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Mechanical models for 2D fiber networks and textiles

Modèles mécaniques de réseaux de fibres 2D et de textiles

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Introduction

Motivation and main results

The mechanics of textile materials has been extensively studied in the past 50 years, but it still provides a wealth of interesting problems both for modelization and mathematical analysis. The availability of nonstandard materials, such as smart materials, the increased computational capabilities, also related to computer graphics, have recently spurred a renewed interest in basic theoretical research in this area. Moreover, the explosive increase of research on biological tissues and materials has provided a new field to which the mechanics of fiber-reinforced materials, initially developed for textiles, can be successfully applied.

In this work we discuss three basic problems related to the mechanical behavior of textile materials.

First, we extend the model of Wang and Pipkin for textiles, described as networks of inextensible fibers with resistance to shear and bending, to a model in which resistance to twist of the individual fibers is taken into account, by including torsion terms in the elastic stored energy.

Second, we study how the geometry of the weave pattern affects the symmetry properties of the elastic and bending energy of a woven fabric. For networks made by two families of fibers, four basic types of weave patterns are possible, in dependence of the angle between the fibers and their material properties. The symmetry properties of the pattern determine the material symmetry group of the network, under which the stored energy is invariant. In this context, we derive representations for the elastic and bending energy of a woven fabric that are invariant under the symmetry group of the network, and discuss the relation of the resulting group invariants with the curvature of the fibers.

Third, we develop a model for textiles viewed as surfaces with microstructure, using a modification of the classical Cosserat model for shells, in which the microstructure accounts for the undulations of the threads at the microscopic scale. Describing the threads as Euler's elasticas, we derive an explicit expression for the microscopic elastic energy that allows to derive a simple model for the macroscopic mechanical behavior of textiles.

Textiles

In textile fabrics different scales can be identified: the macroscopic scale, corresponding to the textile proper; an intermediate scale, at which the undulation of the yarn (or thread) can be distinguished; and the microscopic scale, that corresponds to the yarn and its constituents: the fibers. We remark that, in most of this work, the term fiber is used as a synonym of yarn or thread, while in the technical literature the fiber is a microscopic constituent of a thread.

The techniques used to manufacture a textile can produce different structures, such as knitted materials or woven textiles. A knitted textile is made of a single yarn that forms a loop into a previous loop formed by the yarn itself. A woven textile is formed by the interlace of two families of yarns: the weft and the warp.

Basic mathematical models for textiles

The first mathematical model for a woven material was proposed by Peirce [**49**] at the beginning of the twentieth century. This model essentially describes the geometrical structure of a plain weave, namely a tissue in which warp and weft interlace alternately. The yarns forming the textile were described as flexible, circular cylinders, interlaced together in regular patterns to form the fabric.

Another model that focuses on the geometrical description of the structure of the textile, describing some of its mechanical features, is due to Kawabata [**32**]. In order to simplify the description of the geometrical structure of the weave, he neglects the undulation of the yarns. This model takes into account both the biaxial and uniaxial tensile properties, such as the shear properties of a plain weave. The compressibility of the yarn under the action of lateral compressive forces is discussed, and it is shown that the compressive properties of yarn have a great influence on the tensile properties of the fabric.

In order to model tissues from a mechanical point of view, several attempts have been made to investigate the mechanical properties of woven fabrics and to describe the relationship between the force exerted on a textile material and its deformation. One of the first attempts was due to Olofsson in 1964 [**44**], who proposed a mechanical model that extends the model of Peirce, incorporating the bending stiffness of the yarns.

Another important model was devised by Grosberg and Kedia [**19**], who studied the load-extension modulus of a cloth, showing that it depends not only on the bending modulus of the yarn and its geometry in the cloth, but also on the history of deformation of the fabric. They found that fabrics which still retain their stressed condition, which arose when the cloth was made, have a much higher modulus. Many other models have been proposed after them, and the literature in this field is still growing.

For the sake of completeness, we mention a third family of models, still different from the above, the so-called energetic models, of which we only discuss the one due to De Jong and Postle [**14**], see also Aimene et al. [**2**]. According to these authors, the uniaxial extension curve of the crimped thread is related to three distinct deformation mechanisms:
I- the loss of textile weave crimp or yarn undulation (at the macrolevel);
II- the loss of the undulations of the threads inside the fabric (at the mesoscopic level);
III- the extension of the yarn,
and they were able to develop a formula for the energy that accounts for this behavior.

We finally mention the so-called microstructural models: these constitute a macroscopic approach to textiles that treats the tissue as a deformable surface as a whole, and accounts for the influence of the fine structure through additional, microstructural fields. The resulting models are able to capture those characteristics of the fine structure of a textile that determine its response, for instance the stress-strain curve. For example, the model proposed by Magno, Ganghoffer and others [**39**], [**38**], is principally related with the second of the extension mechanisms discussed by De Jong and Postle, and allows to determine the axial deformation and the axial stress.

The existing models for inextensible networks

We briefly review below some popular theories that describe the mechanical behavior of cloth and cable networks as networks of inextensible fibers.

In 1955, Rivlin [**59**] proposed a theory of networks formed by inextensible cords. He considered the mechanics of a plane net made by two families of parallel inextensible cords, and obtained general solutions of traction boundary value problems.

Later, between 1980 and 1986, in a series of papers, Pipkin, [**54**] [**52**] [**51**] [**50**], further developed the theory of inextensible networks.

In 1986, Wang and Pipkin, [**66**] [**67**], formulated a theory of inextensible nets with bending stiffness. The resulting continuum theory is a special case of finite-deformation plate theory, in which each fiber has a bending couple proportional to its curvature.

In 2001, a theory of bending and twisting effects in three-dimensional deformations of an inextensible network was presented by Luo and Steigmann [**37**]. They derived the Euler-Lagrange equations and boundary conditions by a minimum-energy principle.

Many other models have been presented in the literature, among which we only mention Boisse et al. [**9**].

Some open problems

The mechanics of textile materials is an active and fast growing research field, and some of the open problems in this area are related to the predictive capabilities of the existing models, a problem that can actually be addressed nowadays due to the increased efficiency of numerical methods, specifically finite element methods, for surfaces. Hence, a large body of research is devoted to numerical simulations, which pose interesting problems due to the fact that the limited resistance to bending of textiles allows large deformations and wrinkling.

In parallel, many basic theoretical issues in the mechanics of textile surfaces are still open. For instance, the availability of new materials, such as shape-memory alloys, electro-active polymers, as well as biological tissues, requires a careful rethinking of the basic models for surfaces made by networks of fibers, to account for the non standard behavior of the fibers themselves.

On the other hand, also in view of developing numerically tractable models, simple macroscopic models are needed, which however retain the basic information on the fine-scale structure. This class of models, which we refer to as microstructural models, has a long history, but their application to textiles formed by deformable fibers is quite recent, and the difficulty here is related to the description of the large deformations, and wrinkling, characteristic of textiles.

Main results

The basic original results of this work are related to some of the open problems discussed above.

First, we focus on an extension of the model of a textile as a network of inextensible fibers proposed by Pipkin and Wang. These authors assume that the resistance to bending of each fiber of the network, which is intended to model the thread of a woven fabric, determines the global response of the fabric. We extend this approach to cover materials formed by fibers that resist not only to bending, i.e., changes of the curvature, but also to variations of the torsion. Such materials therefore have a higher rigidity than those with bending stiffness only, and their interest lies in the possibility of designing cylindrical structures, such as hoses or artificial blood vessels, made of helical fibers whose stress- free state is helical. Precisely, in this context, we extend the approach of Wang and Pipkin and derive the basic PDEs of a model of a surface formed by fibers with resistance to twisting, and discuss the constitutive theory of such models.

Second, we discuss the issue of material symmetry for surfaces made by networks of fibers. The problem here is to characterize the restrictions on the stored energy function due to the geometry of the network, namely the angle between the fibers and their interchangeability, which is in turn related to the difference in their material properties. When the weave pattern is simple (alternating intersections between the fibers) four basic structures for the network are possible: the square structure (orthogonal equivalent fibers), the rectangular structure (orthogonal unequivalent fibers), the rhombic structure (non-orthogonal equivalent fibers), the parallelogram structure (non-orthogonal unequivalent fibers). Each of these structures is characterized by its peculiar symmetry group. We extend to our case a technique, based on the so-called Rhychlewski's theorem, that allows to explicitly compute the basic invariants of the action of the symmetry groups and, by consequence, the general form of an invariant stored energy density. Since Rhychlewski's theorem only allows to compute the invariants of the rectangular and parallelogram structures, we prove a generalization of this result that allows to deal with other, more general symmetries of fibered networks. Finally, we apply this result to compute all invariants that depend up to the third derivatives of the deformation, and obtain the general form of an invariant energy depending on the shear between the fibers, and their curvature and torsion.

Third, we develop a macroscopic model of a textile fabric that accounts for some of the fine-scale features of the crimp of the fibers. Namely, in the context of the classical theory of materials with microstructure, we assume that a textile can be characterized as a network of inextensible fibers, just as in Wang and Pipkin, but with additional director fields, parallel to the fibers, whose modulus is proportional to the curvature of the threads at the microscopic scale. Using a variational principle we derive the partial differential equations of the model, and discuss a simple example.

Future research

Current and future research problems related to the thesis regard essentially topics 1 and 3 above. Specifically, it will be interesting to characterize explicitly cylindrical structures formed by fibers with resistance to twisting, in order to study necking and global bending effects on the cylinder itself.

Also, it will be important to characterize specific constitutive laws for the microstructural model, in order to study the regularizing effect of the microstructure itself on the wrinkling of fibers.

Finally, a very important outcome of this work would be a set of numerical simulations of the above models, showing consistency with experiments.

Contents

The thesis is composed of seven chapters and two appendices. The first four chapters deal with standard results and have a bibliographical character, while Chapters 5,6,7 present three different aspects of our model. Below is a short description of the contents.

In Chapter 1 we recall some standard results on differential geometry of surfaces in three dimensional space. In particular we focus on the definition and properties of the Weingarten map and its relation with the Mean and Gaussian curvature.

In Chapter 2 we introduce the notation and the basic assumption of our model; we focus on a surface made by two families of inextensible fibers and describe the kinematics of the resulting textile material. The results of Chapter 1 are applied in order to evaluate the Weingarten matrix for a fibered surface as well as the Gaussian and the Mean curvature.

In Chapter 3 we briefly describe two basic models for woven fabrics proposed in the literature. The models are somewhat paradigmatic of two opposite approaches to the study of mechanics of textiles: the first of the two, due to Peirce [**49**], in fact, is the very first attempt at a mathematical description of the behavior of a plain weave, through the elementary description of the mechanical behavior of the single thread forming the textile. The second model we describe here, proposed by Magno, Ganghoffer and others [**39**], [**38**], on the contrary, is a macroscopic approach that treats the tissue as a deformable surface as a whole, and accounts for the influence of the fine structure through additional, microstructural fields. The resulting model is able to capture those characteristics of the fine structure of a textile, that determine its response, for instance the stress strain curve.

In Chapter 4 we present a short review of the existing theory of inextensible networks. We focus on the work of Wang and Pipkin [**66**] and Luo and Steigmann [**37**]. Both models describe the mechanical behavior of a sheet composed by two families of inextensible fibers. In the first one the resistance to shear and the bending stiffness are taken into account, while in the second one the resistance to twist is added.

In Chapter 5 we present our model for two families of inextensible fibers forming a surface, published in [**29**]. The bending stiffness and the resistance to torsion of the fibers are taken into account, in order to describe the static behavior of textile fabric. We first consider a strain energy density in additive form, such that the contributions due to shear, torsion and bending effects are taken into account separately, and then generalize the result to an arbitrary dependence on curvature and torsion.

In the first part of Chapter 6 we present a short review of the usual approach to the constitutive theory of fiber-reinforced materials (Holzapfel [**26**]). First we recall some ideas on transversely isotropic materials, then we focus on materials made of two families of fibers, discussing the expression that the free energy assumes as a function of suitable set of invariants under the action of the material symmetry group of the net (Spencer [**64**], Smith and Rivlin [**63**], Green and Adkins [**17**], Zhang and Rychlewski [**70**]). In the second part of this chapter we generalize the approach based on invariants to determine the expression of the free energy of a textile made of two families of intersecting yarns.

The aim of Chapter 7 is to determine the equations that describe the effect of shear, bending and microundulations of the fibers on the deformation of the sheet. The field equations are obtained via a variational principle for a strain energy density in additive form, such that the contributions due to shear, bending and microundulations are separately taken into account. In order to model the macroscopic effects of the microundulations, we associate to each family of fibers a vectorial microstructure.

Finally, the first appendix discusses the equivalence between the constitutive relations resulting from a variational approach based on curvature and torsion, and an approach based on the derivatives of the deformation.

The second appendix is a glossary of technical terms related to textile fabrics.

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CHAPTER 1

Differential geometry of surfaces

Contents

In this chapter we recall some basic ideas on the differential geometry of surfaces. We refer to the book of Thorpe [**65**].

Although many of the results below are also valid in $\mathbb{R}^n$, we focus on surfaces in $\mathbb{R}^3$ because we want to apply these ideas to the study of sets of fibers in the three-dimensional space.

1.1. Differentiable surfaces in $\mathbb{R}^3$

Definition 1.1.1. A subset $\mathcal{S}$ of $\mathbb{R}^3$ is a **surface** if, for every point $\boldsymbol{x}$ of $\mathcal{S}$, there is an open set V in $\mathbb{R}^2$ and an open set W in $\mathbb{R}^3$ containing $\boldsymbol{x}$ such that $\mathcal{S} \cap W$ is homeomorphic to V.

The homeomorphisms $\boldsymbol{r}$ from V to $\mathcal{S} \cap W$ are called local **parametrizations** of $\mathcal{S}$.

An **atlas** on $\mathcal{S}$ is a collection of open sets $\{V_i |\ i \in I\}$ (with I set of indices), together with homeomorphisms $\boldsymbol{r}_i : V_i \to \mathbb{R}^3$ such that $\boldsymbol{r}_i(V_i)$ cover $\mathcal{S}$, and satisfy the following compatibility condition: for any pair $(V_i, \boldsymbol{r}_i)$, $(V_j, \boldsymbol{r}_j)$ the transition functions

$$\boldsymbol{r}_i \circ \boldsymbol{r}_j^{-1} : \boldsymbol{r}_j(V_i \cap V_j) \to \boldsymbol{r}_i(V_i \cap V_j)$$

and

$$\boldsymbol{r}_j \circ \boldsymbol{r}_i^{-1} : \boldsymbol{r}_i(V_i \cap V_j) \to \boldsymbol{r}_j(V_i \cap V_j)$$

are homeomorphisms between open subsets of euclidean space.

Given a map $\boldsymbol{r}$ from V to $\mathbb{R}^3$, and cartesian coordinates (X^1, X^2) in $V \subseteq \mathbb{R}^2$, and (x^1, x^2, x^3) in $\mathbb{R}^3$, we write

$$\boldsymbol{r}(X^1, X^2) = (r^1(X^1, X^2), r^2(X^1, X^2), r^3(X^1, X^2))$$

we use in the sequel the following notation for the derivatives

$$\frac{\partial \boldsymbol{r}}{\partial X^i} = \boldsymbol{r}_{,i}\ , \qquad i = 1, 2.$$

Definition 1.1.2. A local parametrization $\boldsymbol{r} : V \to \mathbb{R}^3$ is called **regular** if it is smooth and the vectors $\boldsymbol{r}_{,1}$ and $\boldsymbol{r}_{,2}$ are linearly independent for all (X^1, X^2) in V.

Equivalently:

Definition 1.1.3. A local parametrization $\boldsymbol{r} : V \to \mathbb{R}^3$ is called **regular** if $\boldsymbol{r}$ is smooth and the vector product $\boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2}$ is non zero at every point of $\boldsymbol{r}(V) \subseteq \mathbb{R}^3$.

We also have the following equivalent characterization:

Definition 1.1.4. A local parametrization $\boldsymbol{r} : V \to \mathbb{R}^3$ is called **regular** if r^1, r^2, r^3 are differentiable and the Jacobian matrix

$$J(r^1(X^1, X^2), r^2(X^1, X^2), r^3(X^1, X^2)) = \begin{bmatrix} \dfrac{\partial r^1}{\partial X^1} & \dfrac{\partial r^1}{\partial X^2} \\[2ex] \dfrac{\partial r^2}{\partial X^1} & \dfrac{\partial r^2}{\partial X^2} \\[2ex] \dfrac{\partial r^3}{\partial X^1} & \dfrac{\partial r^3}{\partial X^2} \end{bmatrix}$$

has maximal rank at every point (X^1, X^2) of V.

1.2. Tangent vectors

Consider a smooth curve $\boldsymbol{c}_0$ in $\mathbb{R}^2$, with equation $(X^1(t), X^2(t))$ and its image

$$\boldsymbol{c}(t) = \boldsymbol{r}(X^1(t), X^2(t)),$$

which is a curve on $\mathcal{S} \subset \mathbb{R}^3$.

DEFINITION 1.2.1. A vector $\boldsymbol{v} \in \mathbb{R}^3$ is said to be **tangent** to the surface $\mathcal{S}$ at a point $\boldsymbol{x}_0$ if there exists a curve $\boldsymbol{c}$ on $\mathcal{S}$, passing through $\boldsymbol{x}_0$, such that $\boldsymbol{v}$ is tangent to $\boldsymbol{c}$ in $\boldsymbol{x}_0$ i.e. $\boldsymbol{v} = \dfrac{d\boldsymbol{c}}{dt}$.

The set of tangent vectors at $\boldsymbol{x}_0$ is called the **tangent space** $T_{\boldsymbol{x}_0}\mathcal{S}$ to the surface $\mathcal{S}$ at $\boldsymbol{x}_0$.

PROPOSITION 1.2.2. *The set of the tangent vectors to a surface $\mathcal{S}$ at a point $\boldsymbol{x}_0 = \boldsymbol{r}({X^1}_0, {X^2}_0)$ is a 2-dimensional linear space; the vectors $\boldsymbol{r}_{,1}({X^1}_0, {X^2}_0)$ and $\boldsymbol{r}_{,2}({X^1}_0, {X^2}_0)$ are a basis of this space.*

DEFINITION 1.2.3. Given a point $\boldsymbol{x}_0$ on the surface $\mathcal{S}$, the unit vector

$$\boldsymbol{N}({X^1}_0, {X^2}_0) = \frac{\boldsymbol{r}_{,1}({X^1}_0, {X^2}_0) \times \boldsymbol{r}_{,2}({X^1}_0, {X^2}_0)}{||\boldsymbol{r}_{,1}({X^1}_0, {X^2}_0) \times \boldsymbol{r}_{,2}({X^1}_0, {X^2}_0)||} \tag{1.2.1}$$

is the **unit normal** of the surface at $\boldsymbol{x}_0$.

$\boldsymbol{N}$ is not independent of the choice of local parametrization $\boldsymbol{r}$. In fact, if $\widetilde{\boldsymbol{r}} : V \to \mathbb{R}^3$ is another local parametrization in the atlas of $\mathcal{S}$ containing $\boldsymbol{x}_0$, it is possible to show that

$$\widetilde{\boldsymbol{r}}_{,1} \times \widetilde{\boldsymbol{r}}_{,2} = \det(J)\, \boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2}$$

where J is the jacobian matrix of the transition map from $\boldsymbol{r}$ to $\widetilde{\boldsymbol{r}}$. It follows that the unit normal corresponding to $\widetilde{\boldsymbol{r}}$ is parallel to $\boldsymbol{N}$, but the orientation may vary according to the sign of the determinant of J.

DEFINITION 1.2.4. An **orientable** surface is a surface with an atlas having the property that, for any transition function Φ between any two surface patches in the atlas, then $\det(J(\Phi)) > 0$.

Finally, we recall the classical divergence theorem for vector fields tangent to a surface.

Proposition 1.2.5. *Let $\boldsymbol{f}$ be a smooth tangential vector field on the surface $\mathcal{S}$, i.e., $\boldsymbol{f}(X^1, X^2) \in T_{\boldsymbol{r}(X^1,X^2)}\mathcal{S}$ for $(X^1, X^2) \in V \subset \mathbb{R}^2$ or, equivalently, $\boldsymbol{f} = f_1\boldsymbol{r}_{,1} + f_2\boldsymbol{r}_{,2}$. Then for any $\mathcal{P} \subset V$ with smooth boundary,*

$$\int_{\partial\mathcal{P}} (f_1 dX^2 - f_2 dX^1) = \int_{\mathcal{P}} \left(\frac{\partial f_1}{\partial X^1} + \frac{\partial f_2}{\partial X^2} \right) dX^1 dX^2. \tag{1.2.2}$$

1.3. The first fundamental form

For a curve $\boldsymbol{c}(t) = \boldsymbol{r}(X^1(t), X^2(t))$ on $\mathcal{S}$, the arc-length is given by

$$s = \int_{t_0}^{t} ||\dot{\boldsymbol{c}}(z)|| dz$$

(where the overwritten dot denotes the derivative with respect to t). Letting

$$\begin{aligned} E &= E(X^1, X^2) = \boldsymbol{r}_{,1} \cdot \boldsymbol{r}_{,1} \\ F &= F(X^1, X^2) = \boldsymbol{r}_{,1} \cdot \boldsymbol{r}_{,2} \\ G &= G(X^1, X^2) = \boldsymbol{r}_{,2} \cdot \boldsymbol{r}_{,2} \end{aligned} \tag{1.3.1}$$

where E, F, G are functions that depend only on the surface and not on the curve $\boldsymbol{c}$ on the surface, we find

$$||\dot{\boldsymbol{c}}||^2 = (\boldsymbol{r}_{,1}\ \dot{X}^1 + \boldsymbol{r}_{,2}\ \dot{X}^2) \cdot (\boldsymbol{r}_{,1}\ \dot{X}^1 + \boldsymbol{r}_{,2}\ \dot{X}^2) = E\ (\dot{X}^1)^2 + 2F\ \dot{X}^1\dot{X}^2 + G\ (\dot{X}^2)^2$$

so that the arc-length becomes

$$s = \int_{t_0}^{t} \left[E\ (\dot{X}^1)^2 + 2F\ \dot{X}^1\dot{X}^2 + G\ (\dot{X}^2)^2 \right]^{\frac{1}{2}} dt.$$

This equation sometimes is written as

$$ds^2 = E\ (dX^1)^2 + 2F\ dX^1 dX^2 + G\ (dX^2)^2. \tag{1.3.2}$$

The expression on the right hand side of the equation above is called **the first fundamental form** of the surface.

Notice that if

$$\boldsymbol{v} = v_1\ \boldsymbol{r}_{,1}({X^1}_0, {X^2}_0) + v_2\ \boldsymbol{r}_{,2}({X^1}_0, {X^2}_0)$$

and

$$\boldsymbol{w} = w_1\, \boldsymbol{r}_{,1}({X^1}_0, {X^2}_0) + w_2\, \boldsymbol{r}_{,2}({X^1}_0, {X^2}_0)$$

are two tangent vectors to the surface $\mathcal{S}$ at a point $\boldsymbol{x}_0 = \boldsymbol{r}({X^1}_0, {X^2}_0)$, we have

$$\boldsymbol{v} \cdot \boldsymbol{w} = E\, v_1 w_1 + F\, (v_1 w_2 + v_2 w_1) + G\, v_2 w_2$$

so that the scalar product is defined on the tangent space $T_{\boldsymbol{x}_0}\mathcal{S}$ by the first fundamental form.

In other words, E, F, G are the coefficients of the metric tensor $g_{i,j} \equiv \boldsymbol{r}_{,i} \cdot \boldsymbol{r}_{,j}$ on the tangent space $T_{\boldsymbol{x}}\mathcal{S}$, as expressed by (1.3.1).

1.4. Surface area

Given a surface with local parametrization

$$r^1 = r^1(X^1, X^2) \qquad r^2 = r^2(X^1, X^2) \qquad r^3 = r^3(X^1, X^2)$$

for (X^1, X^2) in a region V of $\mathbb{R}^2$, we would like to evaluate the area of the part of surface delimited by the coordinate curves $X^1 = X^1_0$, $X^1 = X^1_1$, $X^2 = X^2_0$, $X^2 = X^2_1$. (Figure 1.1)

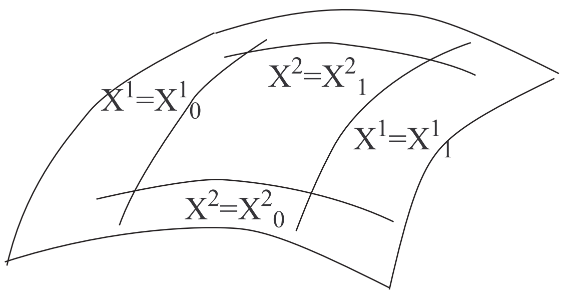

FIGURE 1.1. The area on a surface

First notice that the area $d\sigma$ of the infinitesimal parallelogram bounded by the infinitesimal vectors $\boldsymbol{r}_{,1}\, dX^1$, $\boldsymbol{r}_{,2}\, dX^2$ is

$$d\sigma = ||\boldsymbol{r}_{,1}\, dX^1 \times \boldsymbol{r}_{,2}\, dX^2|| = ||\boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2}|| dX^1 dX^2\ ,$$

which motivates the following definition:

Definition 1.4.1. The area A of the portion of $\mathcal{S}$ delimited by the above parameter curves is

$$A = \int_{X^1{}_0}^{X^1{}_1} \int_{X^2{}_0}^{X^2{}_1} ||\boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2}|| \, dX^1 dX^2.$$

Using the identity $||(\boldsymbol{v} \times \boldsymbol{w})||^2 = (\boldsymbol{v} \cdot \boldsymbol{v})(\boldsymbol{w} \cdot \boldsymbol{w}) - (\boldsymbol{v} \cdot \boldsymbol{w})^2$ we obtain

$$||\boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2}|| = (EG - F^2)^{\frac{1}{2}}$$

so that the area can be expressed as a function of E, F, G

$$A = \int_{X^1{}_0}^{X^1{}_1} \int_{X^2{}_0}^{X^2{}_1} (EG - F^2)^{\frac{1}{2}} dX^1 dX^2.$$

1.5. The Weingarten map

Let $\boldsymbol{v}$ be a tangent vector on $\mathcal{S}$ at $\boldsymbol{x}_0$. Recalling definition (1.2.1), there exists a curve $\boldsymbol{c}(t) = (X^1(t), X^2(t))$ on the surface $\mathcal{S}$ that passes through $\boldsymbol{x}_0$ at t_0, and such that $\boldsymbol{v}$ is tangent to the curve at $\boldsymbol{x}_0$. The derivative

$$\frac{d}{dt}\boldsymbol{N}(X^1(t), X^2(t))_{|t=t_0} = \boldsymbol{N}_{,1}(X^1{}_0, X^2{}_0)\frac{dX^1}{dt}(t_0) + \boldsymbol{N}_{,2}(X^1{}_0, X^2{}_0)\frac{dX^2}{dt}(t_0) \tag{1.5.1}$$

is a vector that describes the **rate of change of the normal vector** along the curve $\boldsymbol{c}$ in a neighborhood of $\boldsymbol{x}_0$.

If

$$\boldsymbol{v} = v_1 \boldsymbol{r}_{,1} + v_2 \boldsymbol{r}_{,2} \tag{1.5.2}$$

then

$$\frac{dX^1}{dt}(t_0) = v_1 \qquad \frac{dX^2}{dt}(t_0) = v_2$$

and (1.5.1) becomes

$$\frac{d}{dt}\boldsymbol{N}(X^1(t), X^2(t))_{|t=t_0} = v_1 \boldsymbol{N}_{,1}(X^1{}_0, X^2{}_0) + v_2 \boldsymbol{N}_{,2}(X^1{}_0, X^2{}_0) =: \nabla_{\boldsymbol{v}} \boldsymbol{N}; \tag{1.5.3}$$

hence, the vector (1.5.1) depends only on $\boldsymbol{v}$ and not on the particular choice of the curve $\boldsymbol{c}$. The vector $\nabla_{\boldsymbol{v}} \boldsymbol{N}$ is the derivative of the vector field $\boldsymbol{N}$ in the direction of the vector $\boldsymbol{v}$.

We have:

DEFINITION 1.5.1. The **Weingarten map** of the surface $\mathcal{S}$ at $\boldsymbol{x}_0$ (cf. Figure 1.2)

$$W_{\boldsymbol{x}_0} \ : \ \boldsymbol{v} \in T_{\boldsymbol{x}_0}\mathcal{S} \longrightarrow W_{\boldsymbol{x}_0}(\boldsymbol{v}) = -\frac{d}{dt}\boldsymbol{N}(X^1(t), X^2(t))_{|t=t_0} = -\nabla_{\boldsymbol{v}}\boldsymbol{N} \in T_{\boldsymbol{x}_0}\mathcal{S}.$$

In fact, notice that for a point $\boldsymbol{x}_0$ of the surface $\mathcal{S}$ and a vector $\boldsymbol{v} \in T_{\boldsymbol{x}_0}\mathcal{S}$, the derivative $\nabla_{\boldsymbol{v}}\boldsymbol{N}$ is tangent to $\mathcal{S}$

$$\nabla_{\boldsymbol{v}}(1) = \nabla_{\boldsymbol{v}}(\boldsymbol{N}\cdot\boldsymbol{N}) = (\nabla_{\boldsymbol{v}}\boldsymbol{N})\cdot\boldsymbol{N} + \boldsymbol{N}\cdot(\nabla_{\boldsymbol{v}}\boldsymbol{N}) = 2(\nabla_{\boldsymbol{v}}\boldsymbol{N})\cdot\boldsymbol{N} = 0$$

and, by consequence, $\nabla_{\boldsymbol{v}}\boldsymbol{N}\perp\boldsymbol{N}$, so that $\nabla_{\boldsymbol{v}}\boldsymbol{N} \in T_{\boldsymbol{x}_0}\mathcal{S}$.

