otimes dP$-a.e.

Proof.

Applying Itô's formula (3.44) to $(\widehat{u}^+)^2$, we have $\forall t\in[0,T]$,

$$\int_{\mathcal{O}}(\widehat{u}_t^+(x))^2dx+2\int_0^t\mathcal{E}((\widehat{u}_s^+))ds=\int_{\mathcal{O}}(\widehat{\xi}^+(x))^2dx+2\int_0^t\int_{\mathcal{O}}\widehat{u}_s^+(x)\widehat{f}_s(x,\widehat{u}_s(x),\nabla\widehat{u}_s(x))dxds$$
$$-2\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\partial_i\widehat{u}_s^+(x)\widehat{g}_s^i(x,\widehat{u}_s(x),\nabla\widehat{u}_s(x))dxds+\int_0^t\int_{\mathcal{O}}I_{\{\widehat{u}_s>0\}}|\widehat{h}_s(x,\widehat{u}_s(x),\nabla\widehat{u}_s(x))|^2dxds$$
$$+2\sum_{i=1}^{\infty}\int_0^t\int_{\mathcal{O}}\widehat{u}_s^+(x)\widehat{h}_s^j(x,\widehat{u}_s(x),\nabla\widehat{u}_s(x))dxdB_s^j+2\int_0^t\int_{\mathcal{O}}\widehat{u}_s^+(x)\widehat{\nu}(dxds),\qquad a.s.$$

The last term is negative because that:

$$\int_0^t\int_{\mathcal{O}}\widehat{u}_s^+(x)\widehat{\nu}(dxds)=\int_0^t\int_{\mathcal{O}}(S^1-u^2)^+\nu^1(dxds)-\int_0^t\int_{\mathcal{O}}(u^1-S^2)^+\nu^2(dxds)\le 0$$

Then we can do the similar calculus as in Proposition 3.23 and get

$$E\left(\left\|\widehat{u}^+\right\|_{2,\infty;t}^2+\left\|\nabla\widehat{u}^+\right\|_{2,2;t}^2\right)\leq k\left(t\right)E\left(\left\|\widehat{\xi}^+\right\|_2^2+\left(\left\|\widehat{f}^{\widehat{u},0+}\right\|_{\#;t}^*\right)^2+\left\|\widehat{g}^{\widehat{u},0}\right\|_{2,2;t}^2+\left\|\widehat{h}^{\widehat{u},0}\right\|_{2,2;t}^2\right).$$

This deduce the result, since $\widehat{\xi}\leq 0$ and $\widehat{f}^0\leq 0$ and $\widehat{g}^0=\widehat{h}^0=0$. ■

3.5.3 Maximum principle

In all this subsection, we work under Assumptions **(H)**, **(OL)**, **(HIL)**, **(HOL)**,**(HI2p)**, **(HO∞p)** and **(HDθp)**. By the following property which has been proved in [23], Lemma 2:

$$\|u\|_{1,1;T}\leq c\,\|u\|_{\theta;T}^*$$

for some constant $c>0$, we know that **(HDθp)** is stronger than **(3.14)**.
We first consider the case of a solution u such that $u\leq 0$ on $\partial\mathcal{O}$.

Theorem 3.41. *Suppose that Assumptions **(H)**, **(OL)**, **(HIL)**, **(HOL)**,**(HI2p)**, **(HO∞p)** and **(HDθp)** hold for some $\theta\in[0,1[$, $p\geq 2$ and that the constants of the Lipschitz conditions satisfy*

$$\alpha+\frac{\beta^2}{2}+72\beta^2<\lambda.$$

Let $(u,\nu)\in\mathcal{R}_{loc}\left(\xi,f,g,h,S\right)$ be such that $u^+\in\mathcal{H}$. Then one has

$$\begin{aligned}E\left\|u^+\right\|_{\infty,\infty;t}^p &\leq k(t)c(p)E\big(\left\|\xi^+-S_0'\right\|_\infty^p+(\left\|\bar f^{0,+}\right\|_{\theta;t}^*)^p+(\left\||\bar g^0|^2\right\|_{\theta;t}^*)^{\frac p2}+(\left\||\bar h^0|^2\right\|_{\theta;t}^*)^{\frac p2}\\ &+\left\|(S_0')^+\right\|_\infty^p+(\left\|f^{',+}\right\|_{\theta;t}^*)^p+(\left\||g'|^2\right\|_{\theta;t}^*)^{\frac p2}+(\left\||h'|^2\right\|_{\theta;t}^*)^{\frac p2}\big)\end{aligned}$$

where $k\left(t\right)$ is constant that depends of the structure constants and $t\in[0,T]$.

Proof. Set $(y,\nu')=\mathcal{R}(\xi^+,\check f,g,h,S)$ the solution with zero Dirichlet boundary conditions, where the function $\check f$ is defined by $\check f=f+f^{0,-}$, with $f^{0,-}=0\vee(-f^0)$. The assumption on the Lipschitz constants ensure the application of the Section 3.4, which give the estimate,

$$E\left\|y-S'\right\|_{\infty,\infty;t}^p\leq k(t)E(\left\|\xi^+-S_0'\right\|_\infty^p+(\left\|\bar f^{0,+}\right\|_{\theta;t}^*)^p+(\left\||\bar g^0|^2\right\|_{\theta;t}^*)^{\frac p2}+(\left\||\bar h^0|^2\right\|_{\theta;t}^*)^{\frac p2}).$$

where $\bar f^{0,+}=\check f^0-f'=f^{0,+}-f'$. On the boundary, $y=0$ and $u\leq 0$, hence, $u-y\leq 0$ on the boundary, i.e. $(u-y)^+\in\mathcal{H}$. Moreover, the other conditions of the comparison theorem are satisfied so that we can apply it and deduce that $u-S'\leq y-S'$. This implies that $(u-S')^+\leq(y-S')^+$ and the above estimate of $y-S'$ leads to the following estimate:

$$E\left\|(u-S')^+\right\|_{\infty,\infty;t}^p\leq k(t)E(\left\|\xi^+-S_0'\right\|_\infty^p+(\left\|\bar f^{0,+}\right\|_{\theta;t}^*)^p+(\left\||\bar g^0|^2\right\|_{\theta;t}^*)^{\frac p2}+(\left\||\bar h^0|^2\right\|_{\theta;t}^*)^{\frac p2}).$$

with the estimate of S'

$$E\left\|(S')^+\right\|_{\infty,\infty;t}^p\leq k(t)E(\left\|(S_0')^+\right\|_\infty^p+(\left\|f^{',+}\right\|_{\theta;t}^*)^p+(\left\||g'|^2\right\|_{\theta;t}^*)^{\frac p2}+(\left\||h'|^2\right\|_{\theta;t}^*)^{\frac p2}).$$

Therefore,

$$\begin{aligned} E\left\|u^+\right\|_{\infty,\infty;t}^p &\leq k(t)c(p)E\big(\left\|\xi^+ - S_0'\right\|_\infty^p + (\left\|\bar f^{0,+}\right\|_{\theta;t}^*)^p + (\left\||\bar g^0|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} + (\left\||\bar h^0|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} \\ &+ \left\|(S_0')^+\right\|_\infty^p + (\left\|f^{',+}\right\|_{\theta;t}^*)^p + (\left\||g'|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} + (\left\||h'|^2\right\|_{\theta;t}^*)^{\frac{p}{2}}\big). \end{aligned}$$

■

Let us generalize the previous result by onsidering a real Itô's process of the form

$$M_t = m + \int_0^t b_s ds + \sum_{j=1}^{+\infty} \int_0^t \sigma_{j,s} dB_s^j$$

where m is a random variable and $b = (b_t)_{t\geq 0}$, $\sigma = (\sigma_{1,t}, ..., \sigma_{n,t}, ...)_{t\geq 0}$ are adapted processes.

Theorem 3.42. *Suppose that Assumptions* ***(H)***, ***(OL)***, ***(HIL)***, ***(HOL)***,***(HI2p)***, ***(HO∞p)*** *and* ***(HDθp)*** *hold for some $\theta \in [0,1[$, $p \geq 2$ and that the constants of the Lipschitz conditions satisfy*

$$\alpha + \frac{\beta^2}{2} + 72\beta^2 < \lambda.$$

Assume also that m and the processes b and σ satisfy the following integrability conditions

$$E\,|m|^p < \infty,\; E\left(\int_0^t |b_s|^{\frac{1}{1-\theta}}\,ds\right)^{p(1-\theta)} < \infty,\; E\left(\int_0^t |\sigma_s|^{\frac{2}{1-\theta}}\,ds\right)^{\frac{p(1-\theta)}{2}} < \infty,$$

for each $t \in [0,T]$. Let $(u,\nu) \in \mathcal{R}_{loc}(\xi, f, g, h, S)$ be such that $(u-M)^+$ belongs to $\mathcal{H}_T$. Then one has

$$\begin{aligned} E\left\|(u-M)^+\right\|_{\infty,\infty;t}^p &\leq c(p)k(t)E\big[\left\|(\xi-m)^+ - (S_0'-m)\right\|_\infty^p + \left(\left\|\bar f^{0,+}\right\|_{\theta;t}^*\right)^p \\ &+ \left(\left\||\bar g^0|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\left\||\bar h^0|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left\|(S_0'-m)^+\right\|_\infty^p \qquad (3.45) \\ &+ \left(\left\|(f'-b)^+\right\|_{\theta;t}^*\right)^p + \left(\left\||g'|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\left\||h'-\sigma|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}}\big] \end{aligned}$$

where $k(t)$ is the constant from the preceding corollary. The right hand side of this estimate is dominated by the following quantity which is expressed directly in terms of the characteristics of the process M,

$$\begin{aligned} c(p)k(t)E\big[&\left\|(\xi-m)^+ - (S_0'-m)\right\|_\infty^p + \left(\left\|\bar f^{0,+}\right\|_{\theta;t}^*\right)^p + \left(\left\||\bar g^0|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\left\||\bar h^0|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} \\ &+ \left\|(S_0'-m)^+\right\|_\infty^p + \left(\left\|f^{',+}\right\|_{\theta;t}^*\right)^p + \left(\left\||g'|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\left\||h'|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} \\ &+ |m|^p + \left(\int_0^t |b_s|^{\frac{1}{1-\theta}}\,ds\right)^{p(1-\theta)} + \left(\int_0^t |\sigma_s|^{\frac{2}{1-\theta}}\,ds\right)^{\frac{p(1-\theta)}{2}}\big]. \end{aligned}$$

Proof. One immediately observes that $u-M$ belongs to $\mathcal{R}_{loc}\left(\xi - m, \check{f}, \check{g}, \check{h}, S - M\right)$, where

$$\check{f}\,(t,\omega,x,y,z) = f\,(t,\omega,x,y+M_t\,(\omega)\,,z) - b_t\,(\omega)\,,$$

$$\check{g}\,(t,\omega,x,y,z) = g\,(t,\omega,x,y+M_t\,(\omega)\,,z)\,,$$

$$\check{h}\,(t,\omega,x,y,z) = h\,(t,\omega,x,y+M_t\,(\omega)\,,z) - \sigma_t\,(\omega)\,.$$

In order to apply the preceding theorem we only have to estimate the zero terms of the following functions:

$$\bar{\check{f}}\,(t,\omega,x,y,z) = \check{f}\,\left(t,\omega,x,y+S'-M,z+\nabla S'\right) - f'(t,\omega,x) + b_t(\omega),$$

$$\bar{\check{g}}\,(t,\omega,x,y,z) = \check{g}\,\left(t,\omega,x,y+S'-M,z+\nabla S'\right) - g'(t,\omega,x),$$

$$\bar{\check{h}}\,(t,\omega,x,y,z) = \check{h}\,\left(t,\omega,x,y+S'-M,z+\nabla S'\right) - h'(t,\omega,x) + \sigma_t(\omega).$$

So we have:

$$\bar{\check{f}}_t^0 = \check{f}_t(S'-M,\nabla S') - f'_t + b_t = f_t(S',\nabla S') - f'_t = \bar{f}^0\,,$$

$$\bar{\check{g}}_t^0 = \check{g}_t(S'-M,\nabla S') - g'_t = g_t(S',\nabla S') - g'_t = \bar{g}^0\,,$$

$$\bar{\check{h}}_t^0 = \check{h}_t(S'-M,\nabla S') - h'_t + \sigma_t = h_t(S',\nabla S') - h'_t = \bar{h}^0\,.$$

Therefore, applying the preceding theorem to $u - M$, we obtain (3.45).
On the other hand, one has the following estimates:

$$\left\|(f'-b)^+\right\|_{\theta;t}^* \le c\left[\left\|f'^{,+}\right\|_{\theta;t}^* + \left(\int_0^t |b_s|^{\frac{1}{1-\theta}}\,ds\right)^{1-\theta}\right],$$

$$\left\||h'-\sigma|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} \le c\left[\left(\left\||h'|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\int_0^t |\sigma_s|^{\frac{2}{1-\theta}}\,ds\right)^{1-\theta}\right].$$

This allows us to conclude the proof. ∎

3.6 Appendix

3.6.1 Proof of Lemma 3.20

Proof. We take the function $f_n(\omega,t,x) := f(\omega,t,x,u,\nabla u) - f^0 + f_n^0$, where $f_n^0,\ n \in \mathbb{N}$, is a sequence of bounded functions such that $E\left(\|f^0 - f_n^0\|_{\#;t}^*\right)^2 \to 0,\ as\ n \to 0$. We consider the following equation

$$du_t^n(x) + Au_t^n(x)dt = f_t^n(x)dt + div\check{g}_t(x)dt + \check{h}_t(x)dB_t$$

where $\check{g}(\omega,t,x) = g(\omega,t,x,u,\nabla u)$ and $\check{h}(\omega,t,x) = h(\omega,t,x,u,\nabla u)$. This is a linear equation in u^n, from [20], we know that u^n uniquely exists.
Applying Itô's formula to $(u^n - u^m)^2$,

$$\|u_t^n - u_t^m\|^2 + 2\int_0^t \mathcal{E}(u_s^n - u_s^m)ds \quad = \quad 2\int_0^t (u_s^n - u_s^m, f_s^n - f_s^m)ds.$$

From (3.5), we have, for $\delta > 0$,

$$2\left|\int_0^t (u_s^n - u_s^m, f_s^n - f_s^m) ds\right| \quad \leq \quad \delta \left\|u^n - u^m\right\|_{\#;t}^2 + C_\delta \left(\left\|f^n - f^m\right\|_{\#;t}^*\right)^2 .$$

Since $\mathcal{E}(u^n - u^m) \geq \lambda \left\|\nabla(u^n - u^m)\right\|_2^2$, we deduce that, for all $t \geq 0$, almost surely,

$$\left\|u_t^n - u_t^m\right\|^2 + 2\lambda \left\|\nabla(u^n - u^m)\right\|_{2,2;t}^2 \leq \delta \left\|u^n - u^m\right\|_{\#;t}^2 + C_\delta \left(\left\|f^n - f^m\right\|_{\#;t}^*\right)^2 . \tag{3.46}$$

Taking the supremum and the expectation, we get

$$E\left(\left\|u^n - u^m\right\|_{2,\infty;t}^2 + \left\|\nabla(u^n - u^m)\right\|_{2,2;t}^2\right) \leq \delta E \left\|u^n - u^m\right\|_{\#;t}^2 + C_\delta E \left(\left\|f^n - f^m\right\|_{\#;t}^*\right)^2 .$$

