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Sur la structure des noyaux sauvages étales des corps de nombres

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Résumé de la thèse

Le but de ce travail est de présenter des résultats à propos des noyaux sauvages étales. Soit p un nombre premier. Les noyaux sauvages étales d'un corps de nombres F (qui sont dénotés par $WK_{2i}^{ét}(F)$ avec $i \in \mathbb{Z}$) sont des généralisations cohomologiques de la p-partie du noyau sauvage classique $WK_2(F)$, qui est le sous-groupe de $K_2(F)$ constitué par les symboles qui sont triviaux pour tout symbole de Hilbert local. Ces noyaux sauvages étales sont des $\mathbb{Z}_p$-modules et l'on sait qu'ils sont finis lorsque $i \geq 1$ (et même, suivant les conventions, si $i = 0$) : on conjecture en plus qu'ils soient toujours finis (conjecture de Schneider). Dans la suite, on va supposer que cette conjecture est satisfaite.

On va s'intéresser en particulier à deux problèmes. Le premier, qui est étudié dans les Chapitres 2 et 3, est la déterminations des structures de groupe qui sont réalisables comme noyaux sauvages étales. En d'autres termes, si l'on se donne un corps de nombres F, un p-groupe abelien fini X et un nombre entier $i \in \mathbb{Z}$, on peut se demander s'il existe une extension finie E/F telle que $WK_{2i}^{ét}(E) \cong X$. Une question semblable a été étudiée pour les p-groupes des classes et il y a un relation précise entre les p-groupes des classes et les noyaux sauvages étales. Par conséquent, on peut espérer traduire les résultats classiques dans le contexte des noyaux sauvages étales. Peut-être est-il intéressant de donner ici une courte récapitulation sur le problème de réalisation classique pour les p-groupes des classes (voir [Ge] et [Ya]). Essentiellement, deux techniques sont utilisées. D'un coté, pour un corps de nombres F fixé, l'on étudie la p-tour des corps des classes de Hilbert de F : Yahagi a montré (voir [Ya] et [So]) que cette tour est infinie si et seulement s'il n'y a pas d'extensions finies E/F dont le p-groupe des classes soit trivial. De plus, si la tour est finie, alors toute structure de p-groupe abélien apparaît comme p-groupe des classes pour quelque extension finie E/F. De l'autre coté, une fois que l'on sait que pour un corps de nombres F fixé, il existe une extension finie dont le p-groupe de classes est trivial, alors on peut se servir de la théorie du corps des classes et de la théorie des genres pour trouver, pour n'importe quel p-groupe abélien fini X, une extension finie E/F telle que le p-groupe des classes de E est isomorphe à X.

En effet, la traduction du résultat de Yahagi dans le contexte des noyaux sauvages étales n'est pas tout à fait immédiate : la relation entre le groupe des classes et le noyau sauvage étale d'un corps de nombres F s'écrit dans le langage de Γ-modules, où Γ est le groupe de Galois sur F de la $\mathbb{Z}_p$-extension cyclotomique de $F(\mu_p)$. La façon la plus naturelle pour s'approcher du problème est donc de considérer le problème de réalisabilité pour les modules d'Iwasawa. Ce problème a été étudié (parmi d'autres auteurs) par Ozaki in [Oz] : il a montré que pour tout Λ-module fini X, il existe un corps de

nombres k tel que le module d'Iwasawa de k (c'est à dire la limite projective des p-groupes des classes le long de la tour cyclotomique) est isomorphe à X. Les techniques utilisées sont inspirées à celles de Yahagi et en fait elles s'appuient d'une façon fondamentale du fait que p ne divise pas le nombre des classes de $\mathbb{Q}$. Pour obtenir la traduction de ce résultat en termes de noyaux sauvages étales il faut considérer plutôt $\mathbb{Q}(\mu_p)$ -plus précisément un sous-corps convenable de $\mathbb{Q}(\mu_p)$. Bien entendu, le nombre des classes de ce sous-corps n'est plus premier avec p (du moment que p peut être irrégulier). D'autre part, si p est régulier, la preuve d'Ozaki peut être adaptée (comme l'on montre dans le Chapitre 2).

Pour traiter le cas mauvais (c'est à dire le cas où le nombre des classes du sous-corps convenable comme dessus n'est pas étranger à p), on considère des analogues des p-corps des classes de Hilbert et des p-tours des classes de Hilbert qui ont été définis par Jaulent et Soriano in [JS] pour $i = 0$ et généralisés par Assim in [As] (mais sous l'hypothèse que le corps de base contient les racines p-ièmes de l'unité). Dans le Chapitre 3, on développe cette théorie dans le cas général : le résultat plus important est que si $WK_{2i}^{ét}(\mathbb{Q}) \neq 0$ et i est impair, alors l'analogue de la p-tour des classes de Hilbert de $\mathbb{Q}$ est infinie. Cette dernière condition est équivalente à la condition $WK_{2i}^{ét}(F) \neq 0$ pour tout corps de nombres F contenant le même sous-corps convenable de $\mathbb{Q}(\mu_p)$ dont on a parlé tout à l'heure. Il s'agit sans doute de la différence la plus importante entre le cas classique des groupes des classes et celui des noyaux sauvages étales : en d'autres termes, la non-finitude de la tour n'implique pas directement l'absence de corps de nombres avec noyau sauvage étale trivial (à cause de la condition sur le sous-corps convenable, bien sûr). Il se peut bien entendu que cette différence soit apparente et que l'on puisse se passer de l'hypothèse sur le sous-corps. On ne s'intéresse pas ici de la question classique sur les conditions suffisantes afin que la tour soit infinie (à la Golod-Shafarevic, voir [JS] et [As]) : de toute façon, comme l'on pourra facilement deviner, une adaptation des résultats classiques ne devrait pas être compliquée.

Le second problème auquel on s'intéresse dans ce travail est étudié en détail dans le Chapitre 4. On regarde de plus près la suite exacte de localisation en K-théorie d'un corps de nombres F

$$0 \longrightarrow K_{2i}(\mathcal{O}_F) \longrightarrow K_{2i}(F) \xrightarrow{\partial} \bigoplus_{v \text{ finite}} K_{2i-1}(k_v) \longrightarrow 0 \tag{1}$$

(où k_v est le corps résiduel de F à la place v et la somme directe est prise sur les places finies de F). On peut se poser la question de déterminer des conditions nécessaires et suffisantes afin que la suite exacte soit scindée, une motivation étant le théorème de Tate-Milnor (voir [Mil], Theorem 2.3) qui affirme que, si E est un corps de fonctions rationnelles à une variable, la suite de localisation pour $K_2(E)$ (qui est tout à fait analogue à (1)) est scindée. Revenant au problème de scission pour les corps de nombres, on est amené naturellement à considérer, pour tout p premier, la p-suite exacte de localisation, c'est à dire la partie p-primaire de la suite (1). Dans [Ba], Banaszak a enoncé un théorème qui affirme que la p-suite de localisation de $K_{2i}(F)$ est scindée si et seulement si $\mathrm{div}(K_{2i}(F))_p = 0$ (pour un groupe abélien M, l'on dénote par $\mathrm{div}(M)$ le sous-groupe des éléments de hauteur infinie). On sait aussi que $\mathrm{div}(K_{2i}(F))_p = WK_{2i}^{ét}(F)$. La trivialité de $WK_{2i}^{ét}(F)$ est bien sûr une condition nécessaire pour que la suite de localisation soit scindée : toutefois la preuve de Banaszak ne semble pas complète. En effet, en cherchant un contre-exemple (c'est à dire un corps de nombres F tel que $WK_{2i}^{ét}(F) = 0$ mais la p-suite de localisation de $K_{2i}(F)$ n'est pas scindée), on trouve une condition nécessaire et suffisante pour que la i-ème suite soit scindée qui est différente de celle de Banaszak. La différence entre cette nouvelle condition et celle de Banaszak ne se voit pas au niveau des petits corps de nombres (c'est à dire par exemple $\mathbb{Q}$ ou les corps quadratiques) : les contre-exemples que l'on exhibe sont en verité difficiles à trouver.

Dans le premier chapitre, on fixe les notations et on rappelle les résultats connus qui servent comme motivation aussi bien que comme outils pour ce travail. En particulier, les groupes de K-théorie et les noyaux sauvages étales sont introduits et l'on décrit brièvement leur propriétés.

Mots-clés en français :
noyaux sauvages étales, théorie d'Iwasawa, théorie des genres, suite de localisation en K-théorie

Keywords :
étale wild kernels, Iwasawa theory, genus theory, K-theory localization sequence

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Contents

Chapter 1

Introduction

1.1 General description of the work

The aim of the present work is to prove some results about étale wild kernels. Let p be an odd prime. Etale wild kernels of a number field F (which are denoted $WK_{2i}^{ét}(F)$ for $i \in \mathbb{Z}$) are cohomological generalizations of the p-part of the classical wild kernel $WK_2(F)$, which is the subgroup of $K_2(F)$ made up by symbols which are trivial for any local Hilbert symbol. Etale wild kernels are $\mathbb{Z}_p$-modules which are known to be finite if $i \geq 1$ (and even if $i = 0$, depending on the chosen convention): actually they are conjectured to be always finite (the Schneider conjecture). In the following we will suppose that this is always the case.

Two problems are studied in detail. The first, which is analyzed in Chapter 2 and Chapter 3, is to determine which group structures are realizable for étale wild kernels. In other words, given a number field F, a finite abelian p-group X and $i \in \mathbb{Z}$, one can ask if there exists a finite extension E/F such that $WK_{2i}^{ét}(E) \cong X$. A similar problem has been studied for p-class groups and there are precise relations between the p-class group and étale wild kernels. Therefore one may expect to translate results from p-class groups to étale wild kernels. It is maybe useful to give here a short account on the classical realizability problem for p-class groups (see [Ge] and [Ya]). Essentially two kind of techniques are used. On the one hand, for a fixed number field F, one studies the Hilbert p-class field tower of F: it has been shown by Yahagi (see [Ya] and [So]) that the Hilbert p-class tower of F is infinite if and only if there is no finite extension E/F whose p-class group is trivial. Furthermore, if the Hilbert p-class tower of F is finite, then every finite abelian p-group structure appears as p-class group of some finite extension E/F. On the other hand, once we know that for a fixed number field F there exists a finite extension whose p-class group is trivial, then class field theory and genus theory are used to exhibit, for any finite abelian p-group

X, a finite extension E/F such that the p-class group of E is isomorphic to X.

Actually, the translation of Yahagi's result in terms of étale wild kernels is not immediate: the relation between the class groups and étale wild kernels of a number field F is expressed in terms of Γ-modules structures, where Γ is the Galois group over F of the cyclotomic $\mathbb{Z}_p$-extension of $F(\mu_p)$. The most natural way to approach the problem is then to consider the realizability problem for Iwasawa modules. This problem is studied (among many others) by Ozaki in [Oz]: he proved that for any finite Λ-module X, there exists a number field k such that the Iwasawa module of k (i.e. the projective limit of p-class groups along the cyclotomic $\mathbb{Z}_p$-extension) is isomorphic to X. The techniques used are inspired to those by Yahagi and actually Ozaki makes fundamental use of the fact that p does not divide the class number of $\mathbb{Q}$. To get the translation of this result in terms of étale wild kernels one has to consider $\mathbb{Q}(\mu_p)$ -more precisely a suitable subfield of $\mathbb{Q}(\mu_p)$ depending on i- instead of $\mathbb{Q}$. Here the problem is that the class number of this suitable subfield is no more coprime with p (as p may be irregular). If this is not the case anyway, the proof of Ozaki can be adapted as it is shown in Chapter 2.

In order to deal with the bad case (i.e. the case where the class number of the suitable subfield above is not coprime with p), one considers analogues of Hilbert p-class fields and Hilbert p-class towers. These have been defined by Jaulent and Soriano in [JS] for $i = 0$ and generalized by Assim in [As] (but only for field containing μ_p). In Chapter 3 we develop this theory in the general case: the main result is that if $WK_{2i}^{ét}(\mathbb{Q}) \neq 0$ and i is odd, then the étale analogue of the Hilbert p-class tower of $\mathbb{Q}$ is infinite. This is equivalent to the fact that, for every number field F containing the suitable subfield of $\mathbb{Q}(\mu_p)$ as above, we have $WK_{2i}^{ét}(F) \neq 0$. This is probably the main difference between the classical class groups case and the étale wild kernels case: in other words, the infiniteness of the tower does not seem to imply directly that there do not exist fields with trivial étale wild kernel (because of the condition on that subfield). Maybe this hypothesis on the subfield is merely a technical one. Here we do not treat the classical question of giving condition for the tower to be infinite (in the spirit of Golod-Shafarevic inequalities, see [JS] and [As]): anyway, as the reader may guess, an adaptation of the classical results to the étale case should not be difficult.

The second problem which is studied in this work is analyzed in Chapter 4. We focus on the K-theory exact localization sequence for a number field F

$$0 \longrightarrow K_{2i}(\mathcal{O}_F) \longrightarrow K_{2i}(F) \xrightarrow{\partial} \bigoplus_{v \text{ finite}} K_{2i-1}(k_v) \longrightarrow 0 \qquad (1.1)$$

(here k_v is the residue field of F at v and the sum is taken over the finite primes of F). One can asks for conditions in order for this exact sequence to be split: one motivation for this question is the Tate-Milnor theorem (see [Mil], Theorem 2.3) which states that, if E is a rational function field of one variable, then the localization sequence for $K_2(E)$ (which is completely analogous to (1.1)) always splits. Coming back to the splitting problem for number fields, one is naturally lead to consider separatedly for each prime p, the p-localization sequence for $K_{2i}(F)$, i.e. the p-primary part of the above localization sequence. In [Ba], Banaszak stated a theorem which says that the p-localization sequence for $K_{2i}(F)$ splits if and only if $\mathrm{div}(K_{2i}(F))_p = 0$ (for an abelian group M, $\mathrm{div}(M)$ denotes the subgroup of divisible elements of M, see Section 1.2). We also know that $\mathrm{div}(K_{2i}(F))_p = WK_{2i}^{\acute{e}t}(F)$. The triviality of $WK_{2i}^{\acute{e}t}(F)$ is easily seen to be a necessary condition in order for the localization sequence to be split but Banaszak's proof of sufficiency seems to be incomplete. Actually looking for a counterexample (i.e. a number field F such that $WK_{2i}^{\acute{e}t}(F) = 0$ but the p-localization sequence for $K_{2i}(F)$ does not split), we found a necessary and sufficient condition for the i-th sequence to be split which is different from that of Banaszak. It turns out for example that in the case $F = \mathbb{Q}$ Banaszak's condition is necessary and sufficient (counterexamples are indeed of subtle nature).

In the rest of this chapter, we fix notation and recall known results which serve at the same time as motivation and tools for our investigations. In particular, K-groups and étale wild kernels are introduced and some of their properties are listed.

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1.2 Basic notation

The following section is devoted to fix notation which will be used throughout the rest of this work. Additional notation which is specific to a chapter will be defined when needed.
Let F be a number field, $n \in \mathbb{N}$ a natural number, p an odd rational prime and B an abelian group. Moreover we will always fix an algebraic closure $\overline{F}$ of F and consider any of the extensions of F which appear as contained in $\overline{F}$. Then

- $\mathbb{Z}_p$ is the ring of p-adic integers, i.e. the projective limit of $\mathbb{Z}/p^n\mathbb{Z}$ with respect to projections;
- $\widehat{\mathbb{Z}}$ is the projective limit of $\mathbb{Z}/n\mathbb{Z}$ with respect to projections;
- $\Lambda = \mathbb{Z}_p[\![T]\!]$ is the Iwasawa algebra in the indeterminate T;
- B_p denotes the p-primary part of B;
- $B[p^n]$ is the subgroup of elements of B whose order divides p^n;
- $\mathrm{Div}(B)$ denotes the maximal divisible subgroup of B;
- $\mathrm{div}(B) = \{b \in B \,|\, \forall\, n \in \mathbb{N}\ \exists\, b_n \in B : b = nb_n\}$ ($\mathrm{div}(B)$ is a sugroup of B which is commonly called the subgroup of (infinitely) divisible elements or the subgroup of elements of infinite height of B);
- $r_1(F)$ (resp. $r_2(F)$) denotes the cardinality of the set of real places (resp. complex places) of F (in particular, if $[F : \mathbb{Q}] = d$, then $r_1(F) + 2r_2(F) = d$);
- $\mathcal{O}_F$ is the ring of integers of F;
- $\mathcal{O}_F^\times$ is the group of units of F;
- Cl_F is the ideal class group of F;
- Cl'_F is the p-split ideal class group of F, i.e. the quotient of Cl_F by the subgroup generated by classes represented by primes above p;
- $\mathfrak{p}$ will generally denote a prime of $\mathcal{O}_F$ and $v_\mathfrak{p}$ is the valuation attached to $\mathfrak{p}$;

- μ_F is the group of roots of unity contained in F;
- μ_n is the group of n-th roots of unity contained in an algebraic closure of F;
- $\mu_{p^\infty} = \cup_{n\in\mathbb{N}}\mu_{p^n}$;
- F^c is the cyclotomic $\mathbb{Z}_p$-extension of F;
- $\Gamma_F = \mathrm{Gal}(F(\mu_p)^c/F)$ and $(\Gamma_F)_p = \mathrm{Gal}(F^c/F)$;
- for any $n \in \mathbb{N}$, F_n will usually denotes the n-th level of F^c;
- $A_{F_n} = (Cl_{F_n})_p$, $A'_{F_n} = (Cl'_{F_n})_p$ and A_{F^c} is the inductive limit of the A_n's with respect to maps of extension of ideals (to ease notation we will also set $A_F = A_{F_0}$);
- X_F (resp. X'_F) is the projective limit of the A_n's (resp. of the A'_n's) with respect to maps of norm of ideals;
- F^{cd} is the maximal pro-p abelian extension of F^c which is split everywhere;
- $\overline{F}^{cd}$ is the maximal pro-p extension of F^c which is split everywhere;
- $\kappa_F : \mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}_p^\times$ is the p-cyclotomic character of F (for any $\zeta \in \mu_{p^\infty}$) and $\sigma \in \mathrm{Gal}(\overline{F}/F)$, then κ_F is defined by $\sigma(\zeta) = \zeta^{\kappa_F}(\sigma)$);
- $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$ is the Teichmüller character (see [Wa], §5.1); we will also denote by $\omega_F : \mathrm{Gal}(F(\mu_p)/F) \to \mathbb{Z}_p^\times$ the character which is obtained composing ω with the natural identification of $\mathrm{Gal}(F(\mu_p)/F)$ with a subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$;
- $\widetilde{Cl}_F$ is the logarithmic class group of Jaulent (see [Ja3])
- if F is a CM field, F^+ denotes its maximal real subfield and if $\Delta = \mathrm{Gal}(F/F^+)$ and B is a Δ-module we set as usual $B^+ = \{b \in B \mid \delta b = b\}$ and $B^+ = \{b \in B \mid \delta b = -b\}$ where $\delta \in \Delta$ is complex conjugation.

If the field F under consideration is clear and no misunderstanding is possible, then we will often let the subscript F drop in the notation of Γ_F, $(\Gamma_F)_p$, κ_F, ... Similar notation is used if F is a p-adic field (whenever it makes sense).
We now define a notation which is classical (see for example [Wa], §6.3). Let Δ be a finite abelian group and denote by $\widehat{\Delta}$ its character group. Let R be a ring which contains the inverse of $|\Delta|$ and all values of χ for any $\chi \in \widehat{\Delta}$. For any $\chi \in \widehat{\Delta}$ set

$$\varepsilon_\chi = \frac{1}{|\Delta|}\sum_{\delta\in\Delta}\chi(\delta)\delta^{-1} \in R[\Delta]$$

The ε_χ's are called orthogonal idempotents of the algebra $R[\Delta]$: they satisfies the following properties

1. $\varepsilon_\chi^2 = \varepsilon_\chi$;
2. $\varepsilon_\chi \varepsilon_\psi = 0$ if $\chi \neq \psi$;
3. $1 = \sum_{\chi \in \widehat{\Delta}} \varepsilon_\chi$;
4. $\varepsilon_\chi \sigma = \chi(\sigma)\varepsilon_\chi$

Let A be an abelian group which is a $R[\Delta]$-module. Then we define $A_\chi = \varepsilon_\chi A$: note in particular that, thanks to property 4 and $\varepsilon_\chi \sigma = \sigma \varepsilon_\chi$ since Δ is abelian, A_χ is the submodule of A on which Δ acts via $\sigma(a) = \chi(\sigma)a$ for all $\sigma \in \Delta$ (in other words A_χ is the eigenspace of σ with eigenvalues $\chi(\sigma)$). Moreover, again using the above properties, we have

$$A = \bigoplus_{\chi \in \widehat{\Delta}} A_\chi$$

Of particular interest to us is the case where $\Delta = \mathrm{Gal}(F(\mu_p)/F)$ (in particular Δ is a cyclic group of order dividing $p-1$): then R can be taken to be $\mathbb{Z}_p$. In this case

$$\widehat{\Delta} = \{\omega_F^i | 0 \leq i < |\Delta|\}$$

and, if A is a $\mathbb{Z}_p[\Delta]$-module, we set $A_{\omega_F^i} = A_i$.
We will make use of Tate twists (see [Ta2]): we recall briefly how they are defined. Set $G_F = \mathrm{Gal}(\overline{F}/F)$. The *Tate module* for F is the $\mathbb{Z}_p[G_F]$-module

$$\mathbb{Z}_p(1) = \varprojlim \mu_{p^n}$$

where the limit is taken over the maps $\mu_{p^{n+1}} \to \mu_{p^n}$ defined by raising to the p-th power. We set $\mathbb{Z}_p(0) = \mathbb{Z}_p$ (which is considered a $\mathbb{Z}_p[G_F]$-module with the trivial action) and, for every $m \in \mathbb{N}$, we define inductively

$$\mathbb{Z}_p(m) = \mathbb{Z}_p(m-1) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$$

$$\mathbb{Z}_p(-m-1) = Hom_{\mathbb{Z}_p}(\mathbb{Z}_p(1), \mathbb{Z}_p(-m))$$

again considered as a $\mathbb{Z}_p[G_F]$-module with the standard action defined on tensor product and homorphisms groups. More generally for every $\mathbb{Z}_p[G_F]$-module M and for every $m \in \mathbb{Z}$, we define

$$M(m) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(m)$$

Actually $M(m)$ is isomorphic (the isomorphism depending on the choice of a $\mathbb{Z}_p$-generator of $\mathbb{Z}_p(1)$) to the module which is equal to M as an abelian group and whose G_F-module structure is given by the rule

$$\sigma \cdot a = \kappa(\sigma)^m . \sigma(a)$$

for every $a \in M$ (the definition of the rule makes sense for $m \in \mathbb{Z}$ because $\kappa(\sigma) \in \mathbb{Z}_p^\times$ is invertible).

We will not make use of deep results in Iwasawa theory (in particular p-adic L-functions and the Main Conjecture will not be used): definitions, notation and results used are classical (and hence will not be recalled here). Of course they can easily be found in Chapter 13 of [Wa].
We will occasionally refer to two old conjectures in algebraic number theory. The first is the Vandiver conjecture which says that p does not divide the class number of the maximal totally real subfield of $\mathbb{Q}(\mu_p)$. The second is the Leopoldt conjecture which can be formulated in many different equivalent ways (see [NSW]): for example it predicts that there are exactly $r_2(F) + 1$ independent $\mathbb{Z}_p$-extensions of a number field F.

We end this section with a remark on cohomology: essentially three types of cohomology will be used. If G is a profinite group and A is a dicrete G-module, then $H^i(G, A)$ denotes the i-th standard group cohomology of G with values in A (see [NSW]). Sometimes, if G is finite, it will be convenient to use Tate cohomology which is as usual denoted by $\widehat{H}^i(G, A)$. On the other hand we will also make use of continuous cohomology (which was first defined by Tate in [Ta2] but see also [NSW]): if G is a profinite group and A is a topological G-module, then $H^i_{cts}(G, A)$ denotes the i-th continuous group cohomology of G with values in A (but we will often let the subscript *cts* drop if no confusion can arise). Finally, we shall use étale cohomology: if X is a scheme and $\mathscr{F}$ is a sheaf on the étale site of X, then $H^i_{\acute{e}t}(X, \mathscr{F})$ denotes the i-th standard group cohomology of X with values in $\mathscr{F}$ (see [Mi1]). Actually since we are going to consider only affine scheme, i.e. of the form $\mathrm{Spec}(R)$ for some ring R, we will also write $H^i_{\acute{e}t}(R, \mathscr{F})$ for $H^i_{\acute{e}t}(\mathrm{Spec}(R), \mathscr{F})$. We will also use the following notation

$$H^i_{\acute{e}t}(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(j)) = \varprojlim H^i_{\acute{e}t}(\mathcal{O}_F[\tfrac{1}{p}], \mu_{p^n}^{\otimes j})$$

It is certainly useful to keep in mind the following result, which is in fact a part of more general theorems whose proofs can be found in [Mi1] (see Example 1.7 of Chapter III) and [Mi2] (see Proposition 2.9 of Chapter II).

Theorem 1.2.1. *Let S be the set of places of F made up by archimedean primes and primes above p. Let $G_{F,S}$ denote the Galois group of the maximal extension of F which is unramified outside S. The following holds:*

- $H^i_{\acute{e}t}(F, \mu_{p^n}^{\otimes j}) = H^i(F, \mu_{p^n}^{\otimes j})$*;*
- $H^i_{\acute{e}t}(\mathcal{O}_F[\tfrac{1}{p}], \mu_{p^n}^{\otimes j}) = H^i(G_{F,S}, \mu_{p^n}^{\otimes j})$.

Here we denote with $\mu_{p^n}^{\otimes j}$ both a module and a sheaf (a notation which is standard).