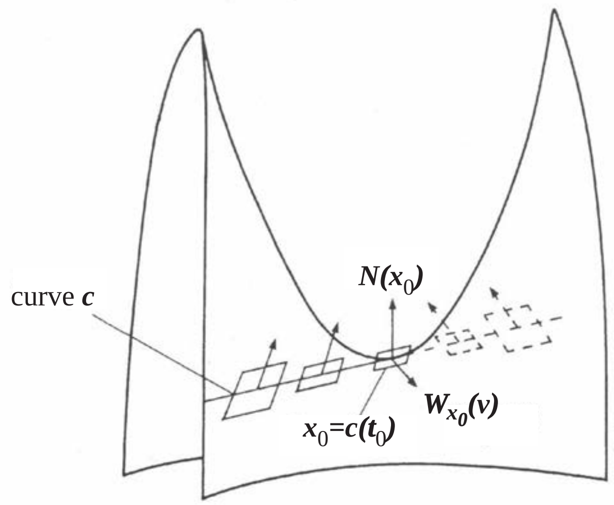

FIGURE 1.2. The Weingarten map on a surface

REMARK 1.5.2. The Weingarten map is a linear map $W_{\boldsymbol{x}_0} \ : \ T_{\boldsymbol{x}_0}\mathcal{S} \longrightarrow T_{\boldsymbol{x}_0}\mathcal{S}$. Moreover

THEOREM 1.5.3. *The Weingarten map is symmetric, i.e.,*

$$W_{\boldsymbol{x}_0}(\boldsymbol{v})\cdot\boldsymbol{w} = \boldsymbol{v}\cdot W_{\boldsymbol{x}_0}(\boldsymbol{w})$$

for all $\boldsymbol{v}, \boldsymbol{w} \in T_{\boldsymbol{x}_0}\mathcal{S}$.

REMARK 1.5.4. If we change atlas on the surface $\mathcal{S}$, the Weingarten map for the new parametrization is the same, unless the sign changes. Moreover, the map changes sign only if the normal vector changes sign.

1.6. Normal curvature of a curve on a surface

Let $\boldsymbol{x}_0$ be a point of $\mathcal{S}$ and $\boldsymbol{c}$ a curve on $\mathcal{S}$ passing through $\boldsymbol{x}_0$. If the curve has equation $\boldsymbol{c}(s) = \boldsymbol{r}(X^1(s), X^2(s))$, where s is the arc length, we define the curvature $\mathcal{K}(s_0)$ and the unit normal vector $\boldsymbol{n}$ to the curve $\boldsymbol{c}$ at $\boldsymbol{x}_0$ through the relation

$$\frac{d^2\boldsymbol{r}}{ds^2}(s_0) = \mathcal{K}(s_0)\boldsymbol{n}. \tag{1.6.1}$$

Definition 1.6.1. The normal curvature k_n of the curve $\boldsymbol{c}$ is the projection (with sign) of $\dfrac{d^2\boldsymbol{r}}{ds^2}$ in the direction of the normal to the surface $\boldsymbol{N}$, viz.

$$k_n = \frac{d^2\boldsymbol{r}}{ds^2} \cdot \boldsymbol{N} = \mathcal{K}\boldsymbol{n} \cdot \boldsymbol{N}.$$

We have:

$$\begin{aligned}\frac{d^2\boldsymbol{r}}{ds} &= \frac{d}{ds}\left(\frac{d\boldsymbol{r}}{ds}\right)\\ &= \boldsymbol{r}_{,11}\left(\frac{dX^1}{ds}\right)^2 + \boldsymbol{r}_{,1}\left(\frac{d^2X^1}{ds^2}\right) + 2\boldsymbol{r}_{,12}\frac{dX^1}{ds}\frac{dX^2}{ds} + \boldsymbol{r}_{,22}\left(\frac{dX^2}{ds}\right)^2 + \boldsymbol{r}_{,2}\left(\frac{d^2X^2}{ds^2}\right)\end{aligned}$$

since $\boldsymbol{r}_{,1}$ and $\boldsymbol{r}_{,2}$ are orthogonal to $\boldsymbol{N}$, the normal curvature becomes

$$k_n = \frac{d^2\boldsymbol{r}}{ds} \cdot \boldsymbol{N} = \big(\boldsymbol{r}_{,11} \cdot \boldsymbol{N}\big)\left(\frac{dX^1}{ds}\right)^2 + 2\big(\boldsymbol{r}_{,12} \cdot \boldsymbol{N}\big)\frac{dX^1}{ds}\frac{dX^2}{ds} + \big(\boldsymbol{r}_{,22} \cdot \boldsymbol{N}\big)\left(\frac{dX^2}{ds}\right)^2$$

and if we let

$$\begin{cases} e &= e(X^1, X^2) &= \boldsymbol{r}_{,11} \cdot \boldsymbol{N} \\ f &= f(X^1, X^2) &= \boldsymbol{r}_{,12} \cdot \boldsymbol{N} \\ g &= g(X^1, X^2) &= \boldsymbol{r}_{,22} \cdot \boldsymbol{N}\ , \end{cases} \tag{1.6.2}$$

then for the curve $\boldsymbol{c}$ above the following proposition holds:

Proposition 1.6.2. *The normal curvature is given by*

$$k_n = e\left(\frac{dX^1}{ds}\right)^2 + 2f\frac{dX^1}{ds}\frac{dX^2}{ds} + g\left(\frac{dX^2}{ds}\right)^2$$

where $(e\,dX^1dX^1 + 2f\,dX^1dX^2 + g\,dX^2dX^2)$ *is called* **the second fundamental form** *of the surface* $\mathcal{S}$.

Notice that if the curve is a function of a parameter t instead of the arc length s, we have

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{dX^1}{dt}\right)^2 + 2F\frac{dX^1}{dt}\frac{dX^2}{dt} + G\left(\frac{dX^2}{dt}\right)^2$$

and the normal curvature becomes

$$k_n = \frac{e\left(\frac{dX^1}{dt}\right)^2 + 2f\frac{dX^1}{dt}\frac{dX^2}{dt} + g\left(\frac{dX^2}{dt}\right)^2}{E\left(\frac{dX^1}{dt}\right)^2 + 2F\frac{dX^1}{dt}\frac{dX^2}{dt} + G\left(\frac{dX^2}{dt}\right)^2}.$$

Notice that the normal curvature of a curve depends only on the tangent vector at $\boldsymbol{x}_0$: in fact $\frac{dX^1}{dt}(t_0)$ and $\frac{dX^2}{dt}(t_0)$ are the component of the tangent vector to the curve in the basis $\boldsymbol{r}_{,1}$, $\boldsymbol{r}_{,2}$. Consequently, we define the **normal curvature** of the surface in the direction of a non zero tangent vector $\boldsymbol{v} \in T_{\boldsymbol{x}_0}\mathcal{S}$

$$k_n(\boldsymbol{v}) = \frac{e(v_1)^2 + 2fv_1v_2 + g(v_2)^2}{E(v_1)^2 + 2Fv_1v_2 + G(v_2)^2}.$$

THEOREM 1.6.3. *The normal curvature is related to the Weingarten map through*

$$k_n(\boldsymbol{v}) = \frac{W_{\boldsymbol{x}_0}(\boldsymbol{v}) \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}$$

so that the eigenvalues k^1, k^2 of the linear application $W_{\boldsymbol{x}_0}$ are a minimum and a maximum for the normal curvature respectively. Moreover, $k^1 = k_n(\boldsymbol{v}_1)$, $k^2 = k_n(\boldsymbol{v}_2)$, where $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are the eigenvectors of $W_{\boldsymbol{x}_0}$. k^1, k^2 are called **principal curvatures** *of the surface, while $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are called* **principal directions**.

DEFINITION 1.6.4. Let k^1, k^2 be the principal curvatures relative to a local parametrization. Then, the **Gaussian curvature** is:

$$K(\boldsymbol{x}_0) = k^1k^2 = \det(W_{\boldsymbol{x}_0})$$

and the **mean curvature** is

$$H(\boldsymbol{x}_0) = \frac{1}{2}(k^1 + k^2) = \frac{1}{2}\mathrm{tr}(W_{\boldsymbol{x}_0}).$$

REMARK 1.6.5. Notice that the Gaussian curvature does not change under reparametrization, while the mean curvature might change its sign. It follows that the Gaussian curvature is well defined for any smooth surface $\mathcal{S}$.

1.7. The Weingarten matrix

Consider the Weingarten map $W_{\boldsymbol{x}_0} : T_{\boldsymbol{x}_0}\mathcal{S} \to T_{\boldsymbol{x}_0}\mathcal{S}$ at $\boldsymbol{x}_0$: we want to evaluate the associated matrix in the basis $(\boldsymbol{r}_{,1}, \boldsymbol{r}_{,2})$. Letting

$$\begin{cases} W(\boldsymbol{r}_{,1}) &= d_{11}\boldsymbol{r}_{,1} + d_{12}\boldsymbol{r}_{,2} \\ W(\boldsymbol{r}_{,2}) &= d_{21}\boldsymbol{r}_{,1} + d_{22}\boldsymbol{r}_{,2} \end{cases} \tag{1.7.1}$$

we want to find the matrix

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}. \tag{1.7.2}$$

From the definition (1.5.1) and from equations (1.5.2),(1.5.3), we have

$$W(\boldsymbol{r}_{,1}) = -\boldsymbol{N}_{,1} \tag{1.7.3}$$

so that

$$W(\boldsymbol{r}_{,1}) \cdot \boldsymbol{r}_{,1} = -\boldsymbol{N}_{,1} \cdot \boldsymbol{r}_{,1} = e.$$

Analogously, it is possible to show that $W(\boldsymbol{r}_{,1}) \cdot \boldsymbol{r}_{,2} = f$, $W(\boldsymbol{r}_{,2}) \cdot \boldsymbol{r}_{,2} = g$. Using equations (1.7.1), we have

$$\begin{cases} e &= d_{11}E + d_{12}F \\ f &= d_{11}F + d_{12}G = d_{21}E + d_{22}F \\ g &= d_{21}F + d_{22}G \end{cases} \tag{1.7.4}$$

or equivalently

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \tag{1.7.5}$$

and the Weingarten matrix is

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}. \tag{1.7.6}$$

Hence, from equations (1.7.6), we find that the Gaussian and mean curvatures become

$$K = \det D = \frac{eg - f^2}{EG - F^2} \tag{1.7.7}$$

$$H = \frac{1}{2}\mathrm{tr} D = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}. \tag{1.7.8}$$

CHAPTER 2

Kinematics of 2-dimensional networks of inextensible fibers

Contents

The aim of this chapter is to introduce the notations adopted in the following chapters and describe the kinematics of the fibered networks that we study later.

We focus on a surface made by two sets of inextensible fibers and define the deformation matrix for the considered set of inextensible fibers.

We refer to Chapter 1 for the definitions and standard results in differential geometry. We evaluate the Weingarten matrix for a fibered surface, as well as the Gaussian and Mean curvatures.

2.1. Configurations

We consider a surface $\mathcal{S}$ in three-dimensional Euclidean space. We assume, from now on, the existence of a global parametrization of $\mathcal{S}$, i.e. a map

$$\boldsymbol{r} : \Sigma_0 \subseteq \mathbb{R}^2 \to \mathbb{R}^3$$

such that $\boldsymbol{r}(\Sigma_0) = \mathcal{S}$.

The domain Σ_0 in $\mathbb{R}^2$ may be identified as a planar reference configuration of a surface, while $\mathcal{S}$ may be identified to the actual configuration of the surface. We recall some notation from the previous chapter. Let (X^1, X^2) be a cartesian coordinate system in $\mathbb{R}^2$ with basis defined by unit vectors $\boldsymbol{A}_1, \boldsymbol{A}_2$ and $\boldsymbol{A}^1, \boldsymbol{A}^2$ the dual basis, such that $\boldsymbol{A}^i \cdot \boldsymbol{A}_j = \delta^i_j$. The parametric representation of the surface in these coordinates is

$$\boldsymbol{r} = \boldsymbol{r}(X^1, X^2).$$

Then

$$\boldsymbol{r}_{,1} = \frac{\partial \boldsymbol{r}}{\partial X^1}\,, \qquad \boldsymbol{r}_{,2} = \frac{\partial \boldsymbol{r}}{\partial X^2}\,, \tag{2.1.1}$$

are tangent vectors to the surface (where the comma stands for the derivative), denoted by

$$\boldsymbol{a}_i = \boldsymbol{r}_{,i}\ \ , \quad i = 1, 2. \tag{2.1.2}$$

The deformation gradient

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}} \boldsymbol{r} \tag{2.1.3}$$

in coordinate representation is

$$\mathbf{F} = \mathbf{r}_{,1} \otimes \boldsymbol{A}^1 + \mathbf{r}_{,2} \otimes \boldsymbol{A}^2 \quad \Leftrightarrow \quad \mathbf{F}\boldsymbol{A}_i = \mathbf{r}_{,i}\,, \quad i = 1, 2. \tag{2.1.4}$$

The above definition involves a specific coordinate system but is independent of it. In fact, consider a coordinate change $\widetilde{X}^i = \widetilde{X}^i(X^j)$, and notice that, letting $\widetilde{\mathbf{r}}(\widetilde{X}^i) = \mathbf{r}(X^i)$ for the representation of $\mathbf{r}$ in the new coordinates, then

$$\frac{\partial \widetilde{\mathbf{r}}}{\partial \widetilde{X}^i} = \frac{\partial \mathbf{r}}{\partial X^j}\frac{\partial X^j}{\partial \widetilde{X}^i}, \qquad \widetilde{\boldsymbol{A}}_i = \frac{\partial X^j}{\partial \widetilde{X}^i}\boldsymbol{A}_j, \qquad \widetilde{\boldsymbol{A}}^i = \frac{\partial \widetilde{X}^i}{\partial X^j}\boldsymbol{A}^j,$$

where summation on repeated indices is understood. Hence,

$$\frac{\partial \widetilde{\mathbf{r}}}{\partial \widetilde{X}^i} \otimes \widetilde{\boldsymbol{A}}^i = \frac{\partial X^j}{\partial \widetilde{X}^i}\frac{\partial \widetilde{X}^i}{\partial X^k}\left(\frac{\partial \mathbf{r}}{\partial \widetilde{X}^j} \otimes \boldsymbol{A}^k\right) = \frac{\partial \mathbf{r}}{\partial X^i} \otimes \boldsymbol{A}^i.$$

Define the right Cauchy–Green tensor for a surface deformation by

$$\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F}, \tag{2.1.5}$$

where a superposed $\top$ denotes transpose: $\boldsymbol{C}$ is a symmetric tensor field on Σ_0 with coordinate representation

$$\boldsymbol{C} = C_{11}\,\boldsymbol{A}^1 \otimes \boldsymbol{A}^1 + C_{12}\,(\boldsymbol{A}^1 \otimes \boldsymbol{A}^2 + \boldsymbol{A}^2 \otimes \boldsymbol{A}^1) + C_{22}\,\boldsymbol{A}^2 \otimes \boldsymbol{A}^2, \tag{2.1.6}$$

and $C_{ij} = \boldsymbol{A}_i \cdot \boldsymbol{C}\boldsymbol{A}_j$.

By (1.2.1), the **unit normal** of the surface $\mathcal{S}$ is

$$\boldsymbol{N} = \frac{\boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2}}{|\boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2}|} \tag{2.1.7}$$

and the **second fundamental form** of $\mathcal{S}$ reads

$$\boldsymbol{r}_{,11} \cdot \boldsymbol{N}\, dX^1 dX^1 \,+\, 2\,\boldsymbol{r}_{,12} \cdot \boldsymbol{N}\, dX^1 dX^2 \,+\, \boldsymbol{r}_{,22} \cdot \boldsymbol{N}\, dX^2 dX^2. \tag{2.1.8}$$

2.2. Basic assumptions

We consider a sheet composed of two families of inextensible fibers. Initially, the fibers lie in a region Σ_0 of the X^1, X^2 plane, and each family of fibers is parallel to one of the coordinate axis (this restriction will be removed later). The fibers are assumed to be continuously distributed, so that every line X^1=constant or X^2=constant in Σ_0 may be regarded as a fiber.

Let $\boldsymbol{A}_1$ be the tangent vector to the curve corresponding to the fiber X^2 =const in the reference configuration and let $\boldsymbol{A}_2$ be the tangent vector to the fiber X^1 =const.

Recall that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, defined in (2.1.2), are the tangent vectors to the curves corresponding to a fiber X^2=constant and X^1=constant, respectively, after the sheet is deformed. We refer to the deformed fibers as the a_1-lines and the a_2-lines respectively.

A basic assumption is that the fibers are **inextensible**, so that the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ have unit length [**50**].

The local distortion of the sheet is measured by the **angle of shear** γ, defined by (cf. Figure 2.1)

$$\sin\gamma = \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 \tag{2.2.1}$$

so that the right Cauchy-Green tensor $\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F}$, defined in (2.1.5), becomes

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{a}_1 & \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 \\ \boldsymbol{a}_2 \cdot \boldsymbol{a}_1 & \boldsymbol{a}_2 \cdot \boldsymbol{a}_2 \end{pmatrix} = \begin{pmatrix} 1 & \sin\gamma \\ \sin\gamma & 1 \end{pmatrix}. \tag{2.2.2}$$

The **normal vector** to the deformed sheet, defined by (2.1.7), reads

$$\boldsymbol{N} = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|} = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\sin\delta|} = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\cos\gamma|}. \tag{2.2.3}$$

where $\delta = \pi/2 - \gamma$ is the angle formed by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

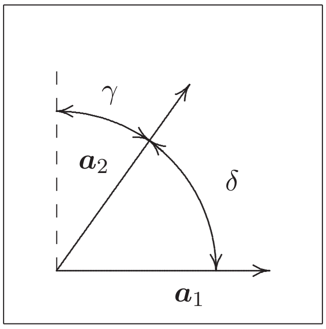

FIGURE 2.1. Angle of shear

Moreover, since the fibers are inextensible, X^1 and X^2 are the arc lengths of the a_1 and a_2 lines, and denoting by $\boldsymbol{n}_i$ and $\boldsymbol{b}_i$ (i=1,2) the normal and binormal unit vectors to the fibers of each family, the **Frenet formulas** yield

$$\begin{cases} \dfrac{\partial \boldsymbol{a}_1}{\partial X^1} = \mathcal{K}_1 \boldsymbol{n}_1 \\ \dfrac{\partial \boldsymbol{n}_1}{\partial X^1} = -\mathcal{K}_1 \boldsymbol{a}_1 + \tau_1 \boldsymbol{b}_1 \\ \dfrac{\partial \boldsymbol{b}_1}{\partial X^1} = -\tau_1 \boldsymbol{n}_1 \end{cases} \tag{2.2.4}$$

with $\mathcal{K}_1$ the curvature, τ_1 the torsion of the a_1-line. Analogous relations hold for the a_2-line.

2.3. The Weingarten matrix for inextensible networks

The aim of this section is to compute the matrix associated to the Weingarten map to characterize the curvature of the surface in a neighborhood of a point $\boldsymbol{x}_0$ of the fabric.

We consider the Weingarten map $W_{\boldsymbol{x}_0} : T_{\boldsymbol{x}_0}\mathcal{S} \to T_{\boldsymbol{x}_0}\mathcal{S}$ at $\boldsymbol{x}_0$, with associated matrix

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \tag{2.3.1}$$

in the basis $(\boldsymbol{a}_1, \boldsymbol{a}_2)$. We then have from (1.7.3)

$$\begin{cases} W_{\boldsymbol{x}_0}(\boldsymbol{a}_1) & = & d_{11}\boldsymbol{a}_1 + d_{12}\boldsymbol{a}_2 = -\boldsymbol{N}_{,1} \\ W_{\boldsymbol{x}_0}(\boldsymbol{a}_2) & = & d_{21}\boldsymbol{a}_1 + d_{22}\boldsymbol{a}_2 = -\boldsymbol{N}_{,2}. \end{cases} \tag{2.3.2}$$

We start by evaluating $\boldsymbol{N}_{,1}$

$$\begin{aligned}
\frac{\partial}{\partial X^1}\left(\frac{\boldsymbol{a}_1\times\boldsymbol{a}_2}{|\sin\delta|}\right) &= \frac{\left[\frac{\partial}{\partial X^1}(\boldsymbol{a}_1\times\boldsymbol{a}_2)\right]|\sin\delta| - (\boldsymbol{a}_1\times\boldsymbol{a}_2)\frac{\partial}{\partial X^1}|\sin\delta|}{|\sin\delta|^2}\\
&= \frac{1}{|\sin\delta|^2}\left\{\left(\frac{\partial\boldsymbol{a}_1}{\partial X^1}\times\boldsymbol{a}_2\right)|\sin\delta| + \left(\boldsymbol{a}_1\times\frac{\partial\boldsymbol{a}_2}{\partial X^1}\right)|\sin\delta|\right.\\
&\quad \left. -(\boldsymbol{a}_1\times\boldsymbol{a}_2)\ sign(\sin\delta)\ \cos\delta\frac{\partial\delta}{\partial X^1}\right\}.
\end{aligned}$$

From $(2.2.4)_1$ and $(2.3.2)_1$ we obtain

$$\begin{aligned}
\boldsymbol{N}_{,1} &= \frac{1}{|\sin\delta|^2}\left\{\mathcal{K}_1\,(\boldsymbol{n}_1\times\boldsymbol{a}_2)\,|\sin\delta| + \left(\boldsymbol{a}_1\times\frac{\partial\boldsymbol{a}_2}{\partial X^1}\right)|\sin\delta|\right.\\
&\quad \left. -(\boldsymbol{a}_1\times\boldsymbol{a}_2)\ sign(\sin\delta)\ \cos\delta\ \frac{\partial\delta}{\partial X^1}\right\} = -(d_{11}\boldsymbol{a}_1 + d_{12}\boldsymbol{a}_2).
\end{aligned} \tag{2.3.3}$$

Multiplying the previous equation by $\boldsymbol{a}_1$ gives

$$\frac{1}{|\sin\delta|}\left\{\mathcal{K}_1(\boldsymbol{b}_1\cdot\boldsymbol{a}_2)\right\} = -(d_{11} + d_{12}\cos\delta). \tag{2.3.4}$$

Multiplying (2.3.3) by $\boldsymbol{a}_2$, we obtain

$$\frac{1}{|\sin\delta|}\left\{\left(\boldsymbol{a}_1\times\frac{\partial\boldsymbol{a}_2}{\partial X^1}\right)\cdot\boldsymbol{a}_2\right\} = -(d_{11}\cos\delta + d_{12}). \tag{2.3.5}$$

Analogously, evaluating $\boldsymbol{N}_{,2}$ using (2.3.2) and Frenet formulas we obtain

$$\frac{1}{|\sin\delta|^2}\left\{\left(\frac{\partial\boldsymbol{a}_1}{\partial X^2}\times\boldsymbol{a}_2\right)\cdot\boldsymbol{a}_1|\sin\delta|\right\} = -(d_{21} + d_{22}\cos\delta) \tag{2.3.6}$$

and

$$\frac{1}{|\sin\delta|^2}\left\{\mathcal{K}_2(\boldsymbol{a}_1\cdot(-\boldsymbol{b}_2))|\sin\delta|\right\} = -(d_{21}\cos\delta + d_{22}). \tag{2.3.7}$$

Finally, from equations (2.3.4), (2.3.5), (2.3.6), (2.3.7) we obtain the components of the Weingarten matrix

$$\begin{cases} d_{11} &= \dfrac{1}{|\sin\delta|^3}\left\{\left(\boldsymbol{a}_1 \times \dfrac{\partial \boldsymbol{a}_2}{\partial X^1}\right)\cdot \boldsymbol{a}_2\ \cos\delta - \mathcal{K}_1(\boldsymbol{b}_1\cdot\boldsymbol{a}_2)\right\} \\ d_{12} &= \dfrac{1}{|\sin\delta|^3}\left\{-\left(\boldsymbol{a}_1 \times \dfrac{\partial \boldsymbol{a}_2}{\partial X^1}\right)\cdot \boldsymbol{a}_2 + \mathcal{K}_1(\boldsymbol{b}_1\cdot\boldsymbol{a}_2)\ \cos\delta\right\} \\ d_{21} &= \dfrac{1}{|\sin\delta|^3}\left\{-\mathcal{K}_2(\boldsymbol{a}_1\cdot\boldsymbol{b}_2)\cos\delta - \left(\dfrac{\partial \boldsymbol{a}_1}{\partial X^2}\times\boldsymbol{a}_2\right)\cdot\boldsymbol{a}_1\right\} \\ d_{22} &= \dfrac{1}{|\sin\delta|^3}\left\{\mathcal{K}_2(\boldsymbol{a}_1\cdot\boldsymbol{b}_2) + \left(\dfrac{\partial \boldsymbol{a}_1}{\partial X^2}\times\boldsymbol{a}_2\right)\cdot\boldsymbol{a}_1\ \cos\delta\right\} \end{cases}$$

or equivalently,

$$\begin{cases} d_{11} &= -\dfrac{1}{|\sin\delta|^2}\left[\left(\boldsymbol{N}\cdot\dfrac{\partial \boldsymbol{a}_2}{\partial X^1}\right)\cos\delta + \dfrac{\mathcal{K}_1}{|\sin\delta|}(\boldsymbol{b}_1\cdot\boldsymbol{a}_2)\right] \\ d_{12} &= \dfrac{1}{|\sin\delta|^2}\left[\boldsymbol{N}\cdot\dfrac{\partial \boldsymbol{a}_2}{\partial X^1} + \mathcal{K}_1(\boldsymbol{b}_1\cdot\boldsymbol{a}_2)\ \dfrac{\cos\delta}{|\sin\delta|}\right] \\ d_{21} &= \dfrac{1}{|\sin\delta|^2}\left\{-\mathcal{K}_2(\boldsymbol{a}_1\cdot\boldsymbol{b}_2)\dfrac{\cos\delta}{|\sin\delta|} + \boldsymbol{N}\cdot\dfrac{\partial \boldsymbol{a}_1}{\partial X^2}\right\} \\ d_{22} &= \dfrac{1}{|\sin\delta|^2}\left\{\dfrac{\mathcal{K}_2}{|\sin\delta|}(\boldsymbol{a}_1\cdot\boldsymbol{b}_2) + \left(\boldsymbol{N}\cdot\dfrac{\partial \boldsymbol{a}_1}{\partial X^2}\right)\ \cos\delta\right\}\ . \end{cases}$$

Relations (2.3.8) are equivalent to the relation (1.7.6), which corresponds to the Weingarten equations

$$\begin{cases} d_{11} &= \dfrac{1}{EG-F^2}\,[eG - fF] \\ d_{12} &= \dfrac{1}{EG-F^2}\,[-eF - fE] \\ d_{21} &= \dfrac{1}{EG-F^2}\,[fG - gF] \\ d_{22} &= \dfrac{1}{EG-F^2}\,[-fF + gE] \end{cases} \tag{2.3.8}$$

where the quantities E, F, G, e, f, g defined by (1.3.1) and (1.6.2) are easily computable

$$\begin{cases} E &= 1 \quad \text{since } \boldsymbol{a}_1 \text{ is a unit vector} \\ F &= \cos\delta \\ G &= 1 \quad \text{since } \boldsymbol{a}_2 \text{ is a unit vector} \\ e &= \dfrac{\partial \boldsymbol{a}_1}{\partial X^1} \cdot \boldsymbol{N} \\ f &= \dfrac{\partial \boldsymbol{r}}{\partial X^1 \partial X^2} \cdot \boldsymbol{N} = \dfrac{\partial \boldsymbol{r}}{\partial X^2 \partial X^1} \cdot \boldsymbol{N} \\ g &= \dfrac{\partial \boldsymbol{a}_2}{\partial X^2} \cdot \boldsymbol{N}. \end{cases} \tag{2.3.9}$$

2.4. Gaussian and Mean curvature

Recalling definition (1.7.8) and relations (2.3.8) we obtain the mean curvature

$$H = \frac{1}{2|\sin\delta|^3}\left\{\mathcal{K}_2(\boldsymbol{a}_1 \cdot \boldsymbol{b}_2) - \mathcal{K}_1(\boldsymbol{a}_2 \cdot \boldsymbol{b}_1) - \left[(\boldsymbol{a}_1 \times \boldsymbol{a}_2) \cdot \left(\frac{\partial \boldsymbol{a}_2}{\partial X^1} + \frac{\partial \boldsymbol{a}_1}{\partial X^2}\right)\right] \cos\delta\right\}.$$

Since $\dfrac{\partial \boldsymbol{a}_2}{\partial X^1} = \dfrac{\partial^2 \boldsymbol{r}}{\partial X^1 \partial X^2} = \dfrac{\partial^2 \boldsymbol{r}}{\partial X^2 \partial X^1} = \dfrac{\partial \boldsymbol{a}_1}{\partial X^2}$,

$$H = \frac{1}{2|\sin\delta|^3}\left\{\mathcal{K}_2(\boldsymbol{a}_1 \cdot \boldsymbol{b}_2) - \mathcal{K}_1(\boldsymbol{a}_2 \cdot \boldsymbol{b}_1) - 2\,(\boldsymbol{a}_1 \times \boldsymbol{a}_2) \cdot \frac{\partial \boldsymbol{a}_2}{\partial X^1}\,\cos\delta\right\}. \tag{2.4.1}$$

Also, from definition (2.3.8) the Gaussian curvature reads

$$K = -\frac{1}{|\sin\delta|^4}\left\{\mathcal{K}_1\mathcal{K}_2(\boldsymbol{a}_2 \cdot \boldsymbol{b}_1)(\boldsymbol{a}_1 \cdot \boldsymbol{b}_2) + \left[\left(\frac{\partial \boldsymbol{a}_2}{\partial X^1} \times \boldsymbol{a}_1\right) \cdot \boldsymbol{a}_2\right]^2\right\}. \tag{2.4.2}$$

Notice that, for instance, the fact that $\mathcal{K}_1$, $\mathcal{K}_2$ are zero, (straight fibers) does not imply that also the Mean and Gaussian curvatures are zero. Indeed, for a ruled surface made by straight inextensible fibers, the fiber curvatures and the Gaussian curvature vanish, but the mean curvature does not.