Dominating the term $E\left\|u^n - u^m\right\|_{\#;t}^2$ by using the estimate (3.4) and taking δ small enough, we obtain the following estimate:

$$E\left(\left\|u_n - u_m\right\|_{2,\infty;t}^2 + \left\|\nabla(u_n - u_m)\right\|_{2,2;t}^2\right) \leq C_\delta E\left(\left\|f^n - f^m\right\|_{\#;t}^*\right)^2 \to 0, \ when\ n, m \to \infty.$$

Therefore u^n has a limit u in $\mathcal{H}$.
See for example [21], we know that for u^n we have the following Itô's formula:

$$\begin{aligned}&\int_{\mathcal{O}} \varphi\left(u_t^n(x)\right) dx + \int_0^t \mathcal{E}\left(\varphi'\left(u_s^n\right), u_s^n\right) ds = \int_{\mathcal{O}} \varphi\left(\xi(x)\right) dx + \int_0^t \left(\varphi'\left(u_s^n\right), f_s^n\right) ds \\ &- \int_0^t \sum_{i=1}^d \left(\partial_i\left(\varphi'\left(u_s^n\right)\right), \breve{g}_s^i\right) ds + \frac{1}{2}\int_0^t \left(\varphi''\left(u_s^n\right), \left|\breve{h}_s\right|^2\right) ds + \sum_{j=1}^\infty \int_0^t \left(\varphi'\left(u_s^n\right), \breve{h}_s^j\right) dB_s^j.\end{aligned}$$

Now, we pass to the limit as n tend to $+\infty$:

$$\begin{aligned}&\left|\int_0^t \left(\varphi'\left(u_s^n\right), f_s^n\right) ds - \int_0^t \left(\varphi'\left(u_s\right), f_s\right) ds\right| \\ \leq\ & \left|\int_0^t \left(\varphi'\left(u_s^n\right) - \varphi'\left(u_s\right), f_s^n\right) ds\right| + \left|\int_0^t \left(\varphi'\left(u_s\right), f_s^n - f_s\right) ds\right| \\ \leq\ & C\left\|u^n - u\right\|_{\#;t} \left\|f^n\right\|_{\#;t}^* + C\left\|u\right\|_{\#;t} \left\|f^n - f\right\|_{\#;t}^* .\end{aligned}$$

The relation (3.4) and the strong convergence of u^n yield that $E\left\|u^n - u\right\|_{\#;t} \to 0$, as $n \to \infty$. So, by extracting a subsequence, we can assume that the right member in the previous inequality tends to 0 $P-$almost surely. So we have

$$\lim_{n\to+\infty} \int_0^t \left(\varphi'\left(u_s^n\right), f_s^n\right) ds = \int_0^t \left(\varphi'\left(u_s\right), f_s\right) ds.$$

The convergence of the other terms are easily deduced from the strong convergence of (u^n) to u in $\mathcal{H}_T$ and yield the desired formula. ■

3.6.2 Proof of Lemma 3.21

Proof. First of all, this equation is a special case of Theorem 3 in [23] hence, we know that w exists is unique and belongs to $\mathcal{H}_T$.
Following M.Pierre [70, 71] and F.Mignot and J.Puel [54], we define

$$\kappa(w, 0) := ess \inf\{u \in \mathcal{P};\ u \geq w\ a.e.,\ u(0) \geq 0\}.$$

We consider the following equation:

$$\begin{cases} \dfrac{\partial v_t^n}{\partial t} = Lv_t^n + n(v_t^n - w_t)^- \\ v_0^n = 0 \end{cases} \tag{3.47}$$

From [54], for almost all $\omega \in \Omega$, we know that $v^n(\omega)$ converges weakly to $v(\omega) := \kappa^\omega(w,0)$ in $L^2(0,T;H_0^1(\mathcal{O}))$ and that $v(\omega) \geq w(\omega)$.
(3.16)-(3.47) yields

$$d(v_t^n - w_t) + A(v_t^n - w_t)dt = (n(v_t^n - w_t)^- - f_t^0)dt$$

so, we have the following relation almost surely, $\forall t \geq 0$,

$$\|v_t^n - w_t\|^2 + 2\int_0^t \mathcal{E}(v_s^n - w_s)ds = 2\int_0^t \int_{\mathcal{O}} (v_s^n - w_s)n(v_s^n - w_s)^- dxds - 2\int_0^t (v_s^n - w_s, f_s^0)ds.$$

The first term is negative and

$$\left| \int_0^t (v_s^n - w_s, f_s^0)ds \right| \leq \delta \|v^n - w\|_{\#;t}^2 + C_\delta \left(\left\| f^0 \right\|_{\#;t}^* \right)^2.$$

Therefore

$$\|v_t^n - w_t\|_2^2 + 2\lambda \|\nabla(v^n - w)\|_{2,2;t}^2 \leq 2\delta \|v^n - w\|_{\#;t}^2 + 2C_\delta \left(\left\| f^0 \right\|_{\#;t}^* \right)^2.$$

Taking the supremum and the expectation, we get

$$E\|v^n - w\|_{2,\infty;t}^2 \leq 2\delta E\|v^n - w\|_{\#;t}^2 + 2C_\delta E\left(\left\| f^0 \right\|_{\#;t}^* \right)^2.$$

Dominating the term $E\|v^n - w\|_{\#;t}^2$ by using the estimate (3.4) and taking δ small enough, we obtain

$$E\|v^n - w\|_{2,\infty;t}^2 + E\|\nabla(v^n - w)\|_{2,2;t}^2 \leq CE\left(\left\| f^0 \right\|_{\#;t}^* \right)^2.$$

By Fatou's lemma, we have

$$E\sup_{t\in[0,T]} \|\kappa_t - w_t\|^2 + E\int_0^T \mathcal{E}(\kappa_t - w_t)dt \leq CE\int_0^T \left(\left\| f_t^0 \right\|_{\#}^* \right)^2 dt. \tag{3.48}$$

We now consider a sequence of random functions $(f^{0,n})_{n\in\mathbb{N}}$ which belongs in $L^2(\Omega) \otimes C_c^\infty(\mathbb{R}^+) \otimes C_c^\infty(\mathcal{O})$ and such that $E\left\| f^{0,n} - f^0 \right\|_{\#;t}^* \to 0$. Let w^n be the solution of

$$\begin{cases} dw_t^n + Aw_t^n dt = f_t^{0,n} dt \\ w_0^n = 0. \end{cases}$$

Then it's well known that w^n is $P-$almost surely continuous in (t,x) (see for example [2]). Then, we consider a sequence of random open sets

$$\vartheta_n = \{|w^{n+1} - w^n| > \epsilon_n\}, \quad \Theta_p = \bigcup_{n=p}^{+\infty} \vartheta_n,$$

and $\kappa_n = \kappa(\frac{1}{\epsilon_n}(w^{n+1} - w^n), 0) + \kappa(-\frac{1}{\epsilon_n}(w^{n+1} - w^n), 0)$. From the definition of κ and the relation (see [71]), we get

$$\kappa(|v|) \leq \kappa(v, v^+(0)) + \kappa(-v, v^-(0).)$$

Moreover, κ_n satisfy the conditions of Lemma 3.3 in [71], i.e. $\kappa_n \in \mathcal{P}$ and $\kappa_n \geq 1$ *a.e.* on ϑ_n, therefore, we get the following relation:

$$E[cap\,(\Theta_p)] \leq \sum_{n=p}^{+\infty} E[cap\,(\vartheta_n)] \leq \sum_{n=p}^{+\infty} E\,\|\kappa_n\|_{\mathcal{K}}^2 \leq 2C \sum_{n=p}^{+\infty} \frac{1}{\epsilon_n^2} E \int_0^T \left(\left\| f_t^{0,n+1} - f_t^{0,n} \right\|_{\#}^* \right)^2 dt,$$

where the last inequality comes from (3.48).
By extracting a subsequence, we can consider that

$$E \int_0^T \left(\left\| f_t^{0,n+1} - f_t^{0,n} \right\|_{\#}^* \right)^2 dt \leq \frac{1}{2^n}$$

and taking $\epsilon_n = \frac{1}{n^2}$ to get

$$E[cap\ (\Theta_p)] \leq \sum_{n=p}^{+\infty} \frac{2Cn^4}{2^n}.$$

Therefore

$$\lim_{p \to +\infty} E[cap\ (\Theta_p)] = 0.$$

Finally, for almost all $\omega \in \Omega$, $w^n(\omega)$ is continuous in (t, x) on $(\Theta_p(\omega))^c$ and $(w^n(\omega))$ converges uniformly to $w(\omega)$ on $(\Theta_p(\omega))^c$ for all p, hence, $w(\omega)$ is continuous in (t, x) on $(\Theta_p(\omega))^c$, then from the definition of quasi-continuity, we know that $w(\omega)$ admits a quasi-continuous version since $cap(\Theta_p)$ tends to 0 almost surely as p tends to $+\infty$. ∎

3.6.3 Technical Lemmas

In this section, we prove technical lemmas that we need in the Step 2 of the proof of Theorem 3.27. For simplicity, we put, for fixed $n \leq m$, $\widehat{u} := \bar{u}^n - \bar{u}^m$, $\widehat{\xi} := \xi^n - \xi^m$, $\widehat{f}(t, \omega, x, y, z) := \bar{f}_{n,m}(t, \omega, x, y, z)$ and similar for $\widehat{g}$ and $\widehat{h}$.

We recall that the initial value $\widehat{\xi}$ and $\widehat{f}^0$, $\widehat{g}^0$, $\widehat{h}^0$ are uniformly bounded.

Lemma 3.43. *Denote*

$$K = \left\|\widehat{\xi}\right\|_{L^\infty(\Omega\times\mathcal{O})} \vee \left\|\widehat{f}^0\right\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})} \vee \left\|\widehat{g}^0\right\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})} \vee \left\|\widehat{h}^0\right\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})}.$$

Then there exist constants c, $c' > 0$ which only depend on K, C, α, β such that, for all real $l \geq 2$, one has

$$E \int_{\mathcal{O}} |\widehat{u}_t(x)|^l dx \leq cK^2 l(l-1) e^{cl(l-1)t}, \tag{3.49}$$

$$E \int_0^t \int_{\mathcal{O}} |\widehat{u}_s(x)|^{l-2} |\nabla \widehat{u}_s(x)|^2 dx ds \leq c' K^2 l(l-1) e^{cl(l-1)t}, \tag{3.50}$$

and

$$E\int_0^t\int_{\mathcal{O}}|\widehat{u}_s(x)|^{l-1}(\nu^n+\nu^m)(dxds)<+\infty. \tag{3.51}$$

Proof. Beginning from the Itô's formula for the difference of two solutions of obstacle problems which has been proved in [25]: we take the same φ_n as in the proof of Lemma 3.29,

$$\begin{aligned}&\int_{\mathcal{O}}\varphi_n(\widehat{u}_t(x))\,dx \;+\;\int_0^t\mathcal{E}\big(\varphi_n'(\widehat{u}_s),\,\widehat{u}_s\big)\,ds=\int_{\mathcal{O}}\varphi_n(\widehat{\xi}(x))dx+\int_0^t\int_{\mathcal{O}}\varphi_n'(\widehat{u}_s(x))\,\widehat{f}(s,x)\,dxds\\&-\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\varphi_n''(\widehat{u}_s(x))\partial_i(\widehat{u}_s(x)))\,\widehat{g}_i(s,x)\,dx\,ds+\sum_{j=1}^{\infty}\int_0^t\int_{\mathcal{O}}\varphi_n'(\widehat{u}_s(x))\,\widehat{h}_j(s,x)\,dxdB_s^j\\&+\frac12\sum_{j=1}^{\infty}\int_0^t\int_{\mathcal{O}}\varphi_n''(\widehat{u}_s(x))\,\widehat{h}_j^2(s,x)\,dx\,ds+\int_0^t\int_{\mathcal{O}}\varphi_n'(\widehat{u}_s(x))\,(\nu^n-\nu^m)(dx\,ds)\,,\quad a.s.\end{aligned} \tag{3.52}$$

The support of ν^n is $\{\bar u^n=S\}$ and the support of ν^m is $\{\bar u^m=S\}$, so the last term is equal to

$$\int_0^t\int_{\mathcal{O}}\varphi_n'(S_s(x)-\bar u_s^m(x))\,\nu^n(dx\,ds)-\int_0^t\int_{\mathcal{O}}\varphi_n'(\bar u_s^n(x)-S_s(x))\,\nu^m(dx\,ds)$$

and the fact that $\varphi_n'(x)\le 0$, when $x\le 0$ and $\varphi_n'(x)\ge 0$, when $x\ge 0$, ensure that the last term is always negative.
By the uniform ellipticity of the operator A, we get

$$\mathcal{E}\big(\varphi_n'(\widehat{u}_s),\,\widehat{u}_s\big)\;\ge\;\lambda\int_{\mathcal{O}}\varphi_n''(\widehat{u}_s)|\nabla\widehat{u}_s|^2\,dx.$$

Let $\epsilon>0$ be fixed. Using the Lipschitz condition on $\widehat{f}$ and the properties of the functions $(\varphi_n)_n$ we get

$$|\varphi_n'(\widehat{u}_s)|\,|\widehat{f}(s,x)|\le l(\varphi_n(\widehat{u}_s)+1)\,|\widehat{f}^0|+(C+c_\epsilon)\,|\widehat{u}_s|^2\varphi_n''(\widehat{u}_s)+\,\epsilon\varphi_n''(\widehat{u}_s)|\nabla(\widehat{u}_s)|^2.$$

Now using Cauchy-Schwarz inequality and the Lipschitz condition on $\widehat{g}$ we get

$$\sum_{i=1}^d\varphi_n''(\widehat{u}_s)\partial_i(\widehat{u}_s)\,\widehat{g}(s,x)\le l(l-1)c_\epsilon K^2+2c_\epsilon(K^2+C^2)l(l-1)|\varphi_n(\widehat{u}_s)|+(\alpha+\epsilon)\,\varphi_n''(\widehat{u}_s)|\nabla(\widehat{u}_s)|^2.$$

In the same way as before

$$\sum_{j=1}^{\infty}\varphi_n''(\widehat{u}_s)\,\widehat{h}(s,x)\le\;2c_\epsilon' l(l-1)K^2+2c_\epsilon'(K^2+C^2)l(l-1)\varphi_n(\widehat{u}_s)+(1+\epsilon)\,\beta^2\,\varphi_n''(\widehat{u}_s)|\nabla(\widehat{u}_s)|^2.$$

Thus taking the expectation, we deduce

$$\begin{aligned}&E\int_{\mathcal{O}}\varphi_n(\widehat{u}_t(x))\,dx+(\lambda-\frac12(1+\epsilon)\beta^2-(\alpha+2\epsilon)\,)\,E\int_0^t\int_{\mathcal{O}}\varphi_n''(\widehat{u}_s)\,|\nabla(\widehat{u}_s)|^2\,dx\,ds\\&\le\;l(l-1)c_\epsilon''K^2\;+\;c_\epsilon''l(l-1)\big(K^2+C^2+C+c_\epsilon\big)E\int_0^t\int_{\mathcal{O}}\varphi_n(\widehat{u}_s(x))\,dx\,ds.\end{aligned}$$

On account of the contraction condition, one can choose $\epsilon > 0$ small enough such that

$$\lambda - \frac{1}{2}(1+\epsilon)\beta^2 - (\alpha + 2\epsilon) > 0$$

and then

$$E\int_{\mathcal{O}} \varphi_n(\widehat{u}_t(x))\,dx \le\, cK^2 l(l-1) \,+\, cl(l-1)E\int_0^t\int_{\mathcal{O}} \varphi_n(\widehat{u}_s(x))\,dx\,ds\,.$$