1.3 Algebraic K-theory of number fields

In this section we briefly recall the definition of algebraic K-theory for rings which are interesting in algebraic number theory, i.e. number fields, ring of integers and finite fields. We also briefly discuss the relationships of algebraic K-theory of those rings with étale cohomology and étale K-theory. We are not going to give an exhaustive treatment or follow the historical evolution of the subject (referring the reader to [Mil], [Ko2] and [We1]).
The definition of K-theory groups is due to Daniel Quillen. Let R be a ring with 1. For any $n \in \mathbb{N}$, let $GL_n(R)$ be the group of invertible $n \times n$ matrix with coefficients in R. Set

$$GL(R) = \varinjlim GL_n(R)$$

the limit being taken with respect to the inclusion $\iota_{n,n+1} : GL_n(R) \to GL_{n+1}(R)$ defined by

$$\iota_{n,n+1}((m_{ij})) = \begin{cases} m_{ij} & \text{if } 1 \leq i,\, j \leq n \\ 1 & \text{if } i = j = n+1 \\ 0 & \text{otherwise} \end{cases}$$

We consider R as a topological ring with the discrete topology. Then $GL(R)$ has a natural induced topology and we consider its classifying space $BGL(R)$. Then we perform the so called Quillen's $+$-construction, which is a topological modification of $BGL(R)$ (and it won't be recalled here since it is rather technical), obtaining a new topological space which we denote by $BGL(R)^+$. Then, for any $n \in \mathbb{N}$,

$$K_i(R) = \pi_i(BGL(R)^+)$$

where $\pi_i(X)$ denotes the n-th homotopy group of the topological space X. For any $i \in \mathbb{N}$, $K_n(R)$ is an abelian group. For $i = 0,\, 1,\, 2$, $K_i(R)$ coincides with the classical K-theory groups which were defined before.
Let R be a finite field: in this case $K_0(R) = \mathbb{Z}$, $K_1(R) = R^\times$ and

$$K_i(R) = \begin{cases} 0 & \text{if } i \text{ is even and nonzero} \\ \text{cyclic of order } |R|^t - 1 & \text{if } i = 2t-1 \text{ is odd} \end{cases}$$

(see [Qu]).
Choose now $R = \mathcal{O}_F$: in this case $K_0(R) = \mathbb{Z} \oplus Cl_F$, $K_1(R) = \mathcal{O}_F^\times$ and

$$K_i(R) = \begin{cases} \text{finite abelian group} & \text{if } i \text{ is even and nonzero} \\ \mathbb{Z}\text{-module of rank } r_1(F) + r_2(F) & \text{if } i \equiv 1 \ (mod\ 4) \text{ and } i > 1 \\ \mathbb{Z}\text{-module of rank } r_2(F) & \text{if } i \equiv 3 \ (mod\ 4) \end{cases}$$

(see [Bo]). In some sense, one can think of even K-groups of $\mathcal{O}_F$ as higher analogues of Cl_F and of odd K-groups as analogues of $\mathcal{O}_F^\times$.
Finally if $R = F$ we have $K_0(R) = \mathbb{Z}$, $K_1(R) = F^\times$ and

$$K_2(F) = F^\times \otimes_{\mathbb{Z}} F^\times / \langle x \otimes (1-x) \,|\, x \in F^\times \setminus \{1\} \rangle$$

An element in $K_2(F)$ is called a symbol: if $x,\, y \in F^\times$, $[x \otimes y]$ is denoted by $\{x,\, y\}$ (actually every symbol is of this form, see [Le]).
Moreover (see [Sou]) $K_i(F) = K_i(\mathcal{O}_F)$ if $i \geq 3$ is odd and there are short exact sequences

$$0 \to K_{2i}(\mathcal{O}_F) \to K_{2i}(F) \xrightarrow{\partial_{2i}} \oplus_{\mathfrak{p}} K_{2i-1}(\mathcal{O}_F/\mathfrak{p}) \to 0 \tag{1.2}$$

(the sum being taken over the finite primes of F) which are called the K-theory exact localization sequences (for algebraic number fields). It shows, in particular, that $\mathrm{Div}(K_{2i}(F)) = 0$. We have the following description of the map $\partial_2 : K_2(F) \to \oplus_{\mathfrak{p}} K_{2i-1}(\mathcal{O}_F/\mathfrak{p})$, which is also called the tame symbol:

$$(\partial_2(x,\, y))_{\mathfrak{p}} = \Big((-1)^{v_{\mathfrak{p}}(x)v_{\mathfrak{p}}(y)} x^{v_{\mathfrak{p}}(y)} y^{-v_{\mathfrak{p}}(x)} \bmod \mathfrak{p}\Big)_{\mathfrak{p}} \tag{1.3}$$

Therefore $K_2(\mathcal{O}_F)$ is also called the tame kernel. There are also analogues of (1.2) if S is a finite set of places of F (including the archimedean ones) and one considers the ring $\mathcal{O}_F^S$ of S-integers of F: namely there are short exact sequences of the form

$$0 \to K_{2i}(\mathcal{O}_F^S) \to K_{2i}(F) \xrightarrow{\partial_{2i}} \oplus_{\mathfrak{p} \notin S} K_{2i-1}(\mathcal{O}_F^S/\mathfrak{p}) \to 0$$

Just as for the first three K-theory groups, higher K-theory groups are interesting invariants of number fields but in general they are very difficult to study. Conjecturally there is a strong relation between étale cohomology and K-theory of ring of integers.

Conjecture 1.3.1. *(Quillen-Lichtenbaum) Let p be an odd prime. For any $i \in \mathbb{N}$ and $j = 1,\, 2$, there are isomorphisms*

$$ch_{i,j} : K_{2i-j}(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow H^j_{\acute{e}t}(\mathcal{O}_F[\tfrac{1}{p}],\, \mathbb{Z}_p(i))$$

The $ch_{i,j}$'s were defined by Soulé ([Sou]) and Dwyer and Friedlander (see [DF]): they are called étale Chern characters. Tate (see [Ta2]) proved that this holds if $i = j = 2$. In this case the étale Chern character is not difficult to describe: we have $ch_{2,2} = h$, where h is the isomorphism

$$h : (K_2(F))_p \longrightarrow H^2(F,\, \mathbb{Z}_p(2))_p$$

defined by

$$h(\{a,\, b\}) = d_F a \cup d_F b$$

where $d_F : F^\times \longrightarrow H^1(F,\, \mathbb{Z}_p(2))$ is the connecting homomorphism of the long exact cohomology sequence associate to the exact sequence

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \varprojlim \overline{F}^\times \longrightarrow \overline{F}^\times \longrightarrow 0$$

(the limit in the middle is taken with respect to p^m-th powers). Moreover

$$h(K_2(\mathcal{O}_F[\tfrac{1}{p}])_p) = H^2_{ét}(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(2))$$

($H^2_{ét}(\mathcal{O}_F[\frac{1}{p}], \mathbb{Z}_p(2))$ is finite and injects in $H^2(F, \mathbb{Z}_p(2))$ by inflation).
Soulé proved that $ch_{i,j}$ is surjective for any i and j. Also Dwyer and Friedlander proved the surjectivity of étale Chern characters making use of étale K-theory (see [DF]). In general, for any $i \in \mathbb{N}$ and $j = 1, 2$, there are surjective maps

$$ch^{ét}_{i,j} : K_{2i-j}(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow K^{ét}_{2i-j}(\mathcal{O}_F[\tfrac{1}{p}])$$

and natural isomorphisms

$$K^{ét}_{2i-j}(\mathcal{O}_F[\tfrac{1}{p}]) \cong H^j_{ét}(\mathcal{O}_F[\tfrac{1}{p}], \mathbb{Z}_p(i)) \tag{1.4}$$

which give $ch_{i,j}$ if composed with $ch^{ét}_{i,j}$. Hence Conjecture 1.3.1 is equivalent to the fact that $ch^{ét}_{i,j}$ are isomomorphisms. The general case of the Quillen-Lichtenbaum conjecture is a consequence of the Bloch-Kato conjecture whose proof seems to be imminent thanks to the work of Rost, Voevodsky, Weibel and others ([Ro], [Vo1], [Vo2], [Vo3]...).
We end this section by recalling a structural result for étale cohomology groups of ring of integers of CM fields containing μ_p. This is actually a generalization of Coates' description of $K_2(\mathcal{O}_F)$.

Theorem 1.3.2. *Suppose that F is a CM field and $\mu_p \subseteq F$. For any $n \in \mathbb{N}$, let F_n be the n-th level of F^c and set $\Gamma_n = \mathrm{Gal}(F^c/F_n)$. Then cohomological restriction induces isomorphisms*

$$H^2_{ét}(\mathcal{O}_{F_n}[\tfrac{1}{p}], \mathbb{Z}_p(i+1))^+ \cong \left(H^2_{ét}(\mathcal{O}_{F^c}[\tfrac{1}{p}], \mathbb{Z}_p(i+1))^+\right)^{\Gamma_n} \quad \text{if } i \geq 1 \text{ is odd}$$

$$H^2_{et}(\mathcal{O}_{F_n}[\tfrac{1}{p}], \mathbb{Z}_p(i+1))^- \cong \left(H^2_{ét}(\mathcal{O}_{F^c}[\tfrac{1}{p}], \mathbb{Z}_p(i+1))^-\right)^{\Gamma_n} \quad \text{if } i \geq 1 \text{ is even}$$

This gives isomorphisms

$$(A^-_{F^c}(i))^{\Gamma_n} \cong H^2_{ét}(\mathcal{O}_{F_n^+}[\tfrac{1}{p}], \mathbb{Z}_p(i+1)) \qquad \text{if } i \text{ is odd}$$

$$(A^-_{F^c}(i))^{\Gamma_n} \cong H^2_{ét}(\mathcal{O}_{F_n}[\tfrac{1}{p}], \mathbb{Z}_p(i+1))^- \qquad \text{if } i \text{ is even}$$

Proof. The proof of this result is maybe well-known but not so easy to find in print: anyway everything can be deduced by [Co] without particular effort so we only sketch the proof. For any i there are Γ-modules H_i and exact sequences

$$0 \longrightarrow (H_i)_{\Gamma_n} \longrightarrow H^2_{et}(\mathcal{O}_{F_n}[\tfrac{1}{p}], \mathbb{Z}_p(i+1))) \longrightarrow (H^2_{ét}(\mathcal{O}_{F^c}[\tfrac{1}{p}], \mathbb{Z}_p(i+1)))^{\Gamma_n} \longrightarrow 0$$

where the last map is cohomological restriction (this result is also proved in [NQD], Proposition 4.4 (ii)). The arguments of the proof of Theorem 11 of [Co] apply and give that $H_i^+ = 0$ if i is odd and $H_i^- = 0$ if i is even. The proof of the last assertion uses the same arguments which are used in the proofs of Theorem 4 and Theorem 6 of [Co]: in this case, instead of Theorem 2 of [Co] (which is due to Tate), one has to use Conjecture 1.4.4 (see next section) which is proved for $i \geq 2$ in Théorème 5 of [Sou] (see Remark 1.4.5). $\square$

If the Quillen-Lichtenbaum conjectures hold for p and F_n^+ we then have (i odd)

$$(A^-(i))^{\Gamma_n} \cong \mathrm{K}_{2i}(\mathcal{O}_{F_n^+})_p$$

1.4 Etale wild kernels

In this section we briefly recall the definition of étale wild kernels of number fields. As in the preceding section we will not follow the historical evolution of the subject (referring the reader to [Sc] and [Ba]). Etale wild kernels are cohomological objects which represent the obstruction to a local-global principle. They are in fact a particular case of Tate-Shafarevic groups (see [NSW]).

Let S be a set of primes of F containing the archimedean ones. Denote by F_S the maximal subextension of $\overline{F}|F$ which is unramified at each prime which does not belong to S. Clearly $F_S|F$ is Galois and we set $G_S = \mathrm{Gal}(F_S|F)$. Let $\mathcal{O}_{F,S}$ be the ring of S-integers of F. In this section M is a finitely generated discrete G_S-module the order of whose torsion subgroup is a unit in $\mathcal{O}_{F,S}$. Furthermore for prime $\mathfrak{p}$ of F denote by $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$. There is an embedding $i_\mathfrak{p} : F \hookrightarrow F_\mathfrak{p}$. We choose an algebraic separable closure $\overline{F_\mathfrak{p}}$ of $F_\mathfrak{p}$ and an embedding $\overline{F} \hookrightarrow \overline{F_\mathfrak{p}}$ which is compatible with $i_\mathfrak{p}$. We set $G_{F_\mathfrak{p}} = \mathrm{Gal}(\overline{F_\mathfrak{p}}|F_\mathfrak{p})$: we have an inclusion $G_{F_\mathfrak{p}} \hookrightarrow G_F$ (which comes from $\overline{F} \hookrightarrow \overline{F_\mathfrak{p}}$) which identifies $G_{F_\mathfrak{p}}$ with the decomposition group of one of the primes of $\overline{F}$ which lies over $\mathfrak{p}$ (in fact, the choice of an embedding $\overline{F} \hookrightarrow \overline{F_\mathfrak{p}}$ corresponds to the choice of a prime of $\overline{F}$ which lies over $\mathfrak{p}$). In this situation, M becomes a $G_{F_\mathfrak{p}}$-module. Composing with the canonical projection (which is in fact restriction) we get a localization map $G_{F_\mathfrak{p}} \longrightarrow G_S$ which allows us to consider, for each $j \geq 0$, the cohomological map

$$\lambda_j = \lambda_S^j(F, M) : H^j(G_S, M) \longrightarrow \prod_{\mathfrak{p} \in S} H^j(G_{F_\mathfrak{p}}, M)$$

In the following we are going to omit the references to S, F and M when no ambiguity can appear. We set

$$\text{Ш}_S^j(F, M) = \mathrm{Ker}(\lambda_S^j(F, M))$$

Now suppose that $\{T_h,\, \tau_{hj}\}_{h\in I}$ is an inverse system (over the inductive set $(I,\, \leq)$) of finite G_S-module whose order is a unit in $\mathcal{O}_{F,S}$. Then

$$T = \varprojlim T_h$$

is a topological G_S-module (with respect to the profinite topology on T). If $h \leq k$, then we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ш^j_S(F,\, T_k) & \longrightarrow & H^j(G_S,\, T_k) & \xrightarrow{\lambda_j} & \prod_{\mathfrak{p}\in S} H^j(G_\mathfrak{p},\, T_k) \\ & & \downarrow & & \tau^*_{hk} \downarrow & & \downarrow \tau^*_{hk} \\ 0 & \longrightarrow & Ш^j_S(F,\, T_h) & \longrightarrow & H^j(G_S,\, T_h) & \xrightarrow{\lambda_j} & \prod_{\mathfrak{p}\in S} H^j(G_\mathfrak{p},\, T_h) \end{array}$$

where τ^*_{hk} is the map induced in cohomology by τ_{hk} and the map

$$Ш^j_S(F,\, T_k) \longrightarrow Ш^j_S(F,\, T_h)$$

is induced by the diagram (it is the restriction of τ^*_{hk} to $Ш^j_S(F,\, T_k)$). From this we can define a morphism

$$\lambda_j = \lambda^j_S(F,\, T) = \varprojlim \lambda^j_S(F,\, T_h) : \varprojlim H^j(G_S,\, T_h) \longrightarrow \varprojlim \prod_{\mathfrak{p}\in S} H^i(G_\mathfrak{p},\, T_h)$$

Then we define

$$Ш^j_{S,\,cts}(F,\, T) = \varprojlim Ш^j_S(F,\, T_h)$$

This choice is motivated by the following.

Proposition 1.4.1. *With the notation just introduced, there is an exact sequence*

$$0 \longrightarrow Ш^j_{S,\,cts}(F,\, T) \longrightarrow H^j_{cts}(G_S,\, T) \longrightarrow \prod_{\mathfrak{p}\in S} H^j_{cts}(G_\mathfrak{p},\, T)$$

Proof. First of all we know that for every $j \geq 0$ and every $h \in I$, both $H^j(G_S,\, T_h)$ and $H^j(G_\mathfrak{p},\, T_h)$ ($\mathfrak{p} \in S$) are finite and the same holds for $Ш^j_S(F,\, T_h)$ because it is a subgroup of $H^j_{cts}(G_S,\, T_h)$. Then we have the equalities

$$H^i_{cts}(G_S,\, T) = \varprojlim H^i_{cts}(G_S,\, T_h) \qquad H^j_{cts}(G_\mathfrak{p},\, T) = \varprojlim H^j_{cts}(G_\mathfrak{p},\, T_h)$$

for every $\mathfrak{p} \in S$. Now observe that inverse systems of finite groups have the so called *Mittag-Leffler property* (see [NSW]): in particular the system $\{Ш^j_S(F,\, T_h),\, \tau^*_{hk}\}_{h\in I}$ (where the τ^*_{hk}'s are the cohomology maps induced by the τ_{hk}'s) has this property. Hence (see the end of the preceding section) we have an exact sequence

$$0 \longrightarrow \varprojlim Ш^j_S(F,\, T_h) \longrightarrow \varprojlim H^j(G_S,\, T_h) \longrightarrow \varprojlim \prod_{\mathfrak{p}\in S} H^j(G_\mathfrak{p},\, T_h)$$

which is exactly the result we were looking for, because inverse limits commute with finite products. $\square$

Definition 1.4.2. *Let p be an odd prime and let $S_{p,\infty}$ be the set made up of primes above p and infinite primes of F. Then, for any $i \in \mathbb{Z}$, we denote $Ш^2_{S_{p,\infty}}(F, \mathbb{Z}_p(i))$ by $WK_{2i}^{ét}(F)$ and call it the i-th étale wild kernel of F.*

Remark 1.4.3. Using Tate's map h defined in the preceding section and local duality, the morphism

$$\lambda_2 : H^2_{cts}(G_S, \mathbb{Z}_p(2)) \longrightarrow \prod_{\mathfrak{p} \in S} H^2_{cts}(G_{\mathfrak{p}}, \mathbb{Z}_p(2))$$

defining $WK_2^{ét}$ becomes, provided that $S_{p,\infty} \subseteq S$,

$$\lambda_2 : K_2(\mathcal{O}_F)_p \longrightarrow \prod_{\mathfrak{p} \in S} (\mu_{F_{\mathfrak{p}}})_p$$

where λ_2 is defined by Hilbert symbol. One can extend this morphism to the whole $K_2(\mathcal{O}_F)$ and its kernel is called the classical wild kernel (often simply denoted $WK_2(F)$), since it is defined by means of Hilbert symbols instead of tame symbols (see (1.3)). By definition $WK_2(F)$ is a subgroup of $K_2(F)$ and in fact

$$WK_2(F)_p = \mathrm{div}(K_2(F)_p)$$

(this statement is due to Tate, see [Ta1], but it is actually proved in [Hu], where also an accurate description of the case $p = 2$ can be found). Similar statements hold for étale wild kernels (even without assuming the Quillen-Lichtenbaum conjectures, which are higher analogues of Tate isomorphism): more precisely $WK_{2i}^{ét}(F)$ can be seen canonically as a subgroup of $K_{2i}(F)$ and

$$WK_{2i}^{ét}(F) = \mathrm{div}(K_{2i}(F)_p)$$

(see [Ba]).

Conjecture 1.4.4. *(Schneider) For any $i \in \mathbb{Z}$, $WK_{2i}^{ét}(F)$ is finite.*

Remark 1.4.5. For any $i \in \mathbb{Z}$, $WK_{2i}^{ét}(F)$ is a finitely generated $\mathbb{Z}_p$-module (see [Sc]). Actually one has (see [Sc], §5, Satz 5)

$$WK_0^{ét}(F) \cong (Cl'_F)_p$$

In particular the Schneider conjecture holds for $i = 0$ (but see also Remark 1.4.7). One can also prove that the Schneider conjecture for $i = -1$ is equivalent to the Leopoldt conjecture (see [Sc], §7, Lemma 1). Soulé proved ([Sou], Théorème 5) that the Schneider conjectures holds for $i \geq 2$: actually the Schneider conjecture for $i \neq 0$ predicts exactly that the maximal divisible subgroup of $H^1(F, \mathbb{Q}_p/\mathbb{Z}_p(i))$ is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^{r_2(F)}$ (resp. $(\mathbb{Q}_p/\mathbb{Z}_p)^{r_1(F)+r_2(F)}$) if i is even (resp. if i is odd).

In the rest of this work, we will always assume that the Schneider conjectures hold for any $i \in \mathbb{Z}$.

Etale wild kernels are objects of cohomological nature and their study can be approached naturally via cohomological methods. There are two interesting descriptions of étale wild kernels in terms of invariants of F (or F^c).

Theorem 1.4.6. *If $i \neq 0$, there is an isomorphism*

$$WK_{2i}^{ét}(F) \cong \left(X'_{F(\mu_p)} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)\right)_{\Gamma_F}$$

Proof. See [Sc], §6, Lemma 1. □

Remark 1.4.7. Many authors prefer to define the i-th étale wild kernel as $X'_{F(\mu_p)}(i)_{\Gamma_F}$ (this definition differs from the original one only for $i = 0$). The anlogous formulation of the Schneider conjectures predicts then that $X'_{F(\mu_p)}(i)_{\Gamma_F}$ is finite for any $i \in \mathbb{Z}$. Recall that $(X'_{F(\mu_p)})_{\Gamma_F} = \widetilde{Cl}_F$ and therefore this last formulation of the Schneider conjecture for $i = 0$ is equivalent to the generalized Gross conjecture (see [Ja3]).

Next we recall a description of étale wild kernels which deals with inductive limits rather than projective limits. The following theorem, which has to be compared with Theorem 1.3.2, is actually a generalization of Theorem 3.5 of [Ko1] (an alternative proof of which can also be found in [Ja1]). For a number field F, we denote by $\dot{C}_F$ the quotient of the idéle group of F defined in [Ko1] after Theorem 1.14. Then $\dot{C}_{F^c}$ denotes the direct limit of $\dot{C}_{F_n}$ where F_n is the n-th level of F^c/F.

Theorem 1.4.8. *Suppose that F is a CM field and $\mu_p \subseteq F$. For any $n \in \mathbb{N}$, let F_n be the n-th level of F^c and set $\Gamma_n = \mathrm{Gal}(F^c/F_n)$. Then cohomological restriction induces isomorphisms*

$$WK_{2i}^{ét}(F_n)^+ \cong \left(WK_{2i}^{ét}(F^c))^+\right)^{\Gamma_n} \qquad \textit{if } i \geq 1 \textit{ is odd}$$

$$WK_{2i}^{ét}(F_n)^- \cong \left(WK_{2i}^{ét}(F^c))^-\right)^{\Gamma_n} \qquad \textit{if } i \geq 1 \textit{ is even}$$

This gives isomorphisms

$$((\dot{C}_{F^c})_p^-(i))^{\Gamma_n} \cong WK_{2i}^{ét}(F_n^+) \qquad \textit{if } i \geq 1 \textit{ is odd}$$

$$((\dot{C}_{F^c})_p^-(i))^{\Gamma_n} \cong WK_{2i}^{ét}(F_n)^- \qquad \textit{if } i \geq 1 \textit{ is even}$$

Proof. The proof follows the original one (see Theorem 3.5,[Ko1]), just as in Theorem 1.3.2. □

Chapter 2

Realizability of abelian p-groups as étale wild kernels

In this chapter we are going to study situations where the realizability problem for étale wild kernels has a positive answer. More precisely, we shall see (Theorem 2.2.6) that if d is the largest divisor of $p-1$ such that $i \equiv 0 \ (mod\, d)$ and the subfield of index d of $\mathbb{Q}(\zeta_p)$ has trivial p-Sylow subgroup of the class group, then, for any abelian p-group X, there is a number field k such that $WK_{2i}^{ét}(k)$ is isomorphic to X. The proof of this result is achieved by using Schneider description of étale wild kernels (see Theorem 1.4.6) and then adapting a result of Ozaki about the realizability problem for finite Iwasawa modules to this situation.

2.1 Generalization of a result by Ozaki

Let d be a divisor of $p-1$ and let $K^{(d)}$ be the subfield of $\mathbb{Q}(\zeta_p)$ such that $[\mathbb{Q}(\zeta_p) : K^{(d)}] = d$. Following the strategy of Ozaki ([Oz]) we are going to prove the following result.

Theorem 2.1.1. *If p does not divide the class number of $K^{(d)}$, then every finite Λ-module structure can be realized as p-split Iwasawa module for some number field k containing $K^{(d)}$.*

In fact, if we forget for a moment about the split condition, Theorem 2.1.1 for $d = p-1$ is Theorem 1 in [Oz]: in this case, $K^{(d)} = \mathbb{Q}$ and the result tells every finite Λ-module structure can be realized as Iwasawa module for some number field k. Thus, the proof of Theorem 2.1.1 consists in a careful rewriting of Ozaki's proof, substituting $\mathbb{Q}$ with $K^{(d)}$ and taking into account the split conditions. As we will see these generalizations are not difficult to deal with and a large part of Ozaki's proof remains essentially unchanged. Still, for the convenience of the reader we rebuild the proof from the beginning.

Remark 2.1.2. Note that if $K^{(d)}$ is not a totally real field, then it is a CM-field. In fact, one proves easily that $K^{(d)}$ is totally real if and only if d is even and it is CM if and only if d is odd. If k is a CM field, we shall denote by k^+ its maximal real subfield. If $\Delta = \mathrm{Gal}(k/k^+)$ and A is a Δ-module we write A^+ for A^Δ.

In the following, we fix d and we shall denote $K^{(d)}$ by K. Let K_n be the n-th layer of K^c. The following notation will be used throughout this section.

- $\mathcal{O}_n$ is the ring of integers of K_n;
- E_n (resp. E'_n) is the group of units (resp. S_n-units) of K_n (where S_n is the set of primes over p in K_n);
- Cl'_n is the p-split class group of K_n (and $(Cl'_n)_p$ is its p-Sylow subgroup).
- $W_n = \mu_{K_n}$ is the group of roots of unity in K_n (thus $K_n = \{\pm 1\}$ if $d \neq 1$ and $K_n = \mu_{p^{n+1}}$ otherwise).

Let Γ be a topological group isomorphic to $\mathbb{Z}_p$ and set $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$. For every $n \geq 0$ set $\Gamma_n = \Gamma/\Gamma^{p^n}$. Let X be a finite Λ-module. Then X is a $\mathbb{Z}/p^{m_0}[\Gamma_{n_0}]$-module for some m_0, $n_0 \geq 0$. We seek for a number field k such that the p-split Iwasawa module X'_k of the cyclotomic $\mathbb{Z}_p$-extension of k is isomorphic to X. In other words we look for a number field k such that, if k_n is the n-th level of the cyclotomic $\mathbb{Z}_p$-extension of k, we have

$$X'_k = \varprojlim (Cl'_{k_n})_p \cong X$$

The following lemma gives us a strategy to accomplish this task (we use the notation just introduced also for an arbitary $\mathbb{Z}_p$-extension k_∞/k).

Lemma 2.1.3. *Assume that a $\mathbb{Z}_p$-extension k_∞/k satisfies the following three conditions:*

- *k_∞/k is totally ramified at every ramified prime;*
- *$(Cl'_{k_{n_0}})_p \cong X$ as Γ_{n_0}-module, viewing $Cl'_{k_{n_0}}$ as Γ_{n_0}-module by some identification* $\mathrm{Gal}(k_\infty/k) = \Gamma$*;*
- *$(Cl'_{k_{n_0}})_p \cong (Cl'_{k_{n_0+1}})_p$.*

Then we have $X'_k \cong X$ as Λ-modules.