2.5. Euler angles

For later use in Chapter 6, we want to correlate the Frenet triads $\{\boldsymbol{a}_1, \boldsymbol{n}_1, \boldsymbol{b}_1\}$, $\{\boldsymbol{a}_2, \boldsymbol{n}_2, \boldsymbol{b}_2\}$ associated to the two families of fibers by three angles. Recalling (2.2.3) we define

$$\begin{aligned} 0 < \delta < \pi \quad &= \text{angle between } \boldsymbol{a}_1 \text{ and } \boldsymbol{a}_2 \text{ with axis } \boldsymbol{N} \\ 0 \le \Phi < 2\pi \quad &= \text{angle between } \boldsymbol{N} \text{ and } \boldsymbol{n}_2 \text{ with axis } \boldsymbol{a}_2 \\ 0 \le \Psi < 2\pi \quad &= \text{angle between } \boldsymbol{N} \text{ and } \boldsymbol{n}_1 \text{ with axis } \boldsymbol{a}_1 \end{aligned} \tag{2.5.1}$$

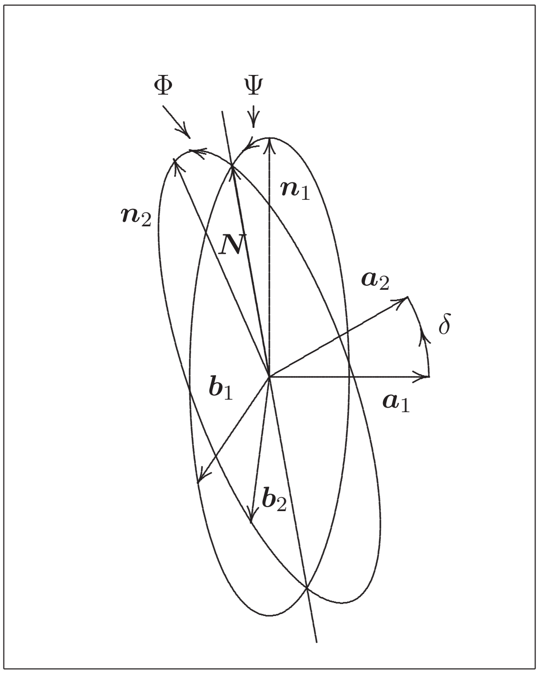

Figure 2.2. Euler angles

Then, the following matrix $\boldsymbol{R}$ relates the two triads

$$\boldsymbol{R} = \begin{pmatrix} \cos\delta & \sin\Psi\ \sin\delta & -\cos\Psi\ \sin\delta \\ \sin\Phi\ \sin\delta & \cos\Phi\ \cos\Psi - \sin\Phi\ \sin\Psi\ \cos\delta & \cos\Phi\ \sin\Psi\ + \sin\Phi\ \cos\Psi\ \cos\delta \\ \cos\Phi\ \sin\delta & -\sin\Phi\cos\Psi - \cos\Phi\sin\Psi\cos\delta & -\sin\Phi\sin\Psi + \cos\Phi\cos\Psi\cos\delta \end{pmatrix} \tag{2.5.2}$$

We can rewrite the mean curvature (2.4.1) in terms of the Euler angles

$$H = -\frac{\mathcal{K}_1 \cos\Phi + \mathcal{K}_2 \cos\Psi}{2|\sin\delta|^2} - \frac{\tau_1 \cos\delta}{|\sin\delta|} + \frac{\mathcal{K}_1 \cos\Phi(\cos\delta)^2}{|\sin\delta|^2} + \frac{\cos\delta}{|\sin\delta|}\frac{\partial\Psi}{\partial X^1}, \tag{2.5.3}$$

while the Gaussian curvature (2.4.2) can be written as

$$K = \frac{\mathcal{K}_1\mathcal{K}_2 \cos\Phi\cos\Psi}{|\sin\delta|^2} - \frac{1}{|\sin\delta|^4}\left[\tau_1(\sin\delta)^2 - \mathcal{K}_1\cos\Phi\cos\delta\sin\delta - \frac{\partial\Phi}{\partial X^1}(\sin\delta)^2\right]^2 \tag{2.5.4}$$

that corresponds to

$$\begin{aligned} K = \frac{\mathcal{K}_1\mathcal{K}_2 \cos\Phi\cos\Psi}{|\sin\delta|^2} - \Bigg[&(\tau_1)^2 + (\mathcal{K}_1)^2\frac{(\cos\Phi\cos\delta)^2}{|\sin\delta|^2} + \left(\frac{\partial\Phi}{\partial X^1}\right)^2 - \frac{2\tau_1\mathcal{K}_1\cos\Phi\cos\delta}{|\sin\delta|} \\ &- 2\tau_1\frac{\partial\Phi}{\partial X^1} + \frac{2\mathcal{K}_1\cos\Phi\cos\delta}{|\sin\delta|}\frac{\partial\Phi}{\partial X^1}\Bigg]. \end{aligned}$$

CHAPTER 3

Some models of textiles in the literature

Contents

In this chapter we briefly describe two basic models for woven fabrics proposed in the literature. The models are somewhat paradigmatic of two opposite approaches to the study of mechanics of textiles: the first of the two, due to Peirce [**49**], in fact, is the very first attempt at a mathematical description of the behavior of a plain weave, through the elementary description of the mechanical behavior of the single thread forming the textile. The second model we describe here, proposed by Magno, Ganghoffer and others [**39**], [**38**], on the contrary, is a macroscopic approach to textiles that treats the tissue as a deformable surface as a whole, and accounts for the influence of the fine structure through additional, microstructural fields. The resulting model is able to capture those characteristics of the fine structure of a textile, that determine its response, for instance the stress strain curve.

More generally, since the beginning of the twentieth century, different models describing the structure of tissues have been proposed, that can be grouped into three different families: geometrical models, such as those proposed by Peirce [**49**], mechanical (or force) models, such as Olofsson's [**44**] or Grosberg's [**19**] models, and, more recently, microstructural models, such as those of Magno and Ganghoffer, and, finally, energy models. As mentioned above, in the next sections we focus on Peirce's and Magno's work, since they are somehow typical of the different approach to textile mechanics.

3.1. A geometrical model: Peirce's model

In 1937, Peirce [**49**] was the first to study the geometry of cloth structures. In particular, he focused on the features of cloth behavior related with the geometry of the textile: for example, when a fabric is stretched in the warp direction, the weft dimension decreases and the weft crimp increases.

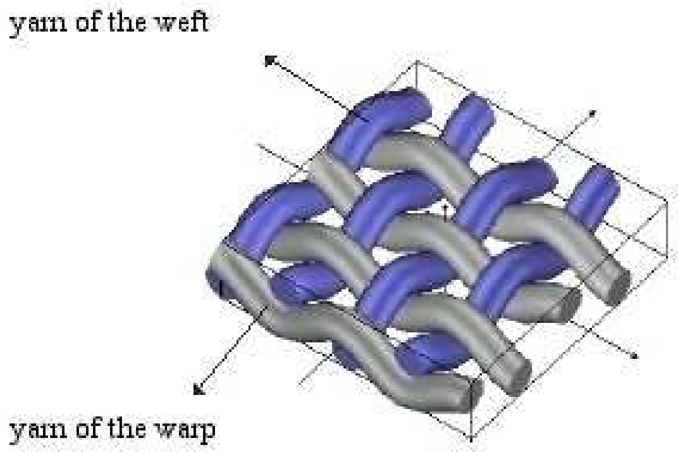

FIGURE 3.1. Threads (yarns) of a plane weave.

The yarns forming the textile were described as flexible, circular cylinders, knitted together in regular patterns to form the fabric.

Peirce considered a weave in which the threads alternately interlace, namely a **plain weave**. In Figures 3.2 and 3.3, a section of the undistorted weave is represented. The section is in the plane of the axis of a warp yarn, which is normal to the plane of the fabric (the central plane equidistant from the two tangent planes that enclose the paths of the warp or weft yarns). The circles represent the sections of the weft's threads, the circular lines around the weft sections and the straight lines connecting them represent the warp's thread.

The geometrical parameters relevant to the description are:

d_1, d_2: the diameter of the warp's, (resp. the weft's) thread;

p_1, p_2: the pick spacing, namely the spacing between the threads' axis of the warp, (resp. weft);
θ_1, θ_2: the maximum angle between the thread axis of the warp (resp. the weft) and the plane of the fabric;
l_1, l_2: the length of the portion of the thread axis between the planes containing the axes of consecutive crossing threads;
h_1, h_2: the maximum displacements of the thread axis, normal to the plane of the cloth;
c_1, c_2: the crimp, namely the ratio of the difference between the length l_1 and the spacing p_2, to the spacing p_2.

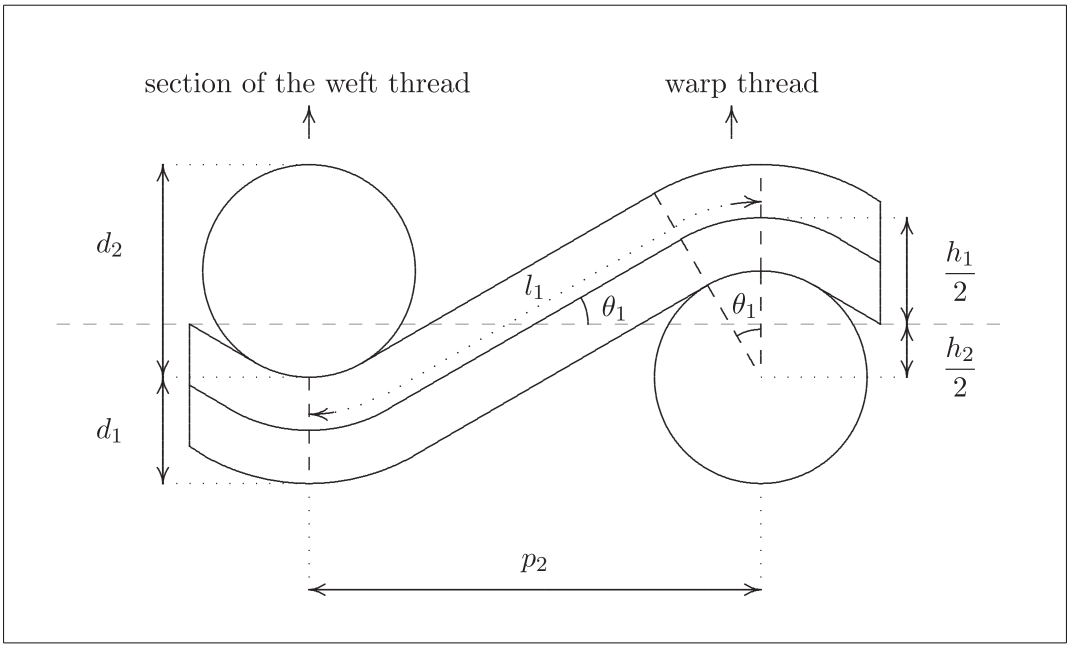

FIGURE 3.2. Peirce's model: section of a plane weave in the plane of the axis of the warp yarn

Introducing the new length parameter $D = d_1 + d_2$, the twelve parameters above reduce to eleven. Moreover, these 11 parameters are not independent, but are related by the following

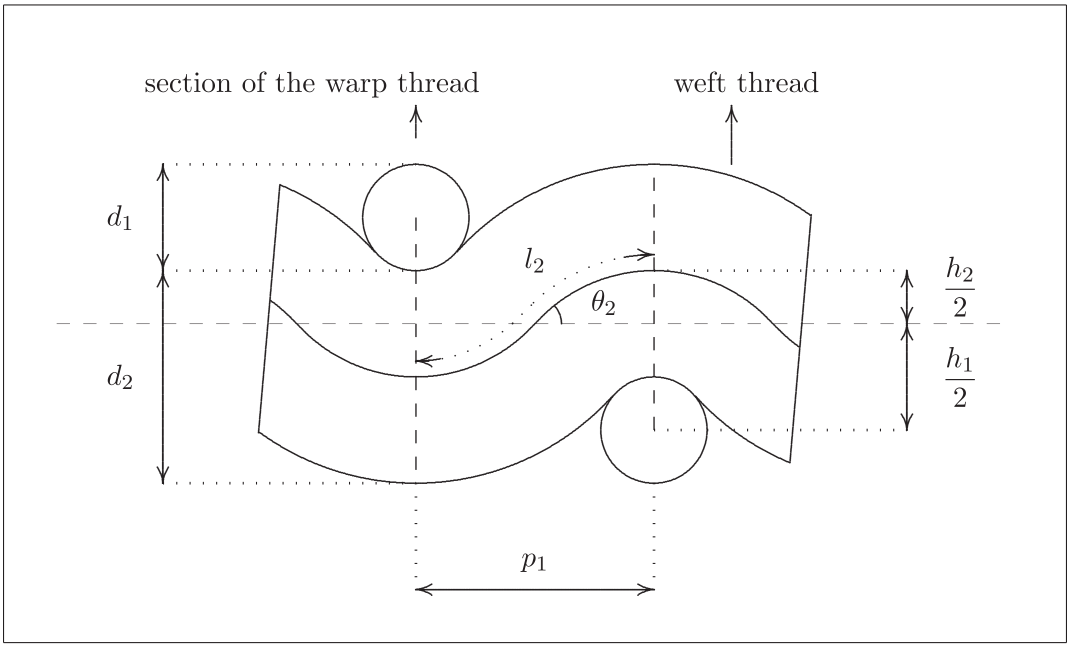

FIGURE 3.3. Peirce's model: section of a plane weave in the plane of the axis of the weft yarn

geometric identities:

$$c_1 = \frac{l_1}{p_2} - 1 \tag{3.1.1}$$

$$c_2 = \frac{l_2}{p_1} - 1 \tag{3.1.2}$$

$$p_2 = (l_1 - D\theta_1)\cos\theta_1 + D\sin\theta_1 \tag{3.1.3}$$

$$p_1 = (l_2 - D\theta_2)\cos\theta_2 + D\sin\theta_2 \tag{3.1.4}$$

$$h_1 = (l_1 - D\theta_1)\sin\theta_1 + D(1 - \cos\theta_1) \tag{3.1.5}$$

$$h_2 = (l_2 - D\theta_2)\sin\theta_2 + D(1 - \cos\theta_2) \tag{3.1.6}$$

$$h_1 + h_2 = D \tag{3.1.7}$$

where the equations (3.1.1), (3.1.2) are the definition of crimp; the pick spacing in (3.1.3), (3.1.4) is obtained by projecting on the cloth plane both the straight and the curve portion of axis of the thread; the total vertical displacement in (3.1.5), (3.1.6) is obtained by projecting both the straight and the curved portion of the thread's axis parallel to the cloth plane.

Equation (3.1.7) couples the weft's and the warp's paths: in fact, the sum of the displacements from the plane of the cloth corresponds to the sum of the diameters $d_1 + d_2$.

Since there are only seven equations relating the eleven parameters: D, p_1, p_2, θ_1, θ_2, l_1, l_2, h_1, h_2, c_1, c_2, it follows that the geometry of the weave is completely determined by four independent quantities.

Usually, the values of θ_1 θ_2 are unknowns. By eliminating l_1 from equations (3.1.4) and (3.1.5), Peirce found a second degree algebraic equation in $t_1 = \tan\frac{\theta_1}{2}$ that admits the following solution

$$t_1 = \frac{p_2 - \sqrt{p_2^2 + h_2^2 - D^2}}{D + h_2}. \tag{3.1.8}$$

Eliminating l_1 from equations (3.1.1) and (3.1.3), he obtained

$$p_2(1 + c_1) = D\theta_1 + \frac{p_2 - D\sin\theta_1}{\cos\theta_1}. \tag{3.1.9}$$

By substituting $t_1 = \tan\frac{\theta_1}{2}$ in (3.1.9), Peirce found an equation that can not be explicitly solved for t_1; however, by substituting (3.1.8) into this equation, a relation in which θ_1 and l_1 are formally eliminated can be found. Unluckily, this equation can not be solved for h_1 explicitly, so that it is not possible to obtain an expression that substituted in (3.1.7) would give the searched analytical relation between D, p_1, p_2, c_1, c_2.

Since it is not possible to write explicitly the equations that relate D, p_1, p_2,c_1, c_2, Peirce solved the problem using a graphical method. Womersley supported Pierce's results by a numerical method.

In order to use the graphical method, it is necessary to express the relation between p, c and h in functional form, and to do this some approximations are performed. For example, it is found that $h_1 = \dfrac{4}{3}p_2\sqrt{c_1}$ (see [**49**] for more details). Peirce also studied special structures: particular cases in which the relations take a simple form that allows direct computation. The analysis of these special structures shows that three independent relations between the weave variables are necessary and sufficient to determine the conformation. In most cases, the relations will not be simply expressible by explicit formulae, so that the graphical or the approximate expressions must be used.

Peirce also studied the crimped form of an elastic thread: the threads were modelled as circular cylinders with bending moment proportional to the curvature of the axis.

Let $\boldsymbol{T}$ be the tension vector and $\boldsymbol{S}$ the shear force vector in the warp at a point P of the axis of the thread, with T and S the norm of the two vectors (Figure 3.4), and write

$$\boldsymbol{t} = \boldsymbol{T} + \boldsymbol{S}$$

for the stress vector in the fiber.

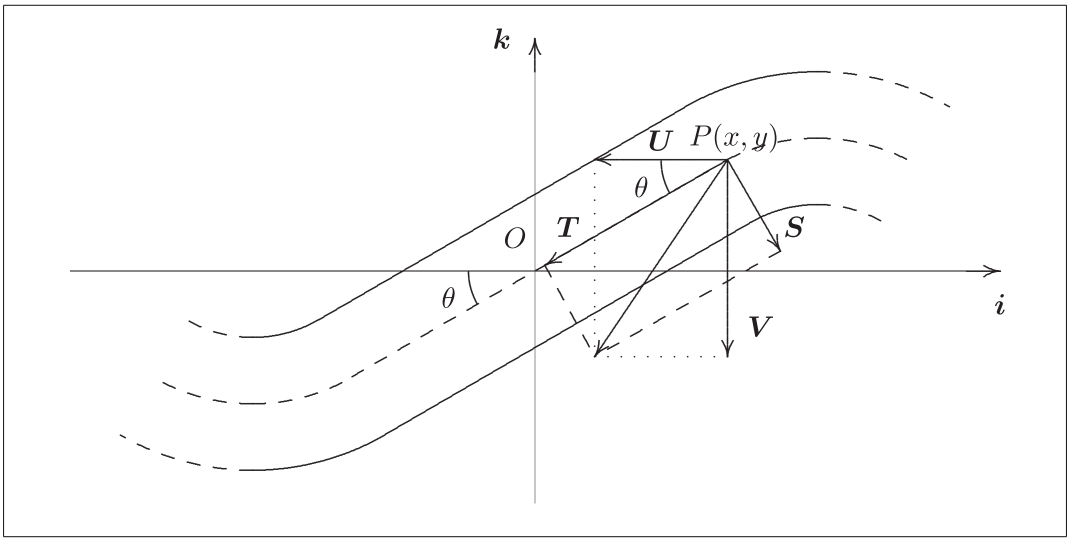

FIGURE 3.4. Forces acting on the warp thread

We have

$$\boldsymbol{S} = S\sin\theta\boldsymbol{i} - S\cos\theta\boldsymbol{k} \qquad \boldsymbol{T} = -T\cos\theta\boldsymbol{i} - T\sin\theta\boldsymbol{k} \tag{3.1.10}$$

and

$$\boldsymbol{S} + \boldsymbol{T} = (S\sin\theta - T\cos\theta)\boldsymbol{i} + (-S\cos\theta - T\sin\theta)\boldsymbol{k} = U\boldsymbol{i} + V\boldsymbol{k} \tag{3.1.11}$$

with $\boldsymbol{U} = U\boldsymbol{i}$ the horizontal force and $\boldsymbol{V} = V\boldsymbol{k}$ the vertical force acting on the thread.

Consider further the balance of couples for a single thread

$$\oint_{\partial\mathcal{B}} \boldsymbol{r} \times \boldsymbol{t} + \oint_{\partial\mathcal{B}} \boldsymbol{c} = \boldsymbol{0} \tag{3.1.12}$$

with $\boldsymbol{t}$ the stress, $\boldsymbol{c}$ the couple acting on the thread, and $\mathcal{B}$ a portion of the thread.

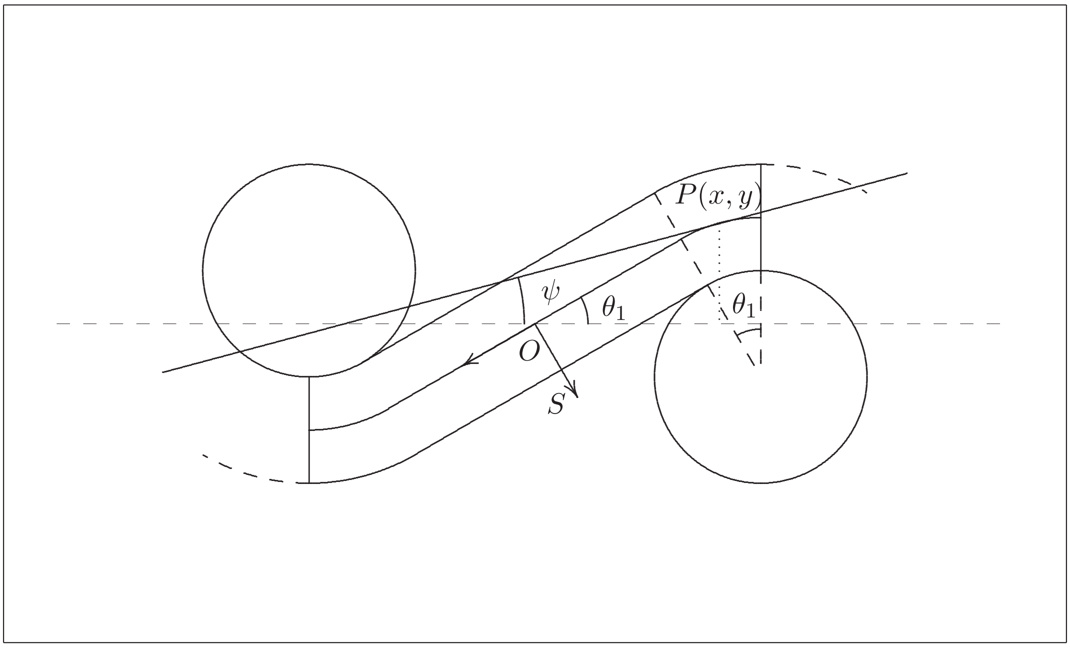

FIGURE 3.5. Peirce's model: the crimped form of an elastic thread

With reference to figure 3.4, this relation can be written as

$$OP \times \boldsymbol{t}_{|P} + \boldsymbol{c}_{|P} - OO \times \boldsymbol{t}_{|O} - \boldsymbol{c}_{|O} = \mathbf{0}. \tag{3.1.13}$$

Since the curvature at the mid point O vanishes, $\boldsymbol{c}_{|O} = \mathbf{0}$ (see below), and the equation above reduces to

$$OP \times \boldsymbol{t}_{|P} + \boldsymbol{c}_{|P} = \mathbf{0}$$

which corresponds to

$$\boldsymbol{c}_{|P} = -(x\boldsymbol{i} + y\boldsymbol{k}) \times (U\boldsymbol{i} + V\boldsymbol{k}) = -(-Vx + Uy)\boldsymbol{j}.$$

Letting $\boldsymbol{c}_{|P} = -m\dfrac{d\psi}{ds}\boldsymbol{j}$, with m be the bending moment and $\dfrac{d\psi}{ds}$ the curvature of the thread, we find

$$-(-Vx + Uy)\boldsymbol{j} = -m\frac{d\psi}{ds}\boldsymbol{j}.$$

Consequently, for any point P on the warp axis between O and the contact point with the weft (figure 3.5), one obtains the relation:

$$m\frac{d\psi}{ds} = -Vx + Uy. \tag{3.1.14}$$

Substituting $ds = \frac{dx}{\cos\psi}$ into (3.1.14) and differentiating with respect to x, this relation reduces to

$$\frac{d^2\psi}{ds^2} - \left(\frac{d\psi}{ds}\right)^2 \tan\psi + \frac{V}{m\cos\psi} - \frac{U\tan\psi}{m\cos\psi} = 0. \tag{3.1.15}$$

In case the external tension on the cloth is zero, the horizontal component U vanishes and $T = S\tan\theta$. The force on the section through P reduces to V. The arc-length of OP is given by

$$OP = \sqrt{\frac{m}{2V}} \int_\theta^\psi \frac{d\psi}{\sqrt{\sin\theta - \sin\psi}}. \tag{3.1.16}$$

Using this equation, Peirce was able to derive an analytical expression that describes the form of the curve [**49**].

The work of Peirce represents a first step to study geometrical models for tissues. He tried to describe the geometry of the threads but also introduced equations that correlate the forces acting on the thread with its deformation.

3.2. The mesoscopic model of Magno and Ganghoffer

In [**39**] and [**38**], Magno and Ganghoffer introduce a model for woven fabric that relates the uniaxial extension of a plain weave fabric to the effect of the undulation of the yarn.

This model yields constitutive laws in parametric form (cf. 3.2.7, 3.2.15) that lead to the J-shaped traction curve of the fabric. Hence, the model is able to capture the geometrical nonlinearities due to the crimp changes.

In a textile fabric, three main scales can be identified: the macroscopic scale, at which the textile can be described as a regular surface, the mesoscopic scale, an intermediate scale at which the undulation of the yarn can be distinguished (yarn waves correspond to this scale), and the microscopic scale that corresponds to the yarn and its constituent, namely the fibers.

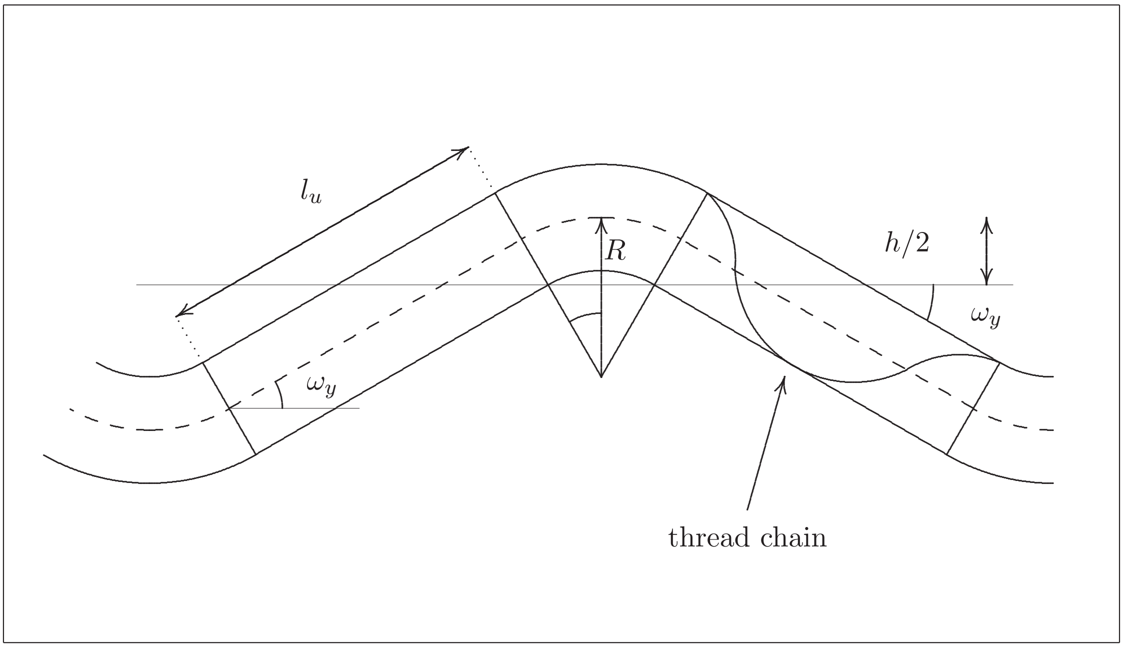

FIGURE 3.6. Magno's model: the woven fabric at the mesoscopic scale, corresponding to the characteristic length of the undulations of the yarn

In real tests, large rotations due to the variation of the undulations within the yarns are observed: hence, the model is formulated in the framework of large rotations and large strains.

A simplifying assumption is that, before deformation, the fabric is planar and, moreover, there is neither any interaction between weft and warp, nor compression of the yarns during the extension of the fabric.

We use here the same notation of [**39**] and [**38**]. Let $\Omega_0 \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^3$ be the reference and the actual configuration respectively: a configuration of the micropolar fabric is defined by a pair (φ, ω) such that $\varphi : \Omega_0 \to \Omega$ is the deformation of the surface, while $\omega : \Omega_0 \to SO(3)$ is the microrotation field. This field defines the orientation of the triad of directors $(\boldsymbol{D}_i)$, and describes the local rotation of the yarns at a scale equivalent to that of the undulation of a single yarn.

A basic assumption is that, for a micropolar material, the deformation gradient $\boldsymbol{F}$ can be decomposed as the product of a left micropolar strain tensor v_ω and the microrotation ω, i.e.,

$$\boldsymbol{F} = v_\omega \omega.$$

Analogously, Magno assumes that the microrotational gradient has a multiplicative decomposition of the form

$$\nabla_x \omega = \kappa \omega \tag{3.2.1}$$

with κ the left microcurvature tensor.

Since v_ω and κ are objective, the authors assume that the free energy density has the form

$$\psi = \psi(v_\omega, \kappa).$$

This implies that the following constitutive laws hold (see Magno [**39**], [**38**])

$$\tau^\top = \frac{\partial \psi}{\partial v_\omega}\, v_\omega^\top \qquad\qquad \mu^\top = \frac{\partial \psi}{\partial \kappa}\, v_\omega^\top \tag{3.2.2}$$

where the Kirchhoff stress $\tau = J\sigma$ and the Kirchhoff couple stress μ depend on the deformation and microcurvature measures.[1]

The micropolar energy density is assumed to be additive

$$\psi = \psi^\sigma(v_\omega) + \psi^\mu(\kappa) \tag{3.2.3}$$

with $\psi^\mu(\kappa)$ the rotational part and $\psi^\sigma(v_\omega)$ the translational part.

In the 1-dimensional case, ψ^μ can be taken to be a quadratic form in the microcurvature κ,

$$\psi^\mu(\kappa) = \frac{\mu_0}{2}\, l_c^2\, (\kappa \cdot \kappa) \tag{3.2.4}$$

where l_c is a characteristic length parameter linked to the couples at the mesoscopic scale.

By $(3.2.2)_2$ and (3.2.4), the relation between couple stress and the microcurvature becomes

$$\mu^\top = \mu_0\, l_c^2\, \kappa\, v_\omega^\top. \tag{3.2.5}$$

Now, according to De Jong and Postle [**14**], the uniaxial extension curve of the crimped thread is related to three distinct deformation mechanisms:

[1]Here $\top$ denotes the transpose.

I- the loss of textile weave crimp or yarn undulation (at the macrolevel);
II- the loss of the undulations of the threads inside the fabric (at the mesoscopic level);
III- the extension of the yarn.

Magno's model is principally related with the second of these extension mechanisms, and allows to determine the axial deformation ε_{xx} and the axial stress.