We obtain by Gronwall's Lemma, that

$$E\int_{\mathcal{O}} \varphi_n(\widehat{u}_t(x))\,dx \le\, c\,K^2 l(l-1)\,\exp\big(c\,l(l-1)\,t\big),$$

and so it is easy to get

$$E\int_0^t\int_{\mathcal{O}} \varphi_n''(\widehat{u}_s(x))\,|\nabla\widehat{u}_s|^2\,dx\,ds \,\le\, c'\,K^2 l\,(l-1)\,\exp\big(cl(l-1)\,t\big).$$

Then, letting $n\to\infty$, by Fatou's lemma we get (3.49) and (3.50).
From (3.52), we know that

$$-\int_0^t\int_{\mathcal{O}} \varphi_n'(\widehat{u}_s(x))(\nu^n - \nu^m)(dxds) \le C.$$

Moreover,

$$\begin{aligned}
& -\int_0^t\int_{\mathcal{O}} \varphi_n'(\widehat{u}_s(x))(\nu^n-\nu^m)(dxds)\\
= & -\int_0^t\int_{\mathcal{O}} \varphi_n'(S_s(x) - \bar{u}_s^m(x))\,\nu^n(dx\,ds) + \int_0^t\int_{\mathcal{O}} \varphi_n'(\bar{u}_s^n(x) - S_s(x))\,\nu^m(dx\,ds)\\
= & \int_0^t\int_{\mathcal{O}} \varphi_n'(\bar{u}_s^m(x) - S_s(x))\,\nu^n(dx\,ds) + \int_0^t\int_{\mathcal{O}} \varphi_n'(\bar{u}_s^n(x) - S_s(x))\,\nu^m(dx\,ds)
\end{aligned}$$

By Fatou's lemma, we obtain

$$\int_0^t\int_{\mathcal{O}} |\bar{u}_s^m(x) - S_s(x)|^{l-1}\nu^n(dxds) + \int_0^t\int_{\mathcal{O}} |\bar{u}^n(x) - S_s(x)|^{l-1}\nu^m(dxds) < +\infty,\ a.s.$$

which is exactly (3.51). ∎

Lemma 3.44. *One has the following formula for $\widehat{u}$: $\forall t\ge 0$, almost surely,*

$$\begin{aligned}
&\int_{\mathcal{O}} |\widehat{u}_t(x)|^l\,dx + \int_0^t \mathcal{E}\left(l\,(\widehat{u}_s)^{l-1}sgn(\widehat{u}_s),\,\widehat{u}_s\right)ds = \int_{\mathcal{O}} \left|\widehat{\xi}(x)\right|^l dx\\
&+l\int_0^t\int_{\mathcal{O}} sgn(\widehat{u}_s)\,|\widehat{u}_s(x)|^{l-1}\,\widehat{f}(s,x)\,dxds - l(l-1)\sum_{i=1}^d\int_0^t\int_{\mathcal{O}} |\widehat{u}_s(x)|^{l-2}\,\partial_i(\widehat{u}_s(x))\,\widehat{g}_i(s,x)\,dx\,ds\\
&+l\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}} sgn(\widehat{u}_s)\,|\widehat{u}_s(x)|^{l-1}\,\widehat{h}_j(s,x)\,dxdB_s^j + \frac{l(l-1)}{2}\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}} |\widehat{u}_s(x)|^{l-2}\,\widehat{h}_j^2(s,x,)\,dx\,ds\\
&+l\int_0^t\int_{\mathcal{O}} sgn(\widehat{u}_s)\,|\widehat{u}_s(x)|^{l-1}\,(\nu^1-\nu^2)(dx\,ds)\,.
\end{aligned} \tag{3.53}$$

Proof. From the Itô's formula for the difference of two solutions (see Theorem 6 in [25]), we have P-almost surely for all $t \in [0,T]$ and $n \in \mathbb{N}$

$$\begin{aligned}&\int_{\mathcal{O}} \varphi_n(\widehat{u}_t(x))\,dx \;+\; \int_0^t \mathcal{E}\big(\varphi_n'(\widehat{u}_s),\,\widehat{u}_s\big)\,ds = \int_{\mathcal{O}} \varphi_n(\widehat{\xi}(x))dx\\&+\int_0^t\int_{\mathcal{O}} \varphi_n'(\widehat{u}_s(x))\,\widehat{f}(s,x)\,dxds - \sum_{i=1}^d \int_0^t\int_{\mathcal{O}} \varphi_n''(\widehat{u}_s(x))\partial_i\widehat{u}_s(x)\,\widehat{g}_i(s,x)\,dx\,ds\\&+\sum_{j=1}^\infty \int_0^t\int_{\mathcal{O}} \varphi_n'(\widehat{u}_s(x))\,\widehat{h}_j(s,x)\,dxdB_s^j + \frac{1}{2}\sum_{j=1}^\infty \int_0^t\int_{\mathcal{O}} \varphi_n''(\widehat{u}_s(x))\,\widehat{h}_j^2(s,x)\,dx\,ds\\&+\int_0^t\int_{\mathcal{O}} \varphi_n'(\widehat{u}_s(x))(\nu^1-\nu^2)(dxds)\,.\end{aligned}$$

Then, passing to the limit as $n \to \infty$, the convergences come from the dominated convergence theorem. ∎

Similar as before, we define the processes $\widehat{v}$ and $\widehat{v}'$ by

$$\begin{aligned}\widehat{v}_t &:= \sup_{s\le t}\left(\int_{\mathcal{O}} |\widehat{u}_s|^l\,dx + \gamma l\,(l-1)\int_0^s\int_{\mathcal{O}} |\widehat{u}_r|^{l-2}\,|\nabla \widehat{u}_r|^2\,dx\,dr\right)\\ \widehat{v}'_t &:= \int_{\mathcal{O}} \left|\widehat{\xi}\right|^l dx + l^2 c_1 \left\||\widehat{u}|^l\right\|_{1,1;t} + l\left\|\widehat{f}^0\right\|_{\theta,t}^* \left\||\widehat{u}|^{l-1}\right\|_{\theta;t}\\&\quad + l^2\left(c_2\,\||\widehat{g}^0|^2\|_{\theta;t}^* + c_3\left\||\widehat{h}^0|^2\right\|_{\theta;t}^*\right)\left\||\widehat{u}|^{l-2}\right\|_{\theta;t},\end{aligned}$$

where above and below γ, c_1, c_2 and c_3 are the constants given by relations (3.34).
We remark first that the last term in (3.53) is non positive, indeed:

$$\begin{aligned}&\int_0^t\int_{\mathcal{O}} sgn(\widehat{u}_s)\left|S_s - u_s^2(x)\right|^{l-1}(\nu^1-\nu^2)(dx\,ds)\\ =\;& \int_0^t\int_{\mathcal{O}} sgn(S_s - u_s^2)\left|S_s - u_s^2(x)\right|^{l-1}\nu^1(dx\,ds)\\ &-\int_0^t\int_{\mathcal{O}} sgn(u_s^1 - S_s)\left|u_s^1(x) - S_s(x)\right|^{l-1}\nu^2(dx\,ds) \le 0.\end{aligned}$$

Then applying the same proof as the one of Lemma 3.33, we obtain:

$$\begin{aligned}&\tau E\sup_{0\le s\le t}\left(\int_{\mathcal{O}} |\widehat{u}_s|^l\,dx + \gamma l\,(l-1)\int_0^s\int_{\mathcal{O}} |\widehat{u}_r|^{l-2}\,|\nabla \widehat{u}_r|^2\,dxdr\right)\\ &\le E\int_{\mathcal{O}} \left|\widehat{\xi}\right|^l dx + l^2 c_1 E\left\||\widehat{u}|^l\right\|_{1,1;t} + lE\left\|\widehat{f}^0\right\|_{\theta,t}^*\left\||\widehat{u}|^{l-1}\right\|_{\theta;t}\\ &\quad + l^2 E\left(c_2\,\||\widehat{g}^0|^2\|_{\theta;t}^* + c_3\left\||\widehat{h}^0|^2\right\|_{\theta;t}^*\right)\left\||\widehat{u}|^{l-2}\right\|_{\theta;t}.\end{aligned}$$

and this yields that the process $\tau\widehat{v}$ is dominated by $\widehat{v}'$.
Starting from here, we can repeat line by line the proofs of Lemmas 15-17 in [21] and apply the Moser iteration as at the end of Subsection 3.4.1 to obtain the desired estimations, namely:

Lemma 3.45. *There exists a function $k_2 : \mathbb{R}_+ \to \mathbb{R}_+$ which involves only the structure constants of our problem and such that the following estimate holds*

$$E\|\widehat{u}\|_{\infty,\infty:t}^{p} \leq k_2\,(t)\, E\left(\left\|\widehat{\xi}\right\|^{p} + \left\|\widehat{f}^0\right\|_{\theta;t}^{*p} + \||\widehat{g}^0|^2\|_{\theta;t}^{*\frac{p}{2}} + \left\||\widehat{h}^0|^2\right\|_{\theta;t}^{*\frac{p}{2}}\right).$$

Lemma 3.46. *There exists a function $k_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which involves only the structure constants of our problem and such that the following estimate holds*

$$E\widehat{v}_t \leq k_1(l,t)E\left(\int_{\mathcal{O}} |\widehat{\xi}|^l dx + \left\|\widehat{f}^0\right\|_{\theta;t}^{*l} + \left\||\widehat{g}^0|^2\right\|_{\theta;t}^{*\frac{l}{2}} + \left\|\left|\widehat{h}^0\right|^2\right\|_{\theta;t}^{*\frac{l}{2}}\right).$$

Chapter 4

Stochastic PDEs driven by $G-$Brownian motion

4.1 Introduction

The aim of this chapter is to study the well-posedness of quasilinear stochastic partial differential equations driven by $G-$Brownian motion in the framework of sublinear expectation spaces (GSPDE for short).

Motivated by uncertainty problems, risk measures and the superhedging in finance, Peng [64, 65, 66] introduced $G-$Brownian motion. The expectation $\mathbb{E}[\cdot]$ associated with $G-$Brownian motion is a sublinear expectation which is called $G-$expectation. The stochastic calculus with respect to the $G-$Brownian motion has been established in [66].

We want to study the solvability of the following stochastic partial differential equation driven by $G-$Brownian motion:

$$\begin{aligned} du_t(x) &= \partial_i \left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right) dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ &\quad + \sum_{j=1}^{d_1} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j, \end{aligned} \tag{4.1}$$

where a is a symmetric bounded measurable matrix which defines a second order operator on $\mathcal{O} \subset \mathbb{R}^d$, with null Dirichlet condition. The initial condition is given as $u_0 = \xi \in L^2(\mathcal{O})$, and f, $g = (g_1, ..., g_d)$ and $h = (h_1, ...h_{d_1})$ are non-linear random functions which satisfy Lipschitz condition with proper Lipschitz coefficients, B is a d_1-dimensional $G-$Brownian motion.

For this purpose, we need to develop the stochastic calculus for Hilbert space valued process with respect to $G-$Brownian motion and to prove the corresponding Burkholder-Davis-Gundy inequality.

The existence and uniqueness result of GSPDE is as follows:

Theorem 4.1. *Under the assumptions of Lipschitz continuity and integrability on f, g and h, there exists a unique solution u of* (4.1) *in a proper space.*

We can also establish an Itô formula and a comparison theorem for the solution of GSPDE.

4.2 Sublinear expectation and Stochastic Calculs under Uncertainty

In this section, we will recall some basic definitions and properties of $G-$expectation and $G-$Brownian motion, which will be needed in the sequel. For the details, one can consult Peng [66].
Briefly speaking a $G-$Brownian motion is a continuous process with independent and stationary increments under a given sublinear expectation. Similar to the Wiener measure, a $G-$Brownian motion can be formulated by a sublinear expectation (i.e. $G-$expectation) on the space of continuous paths from $\mathbb{R}^+$ to $\mathbb{R}^d$.

4.2.1 Sublinear expectation

Let Ω be a given nonempty fundamental space and $\mathcal{H}$ be a linear space of real functions defined on Ω such that

1. $c \in \mathcal{H}$ for each constant c;
2. if $X_1, ..., X_n \in \mathcal{H}$ then $\varphi(X_1, ..., X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of functions φ satisfying
$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad x, y \in \mathbb{R}^n,$$
for some constant $C > 0$, $m \in \mathbb{N}$ depending on φ.

Remark 4.2. *Here one uses $C_{l,Lip}(\mathbb{R})^n$ only for some convenience of technics. In fact the essential requirement is that $\mathcal{H}$ contains all constants and, moreover, $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. In general, $C_{l,Lip}(\mathbb{R}^n)$ can be replaced by other functional spaces, for the details, see [66].*

The set $\mathcal{H}$ is interpreted as the space of random variables defined on Ω. $X = (X_1, ..., X_n)$ is called an $n-$dimensional random vector, denote by $X \in \mathcal{H}^n$.

Definition 4.3. *A sublinear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ with the following properties: for all $X, Y \in \mathcal{H}$, we have*

1. ***Monotonicity:***
$$\mathbb{E}[X] \geq \mathbb{E}[Y], \ if \ X \geq Y.$$

2. ***Constant preserving:***

$$\mathbb{E}[c] = c, \ \ for \ all \ \ c \in \mathbb{R}.$$

3. ***Sub-additivity:***

$$\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y].$$

4. ***Positive homogeneity:***

$$\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \ \ for \ all \ \ \lambda \geq 0.$$

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. It generalizes the classical case of the linear expectation $E[X] = \int_\Omega X dP$, $X \in L^1(\Omega, \mathcal{F}, \mathcal{P})$, over a probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Theorem 4.4. *(Theorem 2.1 in [66]) Let $\mathbb{E}$ be a functional defined on a linear space $\mathcal{H}$ satisfying subadditivity and positive homogeneity. Then there exists a family of linear functionals $\{E_\theta : \ \theta \in \Theta\}$ defined on $\mathcal{H}$ such that*

$$\mathbb{E}[X] = \sup_{\theta \in \Theta} E_\theta[X], \quad for \ \ X \in \mathcal{H}$$

and for each $X \in \mathcal{H}$, there exists $\theta_X \in \Theta$ such that $\mathbb{E}[X] = E_{\theta_X}[X]$. Furthermore, if $\mathbb{E}$ is a sublinear expectation, then the corresponding E_θ is a linear expectation.

Remark 4.5. *'E_θ is a linear expectation' means that it satisfies the additivity: $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$. It is not necessary that E_θ corresponds to a probability.*

We now give the definition of distributions of random variables under sublinear expectations. Let $X = (X_1, ..., X_n) \in \mathcal{H}^n$ be a given $n-$dimensional random vector on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We define a functional on $C_{l,Lip}(\mathbb{R}^n)$ by

$$\mathbb{F}_X[\varphi] := \mathbb{E}[\varphi(X)] : \ \varphi \in C_{l,Lip}(\mathbb{R}^n) \to \mathbb{R}.$$

The triple $(\mathbb{R}^n, C_{l,Lip}(\mathbb{R}^n), \mathbb{F}_X)$ forms a sublinear expectation space. $\mathbb{F}_X$ is called the **distribution** of X under $\mathbb{E}$.
Furthermore, one can prove that there exists a family of probability measure $\{F_X^\theta(\cdot)\}_{\theta \in \Theta}$ defined on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that

$$\mathbb{F}_X(\varphi) = \sup_{\theta \in \Theta} \int_{\mathbb{R}^n} \varphi(x) F_X^\theta(dx), \quad for \ each \ \varphi \in C_{l,Lip}(\mathbb{R}^n).$$

Thus $\mathbb{F}_X[\cdot]$ characterizes the uncertainty of the distributions of X.