Proof. See [Fu]. □

Now fix a topological generator γ of Γ and put $\gamma_n = \gamma \bmod \Gamma^{p^n}$. Let

$$r = \dim_{\mathbb{F}_p} X/(p,\, \gamma_{n_0} - 1) \tag{2.1}$$

Then r is the number of minimal generators of X over $\mathbb{Z}/p^{m_0}[\Gamma_{n_0}]$: this follows from Nakayama's Lemma applied to the $\mathbb{Z}/p^{m_0}[\Gamma_{n_0}]$-module X (note that $(\mathbb{Z}/p^{m_0}[\Gamma_{n_0}],\, (p,\, \gamma_{n_0} - 1))$ is a local ring). Hence there exists an exact sequence of $\mathbb{Z}/p^{m_0}[\Gamma_{n_0}]$-modules

$$0 \to R_{n_0} \to \mathbb{Z}/p^{m_0}[\Gamma_{n_0}]^{\oplus r} \to X \to 0$$

Let ${\pi'}_{n_0+1,\, n_0}$ be the natural map from $\mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus r}$ to $\mathbb{Z}/p^{m_0}[\Gamma_{n_0}]^{\oplus r}$ induced by the natural projection $\Gamma_{n_0+1} \to \Gamma_{n_0}$ and put

$$R_{n_0+1} = {\pi'}^{-1}_{n_0+1,\, n_0}(R_{n_0})$$

Then ${\pi'}_{n_0+1,\, n_0}$ induces an isomorphism

$$\mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus r}/R_{n_0+1} \cong X$$

Define the submodule $\widetilde{R}_{n_0+\delta}$ $(\delta = 0,\, 1)$ of $\mathbb{Z}/p^{m_0}[\Gamma_{n_0+\delta}]^{\oplus r}$ as follows

$$\begin{aligned} \widetilde{R}_{n_0+\delta} \quad = \quad & \Big\{(\alpha_i)_{1 \le i \le r+1} \in \mathbb{Z}/p^{m_0}[\Gamma_{n_0+\delta}]^{\oplus r+1} \,\Big| \\ & (\alpha_i)_{1 \le i \le r} \in R_{n_0+\delta},\, \alpha_{r+1} \cong \sum_{i=1}^{r} \alpha_i \ (mod\ \gamma_{n_0+\delta} - 1)\Big\} \end{aligned}$$

and put

$$\widetilde{X} = \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus r}/\widetilde{R}_{n_0+1}$$

The natural injection $X \hookrightarrow \widetilde{X}$ which is given by

$$(x_i)_{1 \le i \le r}\ mod\ R_{n_0+1} \mapsto (x_1,\, \ldots,\, x_r,\, \sum_{i=1}^{r} x_i)\ mod\ \widetilde{R}_{n_0+1} \tag{2.2}$$

has cokernel isomorphic to $\mathbb{Z}/p^{m_0}$ with trivial action of Γ_{n_0+1}. Then the natural map

$$\pi_{n_0+1,\, n_0} : \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus r+1}/\widetilde{R}_{n_0+1} \to \mathbb{Z}/p^{m_0}[\Gamma_{n_0}]^{\oplus r}/\widetilde{R}_{n_0}$$

gives the isomorphism

$$\widetilde{X} = \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus r+1}/\widetilde{R}_{n_0+1} \cong \mathbb{Z}/p^{m_0}[\Gamma_{n_0}]^{\oplus r+1}/\widetilde{R}_{n_0} \tag{2.3}$$

since $\pi^{-1}_{n_0+1,\, n_0}(\widetilde{R}_{n_0}) = \widetilde{R}_{n_0+1}$.
Let g be the minimal number of generators of $\widetilde{R}_{n_0+1}$ over $\mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]$ and choose once and for all an integer N such that

$$[K_N^+ : \mathbb{Q}] = [K^+ : \mathbb{Q}]p^N \ge g \quad \text{and} \quad N \ge m_0 \tag{2.4}$$

Now we fix an isomorphism $\Gamma \cong \mathrm{Gal}(K^c/K_N)$ and we shall dentify the two groups in the following (note that $\Gamma_t = \mathrm{Gal}(K_{N+t}/K_N)$).
Let $\mathfrak{l}_i$ $(1 \le i \le r+1)$ be distinct degree one primes of K_N which split completely in K_{N+n_0+1}, say $\mathfrak{l}_i = \prod_{\gamma\in\Gamma_{n_0+1}} \gamma\mathfrak{L}_{i,\,n_0+1}$. Furthermore, we assume that $\mathfrak{l}_i$ decomposes completely in $\widetilde{K}_{N+n_0+1} := K_{N+n_0+1}(\mu_p)$. Set $\mathfrak{m} = \prod_{i=1}^{r+1} \mathfrak{l}_i$ and denote by $\mathfrak{L}_{i,\,n_0}$ the prime ideal of K_{N+n_0} below $\mathfrak{L}_{i,\,n_0+1}$.
For $t \ge 0$ denote by L_t the abelian p-extension of K_{N+t} which is maximal with respect to the following conditions

- the conductor of L_t/K_{N+t} divides $\mathfrak{m}$;
- every prime above p splits completely in L_t/K_{N+t};
- the exponent of $\mathrm{Gal}(L_t|K_{N+t})$ is less than or equal to p^{m_0}.

Since $p \nmid |Cl_0|$ (and hence $p \nmid |Cl_n|$ for every $n \ge 0$, see [Wa]), we have a class field theoretic exact sequence of Γ-modules

$$E'_{N+n_0+\delta}/p^{m_0} \overset{\rho_{n_0+\delta}}{\longrightarrow} \left(\mathcal{O}_{N+n_0+\delta}/\mathfrak{m}\right)^{\times}/p^{m_0} \overset{r_{n_0+\delta}}{\longrightarrow} \mathrm{Gal}(L_{n_0+\delta}/K_{N+n_0+\delta}) \to 0$$

for $\delta = 0,\,1$ where $\rho_{n_0+\delta}$ is the natural map and $r_{n_0+\delta}$ is the map induced by the reciprocity map.
The middle term of this exact sequence is isomorphic to $\mathbb{Z}/p^{m_0}[\Gamma_{n_0+\delta}]^{\oplus r+1}$ via the following map

$$[\alpha] \longmapsto \left(\sum_{\gamma\in\Gamma_{n_0+\delta}} \varphi\left(\left(\frac{\alpha}{\widetilde{\gamma}\widetilde{\mathfrak{L}}_{i,\,n_0+\delta}}\right)_{n_0+\delta}\right)\gamma\right)_{1\le i\le r+1} \tag{2.5}$$

Notation is as follows:

- $\widetilde{\gamma} \in \mathrm{Gal}(\widetilde{K}_{N+n_0+\delta}/\widetilde{K}_N)$ is the image of γ via the natural isomorphism $\Gamma_{n_0+\delta} \cong \mathrm{Gal}(\widetilde{K}_{N+n_0+\delta}/\widetilde{K}_N)$ where $\widetilde{K}_{N+n_0+\delta} = K_{N+n_0+\delta}(\mu_p)$;
- $\widetilde{\mathfrak{L}}_{i,n_0+1}$ are fixed primes of $\widetilde{K}_{N+n_0+1}$ lying above $\mathfrak{L}_{i,n_0+1}$ and $\widetilde{\mathfrak{L}}_{i,n_0}$ is the prime below $\widetilde{\mathfrak{L}}_{i,n_0+1}$ in $\widetilde{K}_{n_0+1}$;
- $(\,\cdot\,/\,\cdot\,)_{n_0+\delta}$ is the p^{m_0}-th power residue symbol for $\widetilde{K}_{N+n_0+\delta}$;
- φ is a fixed isomorphism $\mu_{p^{m_0}} \cong \mathbb{Z}/p^{m_0}$.

Note that $\mu_{p^{m_0}} \subseteq \widetilde{K}_N$ by (2.4) hence

$$\left(\frac{\alpha}{\widetilde{\gamma}\widetilde{\mathfrak{L}}_{i,n_0+\delta}}\right) = \left(\frac{\gamma^{-1}\alpha}{\widetilde{\mathfrak{L}}_{i,n_0+\delta}}\right)_{n_0+\delta}$$

(this is a well known property of power residue symbol, see for example [Gr]) and that, $\widetilde{\mathcal{O}}_{N+n_0+\delta}$ being the ring of integers of $\widetilde{K}_{N+n_0+\delta}$,

$$\mathcal{O}_{N+n_0+\delta}/\gamma\mathfrak{L}_{i,n_0+\delta} \cong \widetilde{\mathcal{O}}_{N+n_0+\delta}/\widetilde{\gamma}\widetilde{\mathfrak{L}}_{i,n_0+\delta}$$

since $\mathfrak{l}_i$ decomposes completely in $\widetilde{K}_{N+n_0+1}$.
In the following we shall identify $(\mathcal{O}_{N+n_0+\delta}/\mathfrak{m})^\times/p^{m_0}$ and $\mathbb{Z}/p^{m_0}[\Gamma_{n_0+\delta}]^{\oplus r+1}$ via the above isomorphism. Then we get an exact sequence

$$E'_{N+n_0+\delta}/p^{m_0} \xrightarrow{\rho_{n_0+\delta}} \mathbb{Z}/p^{m_0}[\Gamma_{n_0+\delta}]^{\oplus r+1} \xrightarrow{r_{n_0+\delta}} \mathrm{Gal}(L_{n_0+\delta}/\mathbb{Q}_{N+n_0+\delta}) \to 0$$

where $\rho_{n_0+\delta}$ is given by

$$\rho_{n_0+\delta}(\varepsilon) = \left(\sum_{\gamma\in\Gamma_{n_0+\delta}} \varphi\left(\left(\frac{\varepsilon}{\widetilde{\gamma}\widetilde{\mathfrak{L}}_{i,\,n_0+\delta}}\right)_{n_0+\delta}\right)\gamma\right)_{1\le i\le r+1} \tag{2.6}$$

There is a commutative diagram

$$\begin{array}{ccccccc}
\overline{E'_{N+n_0+1}}/p^{m_0} & \xrightarrow{\rho_{n_0+1}} & \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus r+1} & \xrightarrow{r_{n_0+1}} & \mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle N_{n_0+1,\,n_0}} & & \downarrow{\scriptstyle \pi_{n_0+1,\,n_0}} & & \downarrow{\scriptstyle res_{n_0+1,\,n_0}} & & \\
\overline{E'_{N+n_0}}/p^{m_0} & \xrightarrow{\rho_{n_0}} & \mathbb{Z}/p^{m_0}[\Gamma_{n_0}]^{\oplus r+1} & \xrightarrow{r_{n_0}} & \mathrm{Gal}(L_{n_0}/K_{N+n_0}) & \longrightarrow & 0
\end{array}$$

Diagram 2.1

where $\overline{E'_{N+n_0+1}} = E'_{N+n_0+1}/\{\pm 1\}$ for $\delta = 0,\,1$, $N_{n_0+1,\,n_0}$ is the norm map from K_{N+n_0+1} to K_{N+n_0}, $\pi_{n_0+1,\,n_0}$ is induced and $res_{n_0+1,\,n_0}$ is the restriction. Commutativity is not difficult to check using the properties of the p^{m_0}-th power residue symbol and the reciprocity map.
So far, we have followed Ozaki's proof by simply replacing K by $\mathbb{Q}$ and units by S-units when necessary. The proofs of the next lemmas, however, are slightly modified since they involve more directly the arithmetic (in particular, units) of K.

Definition 2.1.4. *Let k be an abelian number field whose conductor is p^e. We define the cyclotomic units C_k (resp. S-cyclotomic units C'_k) of k as follows*

$$\begin{aligned}
C_k &= \left\langle \pm 1,\, N_{\mathbb{Q}(\zeta_{p^e})/k}\left(\langle \zeta_{p^e},\, \zeta_{p^e}^{\frac{1-a}{2}}\frac{(1-\zeta_{p^e}^a)}{(1-\zeta_{p^e})}\,|(a,\,p)=1\rangle\right)\right\rangle \cap E_k \\
C'_k &= \left\langle \pm 1,\, N_{\mathbb{Q}(\zeta_{p^e})/k}\left(\langle \zeta_{p^e},\,(1-\zeta_{p^e}^a)\,|(a,\,p)=1\rangle\right)\right\rangle \cap E'_k
\end{aligned}$$

Here E_k (resp. E'_k) is the group of units (resp. of S-units, where S consists of the unique prime above p in k) of k.

Remark 2.1.5. The definition of C_k agrees with that of circular units of Sinnott (see [Si], §4) as it is shown for example in [Oz2], Lemma 8 and Lemma 9. In the following we set $C_n = C_{K_n}$ and $C'_n = C'_{K_n}$.

Lemma 2.1.6. *The quotient $C'_{K_n^+}/(C'_n)^+$ is a 2-group.*

Proof. Set
$$A = N_{\mathbb{Q}(\zeta_{p^e})/K_n}\left(\langle \zeta_{p^e},\, (1-\zeta_{p^e}^a)\,|(a,\,p)=1\rangle\right)$$
and $N = N_{K_n/K_n^+}$. Moreover $\Delta = \mathrm{Gal}(K_n/K_n^+)$ and denote by N_Δ the algebraic norm. Then
$$(C'_n)^+ = \langle \pm 1,\, A^\Delta\rangle \cap E_{K_n^+}$$
and
$$C'_{K_n^+} = \langle \pm 1,\, NA\rangle \cap E_{K_n^+}$$
One easily sees that $N_\Delta(A) = NA$ and that
$$|A^\Delta/NA| \quad \text{and} \quad C'_{K_n^+}/(C'_n)^+$$
differ only by a power of 2. Since $A^\Delta/N_\Delta(A) = \widehat{H}^0(\Delta,\, A)$, we know that it is a 2 group (see [Mi3]). $\square$

Lemma 2.1.7. *We have a surjective homomorphism $E_n/C_n \to E'_n/C'_n$. Moreover*
$$C'_n/W_n(C'_n)^+ \longrightarrow E'_n/W_n(E'_n)^+$$
is injective and $E'_n/W_n(E'_n)^+$ has order at most 2.

Proof. Consider the natural homomorphism $E_n \to E'_n/C'_n$: C_n belongs to its kernel, since $C_n \subseteq C'_n$ (and clearly $C_n \subseteq E_n$). Hence we have a homomorphism $E_n/C_n \to E'_n/C'_n$. Now note that
$$\eta_n = N_{\mathbb{Q}(\zeta_{p^{n+1}})/K_n}(1-\zeta_{p^{n+1}})$$
generates the unique prime ideal above p in K_n. In particular every element $u \in E'_n$ can be written $u = \eta_n^b v$ with $b \in \mathbb{Z}$ and $v \in E_n$. This proves surjectivity.
Now observe that $(C'_n)^+ \subseteq (E'_n)^+$ hence $C'_n/W_n(C'_n)^+ \longrightarrow E'_n/W_n(E'_n)^+$ is well defined To prove that it is injective, we have to show that $W_n(E'_n)^+ \cap C'_n = W_n(C'_n)^+$. It is enough to prove $W_n(E'_n)^+ \cap C'_n \subseteq W_n(C'_n)^+$ the other inclusion being clear. We have
$$W_n(E'_n)^+ \cap C'_n \subseteq W_n\left((E'_n)^+ \cap C'_n\right)$$
since $W_n \subseteq C'_n$. Moreover $(E'_n)^+ \cap C'_n \subseteq (C'_n)^+$ hence we get the claim.
Finally, for the last assertion we can suppose that K_n is CM. We know

that $E_n^2 \subseteq W_n E_n^+$ (see [Wa], Theorem 4.12). Now let σ be the generator of $\mathrm{Gal}(K_n/K_n^+)$ and set

$$\eta_n^+ = \eta_n \sigma(\eta_n)$$

Note that

$$\sigma(\eta_n) = \sigma(N_{\mathbb{Q}(\zeta_{p^{n+1}})/K_n}(1-\zeta_{p^{n+1}})) = N_{\mathbb{Q}(\zeta_{p^{n+1}})/K_n}(\sigma(1-\zeta_{p^{n+1}})) =$$

$$= N_{\mathbb{Q}(\zeta_{p^{n+1}})/K_n}(1-\zeta_{p^{n+1}}^{-1}) = N_{\mathbb{Q}(\zeta_{p^{n+1}})/K_n}(-\zeta_{p^n+1}(1-\zeta_{p^{n+1}})) = \zeta' \eta_n$$

where we denoted with σ also a generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n+1})/\mathbb{Q}(\zeta_{p^n+1})^+)$ and $\zeta' \in W_n$. Hence $\eta_n^+ = \zeta' \eta_n^2$. Thus if $u = \eta_n v \in E_n'$ (with $v \in E_n$) we have $u^2 = \zeta'^{-1} \eta_n^+ v^2$ hence

$$(E_n')^2 \subseteq W_n (E_n')^+ E_n^2 \subseteq W_n (E_n')^+$$

$\square$

Lemma 2.1.8. *1. For any $t \geq 0$, we have*

$$\overline{E'_{N+t}}/p^{m_0} \cong W_{N+t}/p^{m_0} \oplus \mathbb{Z}/p^{m_0}[\Gamma_t]^{\oplus [K_N^+:\mathbb{Q}]}$$

as $\mathbb{Z}/p^{m_0}[\Gamma_t]$-modules.

2. The norm map

$$N_{n_0+1,\, n_0} : \overline{E'_{N+n_0+1}}/p^{m_0} \longrightarrow \overline{E'_{N+n_0}}/p^{m_0}$$

is surjective.

Proof. By our hypothesis on the class number of K_n and Theorem 4.1 of [Si] we know that $[E_{N+t} : C_{N+t}]$ is not divisible by p. Hence by Lemma 2.1.7, $p \nmid [E'_{N+t} : C'_{N+t}]$ and we have

$$\overline{E'_{N+t}}/p^{m_0} \cong \left(C'_{N+t}/\{\pm 1\}\right)/p^{m_0} \qquad (2.7)$$

Furthermore

$$C'_{N+t}/p^{m_0} \cong W_{N+t}(C'_{N+t})^+/p^{m_0} \cong W_{N+t} C'_{K^+_{N+t}}/p^{m_0}$$

again by Lemma 2.1.7 and Lemma 2.1.6. Finally

$$W_{N+t} C'_{K^+_{N+t}}/p^{m_0} \cong W_{N+t}/p^{m_0} \oplus C'_{K^+_{N+t}}/p^{m_0}$$

Now we have an isomorphism

$$\mathbb{Z}[\mathrm{Gal}(K^+_{N+t}/\mathbb{Q})] \cong C'_{K^+_{N+t}}/\{\pm 1\}$$

as $\mathbb{Z}[\mathrm{Gal}(K_{N+t}^+/\mathbb{Q})]$-modules: it is defined by

$$\sigma \mapsto \sigma(\eta_{N+t}^+)$$

where

$$\eta_{N+t}^+ = N_{\mathbb{Q}(\zeta_{p^{N+t+1}})/K_{N+t}^+}(1-\zeta_{p^{N+t+1}})$$

Finally we have

$$\mathbb{Z}[\mathrm{Gal}(K_{N+t}^+/\mathbb{Q})] \cong \bigoplus_{\tau\in \mathrm{Gal}(K_{N+t}^+/\mathbb{Q})/\Gamma_t} \mathbb{Z}[\Gamma_t]\tau$$

which completes the proof of 1.
The second claim follows from (2.7) and the fact the the norm map

$$N_{n_0+1,\,n_0} : C'_{N+n_0+1}/\{\pm 1\} \longrightarrow C'_{N+n_0}/\{\pm 1\}$$

is surjective by definition. □

Lemma 2.1.9. *Let $\widetilde{E}'_{N+n_0+1}$ be the S-units group of $\widetilde{K}_{N+n_0+1}$ where S consists of the unique prime above p in $\widetilde{K}_{N+n_0+1}$. Then the natural map*

$$\overline{E'_{N+n_0+1}}/p^{m_0} \to \widetilde{E}'_{N+n_0+1}/p^{m_0}$$

is injective.

Proof. From the exact sequence

$$0 \to \mu_{p^{m_0}} \longrightarrow \widetilde{K}_{n+n_0+1}^\times \xrightarrow{p^{m_0}} (\widetilde{K}_{n+n_0+1}^\times)^{p^{m_0}} \to 0$$

we get the $G = \mathrm{Gal}(\widetilde{K}_{N+n_0+1}/K_{N+n_0+1})$-cohomology sequence

$$K_{n+n_0+1}^\times \xrightarrow{p^{m_0}} (\widetilde{K}_{n+n_0+1}^\times)^{p^{m_0}} \cap K_{n+n_0+1} \longrightarrow H^1(G,\, \mu_{p^{m_0}}) \to 0$$

Now $H^1(G,\, \mu_{p^{m_0}}) = 0$ since $(|G|,\, p) = 1$, hence

$$(K_{n+n_0+1}^\times)^{p^{m_0}} = (\widetilde{K}_{n+n_0+1}^\times)^{p^{m_0}} \cap K_{n+n_0+1}$$

and the lemma follows. □

Lemma 2.1.10. *For any Γ_{n_0+1}-homomorphism*

$$f : \overline{E'_{N+n_0+1}}/p^{m_0} \to \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]$$

there exist infinitely many degree one primes $\widetilde{\mathfrak{L}}$ of $\widetilde{K}_{N+n_0+1}$ such that

$$f(\overline{\varepsilon}) = \sum_{\gamma\in\Gamma_{n_0+1}} \varphi\left(\left(\frac{\varepsilon}{\widetilde{\gamma}\widetilde{\mathfrak{L}}}\right)_{n_0+1}\right)\gamma$$

for any $\varepsilon \in E'_{N+n_0+1}$ *whose class in* $\overline{E'_{N+n_0+1}}/p^{m_0}$ *is* $\overline{\varepsilon}$*. Furthermore, for any fixed finite abelian extension* $M/\widetilde{K}_{N+n_0+1}$ *with*

$$M \cap \widetilde{K}_{N+n_0+1}(\sqrt[p^{m_0}]{E'_{N+n_0+1}}) = \widetilde{K}_{N+n_0+1}$$

and $\tau \in \mathrm{Gal}(M/\widetilde{K}_{N+n_0+1})$ *we can impose the condition*

$$\left(\frac{M/\widetilde{K}_{N+n_0+1}}{\widetilde{\mathfrak{L}}}\right) = \tau$$

on $\widetilde{\mathfrak{L}}$.

Proof. We have to distinguish to cases, namely $d = 1$ and $d \neq 1$: in the first case $W_n = \langle \pm 1, \mu_{p^n+1}\rangle$ while if $d \neq 1$, $W_n = \{\pm 1\}$.
We treat first the case $d = 1$, we have $K = \mathbb{Q}(\zeta_p)$, $\widetilde{K}_n = K_n$ and $\widetilde{\gamma} = \gamma$ so we drop the tilde. Set $q_N = [K_N^+ : \mathbb{Q}]$. From Lemma 2.1.8, there exist $\varepsilon_j \in E'_{N+n_0+1}$ $(1 \leq j \leq q_N)$ such that

$$\overline{E'_{N+n_0+1}}/p^{m_0} = \bigoplus_{j=1}^{q_N} \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]\overline{\varepsilon_j} \oplus \mathbb{Z}/p^{m_0}\overline{\zeta} \tag{2.8}$$

where $\zeta = \zeta_{p^{N+n_0+2}}$ is a primitive p^{N+n_0+2}-th root of unity and $\overline{\varepsilon_j}, \overline{\zeta} \in \overline{E'_{N+n_0+1}}/p^{m_0}$ are the classes of ε_j and ζ respectively.
Assume that

$$f(\overline{\varepsilon_j}) = \sum_{\gamma \in \Gamma_{n_0+1}} c_{j,\gamma}\gamma \quad \text{and} \quad f(\overline{\zeta}) = \sum_{\gamma \in \Gamma_{n_0+1}} d_\gamma \gamma$$

We are going to show that there exist infinitely many degree one primes $\mathfrak{L}$ of K_{N+n_0+1} such that

$$\left(\frac{\varepsilon_j}{\gamma\mathfrak{L}}\right)_{n_0+1} = \varphi^{-1}(c_{j,\gamma}) \quad (1 \leq j \leq q_N,\, \gamma \in \Gamma_{n_0+1}) \tag{2.9}$$

$$\left(\frac{\zeta}{\mathfrak{L}}\right)_{n_0+1} = \varphi^{-1}(d_1) \tag{2.10}$$

Note that if the above conditions hold, then the conditions

$$\left(\frac{\zeta}{\gamma\mathfrak{L}}\right)_{n_0+1} = \varphi^{-1}(d_\gamma) \quad (\gamma \in \Gamma_{n_0+1})$$

also hold since

$$\left(\frac{\zeta}{\gamma\mathfrak{L}}\right)_{n_0+1} = \left(\frac{\zeta}{\mathfrak{L}}\right)_{n_0+1}^{\kappa(\gamma^{-1})} \quad \text{and} \quad d_\gamma = \kappa(\gamma^{-1})d_1$$

Here $\kappa : \Gamma_{n_0+1} \longrightarrow \mathbb{Z}_p^\times$ is the cyclotomic character: in particular $\gamma(\zeta) = \zeta^{\kappa(\gamma)}$. The first equality then comes from the well known properties of the p^{m_0}-power residue symbol while the second equality is a consequence of the fact that f is a Γ_{n_0+1}-homomorphism.
For each $1 \leq j \leq q_N$, consider

$$D_j = K_{N+n_0+1}(\sqrt[p^{m_0}]{\gamma^{-1}\epsilon_j} \,|\, \gamma \in \Gamma_{n_0+1}) \quad \text{and} \quad D_0 = K_{N+n_0+1}(\sqrt[p^{m_0}]{\zeta})$$

It follows from (2.8) that the abelian extensions D_j/K_{N+n_0+1} $(1 \leq j \leq q_N)$ and D_0/K_{N+n_0+1} are independent and that

$$\mathrm{Gal}(D_j/K_{N+n_0+1}) \cong \bigoplus_{\gamma\in\Gamma_{n_0+1}} \mu_{p^{m_0}} \qquad \sigma \mapsto \left(\frac{\sigma\, \sqrt[p^{m_0}]{\gamma^{-1}\varepsilon_j}}{\sqrt[p^{m_0}]{\gamma^{-1}\varepsilon_j}} \right)_{\gamma\in\Gamma_{n_0+1}} \tag{2.11}$$

$$\mathrm{Gal}(D_0/K_{N+n_0+1}) \cong \mu_{p^{m_0}} \qquad \sigma \mapsto \left(\frac{\sigma\, \sqrt[p^{m_0}]{\zeta}}{\sqrt[p^{m_0}]{\zeta}} \right) \tag{2.12}$$

Now call D the compositum of the extension D_j $(1 \leq j \leq q_N)$ and D_0: choose the automorphism $\sigma \in \mathrm{Gal}(D/K_{N+n_0+1})$ which corresponds to the $(q_N + 1)$-uple

$$\Big((\varphi^{-1}(c_{1,\gamma}))_{\gamma\in\Gamma_{n_0+1}}, (\varphi^{-1}(c_{2,\gamma}))_{\gamma\in\Gamma_{n_0+1}}, \dots, (\varphi^{-1}(c_{q_N,\gamma}))_{\gamma\in\Gamma_{n_0+1}}, \varphi^{-1}(d_1)\Big)$$

by (2.11). By Čebotarev density theorem, applied to the abelian extension D/K_{N+n_0+1} and to the automorphism σ, there exist infinitely many primes $\mathfrak{L}$ in K_{N+n_0+1} such that (2.9) and (2.10) are simultaneously satisfied. Note that we can also find infinitely many primes $\mathfrak{L}$ which split completely in $K_{N+n_0+1}/\mathbb{Q}$ and satisfy (2.9) and (2.9) since the set of primes on K_{N+n_0+1} which split completely in $K_{N+n_0+1}/\mathbb{Q}$ has density 1 (for Čebotarev density theorem and this last result, see for example [Mi3]). An analogous kind of reasoning applies to give the last part of the assertion of the lemma, still for $d = 1$.
Now we consider the case $d \neq 1$: the strategy is the same but now we have $\widetilde{K}_n \neq K_n$, $W_n = \{\pm 1\}$ and Lemma 2.1.8 shows that there exist $\varepsilon_j \in E'_{N+n_0+1}$ $(1 \leq j \leq q_N)$ such that

$$\overline{E'_{N+n_0+1}}/p^{m_0} = \bigoplus_{j=1}^{q_N} \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]\overline{\varepsilon_j} \tag{2.13}$$

where we still set $q_N = [K_N^+ : \mathbb{Q}]$ and $\overline{\varepsilon_j}$, $\overline{\zeta} \in \overline{E'_{N+n_0+1}}/p^{m_0}$ is the class of ε_j. Assume that

$$f(\overline{\varepsilon_j}) = \sum_{\gamma\in\Gamma_{n_0+1}} c_{j,\gamma}\gamma$$