At the mesoscopic scale (Figure 3.6), the axial deformation has the form

$$\varepsilon_{xx}(\omega_y) = \frac{l'_u \cos\omega'_y - l_u \cos\omega_y}{l_u \cos\omega_y}, \tag{3.2.6}$$

and by taking the limit as the size of the basic cells tends to zero, a constitutive law of a continuous material equivalent to the initially discrete material is obtained; the length of the cells is supposed to remain constant $l_u = l'_u$ and the deformation, parametrized by the microrotation ω_y in the (x, y)-plane, becomes

$$\varepsilon_{xx}(\omega_y) = \ln\frac{\cos\omega_y}{\cos\omega_y^0}, \tag{3.2.7}$$

with ω_y^0 the initial value of the microrotation.

For microrotations in the (x, z)-plane, the vector associated to the microrotation is

$$\underline{\omega}_{\ y} = \begin{pmatrix} 0 \\ \omega_y \\ 0 \end{pmatrix} \tag{3.2.8}$$

and the microrotation tensor is

$$\underline{\underline{\omega}}_{\ y} = \begin{pmatrix} \cos\omega_y & 0 & -\sin\omega_y \\ 0 & 0 & 0 \\ \sin\omega_y & 0 & \cos\omega_y \end{pmatrix}. \tag{3.2.9}$$

Hence, the gradient of the microrotation has the form

$$\nabla\omega_y = \begin{pmatrix} 0 & 0 & 0 \\ \omega_{y,1} & \omega_{y,2} & \omega_{y,3} \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.2.10}$$

and inserting (3.2.9), (3.2.10) into (3.2.1), we obtain

$$\kappa = \begin{pmatrix} 0 & 0 & 0 \\ \omega_{y,1}\cos\omega_y + \omega_{y,3}\sin\omega_y & 0 & -\omega_{y,1}\sin\omega_y + \omega_{y,3}\cos\omega_y \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.2.11}$$

Also, since only rotations in the (x, z) plane are considered, $\omega_{y,3} = \dfrac{\partial \omega_y}{\partial z} = 0$. Finally, if an approximate expression for the axial gradient of microrotation is chosen, then $\omega_{y,1} = \dfrac{\partial \omega_y}{\partial x} = \dfrac{\omega_y}{R\sin\omega_y}$.

Inserting this expression into (3.2.5), in the case of vanishing extension of the base cells ($v_\omega = 1$), the following expression is obtained (cf. Figure 3.6)

$$\mu_{yx} = \frac{\mu_0\, l_c^2\, \omega_y}{R\sin\omega_y}\cos\omega_y. \tag{3.2.12}$$

Recalling that the axial stress τ_{xx} induced by the moment μ_{yx} is

$$\tau_{xx} = \frac{\mu_{yx}}{h/2} \tag{3.2.13}$$

with

$$h = l_u \sin\omega_y. \tag{3.2.14}$$

The final expression for the axial stress τ_{xx} induced by the moment μ_{yx} reads

$$\tau_{xx} = \left[\frac{2\,\mu_0\, l_c^2}{R\, l_u}\right]\omega_y \frac{\cos\omega_y}{\sin^2\omega_y}. \tag{3.2.15}$$

To summarize, in the work discussed above a constitutive law of a continuous material, equivalent to the initially discrete material, is established, and the constitutive law of the fabric is obtained from a micromechanical analysis performed on an elementary cell (the unit cell), corresponding to the wavelength of the undulation of the threads.

CHAPTER 4

Basic models for inextensible networks

Contents

In this chapter we present a short review of the existing theory of inextensible networks. Such theory describes some features of the mechanical behavior of cloth and cable networks.

In 1955, Rivlin [**59**] proposed a theory of networks formed by inextensible cords. He considered the mechanics of a **plane** net made by two families of parallel inextensible cords, and obtained general solutions of traction boundary value problems.

Between 1980 and 1986, in a series of papers, Pipkin, [**54**] [**52**] [**51**] [**50**], further developed the theory of inextensible networks.

In 1986, Wang and Pipkin, [**66**] [**67**], formulated a theory of inextensible nets with bending stiffness. The resulting continuum theory is a special case of finite-deformation plate theory, in which each fiber has a bending couple proportional to its curvature.

In 2001, a theory of bending and twisting effects in three-dimensional deformations of an inextensible network was presented by Luo and Steigmann [**37**]. They derived the Euler-Lagrange equations and boundary conditions by a minimum-energy principle.

In the following we will focus on Wang and Pipkin's [**66**] and Luo and Steigmann's [**37**] works.

4.1. The model of Wang and Pipkin

4.1.1. Kinematics

We consider a sheet composed of two families of inextensible fibers. Initially, the fibers lie in a plane region and each family of fibers is parallel to one of the coordinate axis. The fibers are assumed to be continuously distributed.

Let $\boldsymbol{r}(X^1, X^2)$ be the position in the current configuration of a point of the fibers with reference coordinates (X^1, X^2).

Also, the same kinematical hypotheses of Section 2.2 hold.

4.1.2. Equilibrium equations

Let $\boldsymbol{t}ds$ be the force exerted across a directed arc $d\boldsymbol{r} = \boldsymbol{a}_1 dX^1 + \boldsymbol{a}_2 dX^2$, whose initial length is ds, and write

$$\boldsymbol{t}ds = \boldsymbol{t}_1 dX^2 - \boldsymbol{t}_2 dX^1 \tag{4.1.1}$$

with $\boldsymbol{t}_1$ the force per unit length exerted across the line $X^1 = X^1_0$ by the material located on the side $X^1 > X^1_0$ on that on the side $X^1 \leq X^1_0$, and $\boldsymbol{t}_2$ the force per unit length exerted across the line $X^2 = X^2_0$ by the material on the side $X^2 > X^2_0$ on the material on the side $X^2 \leq X^2_0$.

Assume that an externally imposed force $\boldsymbol{f}$ per unit reference area acts on the sheet. Then, at equilibrium, for any part $\mathcal{P}$ of the sheet, the sum of the external forces and the forces exerted through the boundary of the sheet is zero, hence the following equation holds

$$\oint_{\partial\mathcal{P}} (\boldsymbol{t}_1 dX^2 - \boldsymbol{t}_2 dX^1) + \iint_{\mathcal{P}} \boldsymbol{f} dX^1 dX^2 = \boldsymbol{0}, \qquad \forall \mathcal{P} \subseteq \Sigma_0. \tag{4.1.2}$$

Equation (4.1.2) holds for any part of the sheet, so that, if $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are smooth, using the divergence theorem, the local form of the **balance of forces**

$$\boldsymbol{t}_{1,1} + \boldsymbol{t}_{2,2} + \boldsymbol{f} = \boldsymbol{0} \tag{4.1.3}$$

follows.

Assume that the material is endowed with bending stiffness. Let $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ be the couples per unit length, defined with the same sign convention adopted for $\boldsymbol{t}_1$, $\boldsymbol{t}_2$, so that the couple $\boldsymbol{c}$ per unit reference length across a directed arc is given by

$$\boldsymbol{c}ds = \boldsymbol{c}_1 dX^2 - \boldsymbol{c}_2 dX^1. \tag{4.1.4}$$

Balance of couples asserts that the torque exerted on any part of the sheet added to the bending couples, vanishes for any part $\mathcal{P}$ of Σ_0

$$\oint_{\partial\mathcal{P}} \boldsymbol{r} \times (\boldsymbol{t}_1 dX^2 - \boldsymbol{t}_2 dX^1) + \iint_{\mathcal{P}} \boldsymbol{r} \times \boldsymbol{f} dX^1 dX^2 + \oint_{\partial\mathcal{P}} (\boldsymbol{c}_1 dX^2 - \boldsymbol{c}_2 dX^1) = \boldsymbol{0}, \qquad \forall \mathcal{P} \subseteq \Sigma_0 \tag{4.1.5}$$

which yields the local equation

$$(\boldsymbol{r} \times \boldsymbol{t}_1 + \boldsymbol{c}_1)_{,1} + (\boldsymbol{r} \times \boldsymbol{t}_2 + \boldsymbol{c}_2)_{,2} + \boldsymbol{r} \times \boldsymbol{f} = \boldsymbol{0}. \tag{4.1.6}$$

If (4.1.3) is satisfied, (4.1.6) becomes

$$\boldsymbol{r}_{,1} \times \boldsymbol{t}_1 + \boldsymbol{r}_{,2} \times \boldsymbol{t}_2 + \boldsymbol{c}_{1,1} + \boldsymbol{c}_{2,2} = \boldsymbol{0}. \tag{4.1.7}$$

4.1.3. Constitutive equations

The small bending stiffness of the fabric can usually be neglected, but it has to be accounted for in deformations that involve wrinkling. In this section, simple constitutive equations for the couple stress vectors that account for bending couples in the deformed sheet are postulated, following Wang and Pipkin.

These authors assume that each fiber in the fabric supports a bending couple proportional to the curvature $\mathcal{K}$ of the fiber, i.e.,

$$\boldsymbol{c}_1 = \Gamma \boldsymbol{a}_1 \times \boldsymbol{a}_{1,1} = \Gamma \mathcal{K}_1 \boldsymbol{b}_1 \qquad \boldsymbol{c}_2 = \Gamma \boldsymbol{a}_2 \times \boldsymbol{a}_{2,2} = \Gamma \mathcal{K}_2 \boldsymbol{b}_2 \tag{4.1.8}$$

where the positive constant Γ is the stiffness coefficient.

Usually, in theories with no couple stress, the balance of couples yields a restriction on the form of the constitutive equation for the stress, equivalent to the requirement of symmetry of the stress tensor. Here, equation (4.1.7) may be treated in a similar fashion: $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are required to be such that (4.1.7) is satisfied identically, so that the rotational equilibrium equation (4.1.6) is automatically satisfied whenever the translational equilibrium equation (4.1.3) is satisfied.

Also the following constitutive relations for the stress vectors hold

$$\begin{cases} \boldsymbol{t}_1 = T_1\boldsymbol{a}_1 + S\boldsymbol{a}_2 - \Gamma\boldsymbol{a}_{1,11} \\ \boldsymbol{t}_2 = T_2\boldsymbol{a}_2 + S\boldsymbol{a}_1 - \Gamma\boldsymbol{a}_{2,22} \end{cases} \tag{4.1.9}$$

with T_1 and T_2 the fiber tensions (reactions to the constraint of inextensibility of the fiber) and S the shear stress. When the stiffness coefficient Γ is zero, these equations reduce to the equations of the theory of networks with no bending stiffness of Pipkin [**50**], and the stress is symmetric by the equality of the shear stress S in the two equations.

However, in the present case, the stress is not symmetric and in general not tangent to the sheet. The terms involving Γ have non vanishing components in the $\boldsymbol{N}$ direction, which are analogous to the shearing stresses on the cross section of a plate.

4.2. The model of Luo and Steigmann

4.2.1. Constitutive assumptions

Luo and Steigmann in [**37**] developed a model for an inextensible network which takes into account both bending and twisting effects in three dimensional deformations. The network is composed by two families of elastic inextensible fibers satisfying the Bernoulli-Euler hypotheses, i.e., the cross sections of each fiber remain plane, undergoes no strain, and are normal to the fiber in every configuration. It is also assumed that fibers are fastened together at their points of intersection to prevent slip of the fibers of one family relative to the other. The same kinematical hypotheses of subsection 2.2 are assumed to hold.

The basic idea is to assume that to each family of fibers in the actual configuration is associated an orthonormal basis $\{\boldsymbol{a}_i^{(1)}\}_{i=1,2,3}$ and $\{\boldsymbol{a}_i^{(2)}\}_{i=1,2,3}$, that describes the configuration and the twist of each family of fibers.

The unit vector $\boldsymbol{a}_1^{(1)}$ corresponds to the tangent vector to the first family of fibers, and $\boldsymbol{a}_1^{(2)}$ is the tangent vector to the second family of fibers, so that

$$\begin{aligned}
&\boldsymbol{a}_1^{(1)} = \boldsymbol{a}_1\\
&\boldsymbol{a}_i^{(1)} \cdot \boldsymbol{a}_j^{(1)} = \delta_{ij}\\
&\boldsymbol{a}_i^{(1)} \cdot \boldsymbol{a}_j^{(1)} \times \boldsymbol{a}_k^{(1)} = e_{ijk}\\
&i, j, k = 1, 2, 3
\end{aligned} \tag{4.2.1}$$

with δ_{ij} the Kronecker delta and e_{ijk} the permutation symbol. Analogously, an orthonormal basis $\{\boldsymbol{a}_i^{(2)}\}$ is associated to family a_2, with:

$$\begin{aligned}
&\boldsymbol{a}_1^{(2)} = \boldsymbol{a}_2\\
&\boldsymbol{a}_i^{(2)} \cdot \boldsymbol{a}_j^{(2)} = \delta_{ij}\\
&\boldsymbol{a}_i^{(2)} \cdot \boldsymbol{a}_j^{(2)} \times \boldsymbol{a}_k^{(2)} = e_{ijk}\\
&i, j, k = 1, 2, 3.
\end{aligned} \tag{4.2.2}$$

Also, let

$$\kappa_i^1 = \frac{1}{2} e_{ijk} \frac{\partial \boldsymbol{a}_k^{(1)}}{\partial X^1} \cdot \boldsymbol{a}_j^{(1)} \qquad\qquad \kappa_i^2 = \frac{1}{2} e_{ijk} \frac{\partial \boldsymbol{a}_k^{(2)}}{\partial X^2} \cdot \boldsymbol{a}_j^{(2)}. \tag{4.2.3}$$

We show that the twist and the curvatures of the a_1 and a_2 lines can be expressed in terms of κ_i^1 and κ_i^2.

In order to express the triad $\boldsymbol{a}_i^{(1)}$ in terms of the Frenet's triad $\{\boldsymbol{a}_1, \boldsymbol{n}_1, \boldsymbol{b}_1\}$ Luo and Steigmann introduce the angle $\theta^1(X^1, X^2)$ through the relations

$$\begin{aligned}
\boldsymbol{a}_1^{(1)} &= \boldsymbol{a}_1\\
\boldsymbol{a}_2^{(1)} &= \cos\theta^1 \boldsymbol{n}_1 + \sin\theta^1 \boldsymbol{b}_1\\
\boldsymbol{a}_3^{(1)} &= -\sin\theta^1 \boldsymbol{n}_1 + \cos\theta^1 \boldsymbol{b}_1,
\end{aligned} \tag{4.2.4}$$

and the angle θ^2 associated to the second family of fibers by means of expressions analogous to (4.2.4) involving the $\boldsymbol{a}_i^{(2)}$.

After some substitutions, the following equations are found for the first family of fibers

$$\begin{aligned}
\kappa_1^1 &= \tau^1 + \frac{\partial \theta^1}{\partial X^1}\\
\kappa_2^1 &= \mathcal{K}^1 \sin\theta^1\\
\kappa_3^1 &= \mathcal{K}^1 \cos\theta^1
\end{aligned} \tag{4.2.5}$$

and for the second family of fibers

$$\begin{aligned}\kappa_1^2 &= \tau^2 + \frac{\partial \theta^2}{\partial X^2}\\ \kappa_2^2 &= \mathcal{K}^2 \sin\theta^2\\ \kappa_3^2 &= \mathcal{K}^2 \cos\theta^2.\end{aligned} \tag{4.2.6}$$

where $\mathcal{K}^1$, $\mathcal{K}^2$ are the curvatures of the fibers.

Introducing the twist angles β^1, β^2 for each family of fibers, through

$$\beta^1 = \tau^1 + \frac{\partial \theta^1}{\partial X^1} \qquad \beta^2 = \tau^2 + \frac{\partial \theta^2}{\partial X^2} \tag{4.2.7}$$

equations $(4.2.5)_1$, $(4.2.6)_1$ can be written as

$$\beta^1 = \kappa_1^1 \qquad \beta^2 = \kappa_1^2. \tag{4.2.8}$$

Hence, the independent variables of this model are the vector $\boldsymbol{r}$, and the angles of twist of the sections θ^1 and θ^2.

4.2.2. The Euler–Lagrange equations

Luo and Steigmann consider a strain energy density per unit reference area of the form

$$W = W(\sin\gamma, \mathcal{K}^1, \mathcal{K}^2, \beta^1, \beta^2).$$

The energy functional $E(\boldsymbol{r}, \boldsymbol{a}_i^{(1)}, \boldsymbol{a}_i^{(2)})$ is assumed to be

$$\begin{aligned}E(\boldsymbol{r}, \boldsymbol{a}_i^{(1)}, \boldsymbol{a}_i^{(2)}) = &\iint_{\Sigma_0} W(\sin\gamma, \mathcal{K}^1, \mathcal{K}^2, \beta^1, \beta^2)dX^1dX^2 - \iint_{\Sigma_0} \boldsymbol{f}\cdot\boldsymbol{r}dX^1dX^2\\ &- \oint_{\partial\Sigma_0}\left(\bar{\boldsymbol{t}}\cdot\boldsymbol{r} + \boldsymbol{m}\cdot\frac{\partial \boldsymbol{r}}{\partial\boldsymbol{\nu}} + \boldsymbol{n}_1^{(1)}\cdot\boldsymbol{a}_2^{(1)} + \boldsymbol{n}_2^{(1)}\cdot\boldsymbol{a}_3^{(1)} + \boldsymbol{n}_1^{(2)}\cdot\boldsymbol{a}_2^{(2)} + \boldsymbol{n}_2^{(2)}\cdot\boldsymbol{a}_3^{(2)}\right) ds\end{aligned} \tag{4.2.9}$$

where the last integral is a line integral along the boundary of the region occupied by the fabric, $\dfrac{\partial \boldsymbol{r}}{\partial\boldsymbol{\nu}}$ denotes the partial derivative of $\boldsymbol{r}$ with respect to arc length in the direction of the outward normal $\boldsymbol{\nu}$ to $\partial\Sigma_0$. $\boldsymbol{f}$ and $\bar{\boldsymbol{t}}$ represent dead loads per unit reference area and unit reference length respectively. Finally, $\boldsymbol{n}_1^{(1)}(s)$, $\boldsymbol{n}_2^{(1)}(s)$, $\boldsymbol{n}_1^{(2)}(s)$, $\boldsymbol{n}_2^{(2)}(s)$ are dead loads per unit length of $\partial\Sigma_0$ and are related to the boundary condition for the twisting couples.

A lengthy calculation shows that the Euler Lagrange equations of the above functional are

$$\begin{cases} \boldsymbol{t}_1 = T^1\boldsymbol{a}_1 + \dfrac{\partial W}{\partial \sin\gamma}\boldsymbol{a}_2 - \dfrac{\partial}{\partial X^1}\left(\dfrac{\partial W}{\partial \mathcal{K}^1}\boldsymbol{n}_1\right) + \dfrac{\partial W}{\partial \beta^1}\mathcal{K}^1\boldsymbol{b}_1 \\ \boldsymbol{t}_2 = T^2\boldsymbol{a}_2 + \dfrac{\partial W}{\partial \sin\gamma}\boldsymbol{a}_1 - \dfrac{\partial}{\partial X^2}\left(\dfrac{\partial W}{\partial \mathcal{K}^2}\boldsymbol{n}_2\right) + \dfrac{\partial W}{\partial \beta^2}\mathcal{K}^2\boldsymbol{b}_2 \\ \dfrac{\partial}{\partial X^1}\left(\dfrac{\partial W}{\partial \beta^1}\right) = 0 \\ \dfrac{\partial}{\partial X^2}\left(\dfrac{\partial W}{\partial \beta^2}\right) = 0 \\ \dfrac{\partial \boldsymbol{t}_1}{\partial X^1} + \dfrac{\partial \boldsymbol{t}_2}{\partial X^2} + \boldsymbol{f} = \boldsymbol{0}\ . \end{cases} \tag{4.2.10}$$

In order to compare this model with the one of Wang and Pipkin, relations (4.1.9), (4.1.3), where the inextensible network has only resistance to shear and bending stiffness, and twist is neglected, consider a special form of the strain energy function

$$W = W_0(\boldsymbol{a}_1 \cdot \boldsymbol{a}_2) + \frac{1}{2}\Gamma\left(\frac{\partial \boldsymbol{a}_1}{\partial X^1}\cdot\frac{\partial \boldsymbol{a}_1}{\partial X^1} + \frac{\partial \boldsymbol{a}_2}{\partial X^2}\cdot\frac{\partial \boldsymbol{a}_2}{\partial X^2}\right) \tag{4.2.11}$$

or equivalently

$$W = W_0(\sin\gamma) + \frac{1}{2}\Gamma\left[\left(\mathcal{K}^1\right)^2 + \left(\mathcal{K}^1\right)^2\right] \tag{4.2.12}$$

with Γ the stiffness coefficient of equations (4.1.8). Using (4.2.12) in (4.2.10), we obtain

$$\begin{cases} \boldsymbol{t}_1 = T^1\boldsymbol{a}_1 + \dfrac{dW_0}{d\sin\gamma}\boldsymbol{a}_1 - \Gamma\boldsymbol{a}_{1,11} \\ \boldsymbol{t}_2 = T^2\boldsymbol{a}_2 + \dfrac{dW_0}{d\sin\gamma}\boldsymbol{a}_2 - \Gamma\boldsymbol{a}_{2,22} \\ \dfrac{\partial \boldsymbol{t}_1}{\partial X^1} + \dfrac{\partial \boldsymbol{t}_2}{\partial X^2} + \boldsymbol{f} = \boldsymbol{0} \end{cases} \tag{4.2.13}$$

that are equivalent to equations (4.1.9).

CHAPTER 5

Inextensible networks with bending and torsional effects

Contents

In this chapter we discuss the model for two families of inextensible fibers forming a surface published in [**29**]. The bending stiffness and the resistance to torsional changes of the fibers are taken into account, in order to describe the static behavior of textile fabric. We assume that each fiber supports a torsional couple proportional to the torsion of the fiber. We obtain a set of equations accounting for the effect of the torsion of the fibers on the deformation of the sheet. We first consider a strain energy density in an additive form, such that the contributions due to shear, torsion and bending effects are taken into account separately, and then generalize the result to an arbitrary dependence of the energy on curvature and torsion.

5.1. Basic assumptions

We consider a surface $\mathcal{S}$ made of two families of inextensible fibers. Fibers are continuously distributed. The reference configuration of the surface occupies a plane region Σ_0. Let X^1, X^2

be cartesian coordinates in Σ_0: we assume in this section that, in the reference configuration, the two families of inextensible fibers lie parallel to the X^1, X^2 axes. Let $\boldsymbol{A}_1$ be the tangent vector to the curve corresponding to the fiber X^2 =const. in the reference configuration, and $\boldsymbol{A}_2$ the tangent vector to the fiber X^1 =const.

The tangent vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ to the deformed fibers are defined trough relations (2.1.2). Since the fibers are assumed to be inextensible, these vectors are unitary; a Frenet's triad of vectors, satisfying equations (2.2.4), is associated to each family of fibers, respectively $\{\boldsymbol{a}_1, \boldsymbol{n}_1, \boldsymbol{b}_1\}$ to the first family of fibers and $\{\boldsymbol{a}_2, \boldsymbol{n}_2, \boldsymbol{b}_2\}$ to the second family of fibers.

The same notation as in the previous chapter is adopted for the angle of shear γ and for the normal vector $\boldsymbol{N} = \dfrac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}$.

We assume that the fibers forming the network satisfy the Bernoulli-Euler hypotheses: namely, the fiber cross sections remain plane and orthogonal to the central line of the fiber. We also assume that at the intersection points of the two families the fibers are fastened together, so that one family of fibers does not slip relative to the other.

5.2. Torsional effects

In this section we develop a model that describes the effects of the torsion of the fiber on the deformation of the fabric. We assume that to each family of fibers a torsional couple is associated. In particular, we are interested in describing the effects related to the torsion of the central lines of the fibers, but neglecting the rotation of the sections of the fibers: by consequence, we choose couples proportional to the torsions τ_1, τ_2 of each family of fibers. Recall that the torsion of a curve measures how much it deviates from being plane.

Let $\boldsymbol{c}$ be the torsional couple per unit reference length across a directed arc

$$\boldsymbol{c}ds = \boldsymbol{c}_1 dX^2 - \boldsymbol{c}_2 dX^1 \tag{5.2.1}$$

where the same sign convention used for (4.1.1) is adopted.

The balance of forces (4.1.2), (4.1.3) and the balance of couples, viz. relations (4.1.5) and (4.1.6), still hold. The balance of forces reads

$$\oint_{\partial\mathcal{P}} (\boldsymbol{t}_1 dX^2 - \boldsymbol{t}_2 dX^1) + \iint_{\mathcal{P}} \boldsymbol{f} dX^1 dX^2 = \boldsymbol{0}, \qquad \forall\mathcal{P} \subseteq \Sigma_0.$$

Hence in local form

$$\boldsymbol{t}_{1,1} + \boldsymbol{t}_{2,2} + \boldsymbol{f} = \boldsymbol{0}. \tag{5.2.2}$$

The couple balance is

$$\oint_{\partial\mathcal{P}} \boldsymbol{r}\times(\boldsymbol{t}_1 dX^2 - \boldsymbol{t}_2 dX^1) + \iint_{\mathcal{P}} \boldsymbol{r}\times\boldsymbol{f} dX^1 dX^2 + \oint_{\partial\mathcal{P}} (\boldsymbol{c}_1 dX^2 - \boldsymbol{c}_2 dX^1) = \boldsymbol{0} \qquad \forall\mathcal{P}\subseteq\Sigma_0 \tag{5.2.3}$$

which becomes in local form

$$(\boldsymbol{r}\times\boldsymbol{t}_1 + \boldsymbol{c}_1)_{,1} + (\boldsymbol{r}\times\boldsymbol{t}_2 + \boldsymbol{c}_2)_{,2} + \boldsymbol{r}\times\boldsymbol{f} = \boldsymbol{0}. \tag{5.2.4}$$

When (5.2.2) is satisfied, (5.2.4) reduces to

$$\boldsymbol{r}_{,1}\times\boldsymbol{t}_1 + \boldsymbol{r}_{,2}\times\boldsymbol{t}_2 + \boldsymbol{c}_{1,1} + \boldsymbol{c}_{2,2} = \boldsymbol{0}. \tag{5.2.5}$$

We first assume that the vector associated to the couple of the first set of fibers, is related to the binormal by the constitutive relation

$$\boldsymbol{c}_1 = \Lambda_1 \boldsymbol{b}_1 \times \boldsymbol{b}_{1,1} \tag{5.2.6}$$

with Λ_1 a constant coefficient measuring the resistance to torsional changes of the first family of fibers. We show below that such a constitutive relation follows from the assumption that the energy is quadratic in the torsion.

Using $(2.2.4)_3$, we have

$$\boldsymbol{c}_1 = \Lambda_1 \tau_1 \boldsymbol{a}_1.$$

Similarly, letting

$$\boldsymbol{c}_2 = \Lambda_2 \boldsymbol{b}_2 \times \boldsymbol{b}_{2,2} \tag{5.2.7}$$

for the second family of fibers and by Frenet's formulae, we obtain

$$\boldsymbol{c}_2 = \Lambda_2 \tau_2 \boldsymbol{a}_2.$$

From definition (5.2.1), since $\boldsymbol{a}_i = \boldsymbol{r}_{,i}$ (with $i = 1, 2$), and using Frenet's formulas (2.2.4), the couple balance (5.2.4) becomes

$$\boldsymbol{a}_1\times\boldsymbol{t}_1 + \Lambda_1\tau_1\mathcal{K}_1(\boldsymbol{b}_1\times\boldsymbol{a}_1) + \Lambda_1\tau_{1,1}\boldsymbol{a}_1 + \boldsymbol{a}_2\times\boldsymbol{t}_2 + \Lambda_2\tau_2\mathcal{K}_2(\boldsymbol{b}_2\times\boldsymbol{a}_2) + \Lambda_2\tau_{2,2}\boldsymbol{a}_2 = \boldsymbol{0} \tag{5.2.8}$$

and, rearranging terms

$$\boldsymbol{a}_1\times[\boldsymbol{t}_1 - \Lambda_1\tau_1\mathcal{K}_1\boldsymbol{b}_1] + \Lambda_1\tau_{1,1}\boldsymbol{a}_1 = -\boldsymbol{a}_2\times[\boldsymbol{t}_2 - \Lambda_2\tau_2\mathcal{K}_2\boldsymbol{b}_2] - \Lambda_2\tau_{2,2}\boldsymbol{a}_2. \tag{5.2.9}$$

5.3. Special solutions: helical fibers

If we restrict to helical fibers, the torsions remain constant along the fibers, and the derivatives $\tau_{1,1}$ and $\tau_{2,2}$ vanish. Equation (5.2.9) then becomes

$$\boldsymbol{a}_1 \times [\boldsymbol{t}_1 - \Lambda_1\tau_1\mathcal{K}_1\boldsymbol{b}_1] = -\boldsymbol{a}_2 \times [\boldsymbol{t}_2 - \Lambda_2\tau_2\mathcal{K}_2\boldsymbol{b}_2]\,. \tag{5.3.1}$$

Notice that the left and right members of this equation are orthogonal to $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ respectively: let $\widetilde{S}\boldsymbol{a}_1 \times \boldsymbol{a}_2$ be their common value. Then

$$\boldsymbol{a}_1 \times [\boldsymbol{t}_1 - \Lambda_1\tau_1\mathcal{K}_1\boldsymbol{b}_1] = [\boldsymbol{t}_2 - \Lambda_2\tau_2\mathcal{K}_2\boldsymbol{b}_2] \times \boldsymbol{a}_2 = \widetilde{S}\boldsymbol{a}_1 \times \boldsymbol{a}_2 \tag{5.3.2}$$

so that

$$\boldsymbol{a}_1 \times \left[\boldsymbol{t}_1 - \Lambda_1\tau_1\mathcal{K}_1\boldsymbol{b}_1 - \widetilde{S}\boldsymbol{a}_2\right] = \boldsymbol{0}, \tag{5.3.3}$$

and

$$\left[\boldsymbol{t}_2 - \Lambda_2\tau_2\mathcal{K}_2\boldsymbol{b}_2 - \widetilde{S}\boldsymbol{a}_1\right] \times \boldsymbol{a}_2 = \boldsymbol{0}. \tag{5.3.4}$$

Now, (5.3.3) implies that the vector in the square brackets is parallel to $\boldsymbol{a}_1$. Let $\widetilde{T}_1\boldsymbol{a}_1$ be its value. Analogously, the vector $\left[\boldsymbol{t}_2 - \Lambda_2\tau_2\mathcal{K}_2\boldsymbol{b}_2 - \widetilde{S}\boldsymbol{a}_1\right]$ is parallel to $\boldsymbol{a}_2$, and we write $\widetilde{T}_2\boldsymbol{a}_2$ for its value. We conclude that the following equations hold

$$\begin{cases} \boldsymbol{t}_1 = \widetilde{T}_1\boldsymbol{a}_1 + \widetilde{S}\boldsymbol{a}_2 + \Lambda_1\tau_1\mathcal{K}_1\boldsymbol{b}_1 \\ \boldsymbol{t}_2 = \widetilde{T}_2\boldsymbol{a}_2 + \widetilde{S}\boldsymbol{a}_1 + \Lambda_2\tau_2\mathcal{K}_2\boldsymbol{b}_2 \\ \tau_{1,1} = 0 \\ \tau_{2,2} = 0. \end{cases} \tag{5.3.5}$$

Comparing these equations with those obtained by Wang and Pipkin (4.1.9), it can be easily seen that $\widetilde{T}_1$ and $\widetilde{T}_2$ correspond to the fiber tensions T_1, T_2, namely the reactions to the constraint of fiber inextensibility. Also, $\widetilde{S}$ corresponds to the shearing stress S, but new terms $\Lambda_1\tau_1\mathcal{K}_1\boldsymbol{b}_1$, $\Lambda_2\tau_2\mathcal{K}_2\boldsymbol{b}_2$ related to the changes of torsion appear.