Definition 4.6. *Given two sublinear expectation spaces $(\Omega, \mathcal{H}, \mathbb{E})$ and $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$, two random vectors $X \in \mathcal{H}^n$ and $Y \in \widetilde{\mathcal{H}}^n$ are said to be* ***identically distributed****, denoted by $X \overset{d}{=} Y$, if for each test function $\varphi \in C_{l,Lip}(\mathbb{R}^n)$,*

$$\mathbb{F}_X[\varphi] = \widetilde{\mathbb{F}}_Y[\varphi].$$

Remark 4.7. *In the case of sublinear expectations, $X \overset{d}{=} Y$ implies that the uncertainty subsets of distributions of X and Y are the same.*

The following notion of independence plays a key role in the sublinear expectation theory.

Definition 4.8. *In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y = (Y_1, ..., Y_n) \in \mathcal{H}^n$ is said to be independent from another random vector $X \in \mathcal{H}^m$ under $\mathbb{E}[\cdot]$ if for each test function $\varphi \in C_{l,Lip}(\mathbb{R}^{n+m})$ we have*

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}].$$

Remark 4.9. *For a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, Y is independent from X means that the uncertainty of distributions $\{F_Y(\theta, \cdot : \theta \in \Theta)\}$ of Y does not change after the realization of $X = x$. In other words, the "conditional sublinear expectation" of Y with respect to X is $\mathbb{E}[\varphi(x, Y)]_{x=X}$.*

Remark 4.10. *It is important to note that under sublinear expectations the condition "Y is independent from X" does not imply automatically that "X is independent from Y".*

We recall some useful inequalities, see [66]:

Proposition 4.11. *For each $X,\ Y \in \mathcal{H}$, we have*

$$\mathbb{E}[|X + Y|^r] \leq 2^{r-1}(\mathbb{E}[|X|^r] + \mathbb{E}[|Y|^r]),$$

$$\mathbb{E}[XY] \leq (\mathbb{E}[|X|^p])^{1/p} \cdot (\mathbb{E}[|Y|^q])^{1/q},$$

$$(\mathbb{E}[|X + Y|^p])^{1/p} \leq (\mathbb{E}[|X|^p])^{1/p} + (\mathbb{E}[|Y|^q])^{1/q},$$

where $r \geq 1$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.
In particular, for $1 \leq p < p'$, we have $(\mathbb{E}[|X|^p])^{1/p} \leq (\mathbb{E}[|X|^{p'}])^{1/p'}$.

4.2.2 $G-$Brownian motion and $G-$expectation

In this section we will consider the following path spaces: $\Omega = C_0^d(\mathbb{R}^+)$ be the space of all $\mathbb{R}^d-$valued continuous paths $(\omega_t)_{t\geq 0}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{+\infty} 2^{-i}[(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2| \wedge 1)].$$

It is clear that (Ω, ρ) is a complete separable metric space. We also denote $\Omega_T = \{\omega_{\cdot\wedge T} : \omega \in \Omega\}$ for each fixed $T \in [0, \infty)$.
Let $\mathcal{H}$ be a vector lattice of real functions defined on Ω such that if $X_1, ..., X_n \in \mathcal{H}$ then $\varphi(X_1, ..., X_n) \in \mathcal{H}$ for each $\varphi \in C_{b,Lip}(\mathbb{R}^n)$, where $C_{b,Lip}(\mathbb{R}^n)$ denotes the space of all bounded and Lipschitz functions on $\mathbb{R}^n$.

Definition 4.12. *Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space. $(X_t)_{t\geq 0}$ is called a $d-$dimensional **stochastic process** if for each $t \geq 0$, X_t is a $d-$dimensional random vector in $\mathcal{H}$.*

Definition 4.13. *A $d-$dimensional random vector X with each component in $\mathcal{H}$ is said to be $G-$normally distributed under the sublinear expectation $\mathbb{E}[\cdot]$ if for each $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, the function u defined by*

$$u(t,x) := \mathbb{E}[\varphi(x + \sqrt{t}X)], \ t \geq 0, \ x \in \mathbb{R}^d$$

satisfies the following $G-$heat equation defined on $[0, \infty) \times \mathbb{R}^d$,

$$\frac{\partial u}{\partial t} - G(D^2 u) = 0, \quad u|_{t=0} = \varphi,$$

where $D^2 u = (\partial^2_{x_i x_j} u)^d_{i,j=1}$ and

$$G(A) := \frac{1}{2} \sup_{\gamma \in \Theta} tr[\gamma\gamma^T A], \quad A \in \mathbb{S}^d.$$

$\mathbb{S}^d$ denotes the space of $d \times d$ symmetric matrices. Θ is a given non empty, bounded and closed subset of $\mathbb{R}^{d\times d}$ which is the space of all $d \times d$ matrices. We denote by $N(0, \Theta)$ the $G-$normal distribution.

Remark 4.14. *The above $G-$heat equation has a unique viscosity solution. We refer to [12] for the definition, existence, uniqueness and comparison theory of this type of parabolic PDE (see also [67] for this specific situation). If G is non-degenerate, i.e., there exists a $\beta > 0$ such that $G(A) - G(B) \leq \beta Tr[A - B]$ for each $A, B \in \mathbb{S}^d$ with $A \geq B$, then the above $G-$heat equation has a unique $C^{1,2}-$solution (see e.g.[78])*

We consider the canonical process by $B_t(\omega) = \omega_t, \ t \geq 0$, for each $\omega \in \Omega$. We introduce the space of finite dimensional cylinder random variables: for each $T \geq 0$, we set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, ..., B_{t_n}) : \ n \geq 1, \ t_1, ..., t_n \in [0,T], \ \varphi \in C_{b,Lip}(\mathbb{R}^{d\times n})\}.$$

It is clear that $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T) \subset C_b(\Omega_T)$, for $t \leq T$. We set

$$L_{ip}(\Omega) := \bigcup_{n=1}^{+\infty} L_{ip}(\Omega_n) \subset C_b(\Omega).$$

Definition 4.15. *Let $\mathbb{E} : L_{ip}(\Omega) \to \mathbb{R}$ be a sublinear expectation, $\mathbb{E}$ is called $G-$expectation if the $d-$dimensional canonical process $(B_t)_{t\geq 0}$ is a $G-$Brownian motion under $\mathbb{E}$, that is,*

1. *$B_0(\omega) = 0$;*

2. *For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $N(0, s\Theta)-$distributed and independent of $(B_{t_1}, ..., B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq ... \leq t_n \leq t$, i.e. for each $\varphi \in C_{l,Lip}(\mathbb{R}^{d\times n})$,*

$$\mathbb{E}[\varphi(B_{t_1}, ..., B_{t_{m-1}}, B_{t_m} - B_{t_{m-1}})] = \mathbb{E}[\psi(B_{t_1}, ..., B_{t_{m-1}})],$$

where $\psi(x_1, ..., x_{m-1}) = \mathbb{E}[\varphi(x_{t_1}, ..., x_{t_{m-1}}, B_{t_m} - B_{t_{m-1}})]$.

We denote by $L_G^p(\Omega_T)$ (resp. $L_G^p(\Omega)$) the topological completion of $L_{ip}(\Omega_T)$ (resp. $L_{ip}(\Omega)$) under the Banach norm $\mathbb{E}[|\cdot|^p]^{\frac{1}{p}}$, $1 \leq p \leq \infty$. $\mathbb{E}[\cdot]$ can be extended uniquely to a sublinear expectation on $L_G^p(\Omega)$. By Proposition 24 in [18], we know that each element in $L_G^p(\Omega)$ has a quasi-continuous version.

Definition 4.16. *Let $\mathbb{E} : L_{ip}(\Omega) \to \mathbb{R}$ be a $G-$expectation, we define the related* ***conditional expectation*** *of $X \in L_{ip}(\Omega_T)$ under $L_{ip}(\Omega_{t_j})$, $0 \leq t_1 \leq ... \leq t_j \leq t_{j+1} \leq ... \leq t_n \leq T$:*

$$\begin{aligned}\mathbb{E}[X|\Omega_{t_j}] &= \mathbb{E}[\varphi(B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})|\Omega_{t_j}] \\ &= \mathbb{E}[\psi(B_{t_1}, ..., B_{t_j} - B_{t_{j-1}})],\end{aligned}$$

where $\psi(x_1, ..., x_j) = \mathbb{E}[\varphi(x_1, ..., x_j, B_{t_{j+1}} - B_{t_j}, ..., B_{t_n} - B_{t_{n-1}})]$.
Since, for $X, Y \in L_{ip}(\Omega_{t_j})$,

$$\mathbb{E}[|\mathbb{E}[X|\Omega_{t_j}] - \mathbb{E}[Y|\Omega_{t_j}]|] \leq \mathbb{E}[|X - Y|],$$

the mapping $\mathbb{E}[\cdot|\Omega_{t_j}] : L_{ip}(\Omega_T) \to L_{ip}(\Omega_{t_j})$ can be continuously extended to $\mathbb{E}[\cdot|\Omega_{t_j}] : L_G^p(\Omega_T) \to L_G^p(\Omega_{t_j})$, $1 \leq p \leq +\infty$.

4.2.3 $G-$expectation as an upper-Expectation

Let P be the Wiener measure on $\Omega = C([0, T]; \mathbb{R}^d)$. The filtration generated by the canonical process $(B_t)_{t \geq 0}$ is denoted by

$$\mathcal{F}_t := \sigma\{B_u,\ 0 \leq u \leq t\}, \quad \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}.$$

It is clear that $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
Let $\mathcal{A}_{0,\infty}^{\Theta}$ be the collection of all $\Theta-$valued $\{\mathcal{F}_t,\ t \geq 0\}-$adapted processes on the interval $[0, \infty)$, i.e. $\{\theta_t,\ t \geq 0\} \in \mathcal{A}_{0,\infty}^{\Theta}$ if and only if θ_t is $\mathcal{F}_t-$measurable and $\theta_t \in \Theta$, for each $t \geq 0$. For each fixed $\theta \in \mathcal{A}_{0,\infty}^{\Theta}$, let P_θ be the law of the process $(\int_0^t \theta_s dB_s)_{t \geq 0}$ under the Wiener measure P.

We denote by $\mathcal{P}_1 = \{P_\theta :\ \theta \in \mathcal{A}_{0,\infty}^{\Theta}\}$ and $\mathcal{P} = \bar{\mathcal{P}}_1$ the closure of $\mathcal{P}_1$ under the topology of weak convergence. By Proposition 49 in [18], we know that $\mathcal{P}_1$ is tight and then $\mathcal{P}$ is weakly compact. We set

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

From Theorem 1 of [18], we know that c is a Choquet capacity. Then we can introduce the notion of "quasi sure" (q.s.).

Definition 4.17. *A set $A \subset \Omega$ is called polar if $c(A) = 0$. A property is said to hold "quasi-surely" (q.s.) if it holds outside a polar set.*

Remark 4.18. *In other words, $A \in \mathcal{B}(\Omega)$ is polar if and only if $P(A) = 0$ for any $P \in \mathcal{P}$.*

For each $X \in L^0(\Omega)$ (the space of all $\mathcal{B}(\Omega)-$measurable real functions on Ω) such that $E_P(X)$ exists for each $P \in \mathcal{P}$, we set

$$\widehat{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} E_P[X].$$

From Theorem 51 of [18], in fact, $L_G^1(\Omega)$ can be rewritten as the collection of all the q.s. continuous random vectors $X \in L^0(\Omega)$ with $\lim_{n\to+\infty} \widehat{\mathbb{E}}[|X|I_{|X|>n}] = 0$. Furthermore, for all $X \in L_G^1(\Omega)$, $\mathbb{E}[X] = \widehat{\mathbb{E}}[X]$.

Denis, Hu and Peng [18] (see Theorem 31) has obtained the following monotone convergence theorem:

$$X_n \in L_G^1(\Omega),\ X_n \downarrow X,\ q.s.\ \Rightarrow \mathbb{E}[X_n] \downarrow \widehat{\mathbb{E}}[X].$$

By the definition of $\widehat{\mathbb{E}}$, the following result is obvious:

$$X_n \in L^0(\Omega),\ X_n \uparrow X,\ q.s.,\ E_\theta(X_1) > -\infty\ for\ all\ P_\theta \in \mathcal{P} \Rightarrow \widehat{\mathbb{E}}[X_n] \uparrow \widehat{\mathbb{E}}[X].$$

4.2.4 Itô's Integral with respect to $G-$Brownian motion

For $T \in \mathbb{R}^+$, a partition π_T of $[0,T]$ is a finite ordered subset $\pi_T = \{t_0, t_1, ..., t_N\}$ such that $0 = t_0 < t_1 < ... < t_N = T$.

$$\mu(\pi_T) := \max\{|t_{i+1} - t_i| : i = 0, 1, ...N-1\}.$$

We use $\pi_T^N = \{t_0^N, t_1^N, ..., t_N^N\}$ to denote a sequence of partitions of $[0,T]$ such that

$$\lim_{N\to\infty} \mu(\pi_T^N) = 0.$$

Let $p \geq 1$ be fixed. We consider the following type of simple processes: for a given partition π_T of $[0,T]$ we set

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k,t_{k+1})}(t),$$

where $\xi \in L_G^p(\Omega_{t_k}),\ k = 0, 1, ..., N-1$ are given. The collection of these processes is denoted by $M_G^{p,0}(0,T)$. We then can introduce a natural norm $\|\cdot\|_{M_G^p(0,T)}$, under which, $M_G^{p,0}(0,T)$ can be extended to $M_G^p(0,T)$ which is a Banach space:

Definition 4.19. *For each $p \geq 1$, $M_G^p(0,T)$ denotes the completion of $M_G^{p,0}(0,T)$ under the norm*

$$\|\eta\|_{M_G^p(0,T)} = \left(\widehat{\mathbb{E}}[\int_0^T |\eta_t|^p dt]\right)^{\frac{1}{p}}.$$

For simplicity, we begin with the case of $1-$dimensional $G-$Brownian motion.
Let $(B_t)_{t\geq 0}$ be a $1-$dimensional $G-$Brownian motion with $G(\alpha) = \frac{1}{2}(\bar{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$, where $0 \leq \underline{\sigma} \leq \bar{\sigma} < \infty$.