We are going to show that there exist infinitely many degree one primes $\mathfrak{L}$ of K_{N+n_0+1} such that

$$\left(\frac{\varepsilon_j}{\widetilde{\gamma}\widetilde{\mathfrak{L}}}\right)_{n_0+1} = \varphi^{-1}(c_{j,\gamma}) \quad (1 \le j \le q_N,\, \gamma \in \Gamma_{n_0+1}) \tag{2.14}$$

For each $1 \le j \le q_N$, consider

$$\widetilde{D}_j = \widetilde{K}_{N+n_0+1}(\sqrt[p^{m_0}]{\gamma^{-1}\epsilon_j} \,|\, \gamma \in \Gamma_{n_0+1})$$

It follows from (2.8) and Lemma 2.1.9 that the extensions $\widetilde{D}_j/\widetilde{K}_{N+n_0+1}$ $(1 \le j \le q_N)$ are independent and that

$$\mathrm{Gal}(\widetilde{D}_j/\widetilde{K}_{N+n_0+1}) \cong \bigoplus_{\gamma\in\Gamma_{n_0+1}} \mu_{p^{m_0}} \qquad \widetilde{\sigma} \mapsto \left(\frac{\widetilde{\sigma}\,\sqrt[p^{m_0}]{\gamma^{-1}\varepsilon_j}}{\sqrt[p^{m_0}]{\gamma^{-1}\varepsilon_j}}\right)_{\gamma\in\Gamma_{n_0+1}} \tag{2.15}$$

Now call $\widetilde{D}$ the compositum of the extension $\widetilde{D}_j$ $(1 \le j \le q_N)$: choose the automorphism $\widetilde{\sigma} \in \mathrm{Gal}(\widetilde{D}/\widetilde{K}_{N+n_0+1})$ which corresponds to the q_N-uple

$$\Big((\varphi^{-1}(c_{1,\gamma}))_{\gamma\in\Gamma_{n_0+1}},\, (\varphi^{-1}(c_{2,\gamma}))_{\gamma\in\Gamma_{n_0+1}},\, \ldots,\, (\varphi^{-1}(c_{q_N,\gamma}))_{\gamma\in\Gamma_{n_0+1}}\Big)$$

by (2.15). As before, Čebotarev density theorem, applied to the extension $\widetilde{D}/\widetilde{K}_{N+n_0+1}$ and to the automorphism $\widetilde{\sigma}$, tells us that there exist infinitely many degree one primes $\widetilde{\mathfrak{L}}$ in $\widetilde{K}_{N+n_0+1}$ such that (2.9) are simultaneously satisfied. The last part of the assertion of the lemma for $d \ne 1$ follows easily.
□

Now we choose the prime $\widetilde{\mathfrak{L}}_{i,\,n_0+1}$ and $\mathfrak{l}_i$. By (2.4) and Lemma 2.1.8, we can find a surjective homomorphism of $\mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]$- modules

$$h' : \overline{E'_{N+n_0+1}}/p^{m_0} \to \widetilde{R}_{n_0+1}$$

Composing h' with the inclusion $\widetilde{R}_{n_0+1} \hookrightarrow \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus^{r+1}}$ we get a map

$$h : \overline{E'_{N+n_0+1}}/p^{m_0} \longrightarrow \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus^{r+1}}$$

whose image is $\widetilde{R}_{n_0+1}$. Assume that the following condition is true for the primes $\mathfrak{L}_{i,\,n_0+1}$ $(1 \le i \le r+1)$:

Condition A.

$$pr_i \circ h(\bar{\varepsilon}) = \sum_{\gamma\in\Gamma_{n_0+1}} \varphi\left(\left(\frac{\bar{\varepsilon}}{\widetilde{\gamma}\widetilde{\mathfrak{L}}_{i,\,n_0+1}}\right)_{n_0+1}\right)\gamma$$

for $1 \leq i \leq r+1$ where $pr_i : \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus^{r+1}} \longrightarrow \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]$ denotes the projection map onto the i-th component.

Lemma 2.1.10 tells us that there exist degree one primes $\mathfrak{L}_{i,\,n_0+1}$ of $\widetilde{K}_{N+n_0+1}$ satisfying Condition A such that $\widetilde{\mathfrak{L}}_{i,\,n_0+1}$'s lie over distinct rational primes. We choose the prime of K_N (resp. $K_{N+n_0+\delta}$) below $\widetilde{\mathfrak{L}}_{i,\,n_0+1}$ as $\mathfrak{l}_i$ (resp. $\mathfrak{L}_{i,\,n_0+\delta}$ ($\delta = 0,\, 1$)) and put $\mathfrak{m} = \prod_{i=1}^{r+1} \mathfrak{l}_i$. Then we have $\mathrm{Im}(\rho_{n_0+1}) = \mathrm{Im}(h) = \widetilde{R}_{n_0+1}$ by (2.6). Hence r_{n_0+1} induces the isomorphism

$$\widetilde{X} = \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus^{r+1}}/\widetilde{R}_{n_0+1} \cong \mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1})$$

We have also

$$\mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1}) \cong \mathrm{Gal}(L_{n_0}/K_{N+n_0}) \tag{2.16}$$

since $\mathrm{Im}(\rho_{n_0}) = \widetilde{R}_{n_0}$ and

$$\mathrm{Gal}(L_{n_0}/K_{N+n_0}) \cong \mathbb{Z}/p^{m_0}[\Gamma_{n_0}]^{\oplus r+1}/\widetilde{R}_{n_0} \cong \widetilde{X}$$

(look at Diagram 2.1 and use Lemma 2.1.8 as well as $\widetilde{R}_{n_0+1} = \pi^{-1}_{n_0+1,\,n_0}(\widetilde{R}_{n_0})$ and (2.3)). We identify $\mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1})$ with $\widetilde{X}$ via this isomorphism. We regard $X = \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus r+1}/R_{n_0+1}$ as a submodule of the module $\widetilde{X} = \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus r+1}/\widetilde{R}_{n_0+1}$ via the embedding given in (2.2) and define F to be the intermediate field of L_{n_0+1}/K_{N+n_0+1} with

$$X = \mathrm{Gal}(L_{n_0+1}/F) \tag{2.17}$$

Lemma 2.1.11.

1. *There exists a unique cyclic extension k/K_N of degree p^{m_0} with conductor dividing $\mathfrak{m}$ and every prime above p splits. Moreover $F = k_{n_0+1} = kK_{N+n_0+1}$ and $L_{n_0} \cap k_{n_0+1} = kK_{N+n_0} = k_{n_0}$.*

2. *For any $1 \leq i \leq r+1$ and any $\gamma_0 \in \Gamma_{n_0+1}$, the inertia subgroup of $\gamma_0\mathfrak{L}_{i_0,\,n_0+1}$ in $\mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1})$ is generated over $\mathbb{Z}/p^{m_0}$ by the element $r_{n_0+1}((0,\,\ldots,\,\gamma_0,\,\ldots,\,0))$ (γ_0 at i_0-th place and 0 everywhere else).*

3. *The primes $\gamma\mathfrak{L}_{i,\,n_0+\delta}$ ($\gamma \in \Gamma_{n_0+\delta},\, 1 \leq i \leq r+1$) are totally ramified in $k_{n_0+\delta}/K_{N+n_0+\delta}$. In particular*

$$\mathrm{Gal}(L_{n_0+\delta}/K_{N+n_0+\delta}) \cong \mathrm{Gal}(L_{n_0+\delta}/k_{n_0+\delta}) \times \mathrm{Gal}(k_{n_0+\delta}/K_{N+n_0+\delta})$$

   Furthermore the primes $\mathfrak{l}_i$ ($1 \leq i \leq r+1$) are totally ramified in k.

4. *$L_{n_0+\delta}$ is the p-split genus p-class field of $k_{n_0+\delta}/K_{N+n_0+\delta}$ for $\delta = 0,\, 1$.*

Proof. 1. Since $\widetilde{X}/X$ is isomorphic to $\mathbb{Z}/p^{m_0}\mathbb{Z}$ with trivial action (see (2.2)), we deduce that F/K_N is an abelian extension. Moreover

$$\mathrm{Gal}(F/K_N) = \mathrm{Gal}(F/K_{N+n_0+1}) \times I_p$$

where $I_p \subseteq \mathrm{Gal}(F/K_N)$ is the inertia subgroup for the unique prime of K_N lying above p (clearly $I_p\mathrm{Gal}(F/K_{N+n_0+1}) = \mathrm{Gal}(F/K_N)$ and then one looks at cardinalities). Then we choose k as the fixed field of I_p and one verifies immediately that k has the required properties. For the second assertion, it is clear by contruction that

$$F = k_{n_0+1} = kK_{N+n_0+1}$$

For the analogous assertion on k_{n_0}, we just remark that k_{n_0} is the inertia field of the primes lying over p in k_{n_0+1}/K_{N+n_0}. This implies $L_{n_0} \cap k_{n_0+1} \subseteq k_{n_0}$ and the reverse inclusion is clear since the conductor of k_{n_0} divides $\mathfrak{m}$, any prime above p splits in k_{n_0}/K_{N+n_0} and k_{n_0}/K_{N+n_0} has exponent dividing p^{m_0} (because all this happens in k/K_N).

2. In order to identify the inertia subgroup of $\gamma_0\mathfrak{L}_{i_0,\,n_0+\delta}$ is better to consider a convenient version of the morphism r_{n_0+1}: first of all we know that

$$\mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus r+1} \cong (\mathcal{O}_{N+n_0+1}/\mathfrak{m})^\times/p^{m_0}$$

the isomorphism being defined in (2.5). Secondly, it is well known (see [Mi3]) that

$$(\mathcal{O}_{N+n_0+1}/\mathfrak{m})^\times/p^{m_0} \cong (K^\times_{N+n_0+1})_\mathfrak{m}/(K^\times_{N+n_0+1})_{\mathfrak{m},\,1}(K^\times_{N+n_0+1})^{p^{m_0}}_\mathfrak{m}$$

where $(K^\times_{N+n_0+1})_\mathfrak{m}$ (resp. $(K^\times_{N+n_0+1})_{\mathfrak{m},\,1}$) is the subgroup of $K^\times_{N+n_0+1}$ whose elements are coprime with $\mathfrak{m}$ (resp. are congruent to 1 modulo $\mathfrak{m}$). In the following we set

$$(K^\times_{N+n_0+1})_\mathfrak{m}/(K^\times_{N+n_0+1})_{\mathfrak{m},\,1}(K^\times_{N+n_0+1})^{p^{m_0}}_\mathfrak{m} = \overline{(K^\times_{N+n_0+1})_\mathfrak{m}}$$

Under this identification it is easy to see that the subgroup generated by $(0,\, \dots,\, \gamma_0,\, \dots,\, 0)$ gets indentified with the subgroup

$$A = \left\{ [\alpha] \in \overline{(K^\times_{N+n_0+1})_\mathfrak{m}} \,\middle|\, \left(\frac{\alpha}{\widetilde{\gamma}\widetilde{\mathfrak{L}}_{i,\,n_0+1}}\right) = 1 \ \forall\ \gamma \neq \gamma_0,\ \ \forall\ i \neq i_0 \right\}$$

Furthermore we know (see [Gr]) that the reciprocity map identifies $\mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1})$ with

$$(I_{N+n_0+1})_T/(P_{N+n_0+1})_{T,\,\mathfrak{m}}S_{N+n_0+1}(I_{N+n_0+1})^{p^{m_0}}_T$$

where $(I_{N+n_0+1})_T$ (resp. $P_{N+n_0+1})_{T,\,\mathfrak{m}}$) is the group of fractional ideals (resp. principal fractional ideals) of K_{N+n_0+1} which are coprime with the support T of $\mathfrak{m}$ (resp. which are coprime with the support T of $\mathfrak{m}$ and that can be generated by elements congruent to 1 modulo $\mathfrak{m}$). In the following set

$$(I_{N+n_0+1})_T/(P_{N+n_0+1})_{T,\,\mathfrak{m}}S_{N+n_0+1}(I_{N+n_0+1})_T^{p^{m_0}} = \overline{(I_{N+n_0+1})_T}$$

Under this isomorphism it is known (see [Gr]) that the inertia subgroup of $\gamma_0\mathfrak{L}_{i_0,\,n_0+1}$ corresponds to the subgroup generated by

$$(P_{N+n_0+1})_{T,\,\frac{\mathfrak{m}}{\gamma_0\mathfrak{L}_{i_0,\,n_0+1}}}$$

in $\overline{(I_{N+n_0+1})_T}$ which we simply denote by $\overline{(P_{N+n_0+1})_T}$. Now we have

$$r_{n_0+1} : \overline{(K^{\times}_{N+n_0+1})_{\mathfrak{m}}} \to \overline{(I_{N+n_0+1})_T}$$

(which now is the map which sends an element in the principal ideal generated by it) and we have to prove that $r_{n_0+1}(A) = \overline{(P_{N+n_0+1})_T}$. By definition $\overline{(P_{N+n_0+1})_T}$ is the smallest subgroup which contains $(P_{N+n_0+1})_{T,\,\mathfrak{m}/\gamma_0\mathfrak{L}_{i_0,\,n_0+1}}$ and hence it is contained in $r_{n_0+1}(A)$. Conversely, suppose that $[\alpha] \in A$: then α is a p^{m_0}-th power modulo $\gamma\mathfrak{L}_{i,\,n_0+1}$ (for every $\gamma \neq \gamma_0$ and $i \neq i_0$). Then by Chinese Remainder Theorem we can find an element $\beta \in (K^{\times}_{N+n_0+1})_{\mathfrak{m}}$ such that α is congruent to $\beta^{p^{m_0}}$ modulo $\gamma\mathfrak{L}_{i,\,n_0+1}$ (for every $\gamma \neq \gamma_0$ and $i \neq i_0$) and

$$(\alpha) = (\alpha\beta^{-p^{m_0}})(\beta^{p^{m_0}}) \in (P_{N+n_0+1})_{T,\,\frac{\mathfrak{m}}{\gamma_0\mathfrak{L}_{i_0,\,n_0+1}}}S_{N+n_0+1}(I_{N+n_0+1})_T^{p^{m_0}}$$

3. Simple computations (using defintions and the fact that $c\gamma \notin (\gamma_{n_0+1} - 1)$ for every $c \in \mathbb{Z}/p^{m_0}$) show that $(0,\,\ldots,\,\gamma,\,\ldots,\,0)$ has order p^{m_0} modulo $\widetilde{R}_{n_0+1}$ and that

$$\left(\left(\mathbb{Z}/p^{m_0}\mathbb{Z}(0,\,\ldots,\,\gamma,\,\ldots,\,0) + \widetilde{R}_{n_0+1}\right)\Big/\widetilde{R}_{n_0+1}\right) \cap X = 0$$

Hence the prime $\gamma\mathfrak{L}_{i,\,n_0+1}$ is totally ramified in k_{n_0+1}/K_{N+n_0+1} and L_{n_0+1}/k_{n_0+1} is an unramified extension where every prime above p splits. In particular the restriction

$$\mathrm{Gal}(L_{n_0+\delta}/K_{N+n_0+\delta}) \to \mathrm{Gal}(k_{n_0+\delta}/K_{N+n_0+\delta})$$

splits (the inertia subgroup of $\gamma\mathfrak{L}_{i,\,n_0+1}$ being a subgroup of the group $\mathrm{Gal}(L_{n_0+\delta}/K_{N+n_0+\delta})$ whose restriction is $\mathrm{Gal}(k_{n_0+\delta}/K_{N+n_0+\delta})$). The remaining assertions easily follow from 1.

4. Let L' be the p-split genus p-class field of k_{n_0+1}/K_{N+n_0+1} (which is of course a Galois extension of K_{N+n_0+1}). Clearly $L_{n_0+1} \subseteq L'$. Now we show $L' \subseteq L_{n_0+1}$: the subgroup of $\mathrm{Gal}(L'/K_{N+n_0+1})$ which is generated by the inertia subgroup of the primes $\gamma\mathfrak{L}_{i,\,n_0+1}$ in L'/K_{N+n_0+1} coincides with the whole $\mathrm{Gal}(L'/K_{N+n_0+1})$, since the $\gamma\mathfrak{L}_{i,\,n_0+1}$'s are the only ramified prime and $p \nmid h_{K_{N+n_0+1}}$. Now each of these inertia subgroups has exponent p^{m_0} (in fact it is isomorphic to the inertia subgroup of the same prime in k_{n_0+1}/K_{N+n_0+1} which is the whole $\mathrm{Gal}(k_{n_0+1}/K_{N+n_0+1})$). Hence $\mathrm{Gal}(L'/K_{N+n_0+1})$ has exponent p^{m_0} and since the conductor of L'/K_{N+n_0+1} divides $\mathfrak{m}$ and L'/K_{N+n_0+1} is split at every prime over p, we deduce $L' \subseteq L_{n_0+1}$. The proof is analogous for L_{n_0}.

$\square$

In the following we are going to prove that, in fact, $L_{n_0+\delta}$ is the whole p-split Hilbert p-class field of $k_{n_0+\delta}$ for $\delta = 0,\,1$. Once we have this, we know that the cyclotomic $\mathbb{Z}_p$-extension over k is the desired $\mathbb{Z}_p$-extension (i.e. its p-split Iwasawa module is isomorphic to X) because of Lemma 2.1.3, Lemma 2.1.11, (2.16) and (2.17).
Let $H_{n_0+\delta}$ be the p-split Hilbert p-class field of $k_{n_0+\delta}$ for $\delta = 0,\,1$ and σ be a generator of $\mathrm{Gal}(k_{n_0+1}/K_{N+n_0+1})$. Then we have

$$\mathrm{Gal}(L_{n_0+1}/k_{n_0+1}) \cong \mathrm{Gal}(H_{n_0+1}/k_{n_0+1})/(\sigma-1)$$

by Lemma 2.1.11. Denote by $\overline{\mathfrak{L}}_{i,\,n_0+1}$ the unique prime of k_{n_0+1} lying over $\mathfrak{L}_{i,\,n_0+1}$ (Lemma 2.1.11). Set for a moment $A = (Cl'_{k_{n_0+1}})_p$ and $G = \langle\sigma\rangle = \mathrm{Gal}(k_{n_0+1}/K_{N+n_0+1})$. First of all note that

$$|A^G| = |A|/|(\sigma-1)A| \tag{2.18}$$

Moreover

$$\left\{\left(\overline{\mathfrak{L}}_{i,\,n_0+1},\, H_{n_0+1}/k_{n_0+1}\right),\, 1 \le i \le r+1\right\} \subseteq A^G$$

by well known properties of the Artin symbol. We have an injective map

$$A^G/(A^G \cap (\sigma-1)A) \longrightarrow A/(\sigma-1)A \cong \mathrm{Gal}(L_{n_0+1}/k_{n_0+1})$$

and, if

$$\left\{\left(\overline{\mathfrak{L}}_{i,\,n_0+1},\, L_{n_0+1}/k_{n_0+1}\right),\, 1 \le i \le r+1\right\}$$

generates $\mathrm{Gal}(L_{n_0+1}/k_{n_0+1})$, this map is an isomorphism. In this case, by (2.18), we must have

$$A^G \cap (\sigma-1)A = 0 \tag{2.19}$$

This means $(\sigma-1)A = 0$ by Nakayama's lemma[1]. Hence, if

$$\left\{\left(\overline{\mathfrak{L}}_{i,\,n_0+1},\, L_{n_0+1}/k_{n_0+1}\right),\, 1 \le i \le r+1\right\}$$

[1] We consider A as an R-module where $R = \mathbb{Z}/p^r\mathbb{Z}[G]$ and p^r is the exponent of A. R is a local ring with maximal ideal $\mathscr{M} = (1-\sigma,\, p)$ and by (2.19) we get $(1-\sigma)\left((1-\sigma)A\right) = (1-\sigma)A$ which implies in particular $\mathscr{M}\left((1-\sigma)A\right) = (1-\sigma)A$ and hence Nakayama's lemma gives us the result.

generates $\mathrm{Gal}(L_{n_0+1}/k_{n_0+1})$, $L_{n_0+1} = H_{n_0+1}$ and this implies also $L_{n_0} = H_{n_0}$ since $H_{n_0}k_{n_0+1} \subseteq H_{n_0+1}$ and $L_{n_0+1} = L_{n_0}k_{n_0+1}$ (by 2.16).

Lemma 2.1.12. *The restriction induces isomorphisms*

$$\mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1})_{\Gamma_{n_0+1}} \cong \mathrm{Gal}(L_0/K_N)$$

and

$$\mathrm{Gal}(L_{n_0+1}/k_{n_0+1})_{\Gamma_{n_0+1}} \cong \mathrm{Gal}(L_0/k)$$

Proof. Let M be the intermediate field of L_{n_0+1}/K_{N+n_0+1} with

$$\mathrm{Gal}(L_{n_0+1}/M) = (\gamma_{n_0+1} - 1)\mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1})$$

γ_{n_0+1} being a generator of Γ_{n_0+1}. Then

$$\mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1})_{\Gamma_{n_0+1}} = \mathrm{Gal}(M/K_{N+n_0+1})$$

and M/K_N is an abelian extension. Clearly we have $L_0K_{N+n_0+1} \subseteq M$. Let $I_p \subseteq \mathrm{Gal}(M/K_N)$ be the inertia subgroup of the unique prime of K_N lying over p. Then

$$\mathrm{Gal}(M/K_N) = \mathrm{Gal}(M/K_{N+n_0+1}) \times I_p$$

and the fixed field of I_p is contained in L_0. Therefore we have $L_0K_{N+n_0+1} = M$ and

$$\mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1})_{\Gamma_{n_0+1}} \cong \mathrm{Gal}(M/K_{N+n_0+1}) \cong \mathrm{Gal}(L_0/K_N)$$

since $I_p = \mathrm{Gal}(M/L_0)$.
To show the second assertion, it is enough to show

$$(\gamma_{n_0+1} - 1)X = (\gamma_{n_0+1} - 1)\widetilde{X}$$

because $(\gamma_{n_0+1} - 1)\widetilde{X} = \mathrm{Gal}(L_{n_0+1}/L_0K_{N+n_0+1})$ by the first assertion and $L_0 \cap k_{n_0+1} = k_0$. Let $\overline{(x_i)} \in \widetilde{X} = \mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]^{\oplus r+1}/\widetilde{R}_{n_0+1}$ be any element. Since

$$\left(0, \ldots, 0, (\gamma_{n_0+1} - 1)\left(\sum_{i=1}^{r} x_i - x_{r+1}\right)\right)$$

we have

$$\begin{aligned}
(\gamma_{n_0+1} - 1)(\overline{(x_i)}) &= \overline{((\gamma_{n_0+1} - 1)x_i)} \\
&= \left((\gamma_{n_0+1} - 1)x_1, \ldots, (\gamma_{n_0+1} - 1)x_r, (\gamma_{n_0+1} - 1)\sum_{i=1}^{r} x_i\right) \\
&= (\gamma_{n_0+1})\left(x_1, \ldots, x_r, \sum_{i=1}^{r} x_i\right) \in (\gamma_{n_0+1})X
\end{aligned}$$

Hence $(\gamma_{n_0+1} - 1)\widetilde{X} \subseteq (\gamma_{n_0+1} - 1)$ and the other inclusion is trivial. $\square$

Let $L_0^{(p)}$ and $L_k^{(p)}$ be the maximal elementary abelian p-subextension of L_0/K_N and L_0/k respectively. Denote by $k^{(p)}$ the intermediate field of k/K_N with $[k^{(p)} : K_N] = p$. Now note that

$$(\mathrm{Gal}(k_{n_0+1}/K_{N+n_0+1})_{\Gamma_{n_0+1}})/p \cong \mathbb{Z}/p \tag{2.20}$$

by Nakayama's lemma. Then we have

$$\begin{aligned} \mathrm{Gal}(L_0^{(p)}/K_N) &\cong (\mathrm{Gal}(L_{n_0+1}/K_{N+n_0+1})_{\Gamma_{n_0+1}})/p \\ &\cong \left((\mathrm{Gal}(L_{n_0+1}/k_{n_0+1}) \times \mathrm{Gal}(k_{n_0+1}/K_{N+n_0+1}))_{\Gamma_{n_0+1}}\right)/p \\ &\cong (\mathbb{Z}/p\mathbb{Z})^{\oplus r+1} \end{aligned}$$

by Lemma 2.1.12, Lemma 2.1.11, (2.1), (2.17) and (2.20). Moreover

$$\mathrm{Gal}(L_k^{(p)}/K_N) = \mathrm{Gal}(L_k^{(p)}/k) \times \mathrm{Gal}(k/K_N) \tag{2.21}$$

because $\mathfrak{l}_i$ is totally ramified in k/K_N and $L_k^{(p)}/k$ is an unramified extension by Lemma 2.1.11. Hence $L_k^{(p)} = kL_0^{(p)}$: an inclusion is clear, the other comes from the fact that the inertia field of $\mathfrak{l}_i$ in $L_k^{(p)}/K_N$ is p-elementary abelian by (2.21) and hence contained in $L_0^{(p)}$ (and $L_k^{(p)}$ is the compositum of this inertia field and k). Furthermore

$$(\mathrm{Gal}(L_{n_0+1}/k_{n_0+1})_{\Gamma_{n_0+1}})/p \cong \mathrm{Gal}(L_k^{(p)}/k) \cong \mathrm{Gal}(L_0^{(p)}/k^{(p)}) \tag{2.22}$$

by Lemma 2.1.12, where isomorphisms are given by restriction. Suppose now that

$$\left\{\left(\bar{\mathfrak{l}}_i,\, L_0^{(p)}/k^{(p)}\right),\, 1 \le i \le r+1\right\} \tag{2.23}$$

($\bar{\mathfrak{l}}_i$ being the unique prime of $k^{(p)}$ lying over $\mathfrak{l}_i$) generates $\mathrm{Gal}(L_0^{(p)}/k^{(p)})$. Then $\mathrm{Gal}(L_{n_0+1}/k_{n_0+1})$ is generated by

$$\left\{\left(\overline{\mathfrak{L}}_{i,\,n_0+1},\, L_{n_0+1}/k_{n_0+1}\right),\, 1 \le i \le r+1\right\} \tag{2.24}$$

over $\mathbb{Z}/p^{m_0}[\Gamma_{n_0+1}]$ since

$$\left(\overline{\mathfrak{L}}_{i,\,n_0+1},\, L_{n_0+1}/k_{n_0+1}\right) \mapsto \left(\bar{\mathfrak{l}}_i,\, L_0^{(p)}/k^{(p)}\right)$$

under the restriction map of (2.22) and then one applies Nakayama's lemma[2]. Hence we are going to choose the $\mathfrak{l}_i$'s in such a way that the set in (2.23) generates $\mathrm{Gal}(L_0^{(p)}/k^{(p)})$.