5.4. Strain energy

We now discuss the general model that accounts for bending and torsion of the fibers. In this case, it is meaningful to consider a strain energy function W of the form

$$W = W(\sin\gamma, \mathcal{K}_1, \mathcal{K}_2, \tau_1, \tau_2). \tag{5.4.1}$$

Recalling that

$$\sin\gamma = \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 \ , \qquad (\mathcal{K}_i)^2 = \boldsymbol{a}_{i,i} \cdot \boldsymbol{a}_{i,i} \ , \qquad (\tau_i)^2 = \boldsymbol{b}_{i,i} \cdot \boldsymbol{b}_{i,i}, \tag{5.4.2}$$

it is clear that W in (5.4.1) is a special case of

$$W = W(\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_{1,1},\ \boldsymbol{a}_{2,2},\ \boldsymbol{b}_{1,1},\ \boldsymbol{b}_{2,2}). \tag{5.4.3}$$

The dissipation inequality, a statement equivalent to the second law of thermodynamics, imposes restrictions on the structure of the constitutive relations. To see this, consider (in this section only) time-dependent deformations $\mathbf{r} = \mathbf{r}(X^i, t)$, and notice that the Frenet basis at each point of a fiber undergoes a rigid motion under such a deformation (being an orthonormal basis by definition). Hence, denoting the time derivative by a superposed dot, there exist vectors $\boldsymbol{\omega}_i$, $i = 1, 2$ such that $\dot{\boldsymbol{a}}_i = \boldsymbol{\omega}_i \times \boldsymbol{a}_i$, $\dot{\boldsymbol{n}}_i = \boldsymbol{\omega}_i \times \boldsymbol{n}_i$ and $\dot{\boldsymbol{b}}_i = \boldsymbol{\omega}_i \times \boldsymbol{b}_i$. The dissipation inequality states that the rate of change of energy cannot be larger than the work done by internal forces, for each process and each portion $\mathcal{P}$ of the surface (we have assumed that the external forces and couples vanish)

$$\frac{d}{dt}\int_{\mathcal{P}} W\, da \leq \int_{\partial\mathcal{P}} \left(\boldsymbol{t}_1 \cdot \dot{\boldsymbol{r}} dX^2 - \boldsymbol{t}_2 \cdot \dot{\boldsymbol{r}} dX^1\right) + \int_{\partial\mathcal{P}} \left(\boldsymbol{c}_1 \cdot \boldsymbol{\omega}_1\, dX^2 - \boldsymbol{c}_2 \cdot \boldsymbol{\omega}_2\, dX^1\right). \tag{5.4.4}$$

Applying the divergence theorem and (5.2.2), and differentiating under the integral sign, and taking into account that the subsurface $\mathcal{P}$ is arbitrary, we find

$$\dot{W} - \boldsymbol{t}_1 \cdot \dot{\boldsymbol{a}}_1 - \boldsymbol{t}_2 \cdot \dot{\boldsymbol{a}}_2 - \boldsymbol{c}_{1,1} \cdot \boldsymbol{\omega}_1 - \boldsymbol{c}_1 \cdot \boldsymbol{\omega}_{1,1} - \boldsymbol{c}_{2,2} \cdot \boldsymbol{\omega}_2 - \boldsymbol{c}_2 \cdot \boldsymbol{\omega}_{2,2} \leq 0. \tag{5.4.5}$$

To account for a dependence of the stored energy on both the curvature and torsion, assume that (5.4.3) holds. Then

$$\dot{W} = \frac{\partial W}{\partial \boldsymbol{a}_1} \cdot \dot{\boldsymbol{a}}_1 + \frac{\partial W}{\partial \boldsymbol{a}_2} \cdot \dot{\boldsymbol{a}}_2 + \frac{\partial W}{\partial \boldsymbol{a}_{1,1}} \cdot \dot{\boldsymbol{a}}_{1,1} + \frac{\partial W}{\partial \boldsymbol{a}_{2,2}} \cdot \dot{\boldsymbol{a}}_{2,2} + \frac{\partial W}{\partial \boldsymbol{b}_{1,1}} \cdot \dot{\boldsymbol{b}}_{1,1} + \frac{\partial W}{\partial \boldsymbol{b}_{2,2}} \cdot \dot{\boldsymbol{b}}_{2,2}.$$

Recalling that $\dot{\boldsymbol{a}}_i = \boldsymbol{\omega}_i \times \boldsymbol{a}_i$, so that $\dot{\boldsymbol{a}}_{i,i} = \boldsymbol{\omega}_{i,i} \times \boldsymbol{a}_i + \boldsymbol{\omega}_i \times \boldsymbol{a}_{i,i}$, we obtain from (5.4.5)

$$\begin{aligned}
&\left(\boldsymbol{a}_1 \times \frac{\partial W}{\partial \boldsymbol{a}_1} - \boldsymbol{a}_1 \times \boldsymbol{t}_1 + \boldsymbol{a}_{1,1} \times \frac{\partial W}{\partial \boldsymbol{a}_{1,1}} + \boldsymbol{b}_{1,1} \times \frac{\partial W}{\partial \boldsymbol{b}_{1,1}} - \boldsymbol{c}_{1,1}\right) \cdot \boldsymbol{\omega}_1 \\
&+ \left(\boldsymbol{a}_2 \times \frac{\partial W}{\partial \boldsymbol{a}_2} - \boldsymbol{a}_2 \times \boldsymbol{t}_2 + \boldsymbol{a}_{2,2} \times \frac{\partial W}{\partial \boldsymbol{a}_{2,2}} + \boldsymbol{b}_{2,2} \times \frac{\partial W}{\partial \boldsymbol{b}_{2,2}} - \boldsymbol{c}_{2,2}\right) \cdot \boldsymbol{\omega}_2 \\
&+ \left(\boldsymbol{a}_1 \times \frac{\partial W}{\partial \boldsymbol{a}_{1,1}} + \boldsymbol{b}_1 \times \frac{\partial W}{\partial \boldsymbol{b}_{1,1}} - \boldsymbol{c}_1\right) \cdot \boldsymbol{\omega}_{1,1} + \left(\boldsymbol{a}_2 \times \frac{\partial W}{\partial \boldsymbol{a}_{2,2}} + \boldsymbol{b}_2 \times \frac{\partial W}{\partial \boldsymbol{b}_{2,2}} - \boldsymbol{c}_2\right) \cdot \boldsymbol{\omega}_{2,2} \leq 0.
\end{aligned}$$

Requiring that the above inequality holds for every time-dependent deformation, yields the constitutive relations

$$\boldsymbol{c}_1 = \boldsymbol{a}_1 \times \frac{\partial W}{\partial \boldsymbol{a}_{1,1}} + \boldsymbol{b}_1 \times \frac{\partial W}{\partial \boldsymbol{b}_{1,1}}, \qquad \boldsymbol{c}_2 = \boldsymbol{a}_2 \times \frac{\partial W}{\partial \boldsymbol{a}_{2,2}} + \boldsymbol{b}_2 \times \frac{\partial W}{\partial \boldsymbol{b}_{2,2}}, \tag{5.4.6}$$

and the restrictions

$$\boldsymbol{a}_1 \times \frac{\partial W}{\partial \boldsymbol{a}_1} - \boldsymbol{a}_1 \times \boldsymbol{t}_1 - \boldsymbol{a}_1 \times \left(\frac{\partial W}{\partial \boldsymbol{a}_{1,1}}\right)_{,1} - \boldsymbol{b}_1 \times \left(\frac{\partial W}{\partial \boldsymbol{b}_{1,1}}\right)_{,1} = 0$$

and

$$\boldsymbol{a}_2 \times \frac{\partial W}{\partial \boldsymbol{a}_2} - \boldsymbol{a}_2 \times \boldsymbol{t}_2 - \boldsymbol{a}_2 \times \left(\frac{\partial W}{\partial \boldsymbol{a}_{2,2}}\right)_{,2} - \boldsymbol{b}_2 \times \left(\frac{\partial W}{\partial \boldsymbol{b}_{2,2}}\right)_{,2} = 0$$

Rewriting the first equality above as

$$\boldsymbol{a}_1 \times \left(\frac{\partial W}{\partial \boldsymbol{a}_1} - \left(\frac{\partial W}{\partial \boldsymbol{a}_{1,1}}\right)_{,1} - \boldsymbol{t}_1\right) = \boldsymbol{b}_1 \times \left(\frac{\partial W}{\partial \boldsymbol{b}_{1,1}}\right)_{,1}, \tag{5.4.7}$$

and taking the scalar products of (5.4.7) with $\boldsymbol{b}_1$ and $\boldsymbol{a}_1$ respectively, we obtain the constitutive restrictions

$$\boldsymbol{n}_1 \cdot \left(\frac{\partial W}{\partial \boldsymbol{a}_1} - \left(\frac{\partial W}{\partial \boldsymbol{a}_{1,1}}\right)_{,1} - \boldsymbol{t}_1\right) = 0, \qquad \boldsymbol{n}_1 \cdot \left(\frac{\partial W}{\partial \boldsymbol{b}_{1,1}}\right)_{,1} = 0. \tag{5.4.8}$$

Moreover, notice that (5.4.7) does not yield any information on the component of $\left(\frac{\partial W}{\partial \boldsymbol{a}_1} - \left(\frac{\partial W}{\partial \boldsymbol{a}_{1,1}}\right)_{,1} - \boldsymbol{t}_1\right)$ parallel to $\boldsymbol{a}_1$, which is therefore indeterminate, so that we may write

$$\boldsymbol{a}_1 \cdot \left(\frac{\partial W}{\partial \boldsymbol{a}_1} - \left(\frac{\partial W}{\partial \boldsymbol{a}_{1,1}}\right)_{,1} - \boldsymbol{t}_1\right) = -T_1,$$

with T_1 an indeterminate multiplier, an analogous relation involving a multiplier T_2 holding for the second family of fibers. It follows that (5.4.7) reduces to the relation

$$-\boldsymbol{b}_1 \cdot \left(\frac{\partial W}{\partial \boldsymbol{a}_1} - \left(\frac{\partial W}{\partial \boldsymbol{a}_{1,1}}\right)_{,1} - \boldsymbol{t}_1\right) = \boldsymbol{a}_1 \cdot \left(\frac{\partial W}{\partial \boldsymbol{b}_{1,1}}\right)_{,1}.$$

Collecting the above results, we obtain finally the constitutive relations

$$\boldsymbol{t}_1 = T_1 \boldsymbol{a}_1 + \frac{\partial W}{\partial \boldsymbol{a}_1} - \left(\frac{\partial W}{\partial \boldsymbol{a}_{1,1}}\right)_{,1} + \left(\boldsymbol{a}_1 \cdot \left(\frac{\partial W}{\partial \boldsymbol{b}_{1,1}}\right)_{,1}\right) \boldsymbol{b}_1 \tag{5.4.9}$$

and

$$\boldsymbol{t}_2 = T_2\boldsymbol{a}_2 + \frac{\partial W}{\partial \boldsymbol{a}_2} - \left(\frac{\partial W}{\partial \boldsymbol{a}_{2,2}}\right)_{,2} + \left(\boldsymbol{a}_2 \cdot \left(\frac{\partial W}{\partial \boldsymbol{b}_{2,2}}\right)_{,2}\right)\boldsymbol{b}_2. \tag{5.4.10}$$

The relations (5.4.6), (5.4.8), and (5.4.9), (5.4.10), are the constitutive relations of a theory which includes both curvature and torsion.

It is not difficult to show that the above relations reduce to (5.3.5) for an energy function of the form

$$W = W_0(\sin\gamma) + \frac{1}{2}(\Lambda_1\tau_1^2 + \Lambda_2\tau_2^2).$$

5.5. A cylindrical shell made of helical fibers

Consider a cylindrical shell made of two families of inextensible helical fibers. Let L and R be the vertical step and the radius of the helices of the two families. A given configuration of the cylinder can be parametrized as

$$\begin{cases} x = R\cos\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}} \\ y = R\sin\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}} \\ z = \frac{L}{2\pi}\frac{X^1+X^2}{\sqrt{R^2+L^2/4\pi^2}} \end{cases}$$

for

$$X^1 - X^2 \in (-\pi\sqrt{R^2+L^2/4\pi^2}, \pi\sqrt{R^2+L^2/4\pi^2}),$$
$$X^1 + X^2 \in (0, 2\pi H\sqrt{R^2+L^2/4\pi^2}/L),$$

with H the height of the cylinder. The curves $X^1 =$ const. and $X^2 =$ const. define two families of transversal helices with radius R and step L, and X^1, X^2 are the arc parameters for each family.

The Frenet basis of the two families are

$$\begin{cases} \boldsymbol{a}_1 = \frac{1}{\sqrt{R^2+L^2/4\pi^2}}\left(-R\sin\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, R\cos\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, \frac{L}{2\pi}\right) \\ \boldsymbol{n}_1 = \left(-\cos\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, -\sin\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, 0\right) \\ \boldsymbol{b}_1 = \frac{1}{\sqrt{R^2+L^2/4\pi^2}}\left(\frac{L}{2\pi}\sin\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, -\frac{L}{2\pi}\cos\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, R\right) \end{cases}$$

and

$$\begin{cases} \boldsymbol{a}_2 = \frac{1}{\sqrt{R^2+L^2/4\pi^2}} \left(R\sin\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, -R\cos\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, \frac{L}{2\pi} \right) \\ \boldsymbol{n}_2 = \left(-\cos\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, -\sin\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, 0 \right) \\ \boldsymbol{b}_2 = \frac{1}{\sqrt{R^2+L^2/4\pi^2}} \left(\frac{L}{2\pi}\sin\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, -\frac{L}{2\pi}\cos\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}}, -R \right) \end{cases}$$

The curvature and torsion of each family is

$$\mathcal{K}_1 = \frac{R}{R^2+L^2/4\pi^2}, \quad \tau_1 = \frac{L/2\pi}{R^2+L^2/4\pi^2},$$

and

$$\mathcal{K}_2 = \frac{R}{R^2+L^2/4\pi^2}, \quad \tau_2 = -\frac{L/2\pi}{R^2+L^2/4\pi^2},$$

and

$$\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = \sin\gamma = \cos\delta = \frac{L^2/4\pi^2 - R^2}{L^2/4\pi^2 + R^2}.$$

Notice that

$$\boldsymbol{n}_1 = \boldsymbol{n}_2 =: \boldsymbol{n}, \quad \mathcal{K}_1 = \mathcal{K}_2 =: \mathcal{K}, \quad \tau_1 = -\tau_2 =: \tau.$$

Notice also the identity

$$\tau = \mathcal{K}\frac{L}{2\pi R} \tag{5.5.1}$$

holds.

5.5.1. Energy, stress and couples

Consider now a quadratic energy accounting for shear, bending and torsional effects

$$W = \frac{A}{2}(\sin\gamma - \sin\gamma^0)^2 + \frac{B}{2}\left[(\mathcal{K}_1 - \mathcal{K}_1^0)^2 + (\mathcal{K}_2 - \mathcal{K}_2^0)^2\right] + \frac{C}{2}\left[(\tau_1 - \tau_1^0)^2 + (\tau_2 - \tau_2^0)^2\right].$$

The corresponding tension for the first family of fibers is (a similar formula holding for the second family)

$$\boldsymbol{t}_1 = T_1\boldsymbol{a}_1 + \frac{\partial W}{\partial \sin\gamma}\boldsymbol{a}_2 - \left(\frac{\partial W}{\partial \boldsymbol{a}_{1,1}}\right)_{,1} + \left(\boldsymbol{a}_1 \cdot \left(\frac{\partial W}{\partial \boldsymbol{b}_{1,1}}\right)_{,1}\right)\boldsymbol{b}_1,$$

which yield

$$\begin{cases} \boldsymbol{t}_1 = T_1\boldsymbol{a}_1 + A(\sin\gamma - \sin\gamma^0)\boldsymbol{a}_2 - B(\mathcal{K}_1 - \mathcal{K}_1^0)(-\mathcal{K}_1\boldsymbol{a}_1 + \tau_1\boldsymbol{b}_1) + C(\tau_1 - \tau_1^0)\mathcal{K}_1\boldsymbol{b}_1, \\ \boldsymbol{t}_2 = T_2\boldsymbol{a}_2 + A(\sin\gamma - \sin\gamma^0)\boldsymbol{a}_1 - B(\mathcal{K}_2 - \mathcal{K}_2^0)(-\mathcal{K}_2\boldsymbol{a}_2 + \tau_2\boldsymbol{b}_2) + C(\tau_2 - \tau_2^0)\mathcal{K}_2\boldsymbol{b}_2. \end{cases}$$

For a cylindrical surfaces made of helices, the above formulas reduce to

$$\begin{cases} \boldsymbol{t}_1 = T_1\boldsymbol{a}_1 + A(\sin\gamma - \sin\gamma^0)\boldsymbol{a}_2 - B(\mathcal{K} - \mathcal{K}^0)(-\mathcal{K}\boldsymbol{a}_1 + \tau\boldsymbol{b}_1) + C(\tau - \tau^0)\mathcal{K}\boldsymbol{b}_1, \\ \boldsymbol{t}_2 = T_2\boldsymbol{a}_2 + A(\sin\gamma - \sin\gamma^0)\boldsymbol{a}_1 - B(\mathcal{K} - \mathcal{K}^0)(-\mathcal{K}\boldsymbol{a}_2 - \tau\boldsymbol{b}_2) - C(\tau - \tau^0)\mathcal{K}\boldsymbol{b}_2. \end{cases}$$

Also, the couples are given by the relations

$$\begin{aligned} \boldsymbol{c}_1 &= \boldsymbol{a}_1 \times \frac{\partial W}{\partial \boldsymbol{a}_{1,1}} + \boldsymbol{b}_1 \times \frac{\partial W}{\partial \boldsymbol{b}_{1,1}} \\ &= \boldsymbol{a}_1 \times [B(\mathcal{K}_1 - \mathcal{K}_1^0)\boldsymbol{n}_1] - \boldsymbol{b}_1 \times [C(\tau_1 - \tau_1^0)\boldsymbol{n}_1] \\ &= B(\mathcal{K} - \mathcal{K}^0)\boldsymbol{b}_1 + C(\tau - \tau^0)\boldsymbol{a}_1, \end{aligned}$$

and

$$\boldsymbol{c}_2 = B(\mathcal{K} - \mathcal{K}^0)\boldsymbol{b}_2 - C(\tau - \tau^0)\boldsymbol{a}_2.$$

5.5.2. Force balance

We may now compute the divergence terms in the equilibrium equation $\boldsymbol{t}_{1,1} + \boldsymbol{t}_{2,2} = 0$:

$$\begin{aligned} \boldsymbol{t}_{1,1} + \boldsymbol{t}_{2,2} =& T_{1,1}\boldsymbol{a}_1 + T_{2,2}\boldsymbol{a}_2 + \big[(T_1 + T_2)\mathcal{K} - 2A(\sin\gamma - \sin\gamma^0)\mathcal{K} \\ &+ 2B(\mathcal{K} - \mathcal{K}^0)(\mathcal{K}^2 + \tau^2) - 2C(\tau - \tau^0)\mathcal{K}\tau\big]\,\boldsymbol{n}, \end{aligned}$$

from which we obtain

$$T_{1,1} = T_{2,2} = 0,$$

and

$$\frac{1}{2}(T_1 + T_2) = A(\sin\gamma - \sin\gamma^0) - B(\mathcal{K} - \mathcal{K}^0)(\mathcal{K}^2 + \tau^2)/\mathcal{K} + C(\tau - \tau^0)\tau. \tag{5.5.2}$$

Consider now a pure traction problem with boundary conditions

$$\boldsymbol{t}_1\nu^1 + \boldsymbol{t}_2\nu^2 = P\boldsymbol{k}, \tag{5.5.3}$$

with P the externally applied traction, $\boldsymbol{k} = (0, 0, 1)$ the unit vector parallel to the axis of the cylinder and $\nu^1 = \nu^2 = \nu$ the unit normal to the boundary of the cylinder, i.e., two circles of radius R, with equations $X^1 + X^2 = 0$ (lower boundary) and $X^1 + X^2 =$

$2\pi H\sqrt{R^2+L^2/4\pi^2}/L$ (upper boundary). Since

$$\boldsymbol{a}_1+\boldsymbol{a}_2=\frac{L/\pi}{\sqrt{R^2+L^2/4\pi^2}}\,\boldsymbol{k},\qquad \boldsymbol{a}_1-\boldsymbol{a}_2=\frac{2R}{\sqrt{R^2+L^2/4\pi^2}}\,\boldsymbol{e}_\theta,$$
$$\boldsymbol{b}_1-\boldsymbol{b}_2=\frac{2R}{\sqrt{R^2+L^2/4\pi^2}}\,\boldsymbol{k},\qquad \boldsymbol{b}_1+\boldsymbol{b}_2=-\frac{L/\pi}{\sqrt{R^2+L^2/4\pi^2}}\,\boldsymbol{e}_\theta,$$

with $\boldsymbol{e}_\theta=\left(-\sin\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}},\cos\frac{X^1-X^2}{\sqrt{R^2+L^2/4\pi^2}},0\right)$ and

$$T_1\boldsymbol{a}_1+T_2\boldsymbol{a}_2=\frac{1}{2}(T_1+T_2)(\boldsymbol{a}_1+\boldsymbol{a}_2)+\frac{1}{2}(T_1-T_2)(\boldsymbol{a}_1-\boldsymbol{a}_2),$$

the boundary conditions yield the equations

$$0=\frac{1}{2}(T_1-T_2), \tag{5.5.4}$$
$$\begin{aligned}P/\nu=&\frac{L/\pi}{\sqrt{R^2+L^2/4\pi^2}}(\frac{1}{2}(T_1+T_2)+A(\sin\gamma-\sin\gamma^0))+B(\mathcal{K}-\mathcal{K}^0)\mathcal{K})\\&+\frac{2R}{\sqrt{R^2+L^2/4\pi^2}}(-B(\mathcal{K}-\mathcal{K}^0)\tau+C(\tau-\tau^0)\mathcal{K}).\end{aligned} \tag{5.5.5}$$

Using (5.5.2) in (5.5.4), we finally obtain

$$\begin{aligned}P/\nu=&\frac{L/\pi}{\sqrt{R^2+L^2/4\pi^2}}(2A(\sin\gamma-\sin\gamma^0)-B(\mathcal{K}-\mathcal{K}^0)\frac{\tau^2}{\mathcal{K}}+C(\tau-\tau^0)\tau)\\&+\frac{2R}{\sqrt{R^2+L^2/4\pi^2}}(-B(\mathcal{K}-\mathcal{K}^0)\tau+C(\tau-\tau^0)\mathcal{K}).\end{aligned} \tag{5.5.6}$$

5.5.3. Couple balance

The couple balance is $\boldsymbol{c}_{1,1}+\boldsymbol{c}_{2,2}+\boldsymbol{a}_1\times\boldsymbol{t}_1+\boldsymbol{a}_2\times\boldsymbol{t}_2=0$. For a surface made of helical fibers, we have

$$\begin{aligned}\boldsymbol{c}_{1,1}+\boldsymbol{c}_{2,2}=&-B(\mathcal{K}-\mathcal{K}^0)\tau\boldsymbol{n}+C(\tau-\tau^0)\mathcal{K}\boldsymbol{n}\\&+B(\mathcal{K}-\mathcal{K}^0)\tau\boldsymbol{n}-C(\tau-\tau^0)\mathcal{K}\boldsymbol{n}=0,\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{a}_1 \times \boldsymbol{t}_1 + \boldsymbol{a}_2 \times \boldsymbol{t}_2 =& \boldsymbol{a}_1 \times T_1\boldsymbol{a}_1 + \boldsymbol{a}_2 \times T_2\boldsymbol{a}_2 + A(\sin\gamma - \sin\gamma^0)(\boldsymbol{a}_1 \times \boldsymbol{a}_2 + \boldsymbol{a}_2 \times \boldsymbol{a}_1)\\ &+ B(\mathcal{K} - \mathcal{K}^0)\tau\boldsymbol{n} - C(\tau - \tau^0)\mathcal{K}\boldsymbol{n}\\ &- B(\mathcal{K} - \mathcal{K}^0)\tau\boldsymbol{n} + C(\tau - \tau^0)\mathcal{K}\boldsymbol{n} = 0\end{aligned}$$

Therefore, the couple balance is identically satisfied and does not yield any information. The boundary conditions for the couple balance for a pure traction problem are

$$\boldsymbol{c}_1\nu^1 + \boldsymbol{c}_2\nu^2 + \boldsymbol{r} \times \boldsymbol{t}_1\nu^1 + \boldsymbol{r} \times \boldsymbol{t}_2\nu^2 = \boldsymbol{r} \times P\boldsymbol{k} + M\boldsymbol{e}_\theta, \tag{5.5.7}$$

where M is an externally imposed boundary moment, and imply that, by (5.5.3),

$$\boldsymbol{c}_1\nu^1 + \boldsymbol{c}_2\nu^2 - M\boldsymbol{e}_\theta = 0,$$

at the boundary of the cylinder. For helical fibers, we obtain

$$M/\nu = \frac{1}{\sqrt{R^2 + L^2/4\pi^2}}\left(-\frac{BL}{\pi}(\mathcal{K} - \mathcal{K}^0) + 2CR(\tau - \tau^0)\right). \tag{5.5.8}$$

Equations (5.5.6) and (5.5.8) are the basic relations that allow to compute the deformation of a cylindrical sheet made of inextensible fibers, given the traction and the moment at the boundary.

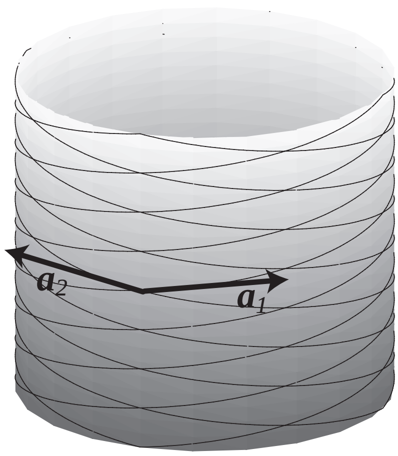

FIGURE 5.1. A cylindrical shell made of helices.

CHAPTER 6

Constitutive theory for textiles made by two families of inextensible fibers

Contents

In this chapter we address the issue of material symmetry for surfaces made by networks of fibers, in order to characterize the restrictions on the elastic energy function due to the geometry of the network and the material properties of the fibers.

When the weave pattern is plain (alternating intersections between the fibers, cf. McBride and Chen [**40**]), four basic structures for the network are possible: the square structure (orthogonal equivalent fibers), the rectangular structure (orthogonal inequivalent fibers), the rhombic structure (non-orthogonal equivalent fibers), and the parallelogram structure (non-orthogonal inequivalent fibers). Each of these structures is characterized by a symmetry group. We extend to this case a technique, based on the so-called Rychlewski's theorem (cf. for instance, Boehler [**7**] [**8**]), that allows to explicitly compute the basic polynomial invariants of the action of the symmetry groups and, by consequence, the general form of an invariant stored energy density (cf. Spencer [**64**], Holzapfel [**26**]). Rychlewski's theorem is extensively used to compute the invariants depending on the Cauchy-Green tensor for fiber-reinforced materials (cf. for instance Hartmann [**24**], Aksel and Itskov [**3**], Schröder and Neff [**60**] the applications of Aimene [**1**] for textiles and of Gasser, Ogden and Holzapfel [**16**] to arterial walls).

The classical form of Rychlewski's theorem, however, does not distinguish between symmetry operations that interchange the fibers, and cannot discriminate, for instance, between the square and the rectangular structure. The result has been extended to cover these cases also (cf. Zhang and Rychlewski [**70**]), but we prove here an alternative generalization that allows to compute the polynomial invariants that depend up to the third derivatives of the deformation, and discuss the general form of an invariant energy depending on the shear between the fibers, and their curvature and torsion.

To compute the invariants, we have used the computer package 'Singular', available online at http://www.singular.uni-kl.de (cf. Greuel, Pfister and Schoenemann [**18**]).

6.1. Material symmetry and objectivity

Consider a deformation $\boldsymbol{r} : \Omega \to \mathbb{R}^3$ with $\Omega \subset \mathbb{R}^2$ for a deformable surface or $\Omega \subset \mathbb{R}^3$ for a 3 three-dimensional body, with deformation gradient $\boldsymbol{F}$.

A rigid body motion in $\mathbb{R}^3$ is a transformation

$$\boldsymbol{x} \mapsto \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{b}, \qquad \boldsymbol{Q} \in \mathrm{SO}(3),\ \boldsymbol{b} \in \mathbb{R}^3.$$

Clearly,

$$\boldsymbol{r}(\boldsymbol{X}) \mapsto \boldsymbol{Q}\boldsymbol{r}(\boldsymbol{X}) + \boldsymbol{b}$$

for $\boldsymbol{X} \in \Omega$, so that $\boldsymbol{F} \mapsto \boldsymbol{Q}\boldsymbol{F}$ is the associated transformation of the deformation gradient.