Definition 4.20. *For each $\eta \in M_G^{2,0}(0,T)$ of the form*

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k,t_{k+1})}(t),$$

we define

$$\mathcal{I}(\eta) = \int_0^T \eta_t dB_t := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}} - B_{t_k}).$$

Proposition 4.21. *The mapping $\mathcal{I} : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ is a continuous linear mapping and thus can be continuously extended to $\mathcal{I} : M_G^2(0,T) \to L_G^2(\Omega_T)$. We have*

$$\widehat{\mathbb{E}}[\int_0^T \eta_t dB_t] = 0, \tag{4.2}$$

and

$$\widehat{\mathbb{E}}[(\int_0^T \eta_t dB_t)^2] \leq \bar{\sigma}^2 \widehat{\mathbb{E}}[\int_0^T \eta_t^2 dt]. \tag{4.3}$$

We now consider the multi-dimensional case. Let $G(\cdot) : \mathbb{S}^d \to \mathbb{R}$ be a given monotonic and sublinear function and let $(B_t)_{t\geq 0}$ be a $d-$dimensional $G-$Brownian motion. For each fixed $\mathbf{a} = (a_1, ..., a_d)^T$, a given vector in $\mathbb{R}^d$, we set $(B_t^{\mathbf{a}})_{t\geq 0} = (\mathbf{a}, B_t)_{t\geq 0}$, where $(\mathbf{a}, B_t)$ denotes the scalar product of $\mathbf{a}$ and B_t. Then $(B_t^{\mathbf{a}})_{t\geq 0}$ is a $1-$dimensional $G_{\mathbf{a}}-$Brownian motion with $G_{\mathbf{a}}(\alpha) = \frac{1}{2}(\sigma^2_{\mathbf{aa}^T}\alpha^+ - \sigma^2_{-\mathbf{aa}^T}\alpha^-)$, where $\sigma^2_{\mathbf{aa}^T} = 2G(\mathbf{aa}^T)$ and $\sigma^2_{-\mathbf{aa}^T} = -2G(-\mathbf{aa}^T)$. Similarly to $1-$dimensional case, we can define Itô's integral by

$$\mathcal{I}(\eta) := \int_0^T \eta_t dB_t^{\mathbf{a}}, \quad for\ \eta \in M_G^2(0,T).$$

We still have, for each $\eta \in M_G^2(0,T)$,

$$\widehat{\mathbb{E}}[\int_0^T \eta_t dB_t^{\mathbf{a}}] = 0,$$

and

$$\widehat{\mathbb{E}}[(\int_0^T \eta_t dB_t^{\mathbf{a}})^2] \leq \sigma^2_{\mathbf{aa}^T} \widehat{\mathbb{E}}[\int_0^T \eta_t^2 dt].$$

Proposition 4.22. *(Exercise 3.9 in [66]) For each $X \in M_G^p(0,T)$, we have*

$$\widehat{\mathbb{E}}[\int_0^T |X_t|^p dt] \leq \int_0^T \widehat{\mathbb{E}}[|X_t|^p] dt. \tag{4.4}$$

4.2.5 Quadratic Variation Process of $G-$Brownian motion

We first consider a $1-$dimensional $G-$Brownian motion $(B_t)_{t\geq 0}$. From the construction of quadratic variation process of $(B_t)_{t\geq 0}$ in Peng [66], we know that the quadratic variation process $(\langle B\rangle_t)_{t\geq 0}$ is an increasing process with $\langle B\rangle_0 = 0$. It characterizes the part of statistic uncertainty of $G-$Brownian motion.

A very interesting point of the quadratic variation process $(\langle B\rangle_t)_{t\geq 0}$ is, just like the $G-$Brownian motion $(B_t)_{t\geq 0}$ itself, the increment $\langle B\rangle_{s+t} - \langle B\rangle_s$ is independent from Ω_s and identically distributed with $\langle B\rangle_t$. In fact, we have

Lemma 4.23. *For each fixed $s, t \geq 0$, $\langle B\rangle_{s+t} - \langle B\rangle_s$ is identically distributed with $\langle B\rangle_t$ and independent from Ω_s.*

We now define the integral of a process $\eta \in M_G^1(0,T)$ with respect to $\langle B \rangle$. We begin with the simple processes, the mapping $\mathcal{Q}_{0,T} : M_G^{1,0}(0,T) \to L_G^1(\Omega_T)$ is defined as following:

$$\mathcal{Q}_{0,T}(\eta) = \int_0^T \eta_t d\langle B \rangle_t := \sum_{k=0}^{N-1} \xi_k(\langle B \rangle_{t_{k+1}} - \langle B \rangle_{t_k}).$$

Proposition 4.24. *For each $\eta \in M_G^{1,0}(0,T)$,*

$$\widehat{\mathbb{E}}[|\mathcal{Q}_{0,T}(\eta)|] \leq \bar{\sigma}^2 \widehat{\mathbb{E}}[\int_0^T |\eta_t| dt]. \tag{4.5}$$

Thus $\mathcal{Q}_{0,T} : M_G^{1,0}(0,T) \to L_G^1(\Omega_T)$ is a continuous linear mapping. Consequently, $\mathcal{Q}_{0,T}$ can be uniquely extended to $M_G^1(0,T)$. We still denote this mapping by

$$\int_0^T \eta_t d\langle B \rangle_t := \mathcal{Q}_{0,T}(\eta), \quad \text{for } \eta \in M_G^1(0,T).$$

Moreover, we still have

$$\widehat{\mathbb{E}}[|\int_0^T \eta_t d\langle B \rangle_t|] \leq \bar{\sigma}^2 \widehat{\mathbb{E}}[\int_0^T |\eta_t| dt], \quad \text{for } \eta \in M_G^1(0,T). \tag{4.6}$$

Proposition 4.25. *Let $0 \leq s \leq t$, $\xi \in L_G^2(\Omega_s)$, $X \in L_G^1(\Omega)$. Then*

$$\begin{aligned} \widehat{\mathbb{E}}[X + \xi(B_t^2 - B_s^2)] &= \widehat{\mathbb{E}}[X + \xi(B_t - B_s)^2] \\ &= \widehat{\mathbb{E}}[X + \xi(\langle B \rangle_t - \langle B \rangle_s)]. \end{aligned}$$

Proposition 4.26. *Let $\eta \in M_G^2(0,T)$. Then*

$$\widehat{\mathbb{E}}[(\int_0^T \eta_t dB_t)^2] = \widehat{\mathbb{E}}[\int_0^T \eta_t^2 d\langle B \rangle_t]. \tag{4.7}$$

We now consider the multi-dimensional case. Let $(B_t)_{t\geq 0}$ be a $d-$dimensional $G-$Brownian motion. For each fixed $\mathbf{a} \in \mathbb{R}^d$, $(B_t^{\mathbf{a}})_{t\geq 0}$ is a $1-$dimensional $G_{\mathbf{a}}-$Brownian motion. Similar to $1-$dimensional case, we can define

$$\langle B^{\mathbf{a}} \rangle_t := \lim_{\mu(\pi_t^N)\to 0} \sum_{k=0}^{N-1} (B^{\mathbf{a}}_{t_{k+1}^N} - B^{\mathbf{a}}_{t_k^N})^2 = (B_t^{\mathbf{a}})^2 - 2\int_0^t B_s^{\mathbf{a}} dB_s^{\mathbf{a}},$$

where $\langle B^{\mathbf{a}} \rangle$ is called the quadratic variation process of $B^{\mathbf{a}}$. The above results also hold for $\langle B^{\mathbf{a}} \rangle$. In particular,

$$\widehat{\mathbb{E}}[|\int_0^T \eta_t d\langle B^{\mathbf{a}} \rangle_t|] \leq \sigma^2_{\mathbf{a}\mathbf{a}^T} \widehat{\mathbb{E}}[\int_0^T |\eta_t| dt] \quad \text{for } \eta \in M_G^1(0,T)$$

and

$$\widehat{\mathbb{E}}[(\int_0^T \eta_t dB_t^{\mathbf{a}})^2] = \widehat{\mathbb{E}}[\int_0^T \eta_t^2 d\langle B^{\mathbf{a}} \rangle_t] \quad \text{for } \eta \in M_G^2(0,T).$$

Let $\mathbf{a} = (a_1, ..., a_d)^T$ and $\bar{\mathbf{a}} = (\bar{a}_1, ..., \bar{a}_d)^T$ be two given vectors in $\mathbb{R}^d$. We then have their quadratic variation process of $\langle B^{\mathbf{a}} \rangle$ and $\langle B^{\bar{\mathbf{a}}} \rangle$. We can define their **mutual variation process** by

$$\begin{aligned} \langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t : &= \frac{1}{4}[\langle B^{\mathbf{a}} + B^{\bar{\mathbf{a}}} \rangle_t - \langle B^{\mathbf{a}} - B^{\bar{\mathbf{a}}} \rangle_t] \\ &= \frac{1}{4}[\langle B^{\mathbf{a}+\bar{\mathbf{a}}} \rangle_t - \langle B^{\mathbf{a}-\bar{\mathbf{a}}} \rangle_t]. \end{aligned}$$

Definition 4.27. *Define the mapping* $M_G^{1,0}(0,T) \to L_G^1(\Omega_T)$ *as follows:*

$$\mathcal{Q}(\eta) = \int_0^T \eta_t d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t := \sum_{k=0}^{N-1} \xi_k (B^{\mathbf{a}}_{t_{k+1}} - B^{\mathbf{a}}_{t_k})(B^{\bar{\mathbf{a}}}_{t_{k+1}} - B^{\bar{\mathbf{a}}}_{t_k}).$$

Then $\mathcal{Q}$ can be uniquely extended to $M_G^1(0,T)$. We still use $\mathcal{Q}(\eta)$ to denote the mapping $\int_0^T \eta(s) d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_s$, $\eta \in M_G^1(0,T)$.

4.2.6 $G-$Itô's formula

Theorem 4.28. *(Proposition 6.3 of [66]) Let α^ν, $\eta^{\nu ij}$ and $\beta^{\nu j} \in M_G^2(0,T)$, $\nu = 1, ..., n$, $i, j = 1, ..., d$ be bounded processes and consider*

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \sum_{i,j=1}^d \int_0^t \eta_s^{\nu_{ij}} d\langle B^i, B^j \rangle_s + \sum_{j=1}^d \int_0^t \beta_s^{\nu_j} dB_s^j,$$

where $X_0^\nu \in \mathbb{R}$, $\nu = 1, ..., n$. Let $\Phi \in C^2(\mathbb{R}^n)$ be a real function with bounded derivatives such that $\{\partial^2_{x^\mu x^\nu} \Phi\}_{\mu,\nu=1}^n$ are uniformly Lipschitz. Then for each $s, t \in [0,T]$, in $L_G^2(\Omega_t)$

$$\begin{aligned} \Phi(X_t) - \Phi(X_s) &= \int_s^t \partial_{x^\nu} \Phi(X_u) \alpha_u^\nu du + \int_s^t \partial_{x^\nu} \Phi(X_u) \eta_u^{\nu_{ij}} d\langle B^i, B^j \rangle_u \\ &+ \int_s^t \partial_{x^\nu} \Phi(X_u) \beta_u^{\nu_j} dB_u^j + \frac{1}{2} \int_s^t \partial^2_{x^\mu x^\nu} \Phi(X_u) \beta_u^{\mu_i} \beta_u^{\nu_j} d\langle B^i, B^j \rangle_u, \end{aligned}$$

where the repeated indices ν, μ, i and j imply the summation.

Proposition 4.29. *(Proposition 6.4 in [66]) Let $X \in M_G^p(0,T)$ with $p \geq 2$ and let $\mathbf{a} \in \mathbb{R}^d$ be fixed. Then we have $\int_0^T X_t dB_t^{\mathbf{a}} \in L_G^p(\Omega_T)$ and*

$$\widehat{\mathbb{E}}[|\int_0^T X_t dB_t^{\mathbf{a}}|^p] \leq C_p \widehat{\mathbb{E}}[|\int_0^T X_t^2 d\langle B^{\mathbf{a}} \rangle_t|^{p/2}]. \tag{4.8}$$

4.3 Quasi-sure stochastic integral for Hilbert space valued processes

In this section, we will define the stochastic integral for Hilbert space valued processes with respect to $G-$Brownian motion which we will use in the study of stochastic partial

differential equations driven by $G-$Brownian motion.
Firstly, we fix $G(\cdot):\mathbb{S}^d\to\mathbb{R}$ a monotonic and sublinear function. By Theorem 2.1 in [66], we know that there exists a bounded, convex and closed subset $\Theta\subset\mathbb{S}^d$ such that $G(A)=\frac{1}{2}\sup_{B\in\Theta}(A,B),\ A\in\mathbb{S}^d$. Furthermore, we know that $G-$normal distribution $N(0,\Theta)$ exists. We consider the associated $G-$Brownian motion $\{B_t:=(B_t^j)_{j\in\{1,\dots,d\}}\}_{t\geq 0}$ and the sublinear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$.

Remark 4.30. *As Θ is bounded, there exists a constant $\bar{\sigma}$ such that $\sum_{j=1}^d d\langle B^j\rangle_t\leq\bar{\sigma}^2dt$.*

Let H be a separable Hilbert space equipped with the norm $\|\cdot\|_H$ and the scalar product $(\cdot,\cdot)_H$. Let $(e_k)_{k\in\mathbb{N}^*}$ be an orthogonal basis of H. For $H-$valued random variable X, which satisfies $\sum_{k=1}^\infty(\widehat{\mathbb{E}}(X,e_k)_H)^2<+\infty$, we define

$$\widehat{\mathbb{E}}[X]=\sum_{k=1}^\infty\widehat{\mathbb{E}}(X,e_k)_He_k.$$

We introduce the following space: for fixed $T\geq 0$,

$$\begin{aligned}L_{ip}(\Omega_T;H):\ &=\ \{\sum_{k=1}^N\varphi_k(B_{t_1},...,B_{t_n})e_k:\forall n\geq 1,\ t_1,...,t_n\in[0,T],\\ &\qquad N\geq 1,\varphi_k\in C_{b,Lip}(\mathbb{R}^{d\times n}),\forall k\in\{1,...,N\}\}.\end{aligned}$$

It is clear that $L_{ip}(\Omega_t;H)\subset L_{ip}(\Omega_T;H)$, for $t\in[0,T]$. We also denote

$$L_{ip}(\Omega;H):=\bigcup_{n=1}^{+\infty}L_{ip}(\Omega_n;H).$$

We denote by $L_G^p(\Omega_T;H)$ (resp. $L_G^p(\Omega;H)$) the topological completion of $L_{ip}(\Omega_T;H)$ (resp. $L_{ip}(\Omega;H)$) under the Banach norm $\mathbb{E}[|\cdot|^p]^{\frac{1}{p}}$, $1\leq p\leq\infty$. The same argument as Proposition 24 in [18] yields that each element in $L_G^2(\Omega;H)$ has a quasi-continuous version.
We denote by $M_G^2([0,T];H)$ the class of $H-$valued progressively measurable processes X such that

$$\|X\|^2_{M_G^2([0,T];H)}=\widehat{\mathbb{E}}\left(\int_0^T\|X_t\|_H^2\,dt\right)<+\infty.$$

To construct the stochastic integral, we start with simple processes and then, in a classical way, we extend it to square integrable progressively measurable processes:
We consider the following type of simple processes: for a given partition $\pi_T=\{t_0,...,t_N\}$ of $[0,T]$ we set

$$X_t(\omega)=\sum_{i=0}^{N-1}\xi_i(\omega)I_{[t_i,t_{i+1})}(t),$$

where $\xi_i\in L_G^2(\Omega_{t_i};H)$, $i=0,1,...,N-1$ are given. The collection of these processes denoted by $M_G^{2,0}([0,T];H)$. We have the following proposition:

Proposition 4.31. $(M_G^2([0,T];H),\|\cdot\|_{M_G^2([0,T];H)})$ *is a Banach space and* $M_G^{2,0}([0,T];H)$ *is dense in it.*

We now can construct stochastic integral in an easy way: assume first that $X = (X^1, ..., X^d)$ belongs to $(M_G^{2,0}([0,T];H))^d$ of the form

$$\forall j \in \{1,...,d\},\ \forall t \in [0,T],\ X_t^j(\omega) = \sum_{i=0}^{N-1} \xi_i^j(\omega) I_{[t_i,t_{i+1})}(t),$$

we define

$$\forall t \in [0,T], \quad I_t^X = \int_0^t X_s dB_s := \sum_{j=1}^{d} \sum_{i=0}^{N-1} \xi_i^j (B^j_{t_{i+1}\wedge t} - B^j_{t_i \wedge t}).$$

Proposition 4.32. *The mapping $I : (M_G^{2,0}([0,T];H))^d \to L_G^2(\Omega_T;H)$ is a continuous linear mapping and can be continuously extended to $I : (M_G^2([0,T];H))^d \to L_G^2(\Omega_T;H)$. We have, for all $t \in [0,T]$,*

$$\widehat{\mathbb{E}}[\int_0^t X_s dB_s] = 0 \tag{4.9}$$

and

$$\widehat{\mathbb{E}}[\|I_t^X\|_H^2] \le \bar{\sigma}^2 \widehat{\mathbb{E}}\left[\int_0^t \|X_s\|^2_{H^d} \, ds\right]. \tag{4.10}$$

where $\bar{\sigma}$ is defined in Remark 4.30.