Let I_i (for $1 \le i \le r+1$) be the inertia subgroup of $\mathrm{Gal}(L_0^{(p)}/K_N)$ for

[2] Let $A = \mathrm{Gal}(L_{n_0+1}/k_{n_0+1})$ (which is a Γ_{n_0+1}-module) and let B be the submodule of A which is generated by the set in (2.24). If $B \to A/(p,\, \gamma_{n_0+1} - 1)$ is surjective, then $B + (p,\, \gamma_{n_0+1} - 1)A = A$ which implies $A = B$ by Nakayama's lemma.

the prime $\mathfrak{l}_i$. Then we have $I_i \cong \mathbb{Z}/p\mathbb{Z}$: for, $\mathfrak{l}_i$ ramifies in $k^{(p)}$ by Lemma 2.1.11 (i.e. I_i is nontrivial) and the inertia group of $\mathfrak{l}_i$ in $\mathrm{Gal}(L_k^{(p)}/K_N)$ is cyclic. Hence I_i (which is the image under restriction the inertia group of $\mathfrak{l}_i$ in $\mathrm{Gal}(L_k^{(p)}/K_N)$) is cyclic of order p, since $L_0^{(p)}/K_N$ is elementary abelian. Therefore

$$\mathrm{Gal}(L_0^{(p)}/K_N) = \bigoplus_{i=1}^{r+1} I_i$$

(since $p \nmid h_{K_N}$ implies that $\prod I_i = \mathrm{Gal}(L_0^{(p)}/K_N)$ and $\mathrm{Gal}(L_0^{(p)}/K_N) \cong (\mathbb{Z}/p)^{\oplus r+1}$). Hence $L_0^{(p)}/K_N$ is the composite of the abelian extensions $K_N^{(p)}(\mathfrak{l}_i)/K_N$ $(1 \le i \le r+1)$ of degree p with conductor $\mathfrak{l}_i$ and the restriction induces the isomorphism

$$\mathrm{Gal}(L_0^{(p)}/k^{(p)}) \cong \bigoplus_{i=1}^{r} \mathrm{Gal}(K_N^{(p)}(\mathfrak{l}_i)/K_N) \tag{2.25}$$

(the restriction is injective since $k^{(p)}$, being ramified at $\mathfrak{l}_{r+1}$, is disjoint with the compositum of the $K_N^{(p)}(\mathfrak{l}_i)$ for $1 \le i \le r$). Assume the following conditions on $\mathfrak{l}_i$ $(1 \le i \le r+1)$:

Condition B. The prime $\mathfrak{l}_2$ is inert in $K_N^{(p)}(\mathfrak{l}_1)$. If $3 \le i \le r+1$, then the prime $\mathfrak{l}_i$ splits in $K_N^{(p)}(\mathfrak{l}_j)$ for all j such that $1 \le j \le i-2$ and is inert in $K_N^{(p)}(\mathfrak{l}_{i-1})$.

Then via the isomorphism of (2.25)

$$\left(\frac{L_0^{(p)}/k^{(p)}}{\bar{\mathfrak{l}}_2}\right) \mapsto (\sigma_1, \ldots), \quad \langle \sigma_1 \rangle = \mathrm{Gal}(K_N^{(p)}(\mathfrak{l}_1)/K_N)$$

and for every $3 \le i \le r+1$

$$\left(\frac{L_0^{(p)}/k^{(p)}}{\bar{\mathfrak{l}}_j}\right) \mapsto (1, \ldots, 1, \sigma_{i-1}, \ldots), \quad \langle \sigma_{i-1} \rangle = \mathrm{Gal}(K_N^{(p)}(\mathfrak{l}_{i-1})/K_N)$$

Therefore the set in (2.23) generates $\mathrm{Gal}(L_0^{(p)}/k^{(p)})$, which implies that $L_{n_0+\delta} = H_{n_0+\delta}$ for $\delta = 0, 1$ under condition B. Condition B is clearly equivalent to the following:

Condition B'. The prime $\widetilde{\mathfrak{L}}_{2,n_0+1}$ is inert in $K_N^{(p)}(\mathfrak{l}_1)\widetilde{K}_{N+n_0+1}$. For every $3 \le i \le r+1$, the prime $\widetilde{\mathfrak{L}}_{i,n_0+1}$ splits in $K_N^{(p)}(\mathfrak{l}_j)\widetilde{K}_{N+n_0+1}$ for all j such that $1 \le j \le i-2$ and is inert in $K_N^{(p)}(\mathfrak{l}_{i-1})\widetilde{K}_{N+n_0+1}$.

Now we choose $\widetilde{\mathfrak{L}}_{1,\,n_0+1}$ such that Condition A is satisfied. This is allowed by Lemma 2.1.10. Next we choose $\widetilde{\mathfrak{L}}_{2,\,n_0+1}$ such that Condition A and Condition B' are satisfied: this is allowed once more by Lemma 2.1.10 since $K_N(\mathfrak{l}_1)\widetilde{K}_{N+n_0+1}$ and $\widetilde{K}_{N+n_0+1}(\sqrt[p^{m_0}]{E'_{N+n_0+1}})$ are linearly disjoint over $\widetilde{K}_{N+n_0+1}$. It is clear therefore that we can perform inductively this kind of choices, in order to get primes $\widetilde{\mathfrak{L}}_{i,\,n_0+1}$ $(1 \le i \le r+1)$ which satisfies both Condition A and Condition B'. This concludes the proof and k is a number field whose p-split Iwasawa module is isomorphic to X.

2.2 Structure of étale wild kernels

In this section we prove the main result of the chapter. First we recall some result about projective limits.

Lemma 2.2.1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a surjective projective system of profinite groups. Then there is a natural isomorphism*

$$\varinjlim \mathrm{Hom}_c(X_n,\, \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}_c(\varprojlim X_n,\, \mathbb{R}/\mathbb{Z})$$

where Hom_c *denotes the group of continuos homomorphisms (and the projective limit has the projective limit topology while $\mathbb{R}/\mathbb{Z}$ has the quotient topology).*

Proof. See [RZ], Lemma 2.9.3 and Lemma 2.9.6. □

Lemma 2.2.2. *Let Γ be a topological group which is isomorphic to $\mathbb{Z}_p$ and set $\Lambda = \mathbb{Z}_p[\![\Gamma]\!]$ and $\Gamma_n = \Gamma^{p^n}$ where the latter denotes the closed subgroup generated by the p^n-th powers. Let X be a (Hausdorff) compact Λ-module. Then*

$$X = \varprojlim X_{\Gamma_n}$$

Proof. We know that X is isomorphic to a projective limit of finite abelian p-groups (see [NSW], Chapter V, Proposition 5.2.4). We are going to use Pontrjagin duality as stated for example in [NSW]: in particular for a Hausdorff, abelian and locally compact topological group A we set

$$A^\vee = \mathrm{Hom}_c(A,\, \mathbb{R}/\mathbb{Z})$$

If A is a profinite group then

$$A^\vee = \mathrm{Hom}_c(A,\, \mathbb{Q}/\mathbb{Z})$$

(see [RZ], Lemma 2.9.6) and if, further, A is pro-p-finite then

$$A^\vee = \mathrm{Hom}_c(A,\, \mathbb{Q}_p/\mathbb{Z}_p)$$

(if $f : A \to \mathbb{Q}/\mathbb{Z}$ is a continuos homorphism, then $\mathrm{Ker}(f)$ is an open subgroup, hence it has finite index which has to be a power of p, see [RZ], Theorem 2.1.3). In the following we are going to use this for $A = X$ or $A = X_{\Gamma_n}$.
Now observe that, since X is a compact Γ-module, $\mathrm{Hom}_c(X, \mathbb{Q}_p/\mathbb{Z}_p)$ is a discrete Γ-module hence

$$\mathrm{Hom}_c(X, \mathbb{Q}_p/\mathbb{Z}_p) = \bigcup_{n\in\mathbb{N}} \mathrm{Hom}_c(X, \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n} = \varinjlim \mathrm{Hom}_c(X, \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n}$$

Then we have

$$X = X^{\vee\vee} = (\mathrm{Hom}_c(X, \mathbb{Q}_p/\mathbb{Z}_p))^{\vee} = \left(\varinjlim \mathrm{Hom}_c(X, \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n}\right)^{\vee} =$$
$$= \left(\varinjlim \mathrm{Hom}_c(X_{\Gamma_n}, \mathbb{Q}_p/\mathbb{Z}_p)\right)^{\vee} = \left(\mathrm{Hom}_c(\varprojlim X_{\Gamma_n}, \mathbb{Q}_p/\mathbb{Z}_p)\right)^{\vee} =$$
$$= (\varprojlim X_{\Gamma_n})^{\vee\vee} = \varprojlim X_{\Gamma_n}$$

where we used Lemma 2.2.1 (the system $\{X_{\Gamma_n}\}$ is clearly surjective). □

Lemma 2.2.3. *Let G be a profinite group and let $\{X_n\}_{n\in\mathbb{N}}$ be a projective system of finite (discrete) G-modules. Set*

$$X = \varprojlim X_n$$

Then

$$\varprojlim (X_n)_G = X_G$$

Proof. X is clearly a topological G-module and it is compact since it is profinite. Now observe that

$$\left(\varinjlim X_n^{\vee}\right)^G = H^0(G, \varinjlim X_n^{\vee}) = \varinjlim H^0(G, X_n^{\vee}) = \varinjlim (X_n^{\vee})^G$$

(see [NSW], Proposition 1.5.1). Then

$$\left(\varprojlim (X_n)_G\right)^{\vee} = \varinjlim ((X_n)_G)^{\vee} = \varinjlim (X_n^{\vee})^G = \left(\varinjlim X_n^{\vee}\right)^G = \left(X^{\vee}\right)^G = (X_G)^{\vee}$$

and taking duals we get the claim. □

Let F be a number field and set $L = F(\mu_p)$ and $\Delta = \mathrm{Gal}(L/F)$. Note that $\Gamma_F = \mathrm{Gal}(L^c/F)$ is naturally isomorphic to $(\Gamma_F)_p \times \Delta$. Observe that L^c coincides with L_n^c: in particular $L_n^{cd} = L^{cd}$.

Definition 2.2.4. *Let k be a number field and let k_n be the n-th stage of the cyclotomic $\mathbb{Z}_p$-extension k^c of k. Then for each $i \in \mathbb{Z}$ we set*

$$WK_{2i}^{\acute{e}t}(k^c) \overset{def}{=} \varprojlim WK_{2i}^{\acute{e}t}(k_n)$$

The following result is known (see [KM]): the last part of the argument (interchanging the twist with the Galois action) is due to Lichtenbaum.

Proposition 2.2.5. *We have*

$$WK_{2i}^{ét}(L^c) \cong X'_L(i) \quad WK_{2i}^{ét}(F^c) \cong \big(X'_L(i)\big)_\Delta$$

Proof. We have

$$WK_{2i}^{ét}(L^c) = \varprojlim WK_{2i}^{ét}(L_n) \cong \varprojlim \big(X'_{L_n}(i)\big)_{\Gamma_n} = \varprojlim \big(X'_L(i)\big)_{\Gamma_n} = X'_L(i)$$

where the last equality comes from Lemma 2.2.2.
For the second statement we have

$$WK_{2i}^{ét}(F^c) = \varprojlim WK_{2i}^{ét}(F_n) \cong \varprojlim \big(X'_{L_n}(i)\big)_{\Gamma_n\times\Delta} =$$

$$= \varprojlim \Big(\big(X'_L(i)\big)_{\Gamma_n}\Big)_\Delta = \Big(\varprojlim \big(X'_L(i)\big)_{\Gamma_n}\Big)_\Delta = \big(X'_L(i)\big)_\Delta$$

and we used Lemma 2.2.3. On the other hand

$$\Big(\big(X'_L(i)\big)_\Delta\Big)^\vee = \big(\mathrm{Hom}_c(X'_L(i),\, \mathbb{Q}/\mathbb{Z})\big)^\Delta = \big(\mathrm{Hom}_c(X'_L,\, \mathbb{Q}/\mathbb{Z})(-i)\big)^\Delta =$$

$$= \mathrm{Hom}_c(X'_L,\, \mathbb{Q}/\mathbb{Z})_{\omega^i}(-i) = \mathrm{Hom}_c(X'_L/\langle\delta-\omega^{-i}(\delta)\rangle X'_L,\, \mathbb{Q}/\mathbb{Z})(-i) =$$

$$= \mathrm{Hom}_c\big((X'_L/\langle\delta-\omega^{-i}(\delta)\rangle X'_L)(i),\, \mathbb{Q}/\mathbb{Z}\big) = \big((X'_L/\langle\delta-\omega^{-i}(\delta)\rangle X'_L)(i)\big)^\vee$$

Hence

$$\big(X'_L(i)\big)_\Delta = (X'_L/\langle\delta-\omega^{-i}(\delta)\rangle X'_L)(i)$$

which shows in particular that if $i \equiv 0$ $(mod\ |\Delta|)$ then

$$WK_{2i}^{ét}(F^c) \cong X'_F(i)$$

since $(X'_L)_\Delta = X'_F$. More generally, for arbitrary i, since $\delta \in \Delta$ acts as multiplication by $\omega^{-i}(\delta)$ on $X'_L/\langle\delta-\omega^{-i}(\delta)\rangle X'_L$, it is easy to see the group norm induces an isomorphism

$$X'_L/\langle\delta-\omega^{-i}(\delta)\rangle X'_L \cong (X'_L)_{\omega^{-i}}$$

which gives

$$\big(X'_L(i)\big)_\Delta \cong (X'_L)_{\omega^{-i}}(i)$$

□

Theorem 2.2.6. *Let d be the greatest common divisor of $p-1$ and i. If p does not divide the class number of $K^{(d)}$, then every finite abelian p-group structure appears as $WK_{2i}^{ét}(k)$ for some number field k.*

Proof. Suppose the hypothesis is satisfied. Choose a finite p-group X which we consider as Λ-module with trivial action and choose k as in the proof of Theorem 2.1.1: in particular $X'_k \cong X$ as trivial Λ-modules. Now

$$WK_{2i}^{\acute{e}t}(k) \cong X'_{k(\zeta_p)}(i)_\Gamma$$

where $\Gamma \cong \mathrm{Gal}(k(\zeta_p)^c/k) = \Delta \times \Gamma_p$ and $\Delta = \mathrm{Gal}(k(\zeta_p)^c/k^c)$ and $\Gamma_p = \mathrm{Gal}(k^c/k)$. As in the proof of the preceding proposition, we have, since $i \equiv 0$ $(mod\ |\Delta|)$ and Γ_p acts trivially on X_k,

$$X'_{k(\zeta_p)}(i)_\Gamma = (X'_{k(\zeta_p)}(i)_\Delta)_{\Gamma_p} \cong (X'_k(i))_{\Gamma_p}$$

Now the action of Γ_p is trivial on X'_k, therefore the action of Γ_p on $X'_k(i)$ is given by

$$\gamma.x = \kappa(\gamma)^i x$$

On the other hand we see that in the construction of k, if p^{m_0} is the exponent of X, then

$$\kappa(\gamma) \equiv 1\ (mod\ p^{m_0})$$

(since k contains $(K^{(d)})_N$ with $N \geq m_0$, see (2.4)). Therefore Γ_p acts trivially on $X'_k(i)$ and we get the result.

$\square$

For example, since $p \nmid h_{\mathbb{Q}}$, for every $i \equiv 0$ $(mod\ p-1)$, every finite abelian p-group structure can be realized as $WK_{2i}^{\acute{e}t}(k)$ for some number field k (in particular this holds for the logarithmic class group, provided the generalized Gross conjecture is true). More generally, for i even, the Vandiver conjecture predicts that p doe not divide the class number of $K^{(d)}$. In the next chapter we are going to study to what extent the condition of the theorem has to be considered as necessary.

Chapter 3

Etale analogues of Hilbert class field

This chapter is devoted to identifying those fields for which not every abelian p-group structure can be realized as étale wild kernel of some finite extension. We follow the strategy which is used in the classical case of class groups: an étale analogue of Hilbert class field is defined and its basic properties are described. Then we pass to étale analogues of Hilbert class field towers and we try to clarify the relation between fields with infinite class field towers and a negative answer to the realizability problem (Section 3.1 and Section 3.2 generalize the results of [JS] and [As]). We end with a partial result on the étale analogue of Hilbert theorem 94.

3.1 Etale analogues of Hilbert class field

Let F be a number field and set $L = F(\mu_p)$. Put $\Gamma_p = \mathrm{Gal}(L^c/L)$ and $\Delta = \mathrm{Gal}(L/F)$: then Γ_p is (non canonically) isomorphic to $\mathbb{Z}_p$ as a topological group and Δ is (canonically) isomorphic to a subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$. Set $\Gamma = \mathrm{Gal}(L^c/F)$: then Γ is a procyclic group and we have a canonical isomorphism

$$\Gamma \cong \Gamma_p \times \Delta$$

Note that L^{cd}/F is a Galois extension and we have a split exact sequence of profinite groups

$$0 \longrightarrow X'_L \longrightarrow \mathrm{Gal}(L^{cd}/F) \xrightarrow{\pi} \Gamma \longrightarrow 0 \tag{3.1}$$

(this sequence splits since the p-cohomological dimension of $\Gamma_p \times \Delta$ is 1, see [Se], I.§3.4). Now, $X'_L(i)_\Gamma$ is the Galois group of an extension $F(i)/L^c$ since

$$X'_L(i)_\Gamma = X'_L(i)\Big/(1-\gamma)X'_L(i) = X'_L\Big/(1-\kappa^i(\gamma)\gamma)X'_L \tag{3.2}$$

if γ is any topological generator of Γ and $(1-\kappa^i(\gamma)\gamma)X'_L$ is a closed subgroup of X'_L. Moreover $F(i)/F$ is Galois too since $(1-\kappa^i(\gamma)\gamma)X'_L$ is a Γ-submodule of X'_L with standard action (since $\gamma(1-\kappa^i(\gamma)\gamma) = (1-\kappa^i(\gamma)\gamma)\gamma$). Furthermore, as before, we get a split exact sequence

$$0 \longrightarrow X'_L(i)_\Gamma \longrightarrow \mathrm{Gal}(F(i)/F) \xrightarrow{\pi_i} \Gamma \longrightarrow 0 \tag{3.3}$$

Let γ be a fixed topological generator of Γ: consider the closed subgroup $\langle\widetilde{\gamma}\rangle$ generated by a preimage $\widetilde{\gamma}$ of γ by π_i in $\mathrm{Gal}(F(i)/F)$ and let $F_{\widetilde{\gamma}}/F$ be the extension which is fixed by $\langle\widetilde{\gamma}\rangle$. Note that $F_{\widetilde{\gamma}}$ is a *complement* of L^c in $F(i)$ over F: in fact $F_{\widetilde{\gamma}}L^c = F(i)$ and $F_{\widetilde{\gamma}} \cap L^c = F$. This easily follows from the next lemma

Lemma 3.1.1. *The closed subgroup $\langle\widetilde{\gamma}\rangle$ generated by $\widetilde{\gamma}$ is an infinite procyclic group and π_i induces an isomorphism of topological groups $(\pi_i)_{|\langle\widetilde{\gamma}\rangle}$: $\langle\widetilde{\gamma}\rangle \longrightarrow \Gamma$ (in particular (3.3) is split by $(\pi_i)_{|\langle\widetilde{\gamma}\rangle}$).*

Proof. Set $G = \mathrm{Gal}(F(i)/F)$ and let $G_p = \pi_i^{-1}(\Gamma_p)$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X'_L(i)_\Gamma & \longrightarrow & G & \xrightarrow{\pi_i} & \Gamma & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X'_L(i)_\Gamma & \longrightarrow & G_p & \xrightarrow{\pi_i} & \Gamma_p & \longrightarrow & 0
\end{array}$$

Note that G_p is the pro-p-Sylow of G_p since Γ_p is the pro-p-Sylow of Γ. Now take a preimage $\widetilde{\gamma}$ of γ by π_i. Then

$$\langle\widetilde{\gamma}\rangle \cap G_p \xrightarrow{\pi_i} \Gamma_p \tag{3.4}$$

is surjective: clearly $\widetilde{\gamma}^{|\Delta|} \in \langle\widetilde{\gamma}\rangle \cap G_p$ and $\langle\widetilde{\gamma}^{|\Delta|}\rangle \xrightarrow{\pi_i} \Gamma_p$ is surjective because the image of $(\pi_i)_{|\langle\widetilde{\gamma}^{|\Delta|}\rangle}$ contains a dense subgroup and it is compact (being the image of the compact $\langle\widetilde{\gamma}^{|\Delta|}\rangle$ by the continuous map π_i) hence closed (since Γ_p is compact). Moreover $\langle\widetilde{\gamma}^{|\Delta|}\rangle$ is pro-p-cyclic hence isomorphic to $\mathbb{Z}_p$: thus the map in (3.4) is an isomorphism, being a surjective map between pro-p-cyclic groups. Now also $\langle\widetilde{\gamma}\rangle \xrightarrow{\pi_i} \Gamma$ is surjective (same argument as above). The claim of the lemma is then easily achieved by noticing that π_i induces an injection from $\langle\gamma\rangle/\langle\widetilde{\gamma}\rangle \cap G_p$ to Γ/Γ_p. $\square$

We also have $[F_{\widetilde{\gamma}} : F] = |X'_L(i)_\Gamma|$ which is the index of the closed subgroup generated by $\widetilde{\gamma}$. Evidently $F_{\widetilde{\gamma}}$ depends on the choice of $\widetilde{\gamma}$. In order to avoid this non natural choice, we define $\widetilde{F}$ as the compositum of the fields $F_{\widetilde{\gamma}}$ as $\widetilde{\gamma}$ runs through the preimages of γ by π_i in $\mathrm{Gal}(F(i)/F)$. Note that $\widetilde{F}/F$ is finite, since $X'_L(i)_\Gamma$ is finite and then $\widetilde{F}/F$ is the compositum of finitely many finite extensions of F. Note that $\widetilde{F}$ does not depend on the choice of the topological generator γ of Γ (use for example Lemma 3.1.1). In the following we will denote by $\widetilde{I}$ (or $\widetilde{I}_F$ if we want to stress on the field involved) the closed subgroup of $\mathrm{Gal}(F(i)/F)$ which corresponds to $\widetilde{F}$: then $\widetilde{I}$ is the intersection of the closed subgroups generated by a preimage of γ by π_i.

Before describing the properties of $\widetilde{F}$ we prove the following proposition about $F(i)$.

Proposition 3.1.2. *Let E/F be a finite extension: then $F(i) \subseteq E(i)$.*

Proof. First of all we prove the assertion if $E \subseteq F(\zeta_p)^c$. In this case $X'_{E(\zeta_p)} = X'_{F(\zeta_p)}$ (and hence $X'_{E(\zeta_p)}(i) = X'_{F(\zeta_p)}(i)$). Now Γ_E is in a natural way a closed subgroup of Γ_F: this means that

$$I_{\Gamma_E} \subseteq I_{\Gamma_F} \quad \text{and hence} \quad I_{\Gamma_E} X'_{F(\zeta_p)}(i) \subseteq I_{\Gamma_F} X'_{F(\zeta_p)}(i)$$

(where $I_{\Gamma_E} \subseteq \mathbb{Z}_p[\Gamma_E]$ and $I_{\Gamma_F} \subseteq \mathbb{Z}_p[\Gamma_F]$ are augmentation ideals). Since $E(i)$ corresponds to $I_{\Gamma_E} X'_{F(\zeta_p)}(i)$ and $F(i)$ to $I_{\Gamma_F} X'_{F(\zeta_p)}(i)$, we deduce that $F(i) \subseteq E(i)$.
Now we drop the assumption $E \subseteq F(\zeta_p)^c$: let $F' = E \cap F(\zeta_p)^c$. From what we have just seen, we know that $F(i) \subseteq F'(i)$ and obviously $F'(\zeta_p)^c \cap E = F'$. Hence we can suppose that $E \cap F(\zeta_p)^c = F$. Now we have to prove that $F(i) \subseteq E(i)$: in other words, we need to show that $\gamma \in \Gamma_E$ acts as multiplication by $\kappa(\gamma)^{-i}$ on $\mathrm{Gal}(F(i)E(\zeta_p)^c/E(\zeta_p)^c)$ (since $E(i)/E^c$ is precisely the maximal subextension of $E(\zeta_p)^{cd}/E(\zeta)^c$ whose Galois group is a Γ_E-module with that action). But this is quite clear: let $\gamma \in \Gamma_E$ and $x \in \mathrm{Gal}(F(i)E(\zeta_p)^c/E(\zeta_p)^c)$. Let $\alpha \in F(i)$: we have

$$\gamma.x(\alpha) = \widetilde{\gamma} x \widetilde{\gamma}^{-1}(\alpha) = (\widetilde{\gamma} x \widetilde{\gamma}^{-1})_{|F(i)}(\alpha) = (x_{|F(i)})^{\kappa(\gamma)^{-i}}(\alpha) = x^{\kappa(\gamma)^{-i}}(\alpha)$$

where $\widetilde{\gamma}$ is a lifting of γ in $\mathrm{Gal}(F(i)E(\zeta_p)^c/E)$. If $\alpha \in E(\zeta_p)^c$, then obviously

$$\gamma.x(\alpha) = \alpha = x^{\kappa(\gamma)^{-i}}(\alpha)$$

hence $\gamma.x = x^{\kappa(\gamma)^{-i}}$ as we claimed. □

The extension $\widetilde{F}/F$ enjoys some remarkable properties (with respect to $WK_{2i}^{ét}(F)$) which we are going to describe in the following. If we think about $WK_{2i}^{ét}(F)$ as an analogue of the p-Sylow of the p-split class group of F, then $\widetilde{F}$ is the analogue of the p-split Hilbert class field of F.

Proposition 3.1.3. *$\widetilde{F}$ is a finite Galois extension of F which is trivial exactly when $X'_L(i)_\Gamma$ is.*

Proof. First of all note that, by definition, $\widetilde{F}/F$ is trivial exactly when $X'_L(i)_\Gamma$ is. We are left to show that $\widetilde{I}$ is a normal subgroup of $\mathrm{Gal}(F(i)/F)$ but this is clear since, for every $\sigma \in \mathrm{Gal}(F(i)/F)$ and every preimage $\widetilde{\gamma}$ of a topological generator γ of Γ by π_i, we have $\sigma\langle\widetilde{\gamma}\rangle\sigma^{-1} = \langle\sigma\widetilde{\gamma}\sigma^{-1}\rangle$ (use for example Lemma 3.1.1) and $\sigma\widetilde{\gamma}\sigma^{-1}$ is still a preimage of γ by π_i. □

Note that Γ acts on $X'_L(i)_\Gamma$ by conjugation and the action is given by

$$\gamma.x = x^{\kappa(\gamma)^{-i}}$$

for every $x \in X'_L(i)_\Gamma$ (use for example (3.2)). This means precisely that, if $\widetilde{\gamma}$ is a preimage of γ by π_i, we have

$$\widetilde{\gamma}x\widetilde{\gamma}^{-1} = x^{\kappa(\gamma)^{-i}}$$

Any another preimage of γ by π_i is of the form $\widetilde{\gamma}x$ with $x \in X'_L(i)_\Gamma$. Note that, for every positive $h \in \mathbb{N}$

$$(\widetilde{\gamma}x)^h = x^{\sum_{j=1}^{h} \kappa(\gamma)^{-ij}}\widetilde{\gamma}^h \tag{3.5}$$

a formula which can be readily proved by induction on h. This lead us to introduce the following definition which will be very useful in the computation of the degree $[\widetilde{F} : F]$.