For a hyperelastic material with elastic energy density $W = W(\boldsymbol{F})$ the objectivity condition is the requirement that the energy is invariant under superimposed rigid body motions, i.e.,

$$W(\boldsymbol{QF}) = W(\boldsymbol{F}), \qquad \boldsymbol{Q} \in \mathrm{SO}(3), \tag{6.1.1}$$

which in turn implies that W can be written in terms of the right Cauchy-Green tensor $\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F}$, so that $W = W(\boldsymbol{C})$ (with abuse of notation we denote by the same symbol W the energy as a function of $\boldsymbol{F}$ and of $\boldsymbol{C}$). Hence, objectivity is related to invariance under transformations acting in the actual configuration.

Consider now a transformation of the reference configuration of the form

$$\boldsymbol{X} \mapsto \boldsymbol{RX} + \boldsymbol{d}, \qquad \boldsymbol{R} \in \mathrm{GL}(n), \boldsymbol{d} \in \mathbb{R}^n, \quad \boldsymbol{X} \in \Omega,$$

with $n = 2$ or $n = 3$, with $\mathrm{GL}(n)$ the group of invertible linear transformations of $\mathbb{R}^n$. Such a transformation is a material symmetry operation if the energy does not change in the new reference configuration. For homogeneous materials, such that the elastic energy is independent of $\boldsymbol{X}$, the induced transformation of the deformation gradient is $\boldsymbol{F} \mapsto \boldsymbol{FR}^\top$, and the set G of $\boldsymbol{R} \in \mathrm{GL}(n)$ that leave the energy invariant, i.e.,

$$W(\boldsymbol{F}) = W(\boldsymbol{FR}^\top) \quad \Leftrightarrow \quad W(\boldsymbol{C}) = W(\boldsymbol{RCR}^\top) \tag{6.1.2}$$

defines the material symmetry group. For instance, the material is isotropic if $W(\boldsymbol{C})$ is invariant under $O(3)$, i.e., $W(\boldsymbol{C}) = W(\boldsymbol{RCR}^\top)$ for every $\boldsymbol{R} \in \mathrm{O}(3)$. Hence, material symmetry is related to invariance under transformations acting in the reference configuration.

6.2. Transversely isotropic materials

Consider a 3-D fiber-reinforced composite made of an isotropic matrix and a family of reinforcing fibers. The fibers are continuously distributed and have the same direction in the reference configuration $\Omega \subset \mathbb{R}^3$ of the material. Denoting by $\boldsymbol{r} : \Omega \to \mathbb{R}^3$ a deformation, we write $\boldsymbol{A}$ for a unit vector parallel to the fiber direction in the reference configuration, and $\boldsymbol{F}$ for the deformation gradient.

Assume that the material is hyperelastic and isotropic with elastic energy density $W = W(\boldsymbol{C})$. The presence of a family of reinforcing fibers in an isotropic matrix gives the material transversely isotropic properties, characterized by the invariance of W under rotations that preserve the fibers direction.

Let G be the group of orthogonal transformations which leave the axis parallel to $\boldsymbol{A}$ invariant, i.e.,

$$G = \{\boldsymbol{R} \in \mathrm{O}(3)/\boldsymbol{RA} = \pm\boldsymbol{A}\}, \tag{6.2.1}$$

whose elements are the rotations with axis parallel to $\boldsymbol{A}$ and the reflections in the planes parallel or orthogonal to $\boldsymbol{A}$.

If we assume the matrix material to be hyperelastic, then it is customary to assume that, in the presence of fibers, the **transverse isotropy condition** holds (Holzapfel [**26**]):

$$W(\boldsymbol{C}) = W(\boldsymbol{RCR}^\top), \qquad \forall \boldsymbol{R} \in G. \tag{6.2.2}$$

Transverse isotropy can also be characterized alternatively in terms of the so-called structural tensor

$$\boldsymbol{M} = \boldsymbol{A} \otimes \boldsymbol{A}.$$

In fact, the group of orthogonal tranformations that leave the structural tensor invariant is clearly G itself, i.e.,

$$G = \left\{\boldsymbol{R} \in \ \mathrm{O}\ (3) : \boldsymbol{RMR}^\top = \boldsymbol{M}\right\}.$$

Assume now that the elastic energy W can be written as a function of $\boldsymbol{C}$ and the structural tensor

$$W = \widehat{W}(\boldsymbol{C}, \boldsymbol{A} \otimes \boldsymbol{A}). \tag{6.2.3}$$

According to Rychlewski's theorem (see section 6.3 below, cf. Zhang and Rychlewski [**70**]) the transverse isotropy condition (6.2.2) is satisfied if and only if the strain energy (6.2.3) is an isotropic function of $\boldsymbol{C}$ and all the structural tensor $\boldsymbol{M}$, i.e.,

$$\widehat{W}(\boldsymbol{QCQ}^\top, \boldsymbol{QMQ}^\top) = \widehat{W}(\boldsymbol{C}, \boldsymbol{M}) \qquad \forall \boldsymbol{Q} \in \mathrm{O}(3). \tag{6.2.4}$$

This results allow to use the isotropic invariant theory to derive representations for the anisotropic energy density function. Recall that an integrity basis for a group action is a set of polynomials invariant under the group action, such that any other invariant polynomial is expressible as a polynomial in the elements of the integrity basis (Betten [**6**], Smith [**62**], Smith and Rivlin [**63**], Liu [**34**] and Xiao [**69**]).

Given a group T of orthogonal transformations such that, for each $\boldsymbol{Q} \in T$,

$$\begin{array}{rccc} \boldsymbol{Q}: & \mathrm{Sym}(3) \times \mathrm{RK}_1(3) & \longrightarrow & \mathrm{Sym}(3) \times \mathrm{RK}_1(3) \\ & (\boldsymbol{C}, \ \boldsymbol{A} \otimes \boldsymbol{A}) & \longmapsto & (\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^\top, \ \boldsymbol{Q}(\boldsymbol{A} \otimes \boldsymbol{A})\boldsymbol{Q}^\top) \end{array}$$

where $\mathrm{Sym}(3)$ is the space of symmetric linear transformations of $\mathbb{R}^3$, and $\mathrm{RK}_1(3)$ is the space of rank-one linear transformations of $\mathbb{R}^3$, then an integrity basis of the action of T on $\boldsymbol{C}$ and $\boldsymbol{A} \otimes \boldsymbol{A}$ is given by the polynomials, in the ring $\mathbb{R}[Y]$ with $Y = \{\mathrm{Sym}(3) \times \mathrm{RK}_1(3)\}$,

$$\begin{aligned} I_1(\boldsymbol{C}) &= \mathrm{tr}\boldsymbol{C}, \\ I_2(\boldsymbol{C}) &= \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{C})^2 - \mathrm{tr}\,(\boldsymbol{C})^2\right], \\ I_3(\boldsymbol{C}) &= \det \boldsymbol{C}, \\ I_4(\boldsymbol{C}, \boldsymbol{A}) &= \boldsymbol{A} \cdot \boldsymbol{C}\boldsymbol{A}, \\ I_5(\boldsymbol{C}, \boldsymbol{A}) &= \boldsymbol{A} \cdot \boldsymbol{C}^2\boldsymbol{A}. \end{aligned} \tag{6.2.5}$$

Notice that I_4 corresponds to the square of the stretch in the fiber direction, and if the fiber is inextensible this invariant is identically constant and equal to one. (For more details see Spencer [**64**], Smith and Rivlin [**63**], Green and Adkins [**17**], Gurtin [**22**]).

6.3. Rychlewski's theorem and its extension

We now formulate and prove two results, the first of which known as Rychlewski's theorem, that allow to compute the anisotropic invariants of a fiber-reinforced material in terms of suitable isotropic invariants of an extended energy function. We henceforth generalize the classical result to cover more general classes of anisotropy.

6.3.1. The classical formulation

We work in arbitrary dimension n in this section. For $\mathbf{A}_1, \ldots, \mathbf{A}_N \in \mathbb{R}^n$ fixed vectors parallel to the fibers, define the structure tensors by

$$\mathbf{M}_i = \mathbf{A}_i \otimes \mathbf{A}_i,$$

and consider the group G which maps each structure tensor into itself (the restricted symmetry group of the structure)

$$\begin{aligned} G &= \{\mathbf{R} \in O(n) \ / \ \mathbf{R}\mathbf{A}_i = \pm \mathbf{A}_i, \ i = 1, \dots, N\} \\ &= \left\{\mathbf{R} \in O(n) \ / \ \mathbf{R}\mathbf{M}_i\mathbf{R}^\top = \mathbf{M}_i, \ i = 1, \dots, N\right\}. \end{aligned}$$

Also, denote by M the orbit of $(\mathbf{M}_1, ..., \mathbf{M}_N)$ under $O(n)$, viz.

$$\begin{aligned} M &= \left\{\left(\mathbf{Q}\mathbf{M}_1\mathbf{Q}^\top, ..., \mathbf{Q}\mathbf{M}_N\mathbf{Q}^\top\right)\right\}_{\mathbf{Q}\in O(n)} \\ &= \{(\mathbf{Q}\mathbf{A}_1 \otimes \mathbf{Q}\mathbf{A}_1, ..., \mathbf{Q}\mathbf{A}_N \otimes \mathbf{Q}\mathbf{A}_N)\}_{\mathbf{Q}\in O(n)} . \end{aligned}$$

Let $\mathbf{C} \in \mathrm{Sym}(n)$ be a symmetric tensor, and consider an elastic energy density of the form $W = W(\mathbf{C})$. We say that W is invariant under G if $W(\mathbf{C}) = W(\mathbf{R}\mathbf{C}\mathbf{R}^\top)$ for all $\mathbf{R} \in G$ and $\mathbf{C} \in \mathrm{Sym}(n)$.

Theorem 6.3.1. **[Rychlewski's theorem]** *(i) If W is invariant under G, then the extended energy function $\widehat{W}$, defined on* $\mathrm{Sym}(n) \times M$ *by*

$$\widehat{W}(\mathbf{C}, \mathbf{P}_i) := W(\mathbf{Q}\mathbf{C}\mathbf{Q}^\top), \tag{6.3.1}$$

with $\mathbf{P}_i \in M$ and $\mathbf{Q} \in O(n)$ is such that $\mathbf{Q}\mathbf{P}_i\mathbf{Q}^\top = \mathbf{M}_i$ for every $i = 1, ..., N$, is an isotropic function of its arguments.

(ii) Given an isotropic function $\widehat{W}$ on $\mathrm{Sym}(n) \times M$*, the function W on* $\mathrm{Sym}(n)$ *defined by*

$$W(\mathbf{C}) := \widehat{W}(\mathbf{C}, \mathbf{M}_i), \tag{6.3.2}$$

is invariant under G.

Proof. (i). First notice that, since W is invariant under G, the definition (6.3.1) makes sense. Since the orthogonal transformation $\mathbf{Q}$ such that $\mathbf{Q}\mathbf{P}_i\mathbf{Q}^\top = \mathbf{M}_i$, for every $i = 1, ..., N$, is not unique, we must verify that the definition (6.3.1) is independent of $\mathbf{Q}$. Let $\mathbf{Q}'$ be another such transformation: then since $\mathbf{Q}'\mathbf{P}_i\mathbf{Q}'^\top = \mathbf{M}_i$, it follows that $\mathbf{Q}'\mathbf{Q}^\top\mathbf{M}_i\mathbf{Q}\mathbf{Q}'^\top = \mathbf{M}_i$ for every $i = 1, ..., N$, so that $\mathbf{Q}'\mathbf{Q}^\top = \mathbf{R} \in G$. Hence,

$$\widehat{W}(\mathbf{C}, \mathbf{P}_i) = W(\mathbf{Q}'\mathbf{C}\mathbf{Q}'^\top) = W(\mathbf{R}\mathbf{Q}\mathbf{C}\mathbf{Q}^\top\mathbf{R}^\top) = W(\mathbf{Q}\mathbf{C}\mathbf{Q}^\top) = \widehat{W}(\mathbf{C}, \mathbf{P}_i),$$

by the invariance under G, and this proves the assertion. Now, for every $\mathbf{U} \in O(n)$, noting that $\mathbf{QU}^\top \left(\mathbf{UP}_i\mathbf{U}^\top\right)\mathbf{UQ}^\top = \mathbf{M}_i$ and using (6.3.1), we have

$$\widehat{W}(\mathbf{UCU}^\top, \mathbf{UP}_i\mathbf{U}^\top) = W(\mathbf{QU}^\top \left(\mathbf{UCU}^\top\right) \mathbf{UQ}^\top) = W(\mathbf{QCQ}^\top) = \widehat{W}(\mathbf{C}, \mathbf{P}_i),$$

which proves that $\widehat{W}$ is an isotropic function of its arguments.

(ii). By hypothesis $\widehat{W}$ is isotropic, hence

$$\widehat{W}(\mathbf{UCU}^\top, \mathbf{UP}_i\mathbf{U}^\top) = \widehat{W}(\mathbf{C}, \mathbf{P}_i),$$

for every $\mathbf{U} \in O(n)$. Choosing $\mathbf{U} = \mathbf{R} \in G$, by (6.3.2) we have

$$W(\mathbf{RCR}^\top) = \widehat{W}(\mathbf{RCR}^\top, \mathbf{M}_i) = \widehat{W}(\mathbf{C}, \mathbf{R}^\top\mathbf{M}_i\mathbf{R}) = \widehat{W}(\mathbf{C}, \mathbf{M}_i) = W(\mathbf{C}),$$

i.e., W is invariant under G, and this completes the proof.

The above result shows that an energy function invariant under G can be expressed as a function of the isotropic invariants of $(\mathbf{C}, \mathbf{M}_i)$.

6.3.2. Extension of the classical result

The restricted symmetry group G of a structure is in general too small to describe its symmetry properties, since it does not allow for symmetry operations that interchange the structure tensors. The full symmetry group of a given structure is defined as

$$\begin{aligned} H = &\Big\{\mathbf{R} \in O(n) \;/\; \mathbf{RA}_i = (-1)^{\lambda(i)}\mathbf{A}_{\sigma(i)},\ i = 1, \dots, N, \\ &\quad \text{for some } \sigma \in S_N, \lambda(i) \in \{0,1\}\} \\ = &\Big\{\mathbf{R} \in O(n) \;/\; \mathbf{RM}_i\mathbf{R}^\top = \mathbf{M}_{\sigma(i)},\ i = 1, \dots, N, \text{ for some } \sigma \in S_N\Big\}. \end{aligned} \tag{6.3.3}$$

By definition, the maps

$$\boldsymbol{R} \mapsto (\lambda(1), \dots, \lambda(N)) \in \{0,1\}^N, \qquad \boldsymbol{R} \mapsto \sigma \in S_N,$$

are group homomorphisms of H into $\mathbb{R}_2^N$ (as an additive group) and S_N, the symmetric group on N elements, i.e., the group of all permutations of the indices $i = 1, \dots, N$. We denote by H' the image of H in S_N.

The following result generalizes the theorem of the previous section.

Theorem 6.3.2. *(i) If W is invariant under H, then the extended energy function $\widehat{W}$, defined on Sym$(n) \times M$ by*

$$\widehat{W}(\mathbf{C}, \mathbf{P}_i) := W(\mathbf{QCQ}^\top), \tag{6.3.4}$$

with $\mathbf{P}_i \in M$ and $\mathbf{Q} \in O(n)$ is such that $\mathbf{QP}_i\mathbf{Q}^\top = \mathbf{M}_i$ for every $i = 1, ..., N$, is an isotropic function of its arguments, and is invariant under H', i.e.,

$$\widehat{W}(\mathbf{C}, \mathbf{P}_{\sigma(i)}) = \widehat{W}(\mathbf{C}, \mathbf{P}_i), \qquad \text{for every } \sigma \in H'. \tag{6.3.5}$$

(ii) Given an isotropic function $\widehat{W}$ on $\mathrm{Sym}(n) \times M$*, invariant under H' (in the sense of (6.3.5)), the function W on* $\mathrm{Sym}(n)$ *defined by*

$$W(\mathbf{C}) := \widehat{W}(\mathbf{C}, \mathbf{M}_i), \tag{6.3.6}$$

is invariant under H.

Proof. (i). We need to prove only (6.3.5). Notice first that if $\mathbf{QP}_i\mathbf{Q}^\top = \mathbf{M}_i$, then also $\mathbf{QP}_{\sigma(i)}\mathbf{Q}^\top = \mathbf{M}_{\sigma(i)}$. Moreover, since $\sigma \in H'$, there exists $\mathbf{R} \in H$ such that $\mathbf{RM}_i\mathbf{R}^\top = \mathbf{M}_{\sigma(i)}$, so that $\mathbf{R}^\top\mathbf{QP}_{\sigma(i)}\mathbf{Q}^\top\mathbf{R} = \mathbf{M}_i$. Hence, using (6.3.4),

$$\widehat{W}(\mathbf{C}, \mathbf{P}_{\sigma(i)}) = W(\mathbf{R}^\top\mathbf{QCQ}^\top\mathbf{R}) = W(\mathbf{QCQ}^\top) = \widehat{W}(\mathbf{C}, \mathbf{P}_i),$$

which proves the thesis.

(ii). As before, by hypothesis $\widehat{W}$ is isotropic, hence

$$\widehat{W}(\mathbf{UCU}^\top, \mathbf{UP}_i\mathbf{U}^\top) = \widehat{W}(\mathbf{C}, \mathbf{P}_i),$$

for every $\mathbf{U} \in O(n)$. Choosing $\mathbf{U} = \mathbf{R} \in H$, by (6.3.2) we have

$$\begin{aligned} W(\mathbf{RCR}^\top) =& \widehat{W}(\mathbf{RCR}^\top, \mathbf{M}_i) = \widehat{W}(\mathbf{C}, \mathbf{R}^\top\mathbf{M}_i\mathbf{R}) \\ =& \widehat{W}(\mathbf{C}, \mathbf{M}_{\sigma(i)}) = \widehat{W}(\mathbf{C}, \mathbf{M}_i) = W(\mathbf{C}), \end{aligned}$$

i.e., W is invariant under H, and this completes the proof.

The above result shows that an energy function invariant under the full symmetry group H can be expressed as a function of the isotropic invariants of $(\mathbf{C}, \mathbf{M}_i)$ that is, in addition, invariant under permutations in H'.

Also, it is straightforward to extend Theorem 6.3.2 to functions depending on higher-order tensors, after properly defining the action of the symmetry group H on these spaces (cf. Section 6.5).

Finally, notice that an immediate Corollary of Theorem 6.3.2 is that the extended energy function $\widehat{W}$ is such that

$$\widehat{W}(\boldsymbol{C}, \boldsymbol{M}_i) = \widehat{W}(\boldsymbol{C}, \boldsymbol{R}^\top \boldsymbol{M}_i \boldsymbol{R}), \qquad \boldsymbol{R} \in H. \tag{6.3.7}$$

6.4. Surfaces made of two families of fibers

In what follows we consider a surface $\mathcal{S}$ formed by two families of fibers, as in the previous chapters. The fibers are continuously distributed and those of each family have the same direction in the reference configuration. We recall that $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the unit vectors associated to the fiber directions in the reference configuration, while $\boldsymbol{A}^1$ and $\boldsymbol{A}^2$ are their dual, with associated coordinate system (X^α), $\alpha = 1, 2$ in $\mathbb{R}^2$ (cf. Section 2.1).

6.4.1. The four basic patterns: square, rectangular, rhombic and parallelogram

We now introduce the four basic structures that characterize the symmetry properties of a network of fibers.

6.4.1.1. *The square structure.* Consider first the case in which the two families of fibers are mutually orthogonal in the reference configuration and have the same material properties (Figure 6.1a).

We refer to this structure as the square structure. The full symmetry group H_{sq} describing the anisotropy of this material is generated by the orthogonal transformations of $\mathbb{R}^2$ whose matrix representation in the orthonormal basis $(\boldsymbol{A}_1, \boldsymbol{A}_2)$ is

$$\boldsymbol{R}_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{R}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};$$

the set of permutations associated to the full symmetry group of the square structure is $H'_{sq} = S_2$, and the corresponding homomorphism is

$$\boldsymbol{R}_{\pi/2} \mapsto (12), \qquad \boldsymbol{R}_1 \mapsto (1)(2),$$

with (1)(2) the identity permutation and (12) the transposition of 1 in 2. Also, the homomorphism $H \to \mathbb{R}_2 \times \mathbb{R}_2$ associated to the group generators is

$$\boldsymbol{R}_{\pi/2} \mapsto (\lambda(1) = 0, \lambda(2) = 1), \qquad \boldsymbol{R}_1 \mapsto (\lambda(1) = 1, \lambda(2) = 0).$$

6.4.1.2. *The rectangular structure.* In case the two families of fibers are mutually orthogonal and have different material properties and $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are not interchangeable, we refer to the resulting structure as the rectangular structure, (Figure 6.1b). The full symmetry group describing the anisotropy of this material is generated by

$$\boldsymbol{R}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{R}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the matrix representation is taken in the orthonormal basis $(\boldsymbol{A}_1, \boldsymbol{A}_2)$. In this case, the fibers are not interchangeable, so that $H'_{rt} = \{(1)(2)\}$ reduces to the identity permutation. The homomorphism $H \to \mathbb{R}_2 \times \mathbb{R}_2$ associated to the group generators is

$$\boldsymbol{R}_1 \mapsto (\lambda(1) = 1, \lambda(2) = 0), \qquad \boldsymbol{R}_2 \mapsto (\lambda(1) = 0, \lambda(2) = 1).$$

6.4.1.3. *The rhombic structure.* In case the two families of fibers, with the same material properties, are not orthogonal, we refer to the resulting structure as to the rhombic structure (Figure 6.1c). The symmetry group describing the anisotropy of this material is generated by

$$\boldsymbol{R}_3 = \begin{pmatrix} \cos\delta & \sin\delta \\ \sin\delta & -\cos\delta \end{pmatrix}, \qquad \boldsymbol{R}_4 = \begin{pmatrix} -\cos\delta & -\sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix}$$

with δ the angle between the fibers, and the matrix representation is taken in the orthonormal basis $(\boldsymbol{A}_1, \boldsymbol{j})$, with $\boldsymbol{j}$ orthogonal to $\boldsymbol{A}_1$. Notice that $\boldsymbol{R}_3$ and $\boldsymbol{R}_4$ are the reflections about the lines bisecting the angle between $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, and the axis orthogonal to it.

In this case the fibers are interchangeable, from definition (6.3.3), it follows that the set of permutations associated to the full symmetry group of the rhombic structure is $H'_{rb} = S_2$, and the corresponding homomorphism is

$$\boldsymbol{R}_3 \mapsto (12), \qquad \boldsymbol{R}_4 \mapsto (12).$$

The homomorphism $H \to \mathbb{R}_2 \times \mathbb{R}_2$ associated to inversion of the fibers is

$$\boldsymbol{R}_3 \mapsto (\lambda(1) = 0, \lambda(2) = 0), \qquad \boldsymbol{R}_4 \mapsto (\lambda(1) = 1, \lambda(2) = 1).$$

6.4.1.4. *The parallelogram structure.* Finally, if the two families of fibers are not orthogonal and have different material properties, we have the parallelogram structure of Figure 6.1d.

The symmetry group describing the anisotropy of this material is

$$H_{pr} = \left\{ \boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, -\boldsymbol{I} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

and H'_{pr} reduces to the identity, while the homomorphism $H \to \mathbb{R}_2 \times \mathbb{R}_2$ is just

$$-\boldsymbol{I} \mapsto (\lambda(1) = 1, \lambda(2) = 1).$$

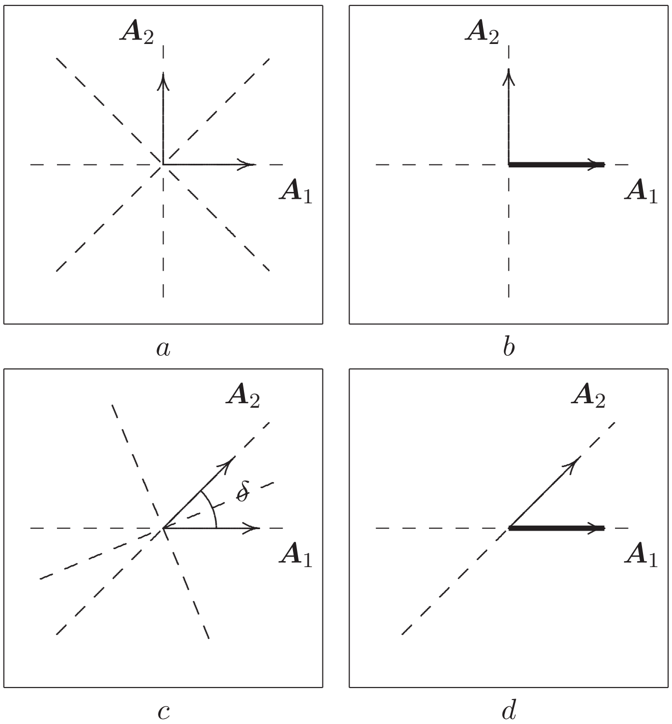

FIGURE 6.1. The four basic structures of a weave pattern. a: square, b: rectangular, c: rhombic, d: parallelogram

6.4.2. Application to a fibered surface without bending and twisting stiffness

We apply the above results to find a representation for a surface strain energy function of the form $W = W(\boldsymbol{C})$ that is invariant under the action of the material symmetry group associated to the fibers. We neglect bending and twisting effects in this section.

Let H be the full symmetry group of the structure. Then, according to the generalized Rychlewski's theorem 6.3.2 the condition

$$W(\boldsymbol{C}) = W(\boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^{\top}) \qquad \forall \boldsymbol{R} \in H$$

is satisfied if and only if the strain energy can be written as an isotropic function $\widehat{W}$ of $\boldsymbol{C}$ and $\boldsymbol{A}_1 \otimes \boldsymbol{A}_1$, $\boldsymbol{A}_2 \otimes \boldsymbol{A}_2$,

$$\widehat{W}(\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^{\top}, \boldsymbol{Q}\boldsymbol{A}_1 \otimes \boldsymbol{A}_1 \boldsymbol{Q}^{\top}, \boldsymbol{Q}\boldsymbol{A}_2 \otimes \boldsymbol{A}_2 \boldsymbol{Q}^{\top}) = \widehat{W}(\boldsymbol{C}, \boldsymbol{A}_1 \otimes \boldsymbol{A}_1, \boldsymbol{A}_2 \otimes \boldsymbol{A}_2) \qquad \forall \boldsymbol{Q} \in O(2), \quad (6.4.1)$$

which is additionally invariant under the permutations of the structure tensors corresponding to H, i.e,

$$\widehat{W}(\boldsymbol{C}, \boldsymbol{A}_1 \otimes \boldsymbol{A}_1, \boldsymbol{A}_2 \otimes \boldsymbol{A}_2) = \widehat{W}(\boldsymbol{C}, \boldsymbol{A}_{\sigma(1)} \otimes \boldsymbol{A}_{\sigma(1)}, \boldsymbol{A}_{\sigma(2)} \otimes \boldsymbol{A}_{\sigma(2)}) \qquad \forall \sigma \in H'. \quad (6.4.2)$$

By analogy with Section 6.2, to form the integrity basis of the tensors $\boldsymbol{C}$, $\boldsymbol{A}_1 \otimes \boldsymbol{A}_1$, $\boldsymbol{A}_2 \otimes \boldsymbol{A}_2$ and to fulfill the requirement (6.4.1), eight invariants are needed. In fact, we know that, in general, all polynomial isotropic functions of three 2×2 symmetric tensors

$$f = f(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{S}_3)$$

can be written as a function of the basic invariants

$$\begin{gathered} \operatorname{tr}(\boldsymbol{S}_i), \quad \operatorname{tr}(\boldsymbol{S}_i^2), \quad \operatorname{tr}(\boldsymbol{S}_i \boldsymbol{S}_j), \quad \operatorname{tr}(\boldsymbol{S}_i^2 \boldsymbol{S}_j), \\ \operatorname{tr}(\boldsymbol{S}_i \boldsymbol{S}_j^2), \quad \operatorname{tr}(\boldsymbol{S}_i^2 \boldsymbol{S}_j^2), \quad \operatorname{tr}(\boldsymbol{S}_1 \boldsymbol{S}_2 \boldsymbol{S}_3). \end{gathered}$$

Specifying to our case and neglecting trivial invariants (such as $\boldsymbol{A}_1 \cdot \boldsymbol{A}_2$), we find that the elastic energy W must be a function of

$$\begin{aligned}
J_1(\boldsymbol{C}) &= \operatorname{tr}\boldsymbol{C} \\
J_2(\boldsymbol{C}) &= \det \boldsymbol{C} \\
J_3(\boldsymbol{C}, \boldsymbol{A}_1) &= \boldsymbol{A}_1 \cdot \boldsymbol{C}\boldsymbol{A}_1 \\
J_4(\boldsymbol{C}, \boldsymbol{A}_1) &= \boldsymbol{A}_1 \cdot \boldsymbol{C}^2\boldsymbol{A}_1 \\
J_5(\boldsymbol{C}, \boldsymbol{A}_2) &= \boldsymbol{A}_2 \cdot \boldsymbol{C}\boldsymbol{A}_2 \\
J_6(\boldsymbol{C}, \boldsymbol{A}_2) &= \boldsymbol{A}_2 \cdot \boldsymbol{C}^2\boldsymbol{A}_2 \\
J_7(\boldsymbol{C}, \boldsymbol{A}_1, \boldsymbol{A}_2) &= (\boldsymbol{A}_1 \cdot \boldsymbol{A}_2)\boldsymbol{A}_1 \cdot \boldsymbol{C}\boldsymbol{A}_2.
\end{aligned} \tag{6.4.3}$$

Now, by (2.1.6), the invariants become

$$\begin{aligned}
J_1 &= C_{11}|\boldsymbol{A}^1|^2 + 2C_{12}\boldsymbol{A}^1 \cdot \boldsymbol{A}^2 + C_{22}|\boldsymbol{A}^2|^2, \\
J_2 &= P(C_{11}, C_{12}, C_{22}) \\
J_3 &= C_{11} \\
J_4 &= C_{11}^2|\boldsymbol{A}^1|^2 + 2C_{11}C_{12}\boldsymbol{A}^1 \cdot \boldsymbol{A}^2 + C_{12}^2|\boldsymbol{A}^2|^2, \\
J_5 &= C_{22} \\
J_6 &= C_{12}^2|\boldsymbol{A}^1|^2 + 2C_{12}C_{22}\boldsymbol{A}^1 \cdot \boldsymbol{A}^2 + C_{22}^2|\boldsymbol{A}^2|^2, \\
J_7 &= (\boldsymbol{A}^1 \cdot \boldsymbol{A}^2)C_{12},
\end{aligned} \tag{6.4.4}$$

with $P(C_{11}, C_{12}, C_{22})$ a second degree-polynomial in its arguments. Hence, two cases are possible:

(i) $\boldsymbol{A}_1 \cdot \boldsymbol{A}_2 = 0 \Leftrightarrow \boldsymbol{A}^1 \cdot \boldsymbol{A}^2 = 0$, i.e., the fibers are orthogonal (square and rectangular structures). Then (6.4.4) reduce to

$$\begin{aligned}
J_1 &= C_{11} + C_{22}, \\
J_2 &= C_{11}C_{22} - C_{12}^2, \\
J_3 &= C_{11} \\
J_4 &= C_{11}^2 + C_{12}^2, \\
J_5 &= C_{22} \\
J_6 &= C_{12}^2 + C_{22}^2,
\end{aligned}$$

so that the independent invariants are just

$$C_{11}, \quad C_{22}, \quad C_{12}^2. \tag{6.4.5}$$

(ii) $\boldsymbol{A}_1 \cdot \boldsymbol{A}_2 \neq 0$, i.e., the fibers are not orthogonal (rhombic and parallelogram structures). In this case, (6.4.4) reduce to the list of independent invariants

$$C_{11}, \quad C_{22}, \quad C_{12}, \tag{6.4.6}$$

which are just the components of $\boldsymbol{C}$ in the basis $\boldsymbol{A}_1$, $\boldsymbol{A}_2$.