Proof. For each i and each j, see Example 2.10 in [66], we have

$$\widehat{\mathbb{E}}[\xi_i^j (B^j_{t_{i+1}} - B^j_{t_i})|\Omega_{t_i}] = \widehat{\mathbb{E}}[-\xi_i^j (B^j_{t_{i+1}} - B^j_{t_i})|\Omega_{t_i}] = 0.$$

Thus, we have

$$\begin{aligned}
\widehat{\mathbb{E}}[\int_0^t X_s dB_s] &= \widehat{\mathbb{E}}[\sum_{j=1}^{d} \left(\int_0^{t_{N-1}\wedge t} X_s^j dB_s^j + \xi_{N-1}^j (B^j_{t_N \wedge t} - B^j_{t_{N-1}\wedge t}) \right)] \\
&= \widehat{\mathbb{E}}[\sum_{j=1}^{d} \left(\int_0^{t_{N-1}\wedge t} X_s^j dB_s^j + \widehat{\mathbb{E}}[\xi_{N-1}^j (B^j_{t_N \wedge t} - B^j_{t_{N-1}\wedge t})|\Omega_{t_{N-1}\wedge t}] \right)] \\
&= \widehat{\mathbb{E}}[\sum_{j=1}^{d} \int_0^{t_{N-1}\wedge t} X_s^j dB_s^j] \\
&= \widehat{\mathbb{E}}[\int_0^{t_{N-1}\wedge t} X_s dB_s].
\end{aligned}$$

Then we repeat this procedure to get (4.9).
Now we give the proof of (4.10):

$$\begin{aligned}
\widehat{\mathbb{E}}[\| I_t^X \|_H^2] &= \sup_{P \in \mathcal{P}} E_P[\| I_t^X \|_H^2] \\
&= \sup_{P \in \mathcal{P}} E_P[\int_0^t \| X_s \|^2_{H^d} \, d\langle B \rangle_s] \\
&= \widehat{\mathbb{E}}[\int_0^t \| X_s \|^2_{H^d} \, d\langle B \rangle_s] \\
&\le \bar{\sigma}^2 \widehat{\mathbb{E}}[\int_0^t \| X_s \|^2_{H^d} \, ds].
\end{aligned}$$

We conclude by using a density argument. ■

Proposition 4.33. *The map* $I^X : (M_G^2([0,T];H))^d \to L_G^2(\Omega_T;H)$ *satisfies the Doob's inequality:*

$$\widehat{\mathbb{E}}(\sup_{t\in[0,T]} \left\|I_t^X\right\|_H^2) \leq 4\bar{\sigma}^2 \widehat{\mathbb{E}} \int_0^T \|X_t\|_{H^d}^2 \, dt. \tag{4.11}$$

Proof. The Doob's inequality under each $P \in \mathcal{P}$ yields

$$\begin{aligned}
\widehat{\mathbb{E}}\left(\sup_{t\in[0,T]} \left\|I_t^X\right\|_H^2\right) &= \widehat{\mathbb{E}}\left(\sup_{t\in[0,T]} [\sum_{k=1}^{\infty}(\int_0^t \xi_s dB_s, e_k)_H^2]\right) \\
&\leq \widehat{\mathbb{E}}\left(\sum_{k=1}^{\infty} \sup_{t\in[0,T]} [(\int_0^t \xi_s dB_s, e_k)_H^2]\right) \\
&= \sup_{P\in\mathcal{P}} E_P\left(\sum_{k=1}^{\infty} \sup_{t\in[0,T]} [\sum_{j=1}^{d} \int_0^t (\xi_s^j, e_k) dB_s^j]^2\right) \\
&\leq 4\sup_{P\in\mathcal{P}} E_P\left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{d}\int_0^T (\xi_s^j, e_k) dB_s^j\right)^2\right) \\
&= 4\sup_{P\in\mathcal{P}} E_P\left(\sum_{k=1}^{\infty}\sum_{j=1}^{d}\int_0^T (\xi_s^j, e_k)^2 d\langle B^j\rangle_s\right) \\
&= 4\widehat{\mathbb{E}}\left(\int_0^T \|X_s\|_{H^d}^2 \, d\langle B\rangle_s\right) \\
&\leq 4\bar{\sigma}^2\widehat{\mathbb{E}}\left(\int_0^T \|X_s\|_{H^d}^2 \, ds\right).
\end{aligned}$$

■

4.4 Quasilinear Stochastic PDEs driven by $G-$Brownian motion

We use the analytical method to prove the existence and uniqueness of the solution of the following quasilinear stochastic partial differential equation driven by a d_1-dimensional $G-$Brownian motion:

$$\begin{aligned}
du_t(x) &= \partial_i\left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right)dt + f(t,x,u_t(x),\nabla u_t(x))dt \\
&\quad + \sum_{j=1}^{d_1} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j.
\end{aligned} \tag{4.12}$$

4.4.1 Preliminaries

We fix $G(\cdot) : \mathbb{S}^{d_1} \to \mathbb{R}$ a monotonic and sublinear function. By Theorem 2.1 in [66], we know that there exists a bounded, convex and closed subset $\Theta \subset \mathbb{S}^{d_1}$ such that $G(A) =$

$\frac{1}{2}\sup_{B\in\Theta}(A,B)$, $A \in \mathbb{S}^{d_1}$. Furthermore, we know that $G-$normal distribution $N(0,\Theta)$ exists. We consider the associated $G-$Brownian motion $\{B_t := (B_t^j)_{j\in\{1,\dots,d_1\}}\}_{t\geq 0}$ and the sublinear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$.
Let $\mathcal{O} \subset \mathbb{R}^d$ be an open domain and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$, it is an Hilbert space equipped with the usual scalar product and norm as follows

$$(u,v) = \int_{\mathcal{O}} u(x)v(x)dx, \qquad \| u \| = (\int_{\mathcal{O}} u^2(x)dx)^{1/2}.$$

Let A be a symmetric second order differential operator, with domain $\mathcal{D}(A)$, given by

$$A := -\sum_{i,j=1}^{d} \partial_i(a^{i,j}\partial_j).$$

We assume that $a = (a^{i,j})_{i,j}$ is a measurable symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{d} a^{i,j}(x)\xi^i\xi^j \leq \Lambda|\xi|^2, \ \forall x\in\mathcal{O}, \ \xi\in\mathbb{R}^d,$$

where λ and Λ are positive constants.

Let $(F,\mathcal{E})$ be the associated Dirichlet form given by $F := \mathcal{D}(A^{1/2}) = H_0^1(\mathcal{O})$ and

$$\mathcal{E}(u,v) := (A^{1/2}u, A^{1/2}v) \ and \ \mathcal{E}(u) = \| A^{1/2}u \|^2, \ \forall u,v\in F,$$

where $H_0^1(\mathcal{O})$ is the first order Sobolev space of functions vanishing at the boundary.

We consider the quasilinear stochastic differential equation (4.12) with initial condition $u(0,\cdot) = \xi(\cdot) \in L^2(\mathcal{O})$, and Dirichlet boundary condition $u(t,x) = 0$, $\forall (t,x)\in\mathbb{R}_+\times\partial\mathcal{O}$.

Assumption (H): f, g and h are random functions satisfying the following Lipschitz conditions:

1. $|f(t,\omega,x,y,z) - f(t,\omega,x,y',z')| \leq C(|y-y'| + |z-z'|)$,
2. $(\sum_{i=1}^{d} |g_i(t,\omega,x,y,z) - g_i(t,\omega,x,y',z')|^2)^{\frac{1}{2}} \leq C|y-y'| + \alpha|z-z'|$,
3. $(\sum_{j=1}^{d_1} |h^j(t,\omega,x,y,z) - h^j(t,\omega,x,y',z')|^2)^{\frac{1}{2}} \leq C|y-y'| + \beta|z-z'|$,
4. the contraction property: $2\alpha + \beta^2\bar{\sigma}^2 < 2\lambda$.

With the uniform ellipticity condition we have the following equivalent conditions:

$$\| f(u,\nabla u) - f(v,\nabla v) \| \leq C \| u-v \| + C\lambda^{-1/2}\mathcal{E}^{1/2}(u-v)$$
$$\| g(u,\nabla u) - g(v,\nabla v) \|_{L^2(\mathcal{O};\mathbb{R}^d)} \leq C \| u-v \| + \alpha\lambda^{-1/2}\mathcal{E}^{1/2}(u-v)$$
$$\| h(u,\nabla u) - h(v,\nabla v) \|_{L^2(\mathcal{O};\mathbb{R}^{\mathbb{N}})} \leq C \| u-v \| + \beta\lambda^{-1/2}\mathcal{E}^{1/2}(u-v)$$

Assumption (I): Moreover we assume that for any $T>0$,

$$f(\cdot,\cdot,\cdot,0,0) := f^0 \in M_G^2([0,T];L^2(\mathcal{O}));$$
$$g(\cdot,\cdot,\cdot,0,0) := g^0 = (g_1^0,\dots,g_d^0) \in M_G^2([0,T];L^2(\mathcal{O}));$$
$$h(\cdot,\cdot,\cdot,0,0) := h^0 = (h_1^0,\dots,h_{d_1}^0) \in M_G^2([0,T];L^2(\mathcal{O})).$$

4.4.2 The existence and uniqueness result

We denote by $M_G^2([0,T];H_0^1(\mathcal{O}))$ the completion of $M_G^{2,0}([0,T];H_0^1(\mathcal{O}))$ under the norm

$$\left(\widehat{\mathbb{E}}\int_0^T \|X_t\|^2_{H_0^1(\mathcal{O})}\,dt\right)^{1/2}.$$

It is clear that $(M_G^2([0,T];H_0^1(\mathcal{O})), \|\cdot\|_{M_G^2([0,T];H_0^1(\mathcal{O}))})$ is a Banach space. We denote by $\mathcal{H}_T^G$ the sub-space of processes $u \in M_G^2([0,T];H_0^1(\mathcal{O}))$ with $L^2(\mathcal{O})-$continuous trajectories. That is for almost all $\omega \in \Omega$, $t \to X_t(\omega)$ is $L^2(\mathcal{O})-$continuous. This space will be endowed with the norm

$$\left(\widehat{\mathbb{E}}\sup_{t\in[0,T]} \parallel X_t \parallel^2 + \widehat{\mathbb{E}}\int_0^T \mathcal{E}(X_t)dt\right)^{1/2}.$$

It is clear that $(\mathcal{H}_T^G, \|\cdot\|_{\mathcal{H}_T^G})$ is a Banach space. It is the basic space in which we are going to look for solutions.

As in the standard case we consider the space of test functions denoted by $\mathcal{D} = \mathcal{C}_c^\infty(\mathbb{R}^+) \times \mathcal{C}_c^2(\mathcal{O})$, where $\mathcal{C}_c^\infty(\mathbb{R}^+)$ is the space of all real valued infinite differentiable functions with compact support in $\mathbb{R}^+$ and $\mathcal{C}_c^2(\mathcal{O})$ the set of C^2-functions with compact support in $\mathcal{O}$.

Definition 4.34. *(Mild solution) We say that $u \in \mathcal{H}_T^G$ is a mild solution of the equation (4.12) if the following equality is verified quasi surely, for each $t \in [0,T]$,*

$$u_t = P_t\xi + \int_0^t P_{t-s}f_s ds + \int_0^t P_{t-s}div g_s ds + \int_0^t P_{t-s}h_s dB_s \tag{4.13}$$

Definition 4.35. *(Weak solution) We say that $u \in \mathcal{H}_T^G$ is a weak solution of the equation (4.12) if the following relation holds quasi surely for each $\varphi \in \mathcal{D}$,*

$$\begin{aligned}(u_t,\varphi_t) - (\xi,\varphi_0) - \int_0^t (u_s,\partial_s\varphi_s)ds + \int_0^t \mathcal{E}(u_s,\varphi_s)ds \\ = \int_0^t (f_s,\varphi_s)ds - \int_0^t (g_s,\nabla\varphi_s)ds + \int_0^t (h_s,\varphi_s)dB_s\end{aligned} \tag{4.14}$$

We start by showing that the quantities appearing in (4.13) are well-defined.

Lemma 4.36. *Let ξ be in $L^2(\mathcal{O})$. Then*

1. *$\Xi :\ t \in [0,T] \to P_t\xi$ admits a continuous version in $L^2([0,T];H_0^1(\mathcal{O}))\cap L^\infty([0,T];L^2(\mathcal{O}))$;*
2. *for all $\varphi \in \mathcal{D}$ and for all $t \in [0,T]$, we have*

$$\int_0^t (\Xi_s,\partial_s\varphi_s)ds = (\Xi_t,\varphi_t) - (\xi,\varphi_0) + \int_0^t \mathcal{E}(\Xi_s,\varphi_s)ds. \tag{4.15}$$

Proof. See Lemma 1.31 in Chapter 1. ∎

Lemma 4.37. *Let f be in $M_G^2([0,T];L^2(\mathcal{O}))$ and adapted. Then*

1. *the process* $\alpha: \ t \in [0,T] \rightarrow \int_0^t P_{t-s} f_s ds$ *admits a version in* $\mathcal{H}_T^G$ *and there exists a constant* C *depending only on* T *and the structure constants of the GSPDE such that*

$$\| \alpha \|_{\mathcal{H}_T^G} \leq C \| f \|_{M_G^2([0,T];H_0^1(\mathcal{O}))} ;$$

2. *for all* $\varphi \in \mathcal{D}$ *and all* $t \in [0,T]$*, we have*

$$\int_0^t (\alpha_s, \partial_s \varphi_s) ds = (\alpha_t, \varphi_t) - \int_0^t (f_s, \varphi_s) dt + \int_0^t \mathcal{E}(\alpha_s, \varphi_s) ds \quad q.s.$$

Proof. Assume firts that $f \in C^1([0,T]) \otimes L_G^2(\Omega_T) \otimes \mathcal{D}(L)$ and is adapted. It is easy to check that $C^1([0,T]) \otimes L_G^2(\Omega_T) \otimes \mathcal{D}(L)$ is dense in $M_G^2([0,T]; H_0^1(\mathcal{O}))$. Fix $\omega \in \Omega_T$, for all $t \in [0,T]$, $\alpha_t(\omega) \in \mathcal{D}(L)$ and $t \rightarrow \alpha_t(\omega)$ is $L^2(\mathcal{O})-$differentiable and satisfies

$$\forall \ t \in [0,T], \ \frac{d\alpha_t}{dt}(\omega) = f_t(\omega) + L\alpha_t(\omega).$$

Integrating by part we get, for all $\varphi \in \mathcal{D}$ and all $t \in [0,T]$,

$$\int_0^t (\alpha_s, \partial_s \varphi_s) ds = (\alpha_t, \varphi_t) - \int_0^t (f_s, \varphi_s) ds + \int_0^t \mathcal{E}(\alpha_s, \varphi_s) ds.$$