Definition 3.1.4. *Let $a \in \mathbb{Z}_p^\times$ and let v_p denote the valuation on $\mathbb{Z}_p$ such that $v_p(p) = 1$. If $a \neq 1$, let $s \in \mathbb{N}$ be such that $v_p(a-1) = s$ and, for every nonzero $n \in \mathbb{N}$, let $d_n(a)$ be the multiplicative order of a modulo p^{n+s}. Moreover set $d_n(1) = p^n$.*

The next lemma explains the relation between d_n and (3.5).

Lemma 3.1.5. *For every $a \in \mathbb{Z}_p^\times$ we have*

$$d_n(a) = \min\{h \in \mathbb{N}, h \geq 1 \mid \sum_{j=1}^{h} a^j \equiv 0 \ (mod\ p^n)\}$$

Suppose further that $a \neq 1$ and $v_p(a-1) = s > 0$: then $d_n(a) = p^n$.

Proof. If $a = 1$, it is clear that $d_n(a) = p^n$. If $a \neq 1$, then the first assertion is an immediate consequence of the following chain of equivalences

$$a^h \equiv 1 \ (mod\ p^{n+s}) \Leftrightarrow (a-1)(\sum_{j=0}^{h-1} a^j) \equiv 0 \ (mod\ p^{n+s}) \Leftrightarrow$$

$$\Leftrightarrow (a-1)(\sum_{j=1}^{h} a^j) \equiv 0 \ (mod\ p^{n+s}) \Leftrightarrow \sum_{j=1}^{h} a^j \equiv 0 \ (mod\ p^n)$$

Now suppose that $a \neq 1$ and $v_p(a-1) = s > 0$: we can take $a \neq 1$. First of all note that the class of a belongs to the cyclic subgroup of order p^n in $(\mathbb{Z}/p^{s+n}\mathbb{Z})^\times$ since a is congruent to 1 modulo p^s and $s > 0$. In particular

$$a^{p^n} \equiv 1 \ (mod\ p^{n+s})$$

On the other hand, the class of a does not belong to the cyclic subgroup of order p^{n-1} in $(\mathbb{Z}/p^{s+n}\mathbb{Z})^\times$ since a is not congruent to 1 modulo p^{s+1}. Hence the class of a generates the cyclic subgroup of order p^n in $(\mathbb{Z}/p^{s+n}\mathbb{Z})^\times$ which implies that $d_n(a) = p^n$. $\square$

Definition 3.1.6. *Suppose that $X'_L(i)_\Gamma \neq 0$, call $e_F(i)$ the exponent of $X'_L(i)_\Gamma$ and set $t_F(i) = v_p(e_F(i))$ (hence $t_F(i) > 0$). Furthermore set $s_F(i) = v_p(\kappa(\gamma)^{-i} - 1)$ and*

$$h_F(i) = \min\{h \in \mathbb{N},\, h \geq 1 \,|\, \sum_{j=1}^{h} \kappa(\gamma)^{-ij} \equiv 0 \ (mod\ e_F(i))\}$$

Finally set $r_F(i) = v_p(h_F(i))$ and let $d_F(i)$ be defined by $h_F(i) = p^{r_F(i)} d_F(i)$ (in particular we have $(d_F(i), p) = 1$).

We remark once and for all that $h_F(i)$ is defined only if $X'_L(i)_\Gamma \neq 0$ (when no misunderstanding is possible, we will not stress on this). As an example, we have $h_F(0) = e_F(0)$ and in general $h_F(i) > 1$. Sometimes we shall simply write $e(i)$, $t(i)$, $h(i)$ and $s(i)$ (if no ambiguity arises). By Lemma 3.1.5 we must have

$$h(i) = d_{t(i)}(\kappa(\gamma)^{-i}) \tag{3.6}$$

and, if $s(i) > 0$, then $h(i) = e(i)$. Note that we also have

$$(\widetilde{\gamma} x)^{h(i)} = \widetilde{\gamma}^{h(i)} \tag{3.7}$$

by (3.5).

Proposition 3.1.7. *Suppose that $X'_L(i)_\Gamma \neq 0$. Then*

$$\widetilde{I} = \langle \widetilde{\gamma}^{h(i)} \rangle$$

In particular $[\widetilde{F} :\, F] = h(i)|X'_L(i)_\Gamma|$. Moreover

$$\mathrm{Gal}(\widetilde{F}/\widetilde{F} \cap L^c) \cong X'_L(i)_\Gamma$$

and hence $[\widetilde{F} \cap L^c :\, F] = h(i)$.

Proof. Recall that we already know that $h(i) = d_{t(i)}(\kappa(\gamma)^{-i})$ (see (3.6)). We have $\widetilde{I} = \langle \widetilde{\gamma}^a \rangle$ for some $a \in \mathbb{Z}$. Let $x \in X_L(i)_\Gamma$ such that the order of x is $e(i)$: then we also have $\widetilde{I} = \langle (\widetilde{\gamma} x)^b \rangle$ with $b \in \mathbb{Z}$. Then there exists $c \in \widehat{\mathbb{Z}}$ such that

$$(\widetilde{\gamma} x)^b = \widetilde{\gamma}^{ac}$$

This means that

$$x^{\sum_{j=1}^{b} \kappa(\gamma)^{-ij}} \in \langle \widetilde{\gamma} \rangle$$

hence

$$\sum_{j=1}^{b} \kappa(\gamma)^{-ij} \equiv 0 \ (mod\ e(i)) \tag{3.8}$$

If $i \neq 0$, this is equivalent (as in the proof of Lemma 3.1.5) to

$$\kappa(\gamma)^{-ib} \equiv 1 \ (mod\ p^{t(i)+s(i)})$$

By definition of $d_{t(i)}(\kappa(\gamma)^{-i})$, we therefore deduce that $b = h(i)q$ for some $q \in \mathbb{Z}$ and the same conclusion holds true in the case $i = 0$ (since $h(0) = e(0)$ and (3.8) holds). Hence

$$\widetilde{I} = \langle(\widetilde{\gamma}x)^b\rangle \subseteq \langle\widetilde{\gamma}^{h(i)}\rangle$$

On the other hand

$$\langle(\widetilde{\gamma}x)^{h(i)}\rangle = \langle\widetilde{\gamma}^{h(i)}\rangle \subseteq \widetilde{I}$$

since (3.7) shows that $\langle\widetilde{\gamma}^{h(i)}\rangle$ is contained in any of the closed subgroups which are generated by a preimage of a topological generator of Γ by π_i, hence in their intersection $\widetilde{I}$. Hence $\widetilde{I} = \langle\widetilde{\gamma}^{h(i)}\rangle$.
For the remaining assertions, just note that

$$\mathrm{Gal}(\widetilde{F}/\widetilde{F} \cap L^c) \cong \widetilde{I}X'_L(i)_\Gamma/\widetilde{I} \cong X'_L(i)_\Gamma/(\widetilde{I} \cap X'_L(i)_\Gamma) \cong X'_L(i)_\Gamma$$

since $\widetilde{I} \cap X'_L(i)_\Gamma \subseteq \langle\widetilde{\gamma}\rangle \cap X'_L(i)_\Gamma$ is trivial (here $\widetilde{\gamma}$ is any of the preimages of a topological generator of Γ by π_i) and $\widetilde{I}$ is normal in $\mathrm{Gal}(F(i)/F)$ by Proposition 3.1.3 (hence the isomorphism theorem applies). □

Remark 3.1.8. The proof of Proposition 3.1.7 in the case $i = 0$ can be found in [JS].

Lemma 3.1.9. *We have* $\widetilde{F}(\zeta_p)^c = F(i)$.

Proof. Clearly $\widetilde{F}(\zeta_p)^c \subseteq F(i)$. To conclude note that

$$[F(i) : F(\zeta_p)^c] = |WK_{2i}^{\acute{e}t}(F)| = [\widetilde{F} : F(\zeta_p)^c \cap \widetilde{F}] = [\widetilde{F}(\zeta_p)^c : F(\zeta_p)^c]$$

by Proposition 3.1.7. □

The following lemmas are very easy but we quote and prove them since they will be used often henceforth. If $a \in \mathbb{Z}_p^\times$, then we denote by $\mathrm{ord}_p(a)$ the order of the class defined by a in $(\mathbb{Z}/p\mathbb{Z})^\times$.

Lemma 3.1.10. *Suppose that* $\widetilde{F} \neq F$. *Then we have*

$$[\widetilde{F} \cap F(\zeta_p) : F] = d_F(i) = \mathrm{ord}_p(\kappa(\gamma_F)^{-i})$$

Proof. The first equality is clear by Proposition 3.1.7. In order to prove the second, set $a = \kappa(\gamma_F)^{-i}$, $q = \mathrm{ord}_p(a)$ and write as before $h(i) = p^{r(i)}d(i)$ for the order of a modulo $p^{t(i)+s(i)}$. In particular

$$(a^{d(i)})^{p^{r(i)}} \equiv a^{d(i)} \equiv 1 \ (mod\ p)$$

which shows that $q|d(i)$. Now observe that a^q belongs to the p-Sylow subgroup of $(\mathbb{Z}/p^{t(i)+s(i)}\mathbb{Z})^\times$. This implies that there exists some m such that $a^{qp^m} \equiv 1 \ (mod\ p^{t(i)+s(i)})$. Hence $h(i)|qp^m$, which implies $d(i)|q$. □

Lemma 3.1.11. *We have*

$$[F(\zeta_p):F] = \mathrm{ord}_p(\kappa(\gamma_F)) = \mathrm{ord}_p(\kappa(\gamma_F)^{-i})\,(\mathrm{ord}_p(\kappa(\gamma_F)),\, i)$$

Proof. Set $a = \kappa(\gamma_F)$ and $q = \mathrm{ord}_p(a)$. The cyclotomic character gives an injective homomorphism

$$\kappa : \mathrm{Gal}(F(\zeta_p)^c/F) \longrightarrow \mathbb{Z}_p^\times$$

Note that $F(\zeta_p)$ is the fixed field of the p-Sylow subgroup of $\mathrm{Gal}(F(\zeta_p)^c/F)$. Hence in order to prove the first equality we have to show that a^q generates the p-Sylow of $\kappa(\mathrm{Gal}(F(\zeta_p)^c/F))$. But this is clear since the p-Sylow subgroup of $\mathbb{Z}_p^\times$ is the subgroup of elements which are congruent to 1 modulo p. To prove the second equality, set $r = \mathrm{ord}_p(a^{-i})$ and note that

$$(a^{-i})^{\frac{q}{(q,\,i)}} = (a^q)^{-\frac{i}{(q,\,i)}} \equiv 1 \ (mod\ p)$$

and this implies

$$r\Big|\frac{q}{(q,\,i)}$$

On the other hand

$$a^{-ir} \equiv 1 \ (mod\ p) \ \Rightarrow\ q|ir \ \Rightarrow\ \frac{q}{(q,\,i)}\Big|\frac{i}{(q,\,i)}r \ \Rightarrow\ \frac{q}{(q,\,i)}\Big|r$$

and this concludes the proof. $\square$

The following proposition describes the properties of $\widetilde{F}/F$ with respect to ramification.

Proposition 3.1.12. *$\widetilde{F}/F$ is a locally cyclotomic extension. If moreover there is only one prime above p in $F(\zeta_p)^c$, then $\widetilde{F}/\widetilde{F}\cap F(\zeta_p)^c$ is unramified everywhere and totally split at every prime above p.*

Proof. The fact that $\widetilde{F}/F$ is locally cyclotomic is clear from the definition. Clearly, if there is only one prime above p in $\widetilde{F}/\widetilde{F}\cap F(\zeta_p)^c$ is everywhere unramified except perhaps at primes above p: furthermore, if it is unramified at primes above p, then it has to be split at those primes. So, supposing that there is only one prime above p in $F(\zeta_p)^c$, we prove that $\widetilde{F}/\widetilde{F}\cap F(\zeta_p)^c$ is unramified at every prime above p. Let $\mathfrak{p}$ be the prime above p in $\widetilde{F}/\widetilde{F}\cap F(\zeta_p)^c$ and let I be the inertia group of $\mathfrak{p}$ in $F(i)/\widetilde{F}\cap F(\zeta_p)^c$ (which is easily seen to be an abelian extension by (3.5) and Proposition 3.1.7). First of all, note that $I\cap X'_L(i)_\Gamma = 0$ (since otherwise there would exist a field F' such that $F(\zeta_p)^c \subseteq F' \subseteq F(i)$ ramified at primes above $\mathfrak{p}$). Now observe that $F(\zeta_p)^c/\widetilde{F}\cap F(\zeta_p)^c$ is totally ramified at $\mathfrak{p}$ (there cannot be inertia) and this, together with $X'_L(i)_\Gamma\cap I = 0$ proves that π_i induces an isomorphism between $\Gamma^{h(i)}$ and I. In particular I is procyclic and we call α a fixed topological generator of I. Let γ be a topologial generator of Γ such that $\gamma^{h(i)} = \pi(\alpha)$. If $\widetilde{\gamma}$ is any preimage of γ by π_i, then

$$\widetilde{I} = \langle \widetilde{\gamma}^{h(i)} \rangle \supseteq \langle \alpha \rangle = I$$

($\pi_i(\alpha) = \pi_i(\widetilde{\gamma}^{h(i)})$ implies $\alpha = \widetilde{\gamma}^{h(i)}$ since $X \cap I = 0$). $\square$

Remark 3.1.13. One can ask if these étale analogues of the Hilbert class field share other interesting properties with the classical case. For example, Hilbert Theorem 94 says that the extension map $Cl'_F \longrightarrow Cl'_{H'_F}$ is trivial, where H'_F is the p-split Hilbert class field of F. In our context the problem is then to see whether the natural map

$$WK_{2i}^{ét}(F) \longrightarrow WK_{2i}^{ét}(\widetilde{F})$$

is always trivial. Unfortunately the answer to this question is negative. A counterexample can be given using a recent result of R. Validire. Suppose that $p = 37$ and $i = 31$ and choose $F = \mathbb{Q}(\zeta_p)$: then $WK_{2i}^{ét}(F)$ is cyclic of order p (since $X'_F(i)_\Gamma$ is). Furthermore X'_F is isomorphic (as a topological group) to $\mathbb{Z}_p$ (see [Wa]). Therefore $\mathrm{Gal}(\overline{F}_{cd}/F^c)$ is pro-p-free and in fact isomorphic to X'_F (see [Gr]). Now $\widetilde{F}/F$ is a locally cyclotomic extension, thanks to Proposition 3.1.12. Now, applying Théorème 4.2.8 of [Va], we see that, if we set $G = \mathrm{Gal}(\widetilde{F}/F)$, the natural map

$$WK_{2i}^{ét}(F) \longrightarrow WK_{2i}^{ét}(\widetilde{F})^G$$

is an isomorphism.

3.2 Etale analogues of class field towers

Now we can define the étale analogue of class field towers as follows. Set $F_{i,0} = F$, $F_{i,1} = \widetilde{F_{i,0}}$ and inductively $F_{i,j+1} = \widetilde{F_{i,j}}$. Moreover

$$F_{i,\infty} = \bigcup_{j=0}^{\infty} F_{i,j}$$

Of course $F_{i,\infty}/F$ can be infinite: in fact it is finite if and only if there exists $n \in \mathbb{N}$ such that $X'_{F_{i,n}(\zeta_p)}(i)_\Gamma = 0$. In this case $F_{i,\infty} = F_{i,n}$.

The following results shows that $d_{F_{i,j}}(i)$ and $r_{F_{i,j}}(i)$ behave quite differently along the tower.

Proposition 3.2.1. *We have $F_{i,j+2} \cap F_{i,j+1}(\zeta_p) = F_{i,j+1}$ for every $j \geq 0$ and in particular, if $F_{i,\infty}/F$ is nontrivial,*

$$[F_{i,\infty} \cap F(\zeta_p) : F] = [F_{i,1} \cap F(\zeta_p) : F] = d_F(i)$$

and $d_{F_{i,j}}(i) = 1$ for every $j \geq 1$ (which implies $[F_{i,j+1} \cap F_{i,j}^c : F_{i,j}] = p^{r_{F_{i,j}}(i)}$) In particular, $F_{i,\infty}/F$ is infinite if and only if for every $j \geq 1$ we have $r_{F_{i,j}}(i) > 1$.

Proof. In order to prove $F_{i,j+2} \cap F_{i,j+1}(\zeta_p) = F_{i,j+1}$ for every $j \geq 0$ we argue as follows. We have

$$\operatorname{ord}_p(\kappa(\gamma_{F_{i,j+1}})) = [F_{i,j+1}(\zeta_p) : F_{i,j+1}] = [F_{i,j}(\zeta_p) : F_{i,j+1} \cap F_{i,j}(\zeta_p)] =$$
$$= \frac{[F_{i,j}(\zeta_p) : F_{i,j}]}{[F_{i,j+1} \cap F_{i,j}(\zeta_p) : F_{i,j}]}$$

(first equality comes from Lemma 3.1.11, the others are easy to check). But

$$[F_{i,j}(\zeta_p) : F_{i,j}] = \operatorname{ord}_p(\kappa(\gamma_{F_{i,j}}))$$

by Lemma 3.1.11 and

$$[F_{i,j+1} \cap F_{i,j}(\zeta_p) : F_{i,j}] = d_{F_{i,j}}(i)$$

by Lemma 3.1.10 (we can suppose $F_{i,j+1}/F_{i,j}$ nontrivial since otherwise the claim of the proposition is trivially verified). Hence

$$\operatorname{ord}_p(\kappa(\gamma_{F_{i,j+1}})) = \frac{\operatorname{ord}_p(\kappa(\gamma_{F_{i,j}}))}{d_{F_{i,j}}(i)} = (\operatorname{ord}_p(\kappa(\gamma_{F_{i,j}})),\, i) \tag{3.9}$$

again by Lemma 3.1.11. Moreover

$$\operatorname{ord}_p(\kappa(\gamma_{F_{i,j+1}})^{-i}) = \frac{\operatorname{ord}_p(\kappa(\gamma_{F_{i,j+1}}))}{(\operatorname{ord}_p(\kappa(\gamma_{F_{i,j+1}})),\, i)} = \frac{(\operatorname{ord}_p(\kappa(\gamma_{F_{i,j}})),\, i)}{(\operatorname{ord}_p(\kappa(\gamma_{F_{i,j}})),\, i)} = 1$$

by Lemma 3.1.11 and (3.9) (the latter is used to get both the numerator and the denominator of the last ratio). Hence by Lemma 3.1.10 (or trivially if $F_{i,j+2} = F_{i,j+1}$), $[F_{i,j+2} \cap F_{i,j+1}(\zeta_p) : F_{i,j+1}] = 1$, which is what we wanted. Now we prove that $F_{i,j+1} \cap F(\zeta_p) = F_{i,1} \cap F(\zeta_p)$. This is immediate by what we have just proved, since for every $j \geq 1$,

$$F_{i,j+1} \cap F(\zeta_p) \subseteq \Big(F_{i,j+1} \cap F_{i,j}(\zeta_p)\Big) \cap F(\zeta_p) = F_{i,j} \cap F(\zeta_p)$$

and the other inclusion is trivial. This shows that

$$[F_{i,\infty} \cap F(\zeta_p) : F] = [F_{i,1} \cap F(\zeta_p) : F] = d_F(i)$$

if $F_{i,\infty}/F$ is nontrivial. The last claim of the proposition follows from the remark after Definition 3.1.6. □

Definition 3.2.2. *In the following we set* $F_i = F_{i,\infty} \cap F(\zeta_p)$*: in particular* $F_i = F$ *if* $F_{i,\infty}/F$ *is trivial, otherwise* $[F_i : F] = d_F(i)$*.*

Remark 3.2.3. Note that, if $F_{i,j+1} \neq F_{i,j}$, then $[F_{i,j+1} \cap F_i^c : F_{i,j} \cap F_i^c] = p^{r_{F_{i,j}}(i)}$. Moreover $F_{i,\infty}/F_i$ is a (pro-)p-extension since $F_{i,j}/F_i$ is a p-extension for every $j \geq 0$.

Theorem 3.2.4. *The following assertions are equivalent:*

- *there exists a finite extension E/F such that $F_i \subseteq E$ and $X'_{E(\zeta_p)}(i)_\Gamma$ is trivial;*
- *$F_{i,\infty}/F$ is finite.*

Proof. (Same strategy as in the proof of Théorème 4 of [JS]). Clearly the second condition implies the first one (with $E = F_{i,\infty}$). In order to prove the other direction, suppose that there exists a finite extension E/F such that $X'_{E(\zeta_p)}(i)_\Gamma = 0$ and $F_i \subseteq E$. By Proposition 3.1.2 and Lemma 3.1.9, we deduce $F(i) \subseteq E(i) = E(\zeta_p)^c$ and $F_{i,1}(\zeta_p)^c = F(i)$. In particular $F_{i,1}(\zeta_p)^c \subseteq E(\zeta_p)^c$. Now suppose inductively that $F_{i,j}(\zeta_p)^c \subseteq E(\zeta_p)^c$: we shall prove that $F_{i,j+1}(\zeta_p)^c \subseteq E(\zeta_p)^c$. Since $F_i \subseteq E$ implies that $EF_{i,j}/E$ is a p-extension, we deduce that $F_{i,j} \subseteq E^c$. Now

$$X'_{E(\zeta_p)^c}(i) = X'_{EF_{i,j}(\zeta_p)^c}(i)$$

Furthermore

$$\Gamma_E = \mathrm{Gal}(E^c/E) \times \mathrm{Gal}(E(\zeta_p)^c/E^c)$$

and

$$\Gamma_{EF_{i,j}} = \mathrm{Gal}((EF_{i,j})^c/EF_{i,j}) \times \mathrm{Gal}(E(\zeta_p)^c/E^c)$$

Hence

$$X'_{E(\zeta_p)^c}(i)_{\Gamma_E} = 0 \Rightarrow X'_{E(\zeta_p)^c}(i)_{\mathrm{Gal}(E(\zeta_p)^c/E^c)} = 0 \Rightarrow X'_{E(\zeta_p)^c}(i)_{\Gamma_{EF_{i,j}}} = 0$$

which implies $(EF_{i,j})(i) = EF_{i,j}(\zeta_p)^c = E(\zeta_p)^c$. Thus, again using Lemma 3.1.9 and Proposition 3.1.2,

$$F_{i,j+1}(\zeta_p)^c = F_{i,j}(i) \subseteq (EF_{i,j})(i) = E(\zeta_p)^c$$

Then by induction, we have $F_{i,j}(\zeta_p)^c \subseteq E(\zeta_p)^c$ for every $j \geq 0$, which means $F_{i,\infty}(\zeta_p)^c \subseteq E(\zeta_p)^c$: in particular we deduce that $F_{i,\infty}(\zeta_p)^c/F(\zeta_p)^c$ is finite since $E(\zeta_p)^c/F(\zeta_p)^c$ is finite. This implies that $F_{i,\infty}/F$ is finite. □

Note that if $i \equiv 0 \ (mod\ p-1)$, then the condition $F_i \subseteq E$ is automatically satisfied since $F_i = F$ and therefore we find the result of Théorème 4 of [JS].

Proposition 3.2.5. *We have $F_{i,\infty}(i) = F_{i,\infty}(\zeta_p)^c$ and $F^c_{i,\infty}/F^c_i$ is everywhere split. If $F_{i,\infty}/F$ is infinite, then $F^c_i \subseteq F_{i,\infty}$ and $F_{i,\infty}/F^c_i$ is infinite.*

Proof. From Lemma 3.1.9, we know that

$$F_{i,j+1}(\zeta_p)^c = F_{i,j}(i)$$

We deduce that

$$F_{i,\infty}(i) = \bigcup_{j \in \mathbb{N}} F_{i,j}(i) = \bigcup_{j \in \mathbb{N}} F_{i,j+1}(\zeta_p)^c = F_{i,\infty}(\zeta_p)^c$$

Note that $F_{i,j+1}^c/F_{i,j}^c$ is everywhere split (because $F_{i,j+1}(\zeta_p)^c/F_{i,j}(\zeta_p)^c$ is everywhere unramified by Lemma 3.1.9 and $[F_{i,j}(\zeta_p)^c : F_{i,j}^c]$ is coprime with p). This shows that $F_{i,\infty}^c/F_i^c$ is everywhere split.
If $F_{i,\infty}/F$ is infinite, then, by Proposition 3.2.1, for every $j \geq 0$ we have

$$[F_{i,j+1} \cap F_i^c : F_i] = \prod_{k=0}^{j} p^{r_{F_{i,k}(i)}} \geq p^j$$

(see the remark after Proposition 3.2.1). This shows that $F_i^c \subseteq F_{i,\infty}$. Moreover

$$[F_{i,j+1} : F_{i,j+1} \cap F_i^c] = |X'_{F_{i,j}(\zeta_p)}(i)_\Gamma|$$

which shows that $F_{i,\infty}/F_i^c$ is infinite. □

Now we are ready to give a necessary condition for the tower to be finite.

Theorem 3.2.6. *If $F_{i,\infty}/F$ is finite and nontrivial, then X'_{F_i} is finite. Moreover if X'_{F_i} is trivial, then $F_{i,\infty}/F$ is trivial.*

Proof. Set $\Delta_i = \mathrm{Gal}(F(\zeta_p)/F_i)$. We have $|\Delta_i| = [F(\zeta_p) : F]/d_F(i)$ and

$$i \equiv 0 \ (mod \ |\Delta_i|) \tag{3.10}$$

since by (3.6), $h(i) = d_{t(i)}(\kappa(\gamma)^{-i})$ and therefore

$$\begin{aligned} ih(i) \equiv 0 \ (mod \ \mathrm{ord}_p(\kappa(\gamma))) \quad &\Leftrightarrow \quad id(i) \equiv 0 \ (mod \ \mathrm{ord}_p(\kappa(\gamma))) \\ &\Leftrightarrow \quad i \equiv 0 \ (mod \ \frac{\mathrm{ord}_p(\kappa(\gamma))}{d(i)}) \end{aligned}$$

and (3.10) follows by Lemma 3.1.11. Now $X'_{F_{i,\infty}(\zeta_p)}(i)_\Gamma = 0$: here

$$\Gamma = \mathrm{Gal}(F_{i,\infty}(\zeta_p)^c/F_{i,\infty}) \cong \Delta_i \times \mathrm{Gal}(F_{i,\infty}(\zeta_p)^c/F_{i,\infty}(\zeta_p))$$

By Nakayama's lemma we deduce that $X'_{F_{i,\infty}(\zeta_p)}(i)_{\Delta_i} = 0$ which means $(X'_{F_{i,\infty}(\zeta_p)}(i))^{\Delta_i} = 0$ (since $|\Delta_i|$ is coprime with p). This means that

$$\left(X'_{F_{i,\infty}(\zeta_p)}\right)^{\Delta_i} = 0$$

by (3.10) which is equivalent to $X'_{F_{i,\infty}} = 0$. This implies in particular that X'_{F_i} is finite.
Note that if $F_{i,\infty}/F$ is nontrivial, then $F_{i,1}^c/F_i^c$ is an abelian p-extension everywhere split (see the proof of Proposition 3.2.5) of degree $|X'_{F(\zeta_p)}(i)_\Gamma|$: therefore if $F_{i,\infty}/F$ is nontrivial, then X'_{F_i} is nontrivial. □

Remark 3.2.7. Theorem 3.2.6 is a generalization of Proposition 3 of [As] which tells that if $\mu_p \subseteq F$ (hence $F_i = F$) and if $F_{i,\infty}/F$ is finite, then X'_F is finite. If $F = \mathbb{Q}(\zeta_p)$, it is well known that X'_F is finite if and only if it is trivial (which is equivalent to p being regular): in fact suppose that X'_F is finite. This means that $X_F^- = 0$ (since X_F^- has no finite Λ-submodule, see [Wa]): hence $|(Cl_F)_p^-| = 1$. But this implies that $|(Cl_F)_p^+| = 1$ and hence $X'_F = 0$. Therefore, if p is irregular, the tower of $\mathbb{Q}(\zeta_p)$ has to be infinite (for any i).