Notice that, if the fibers are supposed inextensible, $C_{11} = C_{22} = 1$ identically. Taking into account permutations, we may henceforth conclude that

(-) for networks with **square symmetry**, formed by two orthogonal families of equivalent fibers, the permutation group H'_{sq} is S_2, so that the energy is a symmetric function of C_{11} and C_{22}, but depends from the square of C_{12}, i.e.,

$$W = W(C_{11} + C_{22}, C_{11}C_{22}, C_{12}^2). \tag{6.4.7}$$

(-) for networks with **rectangular symmetry**, formed by two orthogonal families of non equivalent fibers, the permutation group H'_{rt} is the identity, so that the energy has the form

$$W = W(C_{11}, C_{22}, C_{12}^2). \tag{6.4.8}$$

Notice that, in both cases above, the restriction imposed by invariance forces the energy to depend from the square $(C_{12})^2$, which is just $\sin^2\gamma$, with γ the angle of shear (see def.2.2.1). Physically, this means that a shear of angle γ has the same effect than the shear of angle $-\gamma$ on the shear density.

(-) for networks with **rhombic symmetry**, formed by two non-orthogonal families of equivalent fibers, the permutation group H'_{rb} is again S_2, so that the energy is a symmetric function of C_{11} and C_{22}, i.e.,

$$W = W(C_{11} + C_{22}, C_{11}C_{22}, C_{12}). \tag{6.4.9}$$

(-) for networks with **parallelogram symmetry**, formed by two non-orthogonal families of non equivalent fibers, the permutation group H'_{pr} is the identity, so that the energy has the form

$$W = W(C_{11}, C_{22}, C_{12}), \tag{6.4.10}$$

i.e., is just an arbitrary function of the components of $\boldsymbol{C}$. In both cases above, the situation is radically different from the square and rectangular structure: since the energy depends arbitrarily from C_{12}, now a shear of angle γ does not have the same effect than the shear of angle $-\gamma$.

We remark that the above results are an essential consequence of the extended version 6.3.2 of the classical Rychlewski theorem (section 6.3.1). Indeed, the restricted symmetry group of the square and rectangular structures is the same, and the same holds for the rhombic and parallelogram structures. Hence, these structures cannot be distinguished from each other by the classical Rychlewski theorem.

6.5. Invariant functions of higher gradients

In this section we generalize the previous results to discuss invariance of energy functions depending on higher gradients, in order to take into account curvature and torsion. An argument similar to that following (2.1.4) shows that the following definitions of the higher deformation gradients

$$\nabla\mathbf{F} = \mathbf{r},_{\alpha\beta} \otimes \boldsymbol{A}^\alpha \otimes \boldsymbol{A}^\beta, \qquad \nabla\nabla\mathbf{F} = \mathbf{r},_{\alpha\beta\gamma} \otimes \boldsymbol{A}^\alpha \otimes \boldsymbol{A}^\beta \otimes \boldsymbol{A}^\gamma,$$

are independent of coordinates.

Recalling Section 6.1, we know that the material symmetry group acts in the reference configuration, which corresponds to a right action on the deformation gradient, while the group of changes of observer acts in the actual configuration, i.e., to the left of the deformation gradient. We now extend the above actions to higher gradients of the deformation.

The group of linear transformations of $\mathbb{R}^2$ acts to the right on the deformation gradients as follows

$$\begin{cases} \mathbf{F} \mapsto \mathbf{F}\mathbf{L}^\top = \mathbf{r},_\alpha \otimes \mathbf{L}\boldsymbol{A}^\alpha \\ \nabla\mathbf{F} \mapsto \mathbf{r},_{\alpha\beta} \otimes \mathbf{L}\boldsymbol{A}^\alpha \otimes \mathbf{L}\boldsymbol{A}^\beta, \\ \nabla\nabla\mathbf{F} \mapsto \mathbf{r},_{\alpha\beta\gamma} \otimes \mathbf{L}\boldsymbol{A}^\alpha \otimes \mathbf{L}\boldsymbol{A}^\beta \otimes \mathbf{L}\boldsymbol{A}^\gamma, \end{cases} \tag{6.5.1}$$

for $\mathbf{L} \in GL(\mathbb{R}^2)$, while the group $GL(\mathbb{R}^3)$ acts to the left as follows

$$\begin{cases} \mathbf{F} \mapsto \mathbf{MF} = \mathbf{Mr}_{,\alpha} \otimes \boldsymbol{A}^{\alpha} \\ \nabla\mathbf{F} \mapsto \mathbf{Mr}_{,\alpha\beta} \otimes \boldsymbol{A}^{\alpha} \otimes \boldsymbol{A}^{\beta}, \\ \nabla\nabla\mathbf{F} \mapsto \mathbf{Mr}_{,\alpha\beta\gamma} \otimes \boldsymbol{A}^{\alpha} \otimes \boldsymbol{A}^{\beta} \otimes \boldsymbol{A}^{\gamma}. \end{cases} \tag{6.5.2}$$

Notice now that the full symmetry group H acts to the right on the deformation gradients as in (6.5.1), while $SO(3)$ acts to the left as in (6.5.2).

The right action of the material symmetry group on the deformation gradient induces an action on the partial derivatives as follows:

$$\begin{aligned} \boldsymbol{F}\boldsymbol{R}^{\top} &= (\mathbf{r}_{,\alpha} \otimes \boldsymbol{A}^{\alpha})\mathbf{R}^{\top} = \mathbf{r}_{,\alpha} \otimes \mathbf{R}\boldsymbol{A}^{\alpha} \\ &= (-1)^{\lambda(\alpha)}\mathbf{r}_{,\alpha} \otimes \boldsymbol{A}^{\sigma(\alpha)} = (-1)^{\lambda(\sigma^{-1}(\beta))}\mathbf{r}_{,\sigma^{-1}(\beta)} \otimes \boldsymbol{A}^{\beta} \\ &= (-1)^{\mu(\beta)}\mathbf{r}_{,\tau(\beta)} \otimes \boldsymbol{A}^{\beta}, \end{aligned}$$

for $\boldsymbol{R} \in H$, with $\beta = \sigma(\alpha)$, $\tau = \sigma^{-1}$ and $\mu(\beta) = \lambda(\sigma^{-1}(\beta))$. The above argument yields the transformation rules for the higher partial derivatives

$$\begin{aligned} &\boldsymbol{a}_{\alpha} \mapsto (-1)^{\lambda(\alpha)}\boldsymbol{a}_{\sigma(\alpha)}, \\ &\boldsymbol{a}_{\alpha,\beta} \mapsto (-1)^{(\lambda(\alpha)+\lambda(\beta))}\boldsymbol{a}_{\sigma(\alpha),\sigma(\beta)}, \\ &\boldsymbol{a}_{\alpha,\beta\gamma} \mapsto (-1)^{(\lambda(\alpha)+\lambda(\beta)+\lambda(\gamma))}\boldsymbol{a}_{\sigma(\alpha),\sigma(\beta)\sigma(\gamma)}. \end{aligned} \tag{6.5.3}$$

A trivial application of the theorems in the previous sections leads to the following result.

Proposition 6.5.1. *Let $(\mathbf{A}_{\alpha})$ be a pair of non parallel vectors in $\mathbb{R}^2$, with $\alpha = 1, 2$, with full symmetry group H, and let*

$$W = W(\mathbf{F}, \nabla\mathbf{F}, \nabla\nabla\mathbf{F}) \tag{6.5.4}$$

be an objective function,i.e., invariant under $SO(3)$ with action (6.5.2), and invariant under H with action (6.5.1). Then W can be written in the form

$$W = \widetilde{W}(\boldsymbol{a}_{\alpha} \cdot \boldsymbol{a}_{\beta}, \boldsymbol{a}_{\alpha} \cdot \boldsymbol{a}_{\beta,\gamma}, \boldsymbol{a}_{\alpha,\beta} \cdot \boldsymbol{a}_{\gamma,\delta}, \boldsymbol{a}_{\alpha} \cdot \boldsymbol{a}_{\beta,\gamma\delta}, \boldsymbol{a}_{\alpha\beta} \cdot \boldsymbol{a}_{\gamma,\delta\tau}, \boldsymbol{a}_{\alpha,\beta\gamma} \cdot \boldsymbol{a}_{\delta,\tau\mu}) \tag{6.5.5}$$

where $\boldsymbol{a}_{\alpha} = \boldsymbol{F}\boldsymbol{A}_{\alpha} = \boldsymbol{r}_{,\alpha}$ and $\widetilde{W}$ is invariant under the action of H on its arguments defined by (6.5.3).

Proof. By objectivity, W may be written as a function of the tensors

$$\mathbf{C} = \mathbf{C}_1 = \mathbf{F}^\top \mathbf{F}, \quad \mathbf{C}_2 = \mathbf{F}^\top \nabla \mathbf{F}, \quad \mathbf{C}_3 = \nabla \mathbf{F}^\top \nabla \mathbf{F},$$
$$\mathbf{C}_4 = \mathbf{F}^\top \nabla\nabla \mathbf{F}, \quad \mathbf{C}_5 = \nabla \mathbf{F}^\top \nabla\nabla \mathbf{F}, \quad \mathbf{C}_6 = \nabla\nabla \mathbf{F}^\top \nabla\nabla \mathbf{F},$$

where for instance

$$\nabla \mathbf{F}^\top \nabla\nabla \mathbf{F} = (\mathbf{r},_{\alpha\beta} \cdot \mathbf{r},_{\gamma\delta\tau})\boldsymbol{A}^\alpha \otimes \boldsymbol{A}^\beta \otimes \boldsymbol{A}^\gamma \otimes \boldsymbol{A}^\delta \otimes \boldsymbol{A}^\tau.$$

Now, a straightforward modification of Theorem 6.3.2 implies that a function of $(\mathbf{C}_i)_{i=1,\ldots,6}$ invariant under H may be written as an isotropic function (i.e., invariant under the action of $O(2)$) of the form

$$W = \widehat{W}(\mathbf{C}_i, \mathbf{A}_\alpha), \qquad i \in \{1, \ldots, 6\},$$

such that

$$\widehat{W}(\mathbf{C}_i, \mathbf{A}_\alpha) = \widehat{W}(\mathbf{C}_i, \mathbf{R}\mathbf{A}_\alpha), \qquad \mathbf{R} \in H \; i \in I = \{1, 2, ..., 6\}.$$

But the components of $\mathbf{C}_i$ in the basis $(\mathbf{A}_\alpha)$ are isotropic by construction. In fact, for instance for $\mathbf{C}$, we have

$$\mathbf{A}_\alpha \cdot \mathbf{C}\mathbf{A}_\beta = (\mathbf{Q}\mathbf{A}_\alpha) \cdot (\mathbf{Q}\mathbf{C}\mathbf{Q}^\top)(\mathbf{Q}\mathbf{A}_\beta), \qquad \mathbf{Q} \in O(2).$$

Since the components of a tensor in a given basis completely characterize the tensor itself, it follows that every isotropic invariant of $(\mathbf{C}_i, \mathbf{A}_\alpha)$ may be expressed as a function of the components of the $\mathbf{C}_i$ in the basis $(\mathbf{A}_\alpha)$ and conversely. To compute these components, notice that, for instance for $\mathbf{C} = \mathbf{C}_1$, we have

$$\mathbf{C} = (\mathbf{r},_\alpha \cdot \mathbf{r},_\beta)\mathbf{A}^\alpha \otimes \mathbf{A}^\beta \quad \Leftrightarrow \quad \mathbf{A}_\alpha \cdot \mathbf{C}\mathbf{A}_\beta = \mathbf{r},_\alpha \cdot \mathbf{r},_\beta\,,$$

which proves the existence of $\widetilde{W}$. Finally, the invariance of $\widehat{W}$ under the action $\mathbf{A}_\alpha \mapsto \mathbf{R}\mathbf{A}_\alpha$, for $\mathbf{R} \in H$, yields the invariance of $\widetilde{W}$ under the action (again restricting for instance to $\mathbf{C} = \mathbf{C}_1$)

$$\begin{aligned}\mathbf{r},_\alpha \cdot \mathbf{r},_\beta = \mathbf{A}_\alpha \cdot \mathbf{C}\mathbf{A}_\beta \mapsto (\mathbf{R}\mathbf{A}_\alpha) \cdot \mathbf{C}\mathbf{R}\mathbf{A}_\beta &= (-1)^{\lambda(\alpha)}(-1)^{\lambda(\beta)}\mathbf{A}_{\sigma(\alpha)} \cdot \mathbf{C}\mathbf{A}_{\sigma(\beta)} \\ &= (-1)^{\lambda(\alpha)}(-1)^{\lambda(\beta)}\mathbf{r},_{\sigma(\alpha)} \cdot \mathbf{r},_{\sigma(\beta)}\,,\end{aligned}$$

and the thesis follows.

6.6. Surfaces with two families of fibers with bending and twisting stiffness

We consider as before a surface made of two families of continuously distributed inextensible fibers. We assume that the fibers are inextensible, so that X^1 and X^2 (the coordinates associated to the fiber directions) are arc parameters for the curves of each family, and $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are unit vectors. In the same spirit as the previous chapters, we describe the resistance to bending through a dependence of the energy on the curvature of the fibers, and resistance to twisting through a dependence on the torsion, which involve in turn a dependence on higher derivatives of the deformation. We use here Theorem 6.5.1, a consequence of Rychlewski's theorem, to discuss the restrictions due to material symmetry. Specifically, Theorem 6.5.1 implies that an objective energy function must depend on the scalar products of the partial derivatives of the deformation, which is additionally invariant under the material symmetry group. It is well known that any polynomial energy function invariant under the material symmetry group can be written as a polynomial in a finite number of basic polynomial invariants. Standard algebraic techniques allow to compute these polynomials, and we derive below the basic anisotropic polynomial invariants for the four basic structures of a plain weave. To compute the invariants, we have used the shareware computer package 'Singular' (cf. Greuel, Pfister and Schoenemann [**18**]).

6.6.1. Surfaces with two families of fibers with bending stiffness

Assuming first that the elastic energy is a function of the form $W(\boldsymbol{r}_{,\alpha}, \boldsymbol{r}_{,\beta\gamma})$, Proposition 6.5.1 implies that

$$W = \widetilde{W}\left(\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_{\beta,\gamma}, \boldsymbol{a}_{\alpha,\mu} \cdot \boldsymbol{a}_{\beta,\gamma},\right). \tag{6.6.1}$$

This form of the energy is appropriate to describe a dependence on the curvature and shear of the fibers, which involve derivatives of the deformation up to second order.

6.6.1.1. *The rectangular structure.* The rectangular structure has symmetry group H_{rt} (cf. Section 6.4.1.2): under the hypothesis of inextensibility of the fibers, its action on the arguments of (6.6.1) is listed in Table 1 in the Appendix. It follows that the invariants are

$$
\begin{aligned}
L_1 &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_2)^2 && = \sin^2\gamma \\
L_2 &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})^2 && = \mathcal{K}_2^2\,(\boldsymbol{a}_1 \cdot \boldsymbol{n}_2)^2 \\
L_3 &= (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2)^2 && = \mathcal{K}_1^2\,(\boldsymbol{n}_1 \cdot \boldsymbol{a}_2)^2 \\
L_4 &= \boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1} && = \mathcal{K}_1^2 \\
L_5 &= \boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2} && = \mathcal{K}_2^2 \\
L_6 &= \boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2} && = \mathcal{K}_1\mathcal{K}_2\,(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2) \\
L_7 &= \boldsymbol{a}_{1,2} \cdot \boldsymbol{a}_{1,2} && \\
L_8 &= (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2})^2 && = \mathcal{K}_1^2\,(\boldsymbol{n}_1 \cdot \boldsymbol{a}_{1,2})^2 \\
L_9 &= (\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1})^2 && = \mathcal{K}_2^2\,(\boldsymbol{n}_2 \cdot \boldsymbol{a}_{2,1})^2 \\
L_{10} &= (\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1})\,(\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2}) && = \mathcal{K}_1\mathcal{K}_2\,(\boldsymbol{n}_2 \cdot \boldsymbol{a}_{1,2})\,(\boldsymbol{n}_1 \cdot \boldsymbol{a}_{1,2}) \\
L_{11} &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_2)\,(\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1}) && = \sin\gamma\,(\mathcal{K}_2\boldsymbol{n}_2 \cdot \boldsymbol{a}_{2,1}) \\
L_{12} &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_2)\,(\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,1}) && = \sin\gamma\,(\mathcal{K}_1\boldsymbol{n}_1 \cdot \boldsymbol{a}_{2,1}) \\
L_{13} &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})\,(\boldsymbol{a}_2 \cdot \boldsymbol{a}_{1,1})\,(\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1}) && = \mathcal{K}_1\mathcal{K}_2^2(\boldsymbol{a}_1 \cdot \boldsymbol{n}_2)(\boldsymbol{a}_2 \cdot \boldsymbol{n}_1)\,(\boldsymbol{n}_2 \cdot \boldsymbol{a}_{2,1}) \\
L_{14} &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})\,(\boldsymbol{a}_2 \cdot \boldsymbol{a}_{1,1})\,(\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,1}) && = \mathcal{K}_1^2\mathcal{K}_2(\boldsymbol{a}_1 \cdot \boldsymbol{n}_2)(\boldsymbol{a}_2 \cdot \boldsymbol{n}_1)(\boldsymbol{n}_1 \cdot \boldsymbol{a}_{1,2}) \\
L_{15} &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_2)\,(\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})\,(\boldsymbol{a}_2 \cdot \boldsymbol{a}_{1,1}) && = \mathcal{K}_1\mathcal{K}_2(\sin\gamma)(\boldsymbol{a}_1 \cdot \boldsymbol{n}_2)(\boldsymbol{a}_2 \cdot \boldsymbol{n}_1).
\end{aligned}
\tag{6.6.2}
$$

6.6.1.2. *The square structure.* The material symmetry group of the square structure (cf. Section 6.4.1.1), H_{sq} differs from H_{rt} by four elements, namely the rotations $\boldsymbol{R}_{\pi/2}$ and $\boldsymbol{R}_{-\pi/2}$, and the reflections about the lines parallel to $\boldsymbol{A}_1 \pm \boldsymbol{A}_2$. The action of the generators of H_{sq} on the arguments of (6.6.1) is listed in Table 2 in the Appendix. Under the hypothesis of inextensibility of the fibers, the corresponding invariants are :

$$
\begin{aligned}
K_1 &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_2)^2 \\
K_2 &= [(\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})\,(\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2)]^2 \\
K_3 &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})^2 + (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2)^2 \\
K_4 &= (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1})(\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}) \\
K_5 &= \boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1} + \boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2} \\
K_6 &= |\boldsymbol{a}_{1,2}|^2 \\
K_7 &= \boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2} \\
K_8 &= (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2})\,(\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1})
\end{aligned}
\tag{6.6.3}
$$

$$
\begin{aligned}
K_9 &= (\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2})^2 + (\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,1})^2\\
K_{10} &= (\boldsymbol{a}_1\cdot\boldsymbol{a}_2)\,(\boldsymbol{a}_1\cdot\boldsymbol{a}_{2,2})\,(\boldsymbol{a}_2\cdot\boldsymbol{a}_{1,1})\\
K_{11} &= (\boldsymbol{a}_1\cdot\boldsymbol{a}_2)\,(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2}+\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{1,2})\\
K_{12} &= (\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,1})(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2})^2+(\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,2})(\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{1,2})^2\\
K_{13} &= (\boldsymbol{a}_1\cdot\boldsymbol{a}_2)[(\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,2})(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2})+(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,1})(\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{1,2})]\\
K_{14} &= (\boldsymbol{a}_1\cdot\boldsymbol{a}_{2,2})(\boldsymbol{a}_2\cdot\boldsymbol{a}_{1,1})(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2}+\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{1,2})\\
K_{15} &= (\boldsymbol{a}_1\cdot\boldsymbol{a}_{2,2})^2(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,1})+(\boldsymbol{a}_2\cdot\boldsymbol{a}_{1,1})^2(\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,2})\\
K_{16} &= (\boldsymbol{a}_1\cdot\boldsymbol{a}_{2,2})^2(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2})^2+(\boldsymbol{a}_2\cdot\boldsymbol{a}_{1,1})^2(\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{1,2})^2\\
K_{17} &= (\boldsymbol{a}_1\cdot\boldsymbol{a}_{2,2})(\boldsymbol{a}_2\cdot\boldsymbol{a}_{1,1})[(\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,2})(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2})+(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,1})(\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{1,2})]\\
K_{18} &= (\boldsymbol{a}_1\cdot\boldsymbol{a}_2)[(\boldsymbol{a}_2\cdot\boldsymbol{a}_{1,1})^2(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2})+(\boldsymbol{a}_1\cdot\boldsymbol{a}_{2,2})^2(\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{1,2})]\\
K_{19} &= (\boldsymbol{a}_1\cdot\boldsymbol{a}_{2,2})(\boldsymbol{a}_2\cdot\boldsymbol{a}_{1,1})^3(\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2})+(\boldsymbol{a}_2\cdot\boldsymbol{a}_{1,1})(\boldsymbol{a}_1\cdot\boldsymbol{a}_{2,2})^3(\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{1,2}).
\end{aligned}
$$

When expressed in terms of the curvature of the fibers, the above invariants become

$$
\begin{aligned}
K_1 &= \sin^2\gamma\\
K_2 &= \mathcal{K}_2^2\mathcal{K}_1^2\,[(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)\,(\boldsymbol{n}_1\cdot\boldsymbol{a}_2)]^2\\
K_3 &= \mathcal{K}_2^2\,(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)^2+\mathcal{K}_1^2\,(\boldsymbol{n}_1\cdot\boldsymbol{a}_2)^2\\
K_4 &= \mathcal{K}_1^2\mathcal{K}_2^2\\
K_5 &= \mathcal{K}_1^2+\mathcal{K}_2^2\\
K_6 &= |\boldsymbol{a}_{1,2}|^2\\
K_7 &= \mathcal{K}_1\mathcal{K}_2\,(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\\
K_8 &= \mathcal{K}_1\mathcal{K}_2\,(\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2})\,(\boldsymbol{n}_2\cdot\boldsymbol{a}_{2,1})\\
K_9 &= (\mathcal{K}_1\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2})^2+(\mathcal{K}_2\boldsymbol{n}_2\cdot\boldsymbol{a}_{2,1})^2\\
K_{10} &= \mathcal{K}_1\mathcal{K}_2(\sin\gamma)(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)(\boldsymbol{a}_2\cdot\boldsymbol{n}_1)\\
K_{11} &= \sin\gamma\,(\mathcal{K}_1\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2}+\mathcal{K}_2\boldsymbol{n}_2\cdot\boldsymbol{a}_{1,2})\\
K_{12} &= \mathcal{K}_1^3(\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2})^2+\mathcal{K}_2^3(\boldsymbol{n}_2\cdot\boldsymbol{a}_{1,2})^2\\
K_{13} &= \sin\gamma[\mathcal{K}_1\mathcal{K}_2^2(\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2})+\mathcal{K}_2\mathcal{K}_1^2(\boldsymbol{n}_2\cdot\boldsymbol{a}_{1,2})]\\
K_{14} &= \mathcal{K}_1\mathcal{K}_2(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)(\boldsymbol{a}_2\cdot\boldsymbol{n}_1)(\mathcal{K}_1\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2}+\mathcal{K}_2\boldsymbol{n}_2\cdot\boldsymbol{a}_{1,2})\\
K_{15} &= \mathcal{K}_1^2\mathcal{K}_2^2[(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)^2+(\boldsymbol{a}_2\cdot\boldsymbol{n}_1)^2]\\
K_{16} &= \mathcal{K}_1^2\mathcal{K}_2^2[(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)^2(\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2})^2+(\boldsymbol{a}_2\cdot\boldsymbol{n}_1)^2(\boldsymbol{n}_2\cdot\boldsymbol{a}_{1,2})^2\\
K_{17} &= \mathcal{K}_1\mathcal{K}_2(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)(\boldsymbol{a}_2\cdot\boldsymbol{n}_1)[\mathcal{K}_1\mathcal{K}_2^2(\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2})+\mathcal{K}_1^2\mathcal{K}_2(\boldsymbol{n}_2\cdot\boldsymbol{a}_{1,2})]\\
K_{18} &= \sin\gamma[\mathcal{K}_1^3(\boldsymbol{a}_2\cdot\boldsymbol{n}_1)^2(\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2})+\mathcal{K}_2^3(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)^2(\boldsymbol{n}_2\cdot\boldsymbol{a}_{1,2})]\\
K_{19} &= \mathcal{K}_1^4\mathcal{K}_2(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)(\boldsymbol{a}_2\cdot\boldsymbol{n}_1)^3(\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2})+\mathcal{K}_2^4\mathcal{K}_1(\boldsymbol{a}_2\cdot\boldsymbol{n}_1)(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)^3(\boldsymbol{n}_2\cdot\boldsymbol{a}_{1,2}).
\end{aligned}
\tag{6.6.4}
$$

6.6.1.3. *The rhombic structure.* The rhombic structure has symmetry group H_{rb} (cf. Section 6.4.1.3): under the hypothesis of inextensibility of the fibers, its action on the arguments of (6.6.1) is listed in Table 3 in the Appendix. It follows that the invariants are

$$\begin{aligned}
N_1 &= \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 \\
N_2 &= (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2}) + (\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{1,2}) \\
N_3 &= (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2})\,(\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{1,2}) \\
N_4 &= (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}) + (\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}) \\
N_5 &= (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1})\,(\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}) \\
N_6 &= |\boldsymbol{a}_{1,2}|^2 \\
N_7 &= \boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2} \\
N_8 &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})\,(\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2) \\
N_9 &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})^2 + (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2)^2 \\
N_{10} &= (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1})\,(\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2}) + (\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2})\,(\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1}) \\
N_{11} &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})^2\,(\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2}) + (\boldsymbol{a}_2 \cdot \boldsymbol{a}_{1,1})^2\,(\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1}) \\
N_{12} &= (\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})^2\,(\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}) + (\boldsymbol{a}_2 \cdot \boldsymbol{a}_{1,1})^2\,(\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2})\,.
\end{aligned} \tag{6.6.5}$$

When expressed in terms of the curvature of the fibers, the above invariants become

$$\begin{aligned}
N_1 &= \sin\gamma \\
N_2 &= (\mathcal{K}_1 \boldsymbol{n}_1 \cdot \boldsymbol{a}_{1,2}) + (\mathcal{K}_2 \boldsymbol{n}_2 \cdot \boldsymbol{a}_{2,1}) \\
N_3 &= (\mathcal{K}_1 \boldsymbol{n}_1 \cdot \boldsymbol{a}_{1,2})\,(\mathcal{K}_2 \boldsymbol{n}_2 \cdot \boldsymbol{a}_{1,2}) \\
N_4 &= \mathcal{K}_1^2 + \mathcal{K}_2^2 \\
N_5 &= \mathcal{K}_1^2 \mathcal{K}_2^2 \\
N_6 &= |\boldsymbol{a}_{1,2}|^2 \\
N_7 &= \mathcal{K}_1 \mathcal{K}_2\,(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2) \\
N_8 &= \mathcal{K}_2 \mathcal{K}_1\,(\boldsymbol{a}_1 \cdot \boldsymbol{n}_2)\,(\boldsymbol{n}_1 \cdot \boldsymbol{a}_2) \\
N_9 &= (\mathcal{K}_2)^2\,(\boldsymbol{a}_1 \cdot \boldsymbol{n}_2)^2 + (\mathcal{K}_1)^2\,(\boldsymbol{n}_1 \cdot \boldsymbol{a}_2)^2 \\
N_{10} &= \mathcal{K}_1^3(\boldsymbol{a}_{1,2} \cdot \boldsymbol{n}_1) + \mathcal{K}_2^3(\boldsymbol{a}_{1,2} \cdot \boldsymbol{n}_2) \\
N_{11} &= \mathcal{K}_1 \mathcal{K}_2^2\,(\boldsymbol{a}_1 \cdot \boldsymbol{n}_2)^2\,(\boldsymbol{n}_1 \cdot \boldsymbol{a}_{1,2}) + \mathcal{K}_2 \mathcal{K}_1^2\,(\boldsymbol{a}_2 \cdot \boldsymbol{n}_1)^2\,(\boldsymbol{n}_2 \cdot \boldsymbol{a}_{2,1}) \\
N_{12} &= \mathcal{K}_1^2 \mathcal{K}_2^2[(\boldsymbol{n}_1 \cdot \boldsymbol{a}_{1,1}) + (\boldsymbol{a}_{2,2} \cdot \boldsymbol{n}_2)].
\end{aligned} \tag{6.6.6}$$

6.6.1.4. *The parallelogram structure.* The material symmetry group of the parallelogram structure H_{pr} (cf. Section 6.4.1.4) only contains the identity and the inversion, with basic polynomial invariants

$$\begin{array}{rclcl} Q_1 & = & \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 & = & \sin\gamma \\ Q_2 & = & (\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2})^2 & = & \mathcal{K}_2^2\,(\boldsymbol{a}_1 \cdot \boldsymbol{n}_2)^2 \\ Q_3 & = & (\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2)^2 & = & \mathcal{K}_1^2\,(\boldsymbol{n}_1 \cdot \boldsymbol{a}_2)^2 \\ Q_4 & = & \boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1} & = & \mathcal{K}_1^2 \\ Q_5 & = & \boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2} & = & \mathcal{K}_2^2 \\ Q_6 & = & \boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2} & = & \mathcal{K}_2\mathcal{K}_1\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 \\ Q_7 & = & \boldsymbol{a}_{1,2} \cdot \boldsymbol{a}_{1,2} & & \\ Q_8 & = & \boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2} & = & \mathcal{K}_1\boldsymbol{n}_1 \cdot \boldsymbol{a}_{1,2} \\ Q_9 & = & \boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1} & = & \mathcal{K}_2\boldsymbol{n}_2 \cdot \boldsymbol{a}_{2,1} \end{array} \tag{6.6.7}$$

Notice that the above invariant depend on the shear angle γ, the curvature of the fibers, the angle between the Frenet triads of intersecting fibers, and the derivative $\boldsymbol{a}_{1,2} = \boldsymbol{a}_{2,1}$. If we neglect, as is reasonable for fabrics, any dependence of the elastic energy on the last two sets of variables, it follows that, in this case, the basic polynomial invariants are

(-) for the square symmetry,

$$\sin^2\gamma, \quad \mathcal{K}_1^2\mathcal{K}_2^2, \quad \mathcal{K}_1^2 + \mathcal{K}_2^2. \tag{6.6.8}$$

(-) for the rectangular symmetry,

$$\sin^2\gamma, \quad \mathcal{K}_1^2, \quad \mathcal{K}_2^2. \tag{6.6.9}$$

(-) for the rhombic symmetry,

$$\sin\gamma, \quad \mathcal{K}_1^2\mathcal{K}_2^2, \quad \mathcal{K}_1^2 + \mathcal{K}_2^2. \tag{6.6.10}$$

(-) for the parallelogram symmetry,

$$\sin\gamma, \quad \mathcal{K}_1^2, \quad \mathcal{K}_2^2. \tag{6.6.11}$$

Hence, for rhombic and parallelogram symmetries a shear of angle γ does not have the same effect than the shear of angle $-\gamma$, contrary from what happens for the square and rectangular structures.