Moreover, still integrating by part, we have, for all $t \in [0,T]$,

$$\begin{aligned} \| \alpha_t \|^2 &= 2 \int_0^t (\partial_s \alpha_s, \alpha_s) ds \\ &= 2 \int_0^t (f_s + L\alpha_s, \alpha_s) ds \\ &= 2 \int_0^t (f_s, \alpha_s) ds - 2 \int_0^t \mathcal{E}(\alpha_s) ds. \end{aligned}$$

This yields

$$\| \alpha_t \|^2 + 2 \int_0^t \mathcal{E}(\alpha_s) ds = 2 \int_0^t (f_s, \alpha_s) ds \leq \int_0^t (\| f_s \|^2 + \| \alpha_s \|^2) ds.$$

Taking the supreme, we get quasi-surely

$$\sup_{t \in [0,T]} \| \alpha_t \|^2 \leq \int_0^T \| f_t \|^2 \, dt + \int_0^T \sup_{t \in [0,T]} \| \alpha_t \|^2 \, dt.$$

Thanks to the Grownall's lemma, we have

$$\sup_{t \in [0,T]} \| \alpha_t \|^2 \leq e^T \int_0^T \| f_t \|^2 \, dt, \quad q.s.$$

and

$$2 \int_0^T \mathcal{E}(\alpha_t) dt \leq \int_0^T \| f_t \|^2 + \| \alpha_t \|^2 \, dt \leq (1 + Te^T) \int_0^T \| f_t \|^2 \, dt, \quad q.s.$$

Hence, we deduce that

$$\widehat{\mathbb{E}} \sup_{t\in[0,T]} \| \alpha_t \|^2 + \widehat{\mathbb{E}} \int_0^T \mathcal{E}(\alpha_t)dt \leq (e^T + \frac{1}{2} + \frac{T}{2}e^T)\widehat{\mathbb{E}} \int_0^T \| f_t \|^2 dt.$$

Then for the general case i.e. $f \in M_G^2([0,T]; H_0^1(\mathcal{O}))$, we take $f^n \in C^1([0,T]) \otimes L_G^2(\Omega_T) \otimes \mathcal{D}(L)$ such that $(f^n)_n$ converges to f in $M_G^2([0,T]; H_0^1(\mathcal{O}))$. From the above estimate, we have

$$\| \alpha^n - \alpha^m \|_{\mathcal{H}_T^G} \leq C \| f^n - f^m \|_{M_G^2}, \quad \forall n,\ m \in \mathbb{N}^*.$$

This yields

$$\| \alpha^n - \alpha^m \|_{M_G^2} \leq \| \alpha^n - \alpha^m \|_{\mathcal{H}_T^G} \leq C \| f^n - f^m \|_{M_G^2} \to 0, \ when\ n, m \to \infty.$$

So that we have $\alpha \in \mathcal{H}_T^G$ which is the limit of $(\alpha^n)_n$ in $M_G^2([0,T]; H_0^1(\mathcal{O}))$. ■

Lemma 4.38. *Let g be in $M_G^2([0,T]; L^2(\mathcal{O}))$ and adapted. Then*

1. *the process $\gamma: \ t \to \int_0^t P_{t-s} divg_s ds$ admits a version in $\mathcal{H}_T^G$ and there exists a constant C depending only on T and the structure constants of the GSPDE such that*

$$\| \gamma \|_{\mathcal{H}_T^G} \leq C \| g \|_{M_G^2([0,T]; H_0^1(\mathcal{O}))};$$

2. *for all $\varphi \in \mathcal{D}$ and for all $t \in [0,T]$, we have*

$$\int_0^t (\gamma_s, \partial_s \varphi_s)ds = (\gamma_t, \varphi_t) + \int_0^t (g_s, \partial\varphi_s)ds + \int_0^t \mathcal{E}(\gamma_s, \varphi_s)ds \quad q.s.$$

Proof. Assume first that $g \in C^1([0,T]) \otimes L_G^2(\Omega_T) \otimes \mathcal{D}(L^{3/2})$ and is adapted. It is clear that $divg \in C^1([0,T]) \otimes L^2(\Omega) \otimes \mathcal{D}(L)$ and $C^1([0,T]) \otimes L_G^2(\Omega_T) \otimes \mathcal{D}(L^{3/2})$ is dense in $M_G^2([0,T]; H_0^1(\mathcal{O}))$.

We fix $\omega \in \Omega_T$, for all $t \in [0,T]$, $\gamma_t(\omega) \in \mathcal{D}(L)$ and $t \to \gamma_t(\omega)$ is $L^2(\mathcal{O})-$differentiable and satisfies

$$\forall\ t \in [0,T], \ \frac{d\gamma_t}{dt}(\omega) = divg_t(\omega) + L\gamma_t(\omega).$$

Integrating by part, for all $\varphi \in \mathcal{D}$ and for all $t \in [0,T]$, we get:

$$\begin{aligned}
\int_0^t (\gamma_s, \partial_s \varphi_s)ds &= (\gamma_t, \varphi_t) - \int_0^t (divg_s, \varphi_s)ds + \int_0^t \mathcal{E}(\gamma_s, \varphi_s)ds \\
&= (\gamma_t, \varphi_t) + \int_0^t (g_s, \partial\varphi_s)ds + \int_0^t \mathcal{E}(\gamma_s, \varphi_s)ds.
\end{aligned}$$

Moreover, still integrating by part, we obtain, $\forall t \in [0,T]$,

$$\begin{aligned}
\| \gamma_t \|_{L^2}^2 &= 2\int_0^t (\partial_s \gamma_s, \gamma_s)ds = 2\int_0^t (divg_s + L\gamma_s, \gamma_s)ds \\
&= 2\int_0^t (divg_s, \gamma_s)ds + 2\int_0^t (L\gamma_s, \gamma_s)ds \\
&= -2\int_0^t (g_s, \partial\gamma_s)ds - 2\int_0^t \mathcal{E}(\gamma_s)ds \qquad (4.16)
\end{aligned}$$

Using the inequality $ab \leq c_\epsilon a^2 + \epsilon b^2$ and the uniform elliptic condition, we get

$$\| \gamma_t \|^2 + 2\int_0^t \mathcal{E}(\gamma_s)ds \leq \int_0^t \left(\frac{1}{\epsilon} \| g_s \|^2 + \frac{\epsilon}{\lambda}\mathcal{E}(\gamma_s) \right) ds\,, \quad q.s.$$

therefore,

$$\| \gamma_t \|^2 + (2 - \frac{\epsilon}{\lambda}) \int_0^t \mathcal{E}(\gamma_s)ds \leq \frac{1}{\epsilon} \int_0^t \| g_s \|^2 \, ds, \quad q.s.$$

We can take ϵ small enough such that $(2 - \frac{\epsilon}{\lambda}) > 0$, then taking the supremum, we have the following two relations:

$$\sup_{t\in[0,T]} \| \gamma_t \|^2 \leq \frac{1}{\epsilon} \int_0^T \| g_s \|^2 \, ds, \quad q.s.$$

and

$$\int_0^T \mathcal{E}(\gamma_s)ds \leq \frac{\lambda}{\epsilon(2\lambda - \epsilon)} \int_0^T \| g_s \|^2 \, ds, \quad q.s.$$

These yield

$$\widehat{\mathbb{E}} \sup_{t\in[0,T]} \| \gamma_t \|^2 + \widehat{\mathbb{E}} \int_0^T \mathcal{E}(\gamma_t)dt \leq C\widehat{\mathbb{E}} \int_0^T \| g_t \|^2 \, dt.$$

For the general case, we do the similar density argument as in the previous lemma to get the desired results. ■

Lemma 4.39. *Let h be in $(M_G^2([0,T]; L^2(\mathcal{O})))^d$, then*

1. *the process $t \in [0,T] \to \beta_t = \int_0^t P_{t-s}h_s dB_s$ admits a version in $\mathcal{H}_T^G$ and there exists a constant C depending only on T and the structure constants of the GSPDE such that*

$$\| \beta \|_{\mathcal{H}_T^G} \leq C \| h \|_{M_G^2([0,T];H_0^1(\mathcal{O}))};$$

2. *for all $\varphi \in \mathcal{D}$:*

$$\int_0^T (\beta_t, \partial_t \varphi_t)dt = -\int_0^T (h_t, \varphi_t)dB_t + \int_0^T \mathcal{E}(\beta_t, \varphi_t)dt \quad q.s. \tag{4.17}$$

We denote by $P_{t-s}h_s = (P_{t-s}h_s^1, ..., P_{t-s}h_s^d)$ and $(h_t, \varphi_t) = ((h_t^1, \varphi_t), ..., (h_t^d, \varphi_t))$ for $h = (h^1, ..., h^d)$.

Proof. We denote by $\mathcal{S}$ the set of processes h such that:

$$\forall\, (t, x, \omega) \in [0,T] \times \mathcal{O} \times \Omega,\ h(t, x, \omega) = \sum_{i=0}^{n-1} h_i(x, \omega) I_{[t_i, t_{i+1})}(t),$$

where $n \in \mathbb{N}^*$, $0 \leq t_0 \leq t_1 \leq ... \leq t_n \leq T$ and for all $i \in \{0, 1, ..., n-1\}$,

$$\forall (x, \omega) \in \mathcal{O} \times \Omega, \quad h_i(x, \omega) = \sum_{j=1}^{n_i} \kappa_i^j(\omega) h_i^j(x)$$

where $n_i \in \mathbb{N}^*$ and for all $j \in \{1, ..., n_i\}$, $h_i^j \in Dom(L)$ and $\kappa_i^j \in L_G^2(\Omega_{t_i})$.
As $Dom(L)$ is dense in $L^2(\mathcal{O})$, we can easily prove that $\mathcal{S}$ is a dense subspace in $M_G^{2,0}([0,T]; L^2(\mathcal{O}))$ hence in $M_G^2([0,T]; L^2(\mathcal{O}))$.
First assume that $h \in \mathcal{S}^d$. The process

$$\forall t \in [0,T],\ \beta_t = \int_0^t P_{t-s} h_s dB_s$$

admits a version in $\mathcal{H}_T^G$. A direct calculation yields:

$$\begin{aligned} d\beta_t &= P_{t-t} h_t dB_t + \int_0^t \frac{\partial}{\partial t}(P_{t-s} h_s dB_s) \cdot dt \\ &= h_t dB_t + \int_0^t L(P_{t-s} h_s dB_s) \cdot dt \\ &= h_t dB_t + L\beta_t dt \end{aligned}$$

then

$$\beta_t = \int_0^t h_s dB_s + \int_0^t L\beta_s ds$$

Integrating by part, we have, $\forall \phi \in \mathcal{D}_0$,

$$\begin{aligned} 0 &= \int_0^T (\beta_t, \partial_t \phi_t) dt + \int_0^T (h_t, \phi_t) dB_t + \int_0^T (L\beta_t, \phi_t) dt \\ &= \int_0^T (\beta_t, \partial_t \phi_t) dt + \int_0^T (h_t, \phi_t) dB_t - \int_0^T \mathcal{E}(\beta_t, \phi_t) dt \end{aligned}$$

Itô's formula yields (see Proposition 6.3 in [66]), quasi surely, for all $t \in [0,T]$,

$$\|\beta_t\|^2 + 2\int_0^t \mathcal{E}(\beta_s) ds = 2\int_0^t (\beta_s, h_s) dB_s + \int_0^t \|h_s\|^2 d\langle B\rangle_s \tag{4.18}$$

Hence

$$\widehat{\mathbb{E}}[\sup_{t\in[0,T]} \|\beta_t\|^2] \leq 2\widehat{\mathbb{E}}[\sup_{t\in[0,T]} |\int_0^t (\beta_s, h_s) dB_s|] + \bar{\sigma}^2 \widehat{\mathbb{E}}[\int_0^T \|h_s\|^2 ds]$$

Using Burkholder-David-Gundy's inequality, we get

$$\begin{aligned} \widehat{\mathbb{E}}[\sup_{t\in[0,T]} \|\beta_t\|^2] &\leq 2C\bar{\sigma}^2 \widehat{\mathbb{E}}[\left(\int_0^T (\beta_s, h_s)^2 ds\right)^{1/2}] + \bar{\sigma}^2 \widehat{\mathbb{E}}[\int_0^T \|h_s\|^2 ds] \\ &\leq 2C\bar{\sigma}^2 \widehat{\mathbb{E}}[\sup_{t\in[0,T]} \|\beta_t\| \times \left(\int_0^T \|h_s\|^2 ds\right)^{1/2}] \\ &+ \bar{\sigma}^2 \widehat{\mathbb{E}}[\int_0^T \|h_s\|^2 ds] \end{aligned}$$

so for any $\epsilon > 0$ we have

$$\widehat{\mathbb{E}}[\sup_{t\in[0,T]} \|\beta_t\|^2] \leq C\bar{\sigma}^2 \epsilon \widehat{\mathbb{E}}[\sup_{t\in[0,T]} \|\beta_t\|^2] + (\frac{C\bar{\sigma}^2}{\epsilon} + \bar{\sigma}^2)\widehat{\mathbb{E}}[\int_0^T \|h_s\|^2 ds]$$

Then we can take ϵ small enough such that

$$\widehat{\mathbb{E}}[\sup_{t\in[0,T]} \|\beta_t\|^2] \leq C\bar{\sigma}^2\widehat{\mathbb{E}}[\int_0^T \|h_s\|^2 \, ds]$$

Then, relation (4.18) yields,

$$\widehat{\mathbb{E}}\int_0^T \mathcal{E}(\beta_s)ds \leq C'\bar{\sigma}^2\widehat{\mathbb{E}}[\int_0^T \|h_s\|^2 \, ds].$$

Similar as in the previous lemmas, we conclude by a density argument in the general case. ■

Proposition 4.40. *The mild solution* (4.13) *is equivalent to the weak solution* (4.14).

Proof. We can do a similar argument as in Proposition 1.36 thanks to the fact that Lemma 4.10 in [17] can be easily extended to the quasi surely case. ■

Theorem 4.41. *Under the hypotheses **(H)** and **(I)**, (4.12) admits a unique solution in $M_G^2([0,T];H_0^1(\mathcal{O}))$.*
We denote by $\mathcal{G}(\xi,f,g,h)$ the solution of (4.12) when it exists and is unique.