Remark 3.2.8. Let F'/F be the subextension of $F(\zeta_p)/F$ which has degree $\mathrm{ord}_p(\kappa(\gamma)^{-i})$: thus $F' = F_i$ if $F_{i,\infty}/F$ is nontrivial. Set $\Delta' = \mathrm{Gal}(F'/F)$: then $F_{i,\infty}/F$ is trivial if and only if $X'_{F'}(i)_{\Delta'} = 0$. This is easy to prove since

$$X'_{F(\zeta_p)}(i)_\Gamma = 0 \Longleftrightarrow X'_{F(\zeta_p)}(i)_\Delta = 0 \Longleftrightarrow X'_{F'}(i)_{\Delta'} = 0$$

(where $\Delta = \mathrm{Gal}(F(\zeta_p)/F)$) because $i \equiv 0 \ (mod\ |\mathrm{Gal}(F(\zeta_p)/F')|)$ as in the proof of Theorem 3.2.6.

Theorem 3.2.9. *The following conditions are equivalent*

1. $F_{i,\infty}/F$ *is finite and nontrivial;*
2. $\overline{F_i}^{cd}/(F_i)^c$ *is finite and nontrivial.*

Moreover if $F_{i,\infty}/F$ is finite, then $\overline{F_i}^{cd} = F^c_{i,\infty}$.

Proof. Suppose that $F_{i,\infty}/F$ is infinite: then $F_i^c \subseteq F_{i,\infty}$ and $F_{i,\infty}/F_i^c$ is an infinite subextension of $F_{i,\infty} \subseteq \overline{F_i}^{cd}$ by Proposition 3.2.5.
Now suppose that $F_{i,\infty}/F$ is finite and nontrivial: then

$$X'_{F_{i,\infty}(\zeta_p)}(i)_\Gamma = 0$$

As in the proof of Theorem 3.2.6, one sees that $X'_{F_{i,\infty}} = 0$ and this implies that $\mathrm{Gal}(\overline{F_{i,\infty}}^{cd}/F^c_{i,\infty})$ is trivial, by well-known properties of pro-p-groups, since

$$\mathrm{Gal}(\overline{F_{i,\infty}}^{cd}/F^c_{i,\infty})^{ab} = X'_{F_{i,\infty}}$$

This means that $\overline{F_i}^{cd}/(F_i)^c$ is finite since $\overline{F_i}^{cd} F^c_{i,\infty} \subseteq \overline{F_{i,\infty}}^{cd}$: actually, one sees immediately that $\overline{F_i}^{cd} = F^c_{i,\infty}$. □

The following result deals with the absolute case, namely $F = \mathbb{Q}$: in that case there are no nontrivial finite towers.

Proposition 3.2.10. *Let i be odd and suppose that $X'_{\mathbb{Q}(\zeta_p)}(i)_\Gamma \neq 0$. Then $\mathbb{Q}_{i,\infty}/\mathbb{Q}$ is infinite.*

Proof. Suppose that $\mathbb{Q}_{i,\infty}/\mathbb{Q}$ is finite and nontrivial: then $X'_{\mathbb{Q}_i}$ is finite by Theorem 3.2.6. Since i is odd, $\mathbb{Q}_i$ is a CM-field. This can be easily seen using Lemma 3.1.10 and Lemma 3.1.11 since

$$\mathbb{Q}_i \text{ is a } CM \text{ field } \iff [\mathbb{Q}(\zeta_p) : \mathbb{Q}_i] \text{ is odd}$$

and

$$[\mathbb{Q}(\zeta_p) : \mathbb{Q}_i] = \frac{p-1}{[\mathbb{Q}_i : \mathbb{Q}]} = \frac{p-1}{\mathrm{ord}_p(\kappa(\gamma)^{-i})} = (i,\, p-1) \tag{3.11}$$

Then we know that $(X'_{\mathbb{Q}_i})^-$ has no finite Λ-submodules (see [Wa], Proposition 13.28) and hence $(X'_{\mathbb{Q}_i})^- = 0$. But this implies $(X'_{\mathbb{Q}_i})^+ = 0$ and therefore $X'_{\mathbb{Q}_i} = 0$. This is a contradiction, by Theorem 3.2.6. □

Remark 3.2.11. Proposition 3.2.10 and Theorem 3.2.4 tell us that, if i is odd and $WK_{2i}^{\acute{e}t}(\mathbb{Q}) \neq 0$, then for every number field F containing $\mathbb{Q}_i$, $WK_{2i}^{\acute{e}t}(F)$ is nontrivial. Furthermore note that Vandiver's conjecture implies $WK_{4i}^{\acute{e}t}(\mathbb{Q}) = 0$ for any $i \geq 1$ (in particular the tower is finite, being trivial). If something weaker holds, namely if Greenberg's conjecture holds for real subfields of $\mathbb{Q}(\zeta_p)$, then I do not know whether or not the tower $\mathbb{Q}_{2i,\infty}/\mathbb{Q}$ is finite. For, $\mathbb{Q}_{2i}$ is a totally real field and Greenberg's conjecture then tells that $X'_{\mathbb{Q}_i}$ is finite. But, in general the converse of Theorem 3.2.6 needs not to hold true, hence we cannot conclude that $\mathbb{Q}_{i,\infty}/\mathbb{Q}$ is finite.

Remark 3.2.12. Using Proposition 2.3 in [KM], one can prove that, for any $i \geq 1$ such that $F_{i,\infty}/F$ is infinite, there is no p-extension E/F such that $WK_{2i}^{\acute{e}t}(E) = 0$. Details are as follows: first of all, if $i \equiv 0$ $(mod\ \mathrm{ord}_p(\kappa(\gamma_F)))$, then note that $F_i = F$ and we conclude by Theorem 3.2.4. If instead we have $i \not\equiv 0$ $(mod\ \mathrm{ord}_p\kappa(\gamma_F))$, then the corestriction map

$$WK_{2i}^{\acute{e}t}(E) \longrightarrow WK_{2i}^{\acute{e}t}(F)$$

is surjective: this follows from Proposition 2.3 in [KM] and the fact that a p-group is solvable. Since $WK_{2i}^{\acute{e}t}(F)$ is nontrivial, $WK_{2i}^{\acute{e}t}(E)$ has to be nontrivial as well.

The following is a standard genus-theoretic criterion for $F_{i,\infty}/F$ to be finite (see [JS], Proposition 11).

Proposition 3.2.13. *If X'_{F_i} is (finite) cyclic, then $F_{i,\infty}/F$ is finite.*

Proof. (See Proposition 11 of [JS]) Set $N = (F_i^{cd})^{cd}$: then N is a Galois pro-p-extension of F_i^{cd} which is also Galois over F_i^c. Set $G = \mathrm{Gal}(N/F_i^c)$ and $H = \mathrm{Gal}(F_i^{cd}/F_i^c)$: then G/H acts on H by conjugation. Let σ a generator of G/H: $(1-\sigma)H$ is then the Galois group of the maximal subextension of N/F_i^{cd} which is abelian over F_i^c. By definition of F_i^{cd}, we have $(1-\sigma)H = H$, which implies $H = 0$ (in other words $N = F_i^{cd}$). Now $F_{i,2}F_i^{cd}/F_i^{cd}$ is an abelian p-extension which is everywhere split therefore $F_i^{cd} \subseteq F_{i,2}F_i^{cd} \subseteq N$. Then $F_{i,2}F_i^{cd} = F_i^{cd}$ and therefore $F_{i,2}^c \subseteq F_i^{cd}$: actually, one easily proves that $F_{i,j}^c \subseteq F_i^{cd}$ for every $j \geq 1$, which implies $F_{i,\infty}^c \subseteq F_i^{cd}$. Since F_i^{cd}/F_i^c is finite, we conclude that $F_{i,\infty}/F$ is finite. $\square$

Example. The preceding proposition gives a way to produce finite towers which are nontrivial. Let $F = \mathbb{Q}(\sqrt{d})$ with d a squarefree positive integer. Suppose that

- p remains inert in F;
- $(Cl_F)_p$ is cyclic and nontrivial;
- the natural map $X_F \to (Cl_F)_p$ is an isomorphism;
- $F_i = F$;
- $WK_{2i}^{ét}(F) \neq 0$.

The first three hypotheses imply that X'_F is cyclic and nontrivial. In fact the first implies that there is only one prime $\mathfrak{p}$ above p which is principal and hence the natural map $(Cl_F)_p \to (Cl_F)'_p$ is an isomorphism. Hence we have a commutative diagram

$$\begin{array}{ccc} X_F & \longrightarrow & (Cl_F)_p \\ \downarrow & & \downarrow \\ X'_F & \longrightarrow & (Cl_F)'_p \end{array}$$

where the upper horizontal arrow is an isomorphism thanks to the third hypothesis and the same holds for the right vertical arrow. Now the left vertical arrow is surjective and hence an isomorphism too. Therefore by the second hypothesis X'_F is cyclic (and isomorphic to $(Cl_F)_p$). Then, by Proposition 3.2.13, $F_{i,\infty}/F$ is finite thanks to the fourth hypothesis. The fifth hypothesis then assures that $F_{i,\infty}/F$ is nontrivial.
Now take for instance $p = 3$ and $d = 257$: the field $F = \mathbb{Q}(\sqrt{d})$ satisfies the first three hypotheses (see [KS1] and [KS2], we have $(Cl_F)_3 = Cl_F$ cyclic of order 3, or apply [Fu] because $(Cl_F)_3 \cong (Cl_{F_1})_3$, the isomorphism being given by the norm). If i is even the fourth hypothesis is satisfied. Now observe that, thanks to Corollaire 5 of [JM], the last hypothesis is equivalent to the nontriviality of $\widetilde{Cl}_F$ (which is the logarithmic class group defined by Jaulent, see [Ja3]). There is an exact sequence

$$0 \longrightarrow \widetilde{Cl}_F(p) \longrightarrow \widetilde{Cl}_F \xrightarrow{\varphi} (Cl'_F)_p \longrightarrow \deg_F \mathcal{D}\ell/(\deg_F \mathfrak{p})\mathbb{Z}_p \longrightarrow 0$$

(see [DS], §3, also for the definitions of the right and left-hand terms of this sequence). We have $\widetilde{Cl}_F(p) = 0$ because of the first hypothesis (see [DS], Lemme 4) and it can be shown that also $\deg_F \mathcal{D}\ell/(\deg_F \mathfrak{p})\mathbb{Z}_p = 0$ (see the proof of Proposition 4.2.5 at the end of the next chapter, where a similar computation is explained in detail). Hence φ is an isomorphism, which implies that the last hypothesis is satisfied.

3.3 Examples

1. We show by an example that $\mathbb{Q}_i \neq \mathbb{Q}(\zeta_p)$. Choose $p = 683$ and $i = 31$: then $A_{-i} = A_{651} \neq 0$ since $683|B_{32}$. On the other hand $i|p-1$ (since $682 = 31 \cdot 22$) and hence $d(i) < p-1$ (in fact $d(i) = 22$).

2. We consider the case $p = 37$ and $i = 31$. Then $\mathbb{Q}_{i,\infty}/\mathbb{Q}$ is infinite since $\mathbb{Q}_i = \mathbb{Q}(\zeta_p)$ and
$$(A_n)_5 \cong \mathbb{Z}/p^n\mathbb{Z}$$
(here A_n is the p-Sylow subgroup of $Cl(\mathbb{Q}(\zeta_{p^n}))$) which implies $X_{\mathbb{Q}_i} \cong \mathbb{Z}_p$ (norms are surjective). In particular there is no finite extension of $E/\mathbb{Q}$ such that $WK_{62}^{\acute{e}t}(E) = 0$ and $\mu_p \subseteq E$. We also know that $WK_{62}^{\acute{e}t}(\mathbb{Q}) \cong \mathbb{Z}/37\mathbb{Z}$, provided that the Quillen-Lichtenbaum conjecture holds (in fact it predicts that $\mathrm{K}_{62}(\mathbb{Z}) = \mathbb{Z}/37\mathbb{Z}$). Then $\widetilde{I} = \langle \widetilde{\gamma}^{p-1} \rangle$ (since $h(i) = d(i) = p-1$), which implies in particular that $\mathbb{Q}(\zeta_p) \subseteq \mathbb{Q}_{i,1}$.
Let us have a closer look, just to identify in this case some of the object we described above. First of all note that $\mathbb{Q}_{i,1}/\mathbb{Q}(\zeta_p)$ is the p-Hilbert class field of $\mathbb{Q}(\zeta_p)$ (by Proposition 3.1.12). Now note that
$$WK_{2i}^{\acute{e}t}(\mathbb{Q}(\zeta_p)) \cong WK_{2i}^{\acute{e}t}(\mathbb{Q}) \cong \mathbb{Z}/p\mathbb{Z} \tag{3.12}$$
In fact
$$WK_{2i}^{\acute{e}t}(\mathbb{Q}(\zeta_p)) \cong X_{\mathbb{Q}(\zeta_p)}(i)_{\Gamma_p} = X_{\mathbb{Q}(\zeta_p)}(i)_{\Gamma} \cong WK_{2i}^{\acute{e}t}(\mathbb{Q})$$
since $\mathrm{Gal}(\mathbb{Q}(\zeta_p)^c/\mathbb{Q}^c) \cong (\mathbb{Z}/p\mathbb{Z})^\times$ acts trivially on $X_{\mathbb{Q}(\zeta_p)}(i)$ (recall that $Cl(\mathbb{Q}(\zeta_{p^n}))$ has only the $(p-1-i)$-component). We note in passing that we cannot deduce by a codescent argument that $WK_{2i}(\mathbb{Q}_{i,1}) \neq 0$ since the canonical map
$$WK_{2i}(\mathbb{Q}_{i,1})_{\mathrm{Gal}(\mathbb{Q}_{i,1}/\mathbb{Q}(\zeta_p))} \longrightarrow WK_{2i}(\mathbb{Q}(\zeta_p))$$
is not surjective (see [KM], example 2.5: in this case, it is even the trivial map). We know that
$$[\mathbb{Q}(\zeta_p)_{i,1} : \mathbb{Q}(\zeta_p)] \geq |WK_{2i}^{\acute{e}t}(\mathbb{Q}(\zeta_p))| = p$$
One easily proves that $[\mathbb{Q}(\zeta_p)_{i,1} : \mathbb{Q}(\zeta_p)] = p^2$. Furthermore
$$\mathbb{Q}(\zeta_{p^2}) \subseteq \mathbb{Q}(\zeta_p)_{i,1} \quad \text{but} \quad \mathbb{Q}(\zeta_{p^3}) \neq \mathbb{Q}(\zeta_p)_{i,1}$$

Moreover it is not difficult to see that

$$\mathbb{Q}_{i,1} \subseteq \mathbb{Q}(\zeta_p)_{i,1} \quad \text{but} \quad \mathbb{Q}(\zeta_{p^2}) \neq \mathbb{Q}_{i,1}$$

hence

$$\mathrm{Gal}(\mathbb{Q}(\zeta_p)_{i,1}/\mathbb{Q}(\zeta_p)) \cong (\mathbb{Z}/p\mathbb{Z})^2$$

Hence $\mathbb{Q}(\zeta_p)_{i,1} = \mathbb{Q}_{i,1}(\zeta_{p^2})$.

Chapter 4

Splitting of the K-theory exact localization sequence

As recalled in Section 1.3 there is an exact localization sequence

$$0 \longrightarrow K_{2i}(\mathcal{O}_F) \longrightarrow K_{2i}(F) \xrightarrow{\partial} \bigoplus_{v \text{ finite}} K_{2i-1}(k_v) \longrightarrow 0$$

where k_v is the residue field of F at v and the sum is taken over the finite primes of F. We remark that $K_{2i-1}(k_v)$ is cyclic of order $|k_v|^i - 1$ by Quillen calculation.
The problem studied in this chapter is to determine necessary and sufficient conditions in order for this sequence to split. This problem has positive answer (i.e. the sequence always splits) if E is a rational function field of one variable (Tate-Milnor theorem, see [Mil]). Clearly one can consider the analogous problem on the induced exact sequence on p-primary parts (we call it the p-localization sequence for $K_{2i}(F)$). This has been studied by Banaszak in [Ba]: he stated a theorem which said that the p-localization sequence for $K_{2i}(F)$ splits if and only if $\operatorname{div}(K_{2i}(F))_p = 0$. Recall that for an abelian group M, $\operatorname{div}(M)$ denotes the subgroup of divisible elements of M and that $\operatorname{div}(K_{2i}(F))_p = WK_{2i}^{ét}(F)$ (see Remark 1.4.3). Banaszak's condition is obviously a necessary one, since both the right and the left terms of the sequence have trivial group of divisible elements. However the proof of the converse, in Banaszak's paper, seems to be incomplete. It turns out that there is a counterexample, namely there is a field F such that $WK_{2i}^{ét}(F) = 0$ but the p-localization sequence for $K_{2i}(F)$ does not split. In fact we shall state a necessary and sufficient condition for the i-th sequence to be split and then we will be able to produce a counterexample by using this result. Our structure theorem tells also that in the case $F = \mathbb{Q}$, Banaszak's theorem holds, i.e. the p-localization sequence for $K_{2i}(F)$ splits if and only if $WK_{2i}^{ét}(F) = 0$.

4.1 Obstruction to splitting

In the rest of this chapter i is a positive integer. As in the introduction we are going to consider the p-localization sequence for $K_{2i}(F)$ (where p is as usual an odd prime): since $K_{2i}(\mathcal{O}_F)_p = K_{2i}(\mathcal{O}_F[\frac{1}{p}])_p$ we have the exact sequence

$$0 \longrightarrow K_{2i}(\mathcal{O}_F)_p \longrightarrow K_{2i}(F)_p \xrightarrow{\partial} \bigoplus_{v \nmid p} K_{2i-1}(k_v)_p \longrightarrow 0 \tag{4.1}$$

This exact sequence has (at least conjecturally) a cohomological counterpart. More precisely (see [Ta2] for the case $i = 1$ and [Ba] for the general case) there is a commutative diagram with exact rows

$$\begin{tikzcd}
K_{2i}(\mathcal{O}_F)_p \arrow[r, hook] \arrow[d, "\nu"] & K_{2i}(F)_p \arrow[r, two heads, "\partial"] \arrow[d, "\nu"] & \oplus_{v\nmid p} K_{2i-1}(k_v)_p \arrow[d, "\overline{\nu}"] \\
K^{ét}_{2i}(\mathcal{O}_F[\frac{1}{p}]) \arrow[r, hook] \arrow[d, "\gamma"] & K^{ét}_{2i}(F)_p \arrow[r, two heads, "\partial^{ét}"] \arrow[d, "\gamma"] & \oplus_{v\nmid p} K^{ét}_{2i-1}(k_v) \arrow[d, "\overline{\gamma}"] \\
H^1(\mathcal{O}_F[\frac{1}{p}], W(i+1))/Div \arrow[r, hook] \arrow[d, "\delta"] & H^1(F, W(i+1))/Div \arrow[r, two heads] \arrow[d, "\delta"] & \oplus_{v\nmid p} H^0(k_v, W(i)) \arrow[d, "\delta"] \\
H^2(\mathcal{O}_F[\frac{1}{p}], T(i+1)) \arrow[r, hook] & H^2(F, T(i+1))_p \arrow[r, two heads] & \oplus_{v\nmid p} H^1(k_v, T(i))
\end{tikzcd}$$

Diagram 4.1: Relationship between K-theory and étale cohomology.

We use notation defined in Chapter 2 with some modification (mainly for typographical convenience): $T(i) = \mathbb{Z}_p(i)$, $W(i) = \mathbb{Q}_p/\mathbb{Z}_p(i)$, $\nu = ch^{ét}_{i,0}$, δ is cohomological connecting homomorphism (in fact isomorphism) relative to the exact sequence

$$0 \to T(i) \to \mathbb{Q}_p(i) \to W(i) \to 0$$

and $\delta \circ \gamma$ is the natural isomorphism in (1.4) ($\overline{\nu}$ and $\overline{\gamma}$ are defined similarly). Direct sums are over all finite places of F which do not divide p. Recall that the Quillen-Lichtenbaum conjecture predicts that the ν's are indeed isomorphisms (and Tate (see [Ta2]) proved that this holds if $i = 1$). Note that the kernels of the two ν's are equal and of finite order since $K_{2i}(\mathcal{O}_F)$ is finite.

In the following we shall consider the exact sequence

$$0 \to H^2(\mathcal{O}_F[\tfrac{1}{p}], T(i+1)) \to H^2(F, T(i+1))_p \to \bigoplus_{v \nmid p} H^1(k_v, T(i)) \to 0 \tag{4.2}$$

instead of (4.1): we will refer to it as the i-th cohomological p-localization sequence for F (we will not stress on i unless it is necessary). The following proposition shows that there is no difference between considering (4.1) or (4.2), even without using the Quillen-Lichtenbaum conjecture.

Proposition 4.1.1. *The i-th cohomological p-localization sequence for F splits if and only the p-localization sequence for $K_{2i}(F)$ splits.*

Proof. Banaszak ([Ba], Proposition 2) proved that

$$K_{2i}(\mathcal{O}_F)_p \xrightarrow{\nu} K_{2i}^{\acute{e}t}(\mathcal{O}_F[\tfrac{1}{p}]) \tag{4.3}$$

is split surjective. We are going to prove the analogous result for the map

$$K_{2i}(F)_p \xrightarrow{\nu} K_{2i}^{\acute{e}t}(F)_p \tag{4.4}$$

with the same strategy as Banaszak, taking into account that the groups involved are no more finite (but still torsion). First of all there is a commutative diagram

$$\begin{array}{ccccc} K_{2i}(F, \mathbb{Z}/p^n\mathbb{Z}) & \longrightarrow & K_{2i}(F)[p^n] & \longrightarrow & 0 \\ \big\downarrow & & \nu\big\downarrow & & \\ K_{2i}^{\acute{e}t}(F, \mathbb{Z}/p^n\mathbb{Z}) & \longrightarrow & K_{2i}^{\acute{e}t}(F)[p^n] & \longrightarrow & 0 \end{array}$$

with exact rows and surjective vertical maps (see [Ba], Diagram 1.6). This implies that the kernel C_i of the map $K_{2i}(F)_p \to K_{2i}^{\acute{e}t}(F)_p$ is a *pure* subgroup. i.e. for each $n \in \mathbb{N}$ we have

$$C_i \cap K_{2i}(F)_p^{p^n} = C_i^{p^n}$$

Moreover C_i is finite since it coincides with the kernel of the map in (4.3), as follows easily from the properties of Diagram 4.1 which we listed above. Hence Theorem 7 of [Ka] tells us that the map in (4.4) is split. In fact, this is equivalent to the fact that the map

$$K_{2i}(F)_p \xrightarrow{\delta\circ\gamma\circ\nu} H^2(F, T(i+1))_p \tag{4.5}$$

is split.
Now suppose that (4.2) splits: then using the fact that (4.5) splits, a simple diagram chasing in Diagram 4.1 shows that (4.1) is split too. □

We are going to describe the obstruction to splitting of the cohomological p-localization sequence for F in terms of coinvariants of twisted p-parts of the class groups of the fields $F_n = F(\mu_{p^n})$, following [Hu]. We denote by μ_{p^n} the group of p^n-th roots of unity in an algebraic closure of F (however in the following μ_{p^n} may denote the group of p^n-th roots of unity in an algebraic closure of a field other than F and we shall not stress on that).

For typographical convenience, we set

$$\Omega_{n,i} = \left(H^2(\mathcal{O}_F[\tfrac{1}{p}], T(i+1)) \cap H^2(F, T(i+1))_p^{p^n}\right) \Big/ H^2(\mathcal{O}_F[\tfrac{1}{p}], T(i+1))^{p^n}$$

These groups are the obstructions to the existence of a splitting for the cohomological p-localization sequence for F. Note that, from the definition of $\Omega_{n,\,i}$, we have $\Omega_{0,\,i} = 0$.

Lemma 4.1.2. *The i-th cohomological p-localization sequence for F splits if and only if for every $n \in \mathbb{N}$ we have $\Omega_{n,\,i} = 0$.*

Proof. It is not difficult to realize (see [Ba], proof of Proposition 2) that the i-th cohomological p-localization sequence splits if and only if

$$H^2(F,\, T(i+1))_p[p^n] \longrightarrow \bigoplus_{v \nmid p} H^1(k_v,\, T(i))[p^n]$$

is surjective for every $n \in \mathbb{N}$. Using the snake lemma, we get an exact sequence

$$\begin{tikzcd}
H^2(F,\, T(i+1))_p[p^n] \arrow[r] & \oplus_{v\nmid p} H^1(k_v,\, T(i))[p^n] \arrow[d, "I"] \\
H^2(F,\, T(i+1))_p/p^n;[] \arrow[r] & H^2(\mathcal{O}_F[\tfrac{1}{p}],\, T(i+1))/p^n
\end{tikzcd}$$

Hence the surjectivity of the first map is equivalent to $I = 0$ and therefore to the injectivity of the third map. Since the kernel of this map is exactly $\Omega_{n,\,i}$, we are done. $\square$

We are going to use the following notation: for $n \in \mathbb{N}$, set $F_n = F(\mu_{p^n})$ and $\Gamma_n = \mathrm{Gal}(F_n/F)$. If w is a place in F_n, then denote by $(k_n)_w$ the residue field of F_n at w.

Lemma 4.1.3. *Let $v \nmid p$ be a place in F. For every n, $m \in \mathbb{N}$, there are of isomorphisms of Γ_m-modules*

$$\bigoplus_{w|v} H^1((k_m)_w,\, T(i))[p^n] \cong \bigoplus_{w|v} H^0((k_m)_w,\, \mu_{p^n}^{\otimes i})$$

and

$$H^2(\mathcal{O}_{F_m}[\tfrac{1}{p}],\, T(i))/p^n \cong H^2(\mathcal{O}_{F_m}[\tfrac{1}{p}],\, \mu_{p^n}^{\otimes i})$$

Proof. Both assertions come from the cohomology sequence corresponding to the exact sequence

$$0 \longrightarrow T(i) \xrightarrow{p^n} T(i) \longrightarrow \mu_{p^n}^{\otimes i} \longrightarrow 0$$

together with $H^0((k_n)_w, T(i)) = 0$ (it is a finite $\mathbb{Z}_p$-module on which multiplication by p^n acts injectively) and $H^3(\mathcal{O}_{F_m}[\frac{1}{p}], T(i)) = 0$ (the Galois group of the maximal extension of F_m which is unramified outside p has p-cohomological dimension less or equal to 2). Since conjugation commutes with the connecting homomorphism, this proves that, for any fixed $w_0|v$ in F_n there is a D_v-module homomorphism (D_v being the decomposition group at v in F_n/F)

$$H^1((k_m)_{w_0}, T(i))[p^n] \cong H^0((k_m)_{w_0}, \mu_{p^n}^{\otimes i})$$

and a Γ_m-isomorphism

$$H^2(\mathcal{O}_{F_m}[\tfrac{1}{p}], T(i))/p^n \cong H^2(\mathcal{O}_{F_m}[\tfrac{1}{p}], \mu_{p^n}^{\otimes i})$$

Furthermore

$$\bigoplus_{w|v} H^1((k_m)_w, T(i))[p^n] = \mathrm{Ind}_{D_v}^{\Gamma_m} H^1((k_m)_{w_0}, T(i))[p^n]$$

for any fixed $w_0|v$ in F_n and

$$\bigoplus_{w|v} H^0((k_m)_w, \mu_{p^n}^{\otimes i}) = \mathrm{Ind}_{D_v}^{\Gamma_n} H^0((k_m)_{w_0}, \mu_{p^n}^{\otimes i})$$

This concludes the proof. □

Lemma 4.1.4. *Let $v \nmid p$ be a finite place of F. If w is a place of F_n above v, then, for every $i \geq 0$, the corestriction homomorphisms*

$$H^0((k_n)_w, \mu_{p^n}^{\otimes i}) \longrightarrow H^0(k_v, \mu_{p^n}^{\otimes i})$$

are surjective.

Proof. Note that

$$H^0(k_v, \mu_{p^n}^{\otimes i}) = H^0((k_n)_w, \mu_{p^n}^{\otimes i})^{\mathrm{Gal}((k_n)_w/k_v)}$$

and use [We2], Lemma 3.2 and Remark 3.2.1. □

Lemma 4.1.5. *Let $c : \bigoplus_{w|p} \mu_{p^n}^{\otimes i} \to \mu_{p^n}^{\otimes i}$ be the codiagonal map $(\zeta_w)_w \mapsto \prod_w \zeta_w$. Then c is a surjective map of Γ_n-modules and $H_1(\Gamma_n, \mathrm{Ker}\, c) = 0$.*

Proof. See [Ke], Lemma 6.5, for the case $i = 2$. The general case follows easily (see also [We2], Lemma 3.2 and Remarks 3.2.1). □

Lemma 4.1.6. *For every $n \in \mathbb{N}$, the corestriction map induces an isomorphism*

$$H^2(\mathcal{O}_{F_n}[\tfrac{1}{p}],\, \mu_{p^n}^{\otimes i})_{\Gamma_n} \xrightarrow{\sim} H^2(\mathcal{O}_F[\tfrac{1}{p}],\, \mu_{p^n}^{\otimes i})$$

Proof. Apply [We2], Proposition 2.2 to the group of the maximal extension of F which is unramified outside p, using the fact that it has p-cohomological dimension less or equal to 2. □

We now recall the definition of the map

$$J_{n,\,i} : Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i} \longrightarrow H^2(\mathcal{O}_{F_n}[\tfrac{1}{p}],\, \mu_{p^n}^{\otimes i+1}) \tag{4.6}$$

(see [Hu], [Ke], [We2]). If $\zeta \in \mu_{p^n}$ and $[\mathfrak{A}]$ is the class of the ideal $\mathfrak{A}$ in Cl'_{F_n}, we have

$$J_{n,\,1}([\mathfrak{A}] \otimes \zeta) = h\left(x^{p^n} \ (mod\ K_2(\mathcal{O}_{F_n})_p^{p^n})\right)$$

where $x \in K_2(F_n)_p$ goes to $(\zeta^{w(\mathfrak{A})}\ (mod\, \mathfrak{P}_w))_w$ (here $\mathfrak{P}_w$ is the prime ideal correspondig to w) under the map induced by Hilbert symbols and $h : K_2(\mathcal{O}_{F_n})_p/p^n \to H^2(\mathcal{O}_{F_n}[\frac{1}{p}],\, \mu_{p^n}^{\otimes 2})$ is the map defined in [Ta2]. For $i \geq 2$, $J_{n,\,i}$ is defined observing that

$$H^2(\mathcal{O}_{F_n}[\tfrac{1}{p}],\, \mu_{p^n}^{\otimes i+1}) = H^2(\mathcal{O}_{F_n}[\tfrac{1}{p}],\, \mu_{p^n}^{\otimes 2}) \otimes \mu_{p^n}^{\otimes i-1}$$

Proposition 4.1.7. *(Keune-Weibel) For every $n \in \mathbb{N}$ there is the following exact sequence of Γ_n-modules*

$$0 \to Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i} \xrightarrow{J_{n,\,i}} H^2(\mathcal{O}_{F_n}[\tfrac{1}{p}],\, \mu_{p^n}^{\otimes i+1}) \longrightarrow \bigoplus_{w|p} \mu_{p^n}^{\otimes i} \xrightarrow{c} \mu_{p^n}^{\otimes i} \to 0$$

Moreover, taking coinvariants by Γ_n gives

$$0 \to \left(Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_n} \xrightarrow{J_{n,\,i}} H^2(\mathcal{O}_F[\tfrac{1}{p}],\, \mu_{p^n}^{\otimes i+1}) \to (\bigoplus_{w|p} \mu_{p^n}^{\otimes i})_{\Gamma_n} \xrightarrow{c} (\mu_{p^n}^{\otimes i})_{\Gamma_n} \to 0$$

Proof. See [Ke] Theorem 6.6, for the case $i = 2$. The general case follows easily (for instance see Proposition 4.1 of [We2] and use Lemma 4.1.5 and Lemma 4.1.6). □

Theorem 4.1.8. *The i-th cohomological p-localization sequence for F (or equivalently the p-localization sequence for $K_{2i}(F)$) splits if and only if for every $n \in \mathbb{N}$ we have*

$$\left(Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_n} = 0$$

Proof. Thanks to Lemma 4.1.2 it will be sufficient to show that

$$\left(Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_n} \cong \Omega_{n,\,i}$$

for every $n \in \mathbb{N}$. There is a commutative diagram of Γ_n-modules (those on the bottom line have trivial action) with exact rows

$$\begin{tikzcd} H^2(\mathcal{O}_{F_n}[\frac{1}{p}],\, T(i+1)) \arrow[r, hook] \arrow[d, "cor^2"] & H^2(F_n,\, T(i+1))_p \arrow[r, two heads] \arrow[d, "cor^2"] & \bigoplus_{w \nmid p} H^1((k_n)_w,\, T(i)) \arrow[d, "cor^1"] \\ H^2(\mathcal{O}_{F}[\frac{1}{p}],\, T(i+1)) \arrow[r, hook] & H^2(F,\, T(i+1))_p \arrow[r, two heads] & \bigoplus_{v \nmid p} H^1(k_v,\, T(i)) \end{tikzcd}$$

where *cor* is the (cohomological) corestriction. Following [Hu], Section 3, we consider a part of the commutative diagram induced by snake lemma, namely

$$\begin{tikzcd} \bigoplus_{w \nmid p} H^1((k_n)_w,\, T(i))[p^n] \arrow[r, "I_{n,i}"] \arrow[d, "cor^1"] & H^2(\mathcal{O}_{F_n}[\frac{1}{p}],\, T(i+1))/p^n \arrow[d, "cor^2"] \\ \bigoplus_{v \nmid p} H^1(k_v,\, T(i))[p^n] \arrow[r, "I_i"] & H^2(\mathcal{O}_{F}[\frac{1}{p}],\, T(i+1))/p^n \end{tikzcd}$$

Using Lemma 4.1.3 we can write

$$\begin{tikzcd} \bigoplus_{w \nmid p} \mu_{p^n}^{\otimes i} \arrow[r, "I_{n,i}"] \arrow[d, "cor^1"] & H^2(\mathcal{O}_{F_n}[\frac{1}{p}],\, \mu_{p^n}^{\otimes i+1}) \arrow[d, "cor^2"] \\ \bigoplus_{v \nmid p} H^1(k_v,\, T(i))[p^n] \arrow[r, "I_i"] & H^2(\mathcal{O}_{F}[\frac{1}{p}],\, \mu_{p^n}^{\otimes i+1}) \end{tikzcd}$$

A straightforward verification shows that we can split the map $I_{n,i}$ by

$$\pi : \bigoplus_{w \nmid p} \mu_{p^n}^{\otimes i} \to Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i}$$

which is defined by

$$\pi((\zeta_w)_w) = \sum_{w \nmid p} [\mathfrak{P}_w] \otimes \zeta_w$$

($\mathfrak{P}_w$ is the prime ideal corresponding to w). Note that π is surjective since elements of the form $[\mathfrak{P}_w] \otimes \zeta_w$ (with ζ_w running in μ_{p^n}) generates $Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i}$. In fact we have $I_{n,i} = J_{n,i} \circ \pi$ where $J_{n,i}$ is the map in (4.6). We know that

$$Im\, I_i = \Omega_{n,i}$$

Hence, $\eta = cor^2 \circ J_{n,\,i}$ defines a Γ_n-homomorphism

$$Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i} \to \Omega_{n,\,i}$$

By Lemma 4.1.4, we know that cor^1 is surjective, therefore η is surjective. Moreover taking coinvariants we get a map

$$\eta : \left(Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_n} \longrightarrow \Omega_{n,\,i}$$

This map is injective by Proposition 4.1.7 and this concludes the proof. $\square$

Remark 4.1.9. K. Hutchinson pointed out to me that if $\mu_p \subseteq F$, then the splitting criterion of Theorem 4.1.8 is independent of i. In fact, in this situation, Γ_n is a cyclic p-group and hence by Nakayama's lemma

$$\left(Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_n} = 0 \Leftrightarrow Cl'_{F_n} \otimes \mu_{p^n}^{\otimes i} = 0 \Leftrightarrow Cl'_{F_n}/p^n = 0 \Leftrightarrow (Cl'_{F_n})_p = 0$$

In particular, still supposing that $\mu_p \subseteq F$, if $(Cl'_F)_p \neq 0$, then the p-localization sequence for $K_2(F)$ does not split. Here is another proof of this (partial) result.

Proposition 4.1.10. *Let p be any prime and suppose $\mu_p \subseteq F$. If $(Cl'_F)_p \neq 0$, then the p-localization sequence for $K_2(F)$ does not split.*

Proof. Let $\overline{v}$ be a finite place of F which does not lie over p: since p divides $|(k_{\overline{v}}^{\times})_p|$ (F contains the p-th roots of unity), we can choose an element $\zeta_{\overline{v}}$ of order p in $k_{\overline{v}}^{\times}$. We identify $\zeta_{\overline{v}}$ with the sequence in $\oplus_{v \nmid p} K_1(k_v)_p$ which has 1 everywhere except at $\overline{v}$ where it has $\zeta_{\overline{v}}$. Now suppose that the p-localization sequence for $K_2(F)$ splits: then there exists an element x of order p in $K_2(F)$ such that $\partial(x) = \zeta_{\overline{v}}$. Since F contains the p-th roots of unity, we can find an element $\alpha_{\overline{v}} \in F^{\times}$ and a p-th root of unity $\zeta \in F$ such that $x = \{\zeta, \alpha_{\overline{v}}\}$ (see [Ta2]). Now from the definition of ∂ we see that

$$\overline{v}(\alpha_{\overline{v}}) \not\equiv 0 \ (mod\ p), \quad v(\alpha_{\overline{v}}) \equiv 0 \ (mod\ p) \text{ if } v \neq \overline{v} \text{ and } v \nmid p$$

Hence

$$(\alpha_{\overline{v}}) = \mathfrak{a}^p \mathfrak{a}_p \mathfrak{p}_{\overline{v}}^{\overline{v}(\alpha_{\overline{v}})}$$

where $\mathfrak{a}$ is an ideal of F, $\mathfrak{a}_p$ is a product of prime ideals of F which lie over p and $\mathfrak{p}_{\overline{v}}$ is the prime ideal of F which corresponds to $\overline{v}$. In particular the class of $\mathfrak{p}_{\overline{v}}$ is trivial in $Cl'_F/{Cl'_F}^p$ since $\overline{v}(\alpha_{\overline{v}})$ is invertible modulo p. The same holds for every prime ideal of F (not dividing p) since $\overline{v}$ was chosen arbitrarily. Hence $Cl'_F/{Cl'_F}^p$ has to be trivial, which implies $p \nmid |Cl'_F|$ giving a contradiction. $\square$

4.2 Examples and non-examples

First of all we analyze the simplest case, namely $F = \mathbb{Q}$. Let A_n denote the p-Sylow subgroup of $\mathbb{Q}(\mu_{p^n})$. Let K_n be the n-th level of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Set $\Delta_n = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/K_n)$ and for every $j \in \mathbb{Z}$, let $(A_n)_j$ denote the ω^j-component of A_n where $\omega : \Delta_n \to \mathbb{Z}_p^\times$ denotes the Teichmüller character (notation as in [Wa], §6.3). As in Chapter 2 we set

$$\Gamma_p = \mathrm{Gal}(\mathbb{Q}(\zeta_p)^c/\mathbb{Q}(\zeta_p)) \quad \Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_p)^c/\mathbb{Q})$$

We need the following well known result: for an even more general version of the first statement, see [KM]. For the second statement, see [Ko2], Corollary 5.3.

Proposition 4.2.1. *Suppose that $i, j \geq 1$ and $i \equiv j$ (mod $p-1$). Then $WK_{2i}^{\acute{e}t}(\mathbb{Q}) = 0$ if and only if $WK_{2j}^{\acute{e}t}(\mathbb{Q}) = 0$. Moreover, if A is the p-Sylow of the class group of $\mathbb{Q}(\mu_p)$, then $WK_{2i}^{\acute{e}t}(\mathbb{Q}) = 0$ if and only if $A_{-i} = 0$.*

Proof. From Schneider's isomorphism (see [Sc], §6) we know that

$$WK_{2i}^{\acute{e}t}(\mathbb{Q}) \cong X_{\mathbb{Q}(\mu_p)}(i)_\Gamma$$

where $X_{\mathbb{Q}(\mu_p)}(i)$ denotes the i-th Tate twist of $X_{\mathbb{Q}(\mu_p)}$. Now $\Gamma = \Gamma_p \times \Delta$ where $\Delta = \mathrm{Gal}(\mathbb{Q}(\mu_p)^c/\mathbb{Q}^c) \cong (\mathbb{Z}/p\mathbb{Z})^\times$. In particular, setting $X = X_{\mathbb{Q}(\mu_p)}$, we have

$$X(i)_{\Gamma_p \times \Delta} = (X(i)_\Delta)_{\Gamma_p} \cong X_{-i}(i)_{\Gamma_p}$$

where X_{-i} denotes the ω^{-i}-component of X where $\omega : \Delta \to \mathbb{Z}_p^\times$ is the Teichmüller character. Now by Nakayama's lemma

$$X_{-i}(i)_{\Gamma_p} = 0 \Leftrightarrow X_{-i}(i) = 0 \Leftrightarrow X_{-i} = 0$$

Since $X_{-i} = 0$ if and only if $X_{-j} = 0$, then $WK_{2i}^{\acute{e}t}(\mathbb{Q}) = 0$ if and only if $WK_{2j}^{\acute{e}t}(\mathbb{Q}) = 0$. In order to prove the second assertion it will be enough to prove that $X_{-i} = 0$ if and only if $A_{-i} = 0$. But this comes again from Nakayama's lemma since

$$A_{-i} = 0 \Leftrightarrow (X_{-i})_{\Gamma_p} = 0 \Leftrightarrow X_{-i} = 0$$

□

Remark 4.2.2. Incidentally Proposition 4.2.1 gives, together with a result of Banaszak, a proof using wild kernels of the celebrated result of Kurihara ([Ku]) about sufficient conditions for the Vandiver conjecture to be true: more precisely Kurihara proved that, for every $n \in \mathbb{N}$,

$$K_{4n}(\mathbb{Z})_p = 0 \Rightarrow (A_1)_{-2n} = 0 \tag{4.7}$$

(recall that the Vandiver conjecture predicts precisely that $(A_1)_{-2n} = 0$ for every n). Now, in order to prove (4.7) by means of Proposition 4.2.1, we have to check that if $K_{4n}(\mathbb{Z})_p = 0$, then $WK_{4n}^{\acute{e}t}(\mathbb{Q}) = 0$. We know that $WK_{4n}^{\acute{e}t}(\mathbb{Q})$ is isomorphic to $(\mathrm{div}(K_{4n}(\mathbb{Q})))_p$ (see [Ba], Theorem 3), namely

$$WK_{4n}^{\acute{e}t}(\mathbb{Q}) \cong \left(\bigcap_{r\in\mathbb{N}} K_{4n}(\mathbb{Q})^r \right)_p = \bigcap_{r\in\mathbb{N}} (K_{4n}(\mathbb{Q})_p)^r = \bigcap_{s\in\mathbb{N}} (K_{4n}(\mathbb{Q})_p)^{p^s}$$

This subgroup is contained in $K_{4n}(\mathbb{Z})_p$ since we have the exact localization sequence

$$0 \to K_{4n}(\mathbb{Z})_p \to K_{4n}(\mathbb{Q})_p \to \bigoplus_{q\neq p} K_{4n-1}(\mathbb{F}_q)_p \to 0$$

and, for every prime q, $K_{4n-1}(\mathbb{F}_q)_p$ is a finite (cyclic) group. Hence

$$K_{4n}(\mathbb{Z})_p = 0 \Rightarrow WK_{4n}^{\acute{e}t}(\mathbb{Q}) \Leftrightarrow (A_1)_{-2n} = 0$$

Example. Take $F = \mathbb{Q}$: note that $F_n = \mathbb{Q}(\mu_{p^n})$ and $\Gamma_n = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$. Moreover, for every $n \in \mathbb{N}$, $(Cl'_{\mathbb{Q}(\mu_{p^n})})_p = (Cl_{\mathbb{Q}(\mu_{p^n})})_p = A_n$. For every $n \geq 1$ we have

$$\left(A_n \otimes \mu_{p^n}^{\otimes i}\right)_{\Gamma_n} = \left(\left(A_n \otimes \mu_{p^n}^{\otimes i}\right)_{\Delta_n} \right)_{\mathrm{Gal}(K_n/\mathbb{Q})}$$

By Nakayama's lemma

$$\left(\left(A_n \otimes \mu_{p^n}^{\otimes i}\right)_{\Delta_n} \right)_{\mathrm{Gal}(K_n/\mathbb{Q})} = 0 \Longleftrightarrow \left(A_n \otimes \mu_{p^n}^{\otimes i}\right)_{\Delta_n} = 0$$

Furthermore

$$\left(A_n \otimes \mu_{p^n}^{\otimes i}\right)_{\Delta_n} \cong (A_n)_{p-1-i}$$

Moreover, as in the second part of the proof of Proposition 4.2.1, it is easy to see that for any $n \geq 1$

$$(A_n)_{p-1-i} = 0 \Longleftrightarrow (A_1)_{p-1-i} = 0$$

Hence the p-localization sequence for $K_{2i}(\mathbb{Q})$ is split if and only if $(A_1)_{p-1-i}$ is trivial. Therefore, by Proposition 4.2.1, the p-localization sequence for $K_{2i}(\mathbb{Q})$ is split if and only if $WK_{2i}^{\acute{e}t}(\mathbb{Q})$ is trivial (see also [Ba], Corollary 2).

The condition that $WK_2(F)_p = 0$ implies the splitting of the p-localization sequence for $K_2(F)$ for a large class of field. To give an example, we first recall a structure result of Keune (a similar assertion can be proved with the results of [Sc], §6).

Theorem 4.2.3. *(Keune) There is an isomorphism*

$$WK_2(F)_p \cong \left(Cl'_{F_r} \otimes \mu_{p^r}\right)_{\Gamma_r}$$

for each r such that $\mu(F_w)_p \subseteq \mu_p^r$ for all $w|p$ in F_r and, furthermore, p^r kills the p-primary part of $K_2(\mathcal{O}_F)$.

Proof. See [Ke], Theorem 6.6. □

Proposition 4.2.4. *Let F be quadratic field. Then for each (odd) prime p, the p-localization sequence for $K_2(F)$ splits if and only if $WK_2(F)_p = 0$.*

Proof. Clearly only one of the implication has to be shown (see Introduction). If $WK_2(F)_p = 0$, then by the well known formula (coming from Moore exact sequence) which expresses the relation between the orders of $K_2(\mathcal{O}_F)$ and $WK_2(F)$ (see for example [Ba]), we have

$$|K_2(\mathcal{O}_F)_p| = \frac{\prod_{v|p} |\mu(F_v)_p|}{|\mu(F)_p|} \tag{4.8}$$

Suppose first that $p \neq 3$: then both the denominator and the numerator are trivial (because F is quadratic). Hence $K_2(\mathcal{O}_F)_p$ is trivial and therefore the p-localization sequence for $K_2(F)$ trivially splits. In the case $p = 3$ and $F = \mathbb{Q}(\mu_3)$, $K_2(\mathcal{O}_F)_3$ is again trivial (by a calculation of Tate, see also [Ke], 3.8) and we conclude as before.
Now suppose that $p = 3$ and $F \neq \mathbb{Q}(\mu_3)$. Then the denominator of (4.8) is again trivial. Clearly the numerator must be a divisor of 3 which implies that 3 kills $K_2(\mathcal{O}_F)_3$. Since again $\mu(F_v)_3 \subseteq \mu_3$ for each $v|3$ in F, it follows from Theorem 4.2.3 that for each $r \geq 1$ we have

$$\left(Cl'_{F_r} \otimes \mu_{3^r}\right)_{\Gamma_r} \cong WK_2(F)_3 = 0$$

Hence by Theorem 4.1.8 we see that the p-localization sequence for $K_2(F)$ splits. □

Anyway in general the condition $WK_{2i}^{ét}(F)_p = 0$ is weaker than the condition of Theorem 4.1.8, as we will show in the next example. First we need the following criterion.

Proposition 4.2.5. *Let $F/\mathbb{Q}$ be finite Galois extension such that*

- $\mu_p \subseteq F$;
- $(Cl'_F)_p \cong \mathbb{Z}/p\mathbb{Z}$;
- *every prime over p in $F(\mu_{p^2})/F$ is totally split.*

Then $WK_2(F)_p$ is trivial but the p-localization sequence for $K_2(F)$ does not split (and the same holds for the p-localization sequence for $K_{2i}(F)$, $i \geq 1$).

Proof. We are going to use the language and the results developed in [Ja3]. For any $F/\mathbb{Q}$ finite and Galois (even not satisfying the hypotheses) we have an exact sequence (see [DS], §3)

$$0 \to \widetilde{Cl}_F(p) \to \widetilde{Cl}_F \xrightarrow{\varphi} (Cl'_F)_p \longrightarrow \deg_F \mathcal{D}\ell/(\deg_F \mathfrak{p})\mathbb{Z}_p \to 0$$

where $\mathfrak{p}$ is any prime of F over p. Moreover

$$\deg_F \mathcal{D}\ell = p[F \cap \widehat{\mathbb{Q}}^c : \mathbb{Q}]\mathbb{Z}_p \tag{4.9}$$

where $\widehat{\mathbb{Q}}^c$ is the cyclotomic $\widehat{\mathbb{Z}}$-extension of $\mathbb{Q}$ and

$$\deg_F \mathfrak{p} = \tilde{f}_{\mathfrak{p}} \cdot \deg p = [F_{\mathfrak{p}} \cap \widehat{\mathbb{Q}}_p^c : \mathbb{Q}_p] \cdot p$$

where $\widehat{\mathbb{Q}}_p^c$ is the compositum of the $\mathbb{Z}_q$-extensions of $\mathbb{Q}_p$ for every rational prime q. Now we want to compare $[F \cap \widehat{\mathbb{Q}}^c : \mathbb{Q}]$ and $[F_{\mathfrak{p}} \cap \widehat{\mathbb{Q}}_p^c : \mathbb{Q}_p]$. Suppose that $v_p([F \cap \widehat{\mathbb{Q}}^c : \mathbb{Q}]) = t$ and that the first stage F_1 of the cyclotomic $\mathbb{Z}_p$ extension of F is totally split at every prime $\mathfrak{p}$ above p: this means that $(F_1)_{\mathfrak{p}} = F_{\mathfrak{p}}$. In particular $v_p([F_{\mathfrak{p}} \cap \widehat{\mathbb{Q}}_p^c : \mathbb{Q}_p]) \geq t+1$. In other words

$$\deg_F \mathcal{D}\ell/(\deg_F \mathfrak{p})\mathbb{Z}_p \cong \mathbb{Z}/p^s\mathbb{Z}$$

with $s \geq 1$. Therefore, if $(Cl'_F)_p \cong \mathbb{Z}/p\mathbb{Z}$, we have $\widetilde{Cl}_F = 0$. Since $\mu_p \subseteq F$, we can use the isomorphism (see for example [Ja2])

$$\mu_p \otimes \widetilde{Cl}_F \cong WK_2(F)/pWK_2(F)$$

to deduce that $WK_2(F)_p = 0$. On the other hand $(Cl'_F)_p$ is non trivial, hence Proposition 4.1.10 (or Theorem 4.1.8) tells us that the p-localization sequence for $K_2(F)$ does not split. $\square$

Example. (Computations are performed using the PARI package, [PA]).
We have to find a field satisfying the hypotheses of Proposition 4.2.5. We proceed as follows: we take $p = 3$ and we choose a prime ℓ such that $\ell \equiv 1 \ (mod\ 3)$: this ensures that $\mathbb{Q}(\zeta_\ell)$ has exactly one subextension of degree 3 which we call E. Let K be the subextension of degree 3 of $\mathbb{Q}(\mu_9)$: then EK is an abelian number field whose Galois group is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. Now, if the order of 3 modulo ℓ is not divisible by 3, then E has to be totally split at 3. In particular, if $F' \neq K$ is any of the subextension of degree 3 of EK, then EK/F' is totally split at 3. We may then choose $F = F'(\mu_3)$: then the first and the third hypotheses of Proposition 4.2.5 are satisfied. So we are left to find such an ℓ with the additional requirement that $(Cl'_F)_3$ is cyclic of order 3.
Choose $\ell = 61$: of course we have $61 \equiv 1 \ (mod\ 3)$ and 3 has order 10 modulo 61. Choose F' the subextension of EK defined by the polynomial $X^3 - 183X - 783$ (one can check that F' has conductor $3^2 \cdot 61$): then

$F = F'(\mu_3) = \mathbb{Q}(\theta)$ where θ is a root of the polynomial $X^6 - 793X^3 + 226981$. There is only one (totally ramified) prime above 3 in F and it is principal. Computations give $Cl_F \cong \mathbb{Z}/39\mathbb{Z}$ and then $(Cl'_F)_3 \cong \mathbb{Z}/3\mathbb{Z}$. Then by Proposition 4.2.5, we deduce that $WK_2(F)_3 = 0$ but the 3-localization sequence for $K_2(F)$ does not split and $K_2(\mathcal{O}_F)_3 \cong \mathbb{Z}/3\mathbb{Z}$.

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