To prove this theorem, we need the following Itô formula:

Theorem 4.42. ***(Itô's formula)*** *Assume that f, g, h belong to $M_G^2([0,T];H_0^1(\mathcal{O}))$ and are adapted and $\xi \in L^2(\mathcal{O})$ and consider $u : \mathcal{G}(\xi,f,g,h)$. Let $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^{1,2}$. We denote by Φ' and Φ'' the derivatives of Φ with respect to the space variables and by $\frac{\partial \Phi}{\partial t}$ the partial derivative with respect to time. We assume that these derivatives are bounded and $\Phi'(t,0) = 0$ for all $t \in [0,T]$. Then we have the following relation quasi surely, for all $t \in [0,T]$,*

$$\begin{aligned}
&\int_{\mathcal{O}} \Phi(t,u_t(x))dx + \int_0^t \mathcal{E}(\Phi'(s,u_s),u_s)ds = \int_{\mathcal{O}} \Phi(0,\xi(x))dx + \int_0^t\int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s,u_s(x))dx \\
&+ \int_0^t (\Phi'(s,u_s),f_s)ds - \sum_{i=1}^d \int_0^t\int_{\mathcal{O}} \Phi''(s,u_s(x))\partial_i u_s(x) g_s^i(x)dxds \qquad (4.19) \\
&+ \sum_{j=1}^{d_1} \int_0^t (\Phi'(s,u_s(x)),h_s^j(x))dxdB_s^j + \frac{1}{2}\sum_{j=1}^{d_1}\int_0^t\int_{\mathcal{O}} \Phi''(s,u_s(x))(h_s^j(x))^2 dxd\langle B^j\rangle_s
\end{aligned}$$

Proof. We begin with the regular case, i.e. $f,\ h \in C^1([0,T]) \otimes L_G^2(\Omega_T) \otimes \mathcal{D}(L)$, $g \in C^1([0,T]) \otimes L_G^2(\Omega_T) \otimes \mathcal{D}(L^{3/2})$ and are adapted, and $\xi \in \mathcal{D}(L)$, then u is a semi-martingale and it posses the following form:

$$u_t = \xi - \int_0^t Au_s ds + \int_0^t f_s ds + \int_0^t div g_s ds + \int_0^t h_s dB_s$$

Thanks to Itô's formula for semi-martingale, we have quasi-surely for all $t \in [0,T]$:

$$\int_{\mathcal{O}} \Phi(t,u_t(x))dx = \int_{\mathcal{O}} \Phi(0,\xi(x))dx - \int_0^t (\Phi'(s,u_s), Au_s)ds + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s,u_s(x))ds$$
$$+ \int_0^t (\Phi'(s,u_s), f_s)ds + \int_0^t (\Phi'(s,u_s), h_s)dB_s + \int_0^t \int_{\mathcal{O}} \Phi'(s,u_s(x))divg_s(x)dxds$$
$$+\frac{1}{2}\int_0^t \int_{\mathcal{O}} \Phi''(s,u_s(x))h_s^2(x)dxd\langle B\rangle_s$$

Then, as

$$(\Phi'(s,u_s), Au_s) = \mathcal{E}(\Phi'(s,u_s), u_s)$$

and

$$\int_{\mathcal{O}} \Phi'(s,u_s(x))divg_s(x)dx = -\int_{\mathcal{O}} \Phi''(s,u_s(x))\partial u_s(x) g_s(x)dx$$

we get the desired equality.
The general case is obtained by approximation. We take $f^n,\ h^n \in C^1([0,T]) \otimes L_G^2(\Omega_T) \otimes \mathcal{D}(L)$, $g^n \in C^1([0,T]) \otimes L_G^2(\Omega_T) \otimes \mathcal{D}(L^{3/2})$ and $\xi^n \in \mathcal{D}(L)$ such that $f^n \to f$, $g^n \to g$ and $h^n \to h$ strongly in $M_G^2([0,T]; H_0^1(\mathcal{O}))$ and $\xi^n \to \xi$ strongly in $L^2(\mathcal{O})$. Therefore, $u^n := \mathcal{G}(\xi^n, f^n, g^n, h^n)$ converges strongly to $u = \mathcal{G}(\xi, f, g, h)$ in $M_G^2([0,T]; H_0^1(\mathcal{O}))$. Thanks to the first step, we have the Itô formula for u^n, that is, quasi-surely, $\forall t \in [0,T]$,

$$\int_{\mathcal{O}} \Phi(t,u_t^n(x))dx + \int_0^t \mathcal{E}(\Phi'(s,u_s^n), u_s^n)ds = \int_{\mathcal{O}} \Phi(0,\xi^n(x))dx + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s,u_s^n(x))ds$$
$$+ \int_0^t (\Phi'(s,u_s^n), f_s^n)ds + \int_0^t (\Phi'(s,u_s^n), h_s^n)dB_s + \int_0^t \int_{\mathcal{O}} \Phi'(s,u_s^n(x))divg_s^n(x)dxds$$
$$+\frac{1}{2}\int_0^t \int_{\mathcal{O}} \Phi''(s,u_s^n(x))(h_s^n(x))^2dxd\langle B\rangle_s$$

Then, under each $P \in \mathcal{P}$, thanks to the dominated convergence theorem under P, we can easily get the following Itô formula, (see for example Lemma 1.37)

$$\int_{\mathcal{O}} \Phi(t,u_t(x))dx + \int_0^t \mathcal{E}(\Phi'(s,u_s), u_s)ds = \int_{\mathcal{O}} \Phi(0,\xi(x))dx + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s,u_s(x))ds$$
$$+ \int_0^t (\Phi'(s,u_s), f_s)ds + \int_0^t (\Phi'(s,u_s), h_s)dB_s + \int_0^t \int_{\mathcal{O}} \Phi'(s,u_s(x))divg_s(x)dxds$$
$$+\frac{1}{2}\int_0^t \int_{\mathcal{O}} \Phi''(s,u_s(x))h_s^2(x)dxd\langle B\rangle_s, \quad P-a.s.$$

Finally, as each member in the equality admits a quasi-continuous version, we get the formula quasi-surely. ■

Now we come to the proof of Theorem 4.41:
Proof. Let γ and δ be two positive constants. On $M_G^2([0,T]; H_0^1(\mathcal{O}))$, we introduce the norm

$$\parallel u \parallel_{\gamma,\delta} = \widehat{\mathbb{E}}(\int_0^T e^{-\gamma s}(\delta \parallel u_s \parallel^2 + \parallel \nabla u_s \parallel^2)ds)$$

which clearly defines an equivalent norm on $M_G^2([0,T];H_0^1(\mathcal{O}))$.
We define the application $\Lambda : M_G^2([0,T];H_0^1(\mathcal{O})) \to M_G^2([0,T];H_0^1(\mathcal{O}))$ as following:

$$(\Lambda u)_t = P_t\xi + \int_0^t P_{t-s}f_s(u_s,\nabla u_s)ds + \int_0^t P_{t-s}divg_s(u_s,\nabla u_s)ds + \int_0^t P_{t-s}h_s(u_s,\nabla u_s)dB_s$$

Denoting $\bar{u}_t = \Lambda u_t - \Lambda v_t$ with u and v are in $M_G^2([0,T];H_0^1(\mathcal{O}))$, $\bar{f} = f(u,\nabla u) - f(v,\nabla v)$, $\bar{g} = g(u,\nabla u) - g(v,\nabla v)$ and $\bar{h} = h(u,\nabla u) - h(v,\nabla v)$. Applying Itô's formula to $e^{-\gamma T}\bar{u}_T^2$, we have quasi surely:

$$\begin{aligned}&e^{-\gamma T} \parallel \bar{u}_T \parallel^2 + 2\int_0^T e^{-\gamma s}\mathcal{E}(\bar{u}_s)ds = -\gamma\int_0^T e^{-\gamma s} \parallel \bar{u}_s \parallel^2 ds + 2\int_0^T e^{-\gamma s}(\bar{u}_s,\bar{f}_s)ds\\ &-2\sum_{i=1}^d \int_0^T e^{-\gamma s}(\partial_i\bar{u}_s,\bar{g}_s^i)ds + 2\sum_{j=1}^\infty \int_0^T e^{-\gamma s}(\bar{u}_s,\bar{h}_s^j)dB_s^j + \int_0^T e^{-\gamma s} \parallel |\bar{h}_s| \parallel^2 d\langle B\rangle_s\end{aligned}$$

The following calculus are based on the Lipschitz conditions and Cauchy-Schwarz's inequality:

$$\begin{aligned}2\int_0^T e^{-\gamma s}(\bar{u}_s,\bar{f}_s)ds &\le \frac{1}{\epsilon}\int_0^T e^{-\gamma s} \parallel \bar{u}_s \parallel^2 ds + \epsilon\int_0^T e^{-\gamma s} \parallel \bar{f}_s \parallel^2 ds\\ &\le \frac{1}{\epsilon}\int_0^T e^{-\gamma s} \parallel \bar{u}_s \parallel^2 ds + C\epsilon\int_0^T e^{-\gamma s} \parallel u_s - v_s \parallel^2 ds\\ &+ C\epsilon\int_0^T e^{-\gamma s} \parallel \nabla(u_s - v_s) \parallel^2 ds\end{aligned}$$

and

$$\begin{aligned}2\sum_{i=1}^d \int_0^T e^{-\gamma s}(\bar{g}_s^i,\partial_i\bar{u}_s)ds &\le 2\int_0^T e^{-\gamma s} \parallel \nabla\bar{u}_s \parallel (C \parallel u_s - v_s \parallel + \alpha \parallel \nabla(u_s - v_s) \parallel)ds\\ &\le C\epsilon\int_0^T e^{-\gamma s} \parallel \nabla\bar{u}_s \parallel^2 ds + \frac{C}{\epsilon}\int_0^T e^{-\gamma s} \parallel u_s - v_s \parallel^2 ds\\ &\quad +\alpha\int_0^T e^{-\gamma s} \parallel \nabla\bar{u}_s \parallel^2 ds + \alpha\int_0^T e^{-\gamma s} \parallel \nabla(u_s - v_s) \parallel^2 ds\end{aligned}$$

and

$$\int_0^T e^{-\gamma s} \parallel |\bar{h}_s| \parallel^2 ds \le C(1+\frac{1}{\epsilon})\int_0^T e^{-\gamma s} \parallel u_s - v_s \parallel^2 ds + \beta^2(1+\epsilon)\int_0^T e^{-\gamma s} \parallel \nabla(u_s - v_s) \parallel^2 ds$$

where C, α and β are the constants in the Lipschitz conditions. Using the elliptic condition and taking expectation, we get:

$$\begin{aligned}&(\gamma - \frac{1}{\epsilon})E\int_0^T e^{-\gamma s} \parallel \bar{u}_s \parallel^2 ds + (2\lambda - \alpha - C\epsilon)E\int_0^T e^{-\gamma s} \parallel \nabla\bar{u}_s \parallel^2 ds\\ \le\ &C((1+\frac{1}{\epsilon})\bar{\sigma}^2 + \epsilon + \frac{1}{\epsilon})\int_0^T e^{-\gamma s} \parallel u_s - v_s \parallel^2 ds\\ &+(C\epsilon + \alpha + \bar{\sigma}^2\beta^2(1+\epsilon))E\int_0^T e^{-\gamma s} \parallel \nabla(u_s - v_s) \parallel^2 ds\end{aligned}$$

We choose ϵ small enough and then γ such that

$$C\epsilon + \alpha + \bar{\sigma}^2\beta^2(1+\epsilon) < 2\lambda - \alpha - C\epsilon \textit{ and } \frac{\gamma - 1/\epsilon}{2\lambda - \alpha - C\epsilon} = \frac{C((1+1/\epsilon)\bar{\sigma}^2 + \epsilon + 1/\epsilon)}{C\epsilon + \alpha + \bar{\sigma}^2\beta^2(1+\epsilon)}$$

If we set $\delta = \frac{\gamma - 1/\epsilon}{2\lambda - \alpha - C\epsilon}$, we have the following inequality:

$$\| \bar{u} \|_{\gamma,\delta} \leq \frac{C\epsilon + \alpha + \bar{\sigma}^2\beta^2(1+\epsilon)}{2\lambda - \alpha - C\epsilon} \| u - v \|_{\gamma,\delta} .$$

We conclude thanks to the fixed point theorem. ∎

4.4.3 Comparison theorem

In this subsection we will establish a comparison theorem for the solution of GSPDE (4.12) as following:

Theorem 4.43. *Let f' be another coefficient which satisfies the same hypotheses as f and $\xi' \in L^2_G(\Omega_0; L^2(\mathcal{O}))$. Let u' be the solution of*

$$\begin{aligned} du'_t(x) &= Lu'_t(x)dt + f'(t,x,u'_t(x),\nabla u'_t(x))dt + \sum_{i=1}^{d} \partial_i g_i(t,x,u'_t(x),\nabla u'_t(x))dt \\ &+ \sum_{j=1}^{d_1} h_j(t,x,u'_t(x),\nabla u'_t(x))dB^j_t, \end{aligned}$$

with initial condition $u'_0 = \xi'$.
Assume that $\xi \leq \xi'$ q.e. and for quasi all $\omega \in \Omega$,

$$f(t,x,u_t(x),\nabla u_t(x)) \leq f'(t,x,u'_t(x),\nabla u'_t(x)) \; dt \otimes dx - a.e.$$

then

$$\forall t \in [0,T], \quad u_t \leq u'_t \; q.e.$$

Proof. We put $\widehat{u} = u - u'$, $\widehat{\xi} = \xi - \xi'$, $\widehat{f}_t = f(t,u_t,\nabla u_t) - f'(t,u'_t,\nabla u'_t)$, $\widehat{g}_t = g(t,u_t,\nabla u_t) - g(t,u'_t,\nabla u'_t)$ and $\widehat{h}_t = h(t,u_t,\nabla u_t) - h(t,u'_t,\nabla u'_t)$. The main idea is to evaluate $\widehat{\mathbb{E}} \| \widehat{u}^+_t \|^2$ and then apply Gronwall's lemma.

We approximate $\psi(y) = (y^+)^2$ by a sequence of regular functions: Let φ be an increasing $\mathcal{C}^\infty$ function such that $\varphi(y) = 0$ for any $y \in]-\infty, 1]$ and $\varphi(y) = 1$ for any $y \in [2, \infty[$. We set $\psi_n(y) = y^2\varphi(ny)$, for each $y \in \mathbb{R}$ and all $n \in \mathbb{N}^*$. It is easy to verify that $(\psi_n)_{n\in\mathbb{N}^*}$ converges uniformly to the function ψ and that

$$\lim_{n\to\infty} \psi'_n(y) = 2y^+, \; \lim_{n\to\infty} \psi''_n(y) = 2 \cdot I_{\{y>0\}},$$

for any $y \in \mathbb{R}$. Moreover we have the estimates

$$0 \leq \psi_n(y) \leq \psi(y), \; 0 \leq \psi'_n(y) \leq Cy, \; |\psi''_n(y)| \leq C,$$

for any $y \geq 0$ and all $n \in \mathbb{N}^*$, where C is a constant. We have quasi-surely for all $n \in \mathbb{N}^*$ and each $t \in [0,T]$,

$$\int_{\mathcal{O}} \psi_n\left(\widehat{u}_t\left(x\right)\right) dx + \int_0^t \mathcal{E}\left(\psi_n'\left(\widehat{u}_s\right), \widehat{u}_s\right) ds = \int_{\mathcal{O}} \psi_n\left(\widehat{\xi}\left(x\right)\right) dx + \int_0^t \left(\psi_n'\left(\widehat{u}_s\right), \widehat{f}_s\right) ds$$
$$- \int_0^t \sum_{i=1}^d \left(\psi_n''\left(\widehat{u}_s\right) \partial_i \widehat{u}_s, \widehat{g}_{i,s}\right) ds + \frac{1}{2}\int_0^t \left(\psi_n{}''\left(\widehat{u}_s\right), \left|\widehat{h}_s\right|^2\right) d\langle B\rangle_s + \sum_{j=1}^{d_1} \int_0^t \left(\psi_n'\left(\widehat{u}_s\right), \widehat{h}_{j,s}\right) dB_s^j.$$

Hence, under each $P \in \mathcal{P}$, a similar calculus as in the proof of Theorem 1.39 in Chapter 1 yields

$$E_P \parallel \psi_n(u_t) \parallel \leq C \int_0^t E_P \parallel \psi_n(u_s) \parallel ds$$

Taking the limit, $n \to \infty$, we get

$$E_P \parallel \widehat{u}_t^+ \parallel^2 \leq C \int_0^t E_P \parallel \widehat{u}_s^+ \parallel ds.$$

Therefore,

$$\widehat{\mathbb{E}} \parallel \widehat{u}_s^+ \parallel^2 \leq C \int_0^t \widehat{\mathbb{E}} \parallel \widehat{u}_s^+ \parallel^2 ds.$$

We deduce the result from Gronwall's lemma. ∎

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