6.6.2. Surfaces composed of fibers with bending and twisting stiffness

Assume finally that the elastic energy is a function of $\boldsymbol{r}_{,\alpha}, \boldsymbol{r}_{,\beta\gamma}$ and the third order derivatives $\boldsymbol{r}_{,\delta\mu\nu}$: Proposition 6.5.1 implies that W can be written as

$$\widetilde{W}\left(\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_{\beta,\gamma}, \boldsymbol{a}_{\alpha,\mu} \cdot \boldsymbol{a}_{\beta,\gamma}, \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_{\beta,\gamma\nu}, \boldsymbol{a}_{\alpha,\mu} \cdot \boldsymbol{a}_{\beta,\gamma\nu}, \boldsymbol{a}_{\alpha,\mu\delta} \cdot \boldsymbol{a}_{\beta,\gamma\nu}\right). \tag{6.6.12}$$

This form of the energy is appropriate to describe a dependence on the curvature, torsion and shear of the fibers, which involve derivatives of the deformation up to third order.

Under the assumption of inextensibility of the fibers, and using the symmetry of the indices, the list of arguments of the strain energy function reduces to 48 elements, and the resulting list of invariants under the action of the material symmetry groups is quite complicated.

For the sake of simplicity, we choose a strain energy depending on invariants that are in turn functions of the curvatures $\mathcal{K}_\alpha$ and the torsions τ_α of the fibers only, with $\alpha = 1, 2$. Indeed, the curvatures and torsions can be expressed in terms of the derivatives of the position vector $\boldsymbol{r}$ as follows:

Proposition 6.6.1. *The following relations hold (no summation on repeated indices):*

$$(\mathcal{K}_\alpha)^2 = \boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}, \tag{6.6.13}$$

$$(\mathcal{K}_{\alpha,\alpha})^2 = \frac{\left(\boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}\right)^2}{\boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}}, \tag{6.6.14}$$

$$(\tau_\alpha)^2 = \frac{\boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha\alpha}}{\boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}} - \frac{\left(\boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}\right)^2}{\left(\boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}\right)^2} - \boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}. \tag{6.6.15}$$

Proof. From Frenet formula $(2.2.4)_1$ we have

$$\boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha} = (\mathcal{K}_\alpha)^2. \tag{6.6.16}$$

By differentiating $(2.2.4)_1$ and using $(2.2.4)_2$ we obtain

$$\boldsymbol{a}_{\alpha,\alpha\alpha} = \mathcal{K}_{\alpha,\alpha}\boldsymbol{n}_\alpha + \mathcal{K}_\alpha \boldsymbol{n}_{\alpha,\alpha} = \mathcal{K}_{\alpha,\alpha}\boldsymbol{n}_\alpha - (\mathcal{K}_\alpha)^2 \boldsymbol{a}_\alpha + \mathcal{K}_\alpha \tau_\alpha \boldsymbol{b}_\alpha,$$

from which it follows that

$$\begin{aligned} \boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha\alpha} &= (\mathcal{K}_{\alpha,\alpha})^2 + (\mathcal{K}_\alpha)^4 + (\mathcal{K}_\alpha \tau_\alpha)^2 \\ \boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha} &= \mathcal{K}_{\alpha,\alpha}\mathcal{K}_\alpha \end{aligned} \tag{6.6.17}$$

From $(6.6.17)_2$ and (6.6.16) we obtain

$$(\mathcal{K}_{\alpha,\alpha})^2 = \left(\frac{\boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}}{\mathcal{K}_\alpha} \right)^2 = \frac{(\boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha})^2}{\boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}}. \tag{6.6.18}$$

Moreover, using (6.6.18) and $(6.6.17)_1$:

$$\begin{aligned} (\tau_\alpha)^2 =& \frac{\boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha\alpha} - (\mathcal{K}_{\alpha,\alpha})^2 - (\mathcal{K}_\alpha)^4}{(\mathcal{K}_\alpha)^2} \\ =& \frac{\boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha\alpha}}{\boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}} - \frac{(\boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha})^2}{(\boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha})^2} - \boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}, \end{aligned}$$

which yields the thesis.

The following result characterizes those anisotropic invariants of the four basic structures that are functions of the fibers curvature and torsion only.

Proposition 6.6.2. *The expressions*

$$\mathcal{K}_1^2\mathcal{K}_2^2, \qquad \mathcal{K}_1^2 + \mathcal{K}_2^2, \qquad \tau_1^2\tau_2^2, \qquad \tau_1^2 + \tau_2^2, \tag{6.6.19}$$

are invariant under the action of H_{sq} and H_{rb}. Moreover, the squared curvature and torsion of each fiber

$$\mathcal{K}_1^2, \qquad \mathcal{K}_2^2, \qquad \tau_1^2, \qquad \tau_2^2, \tag{6.6.20}$$

are invariant under the action of H_{rt} and H_{pr}

Proof. We only prove the first assertion regarding the symmetric polynomials in the torsion. The action of H_{sq} on the derivatives in the expressions for $(\tau_1)^2$ and $(\tau_2)^2$, i.e.,

$$\boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha\alpha}, \quad \boldsymbol{a}_{\alpha,\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha}, \quad (\boldsymbol{a}_{\alpha,\alpha\alpha} \cdot \boldsymbol{a}_{\alpha,\alpha})^2,$$

is listed in Table 4.

A straightforward verification shows that $\tau_1^2 + \tau_2^2$ and $\tau_1^2\tau_2^2$ are invariant under the action of H_{sq}. The proof of the remaining assertions follows the same lines and is omitted.

We conclude that

(-) for inextensible networks with the **square symmetry**, formed by two orthogonal families of equivalent fibers, an energy function depending only on $\boldsymbol{C}$ and the fiber curvature and torsion must have the form

$$W = W(\sin^2\gamma, \mathcal{K}_1^2\mathcal{K}_2^2, \mathcal{K}_1^2 + \mathcal{K}_2^2, \tau_1^2\tau_2^2, \tau_1^2 + \tau_2^2). \tag{6.6.21}$$

(-) for inextensible networks with the **rectangular symmetry**, formed by two orthogonal families of non equivalent fibers, the energy has the form

$$W = W(\sin^2\gamma, \mathcal{K}_1^2, \mathcal{K}_2^2, \tau_1^2, \tau_2^2). \tag{6.6.22}$$

(-) for inextensible networks with the **rhombic symmetry**, formed by two non-orthogonal families of equivalent fibers, the energy has the form

$$W = W(\sin\gamma, \mathcal{K}_1^2\mathcal{K}_2^2, \mathcal{K}_1^2 + \mathcal{K}_2^2, \tau_1^2\tau_2^2, \tau_1^2 + \tau_2^2). \tag{6.6.23}$$

(-) for inextensible networks with the **parallelogram symmetry**, formed by two non-orthogonal families of non equivalent fibers, the energy has the form

$$W = W(\sin\gamma, \mathcal{K}_1^2, \mathcal{K}_2^2, \tau_1^2, \tau_2^2). \tag{6.6.24}$$

6.7. Tables of the group actions

The tables below list the action (6.5.3) of the generators of the symmetry groups of the rectangular, square and rhombic structure. The action of the generators of the group H_{sq} on the components of $(\tau_1)^2$ and $(\tau_2)^2$ is also listed below.

Table 1: Action of the generators of H_{rt}

Argument	$\boldsymbol{R}_1$	$\boldsymbol{R}_2$
$\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$	$-\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$	$-\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$
$\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2}$	$-\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2}$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2$	$-\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2$

Continued on next page

Argument	$\boldsymbol{R}_1$	$\boldsymbol{R}_2$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}$
$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2}$
$\boldsymbol{a}_{1,2} \cdot \boldsymbol{a}_{1,2}$	$\boldsymbol{a}_{1,2} \cdot \boldsymbol{a}_{1,2}$	$\boldsymbol{a}_{1,2} \cdot \boldsymbol{a}_{1,2}$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2}$	$-\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2}$	$-\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2}$
$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1}$	$-\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1}$	$-\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1}$

Table 2: Action of the generators of H_{sq}

Argument	$\boldsymbol{R}_{\pi/2}$	$\boldsymbol{R}_1$
$\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$	$-\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$	$-\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$
$\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2}$	$-\boldsymbol{a}_2 \cdot \boldsymbol{a}_{1,1}$	$-\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2}$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2$	$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_1$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}$	$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}$
$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}$	$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{1,1}$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2}$
$\boldsymbol{a}_{1,2} \cdot \boldsymbol{a}_{1,2}$	$\boldsymbol{a}_{2,1} \cdot \boldsymbol{a}_{2,1}$	$\boldsymbol{a}_{1,2} \cdot \boldsymbol{a}_{1,2}$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2}$	$-\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1}$	$-\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2}$
$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1}$	$-\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,2}$	$-\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,1}$

Table 3: Action of the generators of H_{rb}

Argument	$\boldsymbol{R}_3$	$\boldsymbol{R}_4$
$\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$	$\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$	$\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$
$\boldsymbol{a}_1 \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_2 \cdot \boldsymbol{a}_{1,1}$	$-\boldsymbol{a}_2 \cdot \boldsymbol{a}_{1,1}$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_2$	$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_1$	$-\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_1$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}$	$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}$
$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{1,1}$
$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{2,2} \cdot \boldsymbol{a}_{1,1}$	$\boldsymbol{a}_{1,1} \cdot \boldsymbol{a}_{2,2}$

Continued on next page

Argument	$\boldsymbol{R}_3$	$\boldsymbol{R}_4$
$\boldsymbol{a}_{1,2}\cdot\boldsymbol{a}_{1,2}$	$\boldsymbol{a}_{2,1}\cdot\boldsymbol{a}_{2,1}$	$\boldsymbol{a}_{1,2}\cdot\boldsymbol{a}_{1,2}$
$\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2}$	$\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,1}$	$\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,1}$
$\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,1}$	$\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2}$	$\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,2}$

Table 4: Action of the generators of the group H_{sq} on the components of $(\tau_1)^2$ and $(\tau_2)^2$

Component	$\boldsymbol{R}_1$	$\boldsymbol{R}_{\pi/2}$
$\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,1}$	$\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,1}$	$\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,2}$
$\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,2}$	$\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,1}$
$\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,11}$	$-\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,11}$	$-\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,22}$
$\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,22}$	$\boldsymbol{a}_{2,2}\cdot\boldsymbol{a}_{2,22}$	$\boldsymbol{a}_{1,1}\cdot\boldsymbol{a}_{1,11}$
$\boldsymbol{a}_{1,11}\cdot\boldsymbol{a}_{1,11}$	$\boldsymbol{a}_{1,11}\cdot\boldsymbol{a}_{1,11}$	$\boldsymbol{a}_{2,22}\cdot\boldsymbol{a}_{2,22}$
$\boldsymbol{a}_{2,22}\cdot\boldsymbol{a}_{2,22}$	$\boldsymbol{a}_{2,22}\cdot\boldsymbol{a}_{2,22}$	$\boldsymbol{a}_{1,11}\cdot\boldsymbol{a}_{1,11}$

6.8. Plane deformations

Consider now an energy function of the form (6.6.1): we have seen that some anisotropic invariants, for instance $L_7, \ldots, L_{14}$, depend not only on the curvatures, but also on the mixed derivatives $\boldsymbol{a}_{1,2}$. We next show that, at least for plane deformations, the curvature and the shear angle are sufficient to characterize the deformation. A plane deformation is such that there exists a constant vector $\boldsymbol{k}$ such that $\boldsymbol{r}(X^1, X^2)\cdot\boldsymbol{k} = \text{const.}$ for every $(X^1, X^2) \in \Sigma_0$.

THEOREM 6.8.1. *Given two sets of inextensible fibers undergoing plane deformations, then*

$$\boldsymbol{a}_{1,2} = \mathbf{0}.$$

Proof. Choose unit vectors $\boldsymbol{i}, \boldsymbol{j}$ in the plane of deformation and let

$$\boldsymbol{r}(X^1, X^2) = f(X^1, X^2)\,\boldsymbol{i} + g(X^1, X^2)\,\boldsymbol{j};$$

then

$$\boldsymbol{a}_1 = f_{,1}\,\boldsymbol{i} + g_{,1}\,\boldsymbol{j} \qquad \boldsymbol{a}_2 = f_{,2}\,\boldsymbol{i} + g_{,2}\,\boldsymbol{j}. \tag{6.8.1}$$

As a consequence of the inextensibility of the fibers,

$$(f_{,1})^2 + (g_{,1})^2 = 1, \qquad (f_{,2})^2 + (g_{,2})^2 = 1,$$

so that there exist $\varphi = \varphi(X^1, X^2)$ and $\psi = \psi(X^1, X^2)$ such that

$$f_{,1} = \sin\varphi \qquad f_{,2} = \sin\psi \qquad g_{,1} = \cos\varphi \qquad g_{,2} = \cos\psi. \tag{6.8.2}$$

Imposing the Schwartz condition on second order derivatives, we obtain

$$f_{,12} = f_{,21} \qquad g_{,12} = g_{,21},$$

or equivalently

$$\begin{cases} \varphi_{,2}\cos\varphi - \psi_{,1}\cos\psi &= 0 \\ \varphi_{,2}\sin\varphi - \psi_{,1}\sin\psi &= 0. \end{cases} \tag{6.8.3}$$

Viewing this as a linear system in the unknowns $\phi_{,2}$ and $\psi_{,1}$, the determinant is

$$\det\begin{pmatrix} \cos\varphi & \cos\psi \\ \sin\varphi & \sin\psi \end{pmatrix} = \sin(\varphi - \psi). \tag{6.8.4}$$

We have to distinguish two cases: either $\varphi = \psi$ or $\varphi \neq \psi$. If $\varphi = \psi$, then from (6.8.1) and from (6.8.2), it follows that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are parallel (not acceptable solution/configuration). Then, necessarily, $\varphi \neq \psi$. As a consequence, the system (6.8.3) admits only the trivial solution:

$$\varphi_{,2} = 0 \qquad \psi_{,1} = 0$$

then:

$$\varphi = \varphi(X^1) \qquad \text{and} \qquad \psi = \psi(X^2)$$

consequently:

$$\begin{cases} f(X^1, X^2) = \int \sin\varphi(X^1)\,dX^1 + \int \sin\psi(X^2)\,dX^2 \\ g(X^1, X^2) = \int \cos\varphi(X^1)\,dX^1 + \int \cos\psi(X^2)\,dX^2 \end{cases} \tag{6.8.5}$$

which implies that:

$$\begin{cases} f(X^1, X^2) = A(X^1) + B(X^2) \\ g(X^1, X^2) = C(X^1) + D(X^2) \end{cases} \tag{6.8.6}$$

so that finally

$$\begin{cases} \boldsymbol{a}_1 &= \sin\varphi(X^1)\,\boldsymbol{i} + \cos\varphi(X^1)\,\boldsymbol{j} \\ \boldsymbol{a}_2 &= \sin\psi(X^1)\,\boldsymbol{i} + \cos\psi(X^2)\,\boldsymbol{j} \end{cases} \tag{6.8.7}$$

and $\boldsymbol{a}_{1,2} = \boldsymbol{a}_{2,1} = \boldsymbol{0}$, which concludes the proof.

Examples of plane deformations consistent with the constraint of inextensibility are:

1) $\boldsymbol{r}(X^1, X^2) = (-\cos X^2)\ \boldsymbol{i} + (X^1 + \sin X^2)\ \boldsymbol{j}$, corresponding to $\varphi(X^1) \equiv 0$ and $\psi(X^2) \equiv X^2$. Then $\boldsymbol{a}_1 = \boldsymbol{j}$ and $\boldsymbol{a}_2 = \sin X^2 \boldsymbol{i} + \cos X^2 \boldsymbol{j}$; in the actual configuration the first family of fibers is parallel to the X^2 axis.(fig.6.2)
2) $\boldsymbol{r}(X^1, X^2) = (X^2 \sin\beta)\ \boldsymbol{i} + (X^1 + X^2 \cos\beta)\ \boldsymbol{j}$, corresponding to $\psi(X^1) = \frac{\pi}{2}$ and $\psi(X^2) \equiv \beta = const.$

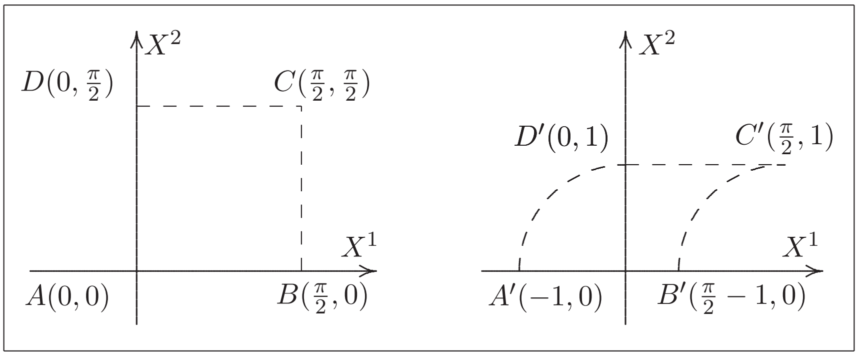

FIGURE 6.2. Plane deformations of inextensible networks: example 1.

6.9. Invariants as functions of the Euler angles

The lists of invariants (6.6.5) and (6.6.7) can also be written in terms of the Euler angles (2.5.2). We only give the expression corresponding to the invariants (6.6.5) of the rectangular structure, since those for the square structure (6.6.7) follow immediately. The list (6.6.5) becomes

$$
\begin{aligned}
L_1 &= \sin^2\gamma \\
L_2 &= (\mathcal{K}_2)^2\,(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)^2 && = (\mathcal{K}_2)^2\,(\sin\Psi\sin\delta)^2 \\
L_3 &= (\mathcal{K}_1)^2\,(\boldsymbol{n}_1\cdot\boldsymbol{a}_2)^2 && = (\mathcal{K}_1)^2\,(\sin\Phi\sin\delta)^2 \\
L_4 &= (\mathcal{K}_1)^2 \\
L_5 &= (\mathcal{K}_2)^2 \\
L_6 &= \mathcal{K}_1\mathcal{K}_2\,(\boldsymbol{n}_1\cdot\boldsymbol{n}_2) && = \mathcal{K}_1\mathcal{K}_2(\cos\Phi\cos\Psi-\sin\Phi\sin\Psi\cos\delta) \\
L_7 &= \boldsymbol{a}_{1,2}\cdot\boldsymbol{a}_{1,2} && = \left[-\frac{\partial\delta}{\partial X^2}\sin\delta-\mathcal{K}_2\sin\Psi\sin\delta\right]^2 \\
& && \quad+\left[\tau_2\sin\Psi\sin\delta-\frac{\partial\cos\Psi\sin\delta}{\partial X^2}\right]^2 \\
& && \quad+\left[\mathcal{K}_2\cos\delta+\tau_2\cos\Psi\sin\delta+\frac{\partial(\sin\Psi\sin\delta)}{\partial X^2}\right]^2 \\
L_8 &= (\mathcal{K}_1\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2})^2 && = \left[(\mathcal{K}_1)^2\cos\delta+\mathcal{K}_1\cos\Phi\sin\delta\left(\frac{\partial\Phi}{\partial X^1}-\tau_1\right)\right. \\
& && \quad\left.+\mathcal{K}_1\frac{\partial\delta}{\partial X^1}\sin\Phi\cos\delta\right]^2 \\
L_9 &= (\mathcal{K}_2\boldsymbol{n}_2\cdot\boldsymbol{a}_{2,1})^2 && = \left[(\mathcal{K}_2)^2\cos\delta+\mathcal{K}_2\cos\Psi\sin\delta\left(\frac{\partial\Psi}{\partial X^2}+\tau_2\right)\right. \\
& && \quad\left.+\mathcal{K}_2\frac{\partial\delta}{\partial X^2}\sin\Psi\cos\delta\right]^2 \\
L_{10} &= \mathcal{K}_1\mathcal{K}_2\,(\boldsymbol{n}_2\cdot\boldsymbol{a}_{1,2})\,(\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2}) && = \mathcal{K}_1\mathcal{K}_2\left[\mathcal{K}_1\cos\delta+\cos\Phi\sin\delta\left(\frac{\partial\Phi}{\partial X^1}-\tau_1\right)+\frac{\partial\delta}{\partial X^1}\sin\Phi\cos\delta\right] \\
& && \quad\left[\mathcal{K}_2\cos\delta+\cos\Psi\sin\delta\left(\frac{\partial\Psi}{\partial X^2}+\tau_2\right)+\frac{\partial\delta}{\partial X^2}\sin\Psi\cos\delta\right] \\
L_{11} &= \sin\gamma\,(\mathcal{K}_2\boldsymbol{n}_2\cdot\boldsymbol{a}_{2,1}) && = \sin\gamma\left[(\mathcal{K}_2)^2\cos\delta+\mathcal{K}_2\cos\Psi\sin\delta\left(\frac{\partial\Psi}{\partial X^2}+\tau_2\right)\right. \\
& && \quad\left.+\mathcal{K}_2\frac{\partial\delta}{\partial X^2}\sin\Psi\cos\delta\right] \\
L_{12} &= \sin\gamma\,(\mathcal{K}_1\boldsymbol{n}_1\cdot\boldsymbol{a}_{2,1}) && = \sin\gamma\left[(\mathcal{K}_1)^2\cos\delta+\mathcal{K}_1\cos\Phi\sin\delta\left(\frac{\partial\Phi}{\partial X^1}-\tau_1\right)\right. \\
& && \quad\left.+\mathcal{K}_1\frac{\partial\delta}{\partial X^1}\sin\Phi\cos\delta\right] \\
L_{13} &= \mathcal{K}_1\mathcal{K}_2^2(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)(\boldsymbol{a}_2\cdot\boldsymbol{n}_1)\,(\boldsymbol{n}_2\cdot\boldsymbol{a}_{2,1}) && = \mathcal{K}_1\mathcal{K}_2^2\sin\Psi\sin\Phi\,(\sin\delta)^2 \\
& && \quad\left[\mathcal{K}_2\cos\delta+\cos\Psi\sin\delta\left(\frac{\partial\Psi}{\partial X^2}+\tau_2\right)+\frac{\partial\delta}{\partial X^2}\sin\Psi\cos\delta\right] \\
L_{14} &= \mathcal{K}_1^2\mathcal{K}_2(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)(\boldsymbol{a}_2\cdot\boldsymbol{n}_1)(\boldsymbol{n}_1\cdot\boldsymbol{a}_{1,2}) && = \mathcal{K}_1^2\mathcal{K}_2\sin\Psi\sin\Phi\,(\sin\delta)^2 \\
& && \quad\left[\mathcal{K}_1\cos\delta+\cos\Phi\sin\delta\left(\frac{\partial\Phi}{\partial X^1}-\tau_1\right)+\frac{\partial\delta}{\partial X^1}\sin\Phi\cos\delta\right] \\
L_{15} &= \mathcal{K}_1\mathcal{K}_2(\sin\gamma)(\boldsymbol{a}_1\cdot\boldsymbol{n}_2)(\boldsymbol{a}_2\cdot\boldsymbol{n}_1) && = \mathcal{K}_1\mathcal{K}_2\sin\gamma\sin\Psi\sin\Phi(\sin\delta)^2.
\end{aligned}
\tag{6.9.1}
$$

CHAPTER 7

A microstructural model for textiles

Contents

In a textile fabric different scales can be identified: a macroscale, corresponding to the textile proper, and a microscale related to the weaving of the weft and the warp. In order to distinguish the undulations related to the weave from the macroscopic curvature of the surface, we will refer to the former as microundulations.

In chapter 3, we have already discussed a microstructural model by Magno and Ganghoffer ([**39**], [**38**]), that introduce a model for woven fabrics correlating the effect of the undulation of the yarn to the uniaxial extension of a plane weaved fabric.

The aim of this chapter is to determine the equations that describe the effect of shearing, bending and microundulations of the fibers on the deformation of the sheet. The field equations are obtained via a variational principle for a strain energy density in additive form, such that the contributions due to shear, bending and microundulations are separately taken into account. In order to model the macroscopic effects of the microundulations, we associate to each family of fibers a vectorial microstructure (in the sense of Capriz [**10**], Pastrone [**48**]).

7.1. Microstructures: the model

In the last twenty years the interest on microstructured materials has increased steadily. Materials like ceramics, composites, alloys, and so on, have increasingly wide applications, and it is important to study carefully their structure, in order to model those physical phenomena that are related to their various characteristic macro- and microscopic scales.

Models for materials with microstructure have a long history: already in 1964, Mindlin [**42**] formulated a first model for solids with crystalline microstructure, which was later expanded and developed by various authors, among others Gurtin and Podio Guidugli [**23**], Capriz [**10**], who extended the classical axioms of continuum mechanics to cover materials with microstructure.

In this section we adapt to our case the model of a solid with vectorial microstructure as introduced by Pastrone [**45**], [**48**].

7.1.1. The field equations

Let the deformation of a body be described by the vector-valued function

$$\boldsymbol{r} = \boldsymbol{r}(X^h, t) \tag{7.1.1}$$

with X^h material coordinates and t the time. We write

$$\boldsymbol{r}_{,h} = \frac{\partial \boldsymbol{r}}{\partial X^h} \qquad \dot{\boldsymbol{r}} = \frac{\partial \boldsymbol{r}}{\partial t}. \tag{7.1.2}$$

We are interested in microstructures that can be described by a collection of vector fields $\boldsymbol{d}_H$ on the body ($H = 1, \dots, N$), which may be introduced, for instance, by a procedure called magnification (see [**48**] for more details). The kinetic energy density of the microstructured body is defined as a quadratic form in the velocities $\dot{\boldsymbol{r}}$, $\dot{\boldsymbol{d}}_H$:

$$T = \frac{1}{2}\left(\rho \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + 2\rho^H \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{d}}_H + \rho^{HK} \dot{\boldsymbol{d}}_H \cdot \dot{\boldsymbol{d}}_K\right) \tag{7.1.3}$$

where ρ is the usual mass density, and ρ^H, ρ^{HK} are coefficients including density and inertia terms, which must satisfy the conditions

$$T \geq 0, \quad \text{and} \quad T = 0 \Leftrightarrow \dot{\boldsymbol{r}} = \dot{\boldsymbol{d}} = 0,$$

which guarantee that T is a positive definite quadratic form.

Without loss of generality, we can reduce the kinetic energy to a diagonal form

$$T = \frac{1}{2}\left(\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + I^{HK}\dot{\boldsymbol{d}}_H \cdot \dot{\boldsymbol{d}}_K\right) \tag{7.1.4}$$

where I^{HK} represents the inertia tensor of the microstructure.

In a Lagrangian formulation, ρ and I^{HK} are defined in the reference configuration, and are functions of X^h only.

We assume that the body admits a generalized strain energy density function:

$$W = W(\boldsymbol{r}_{,h}, \boldsymbol{d}_H, \boldsymbol{d}_{H,J}, X^h). \tag{7.1.5}$$

This energy is assumed to be related to the total mechanical power expended by the relation

$$P_T = \frac{dW}{dt} = \frac{\partial W}{\partial \boldsymbol{r}_{,h}} \cdot \dot{\boldsymbol{r}}_{,h} + \frac{\partial W}{\partial \boldsymbol{d}_H} \cdot \dot{\boldsymbol{d}}_H + \frac{\partial W}{\partial \boldsymbol{d}_{H,i}} \cdot \dot{\boldsymbol{d}}_{H,i}. \tag{7.1.6}$$

where $\dot{\boldsymbol{r}}_{,h}$, $\dot{\boldsymbol{d}}_H$, $\dot{\boldsymbol{d}}_{H,j}$ are the strain velocities and $\frac{\partial W}{\partial \boldsymbol{r}_{,h}}$, $\frac{\partial W}{\partial \boldsymbol{d}_H}$, $\frac{\partial W}{\partial \boldsymbol{d}_{H,j}}$ are the generalized stresses. Neglecting boundary conditions, the total energy can be written in the form

$$E = \int_{\mathcal{B}} (T - W - W_b)dV \tag{7.1.7}$$

where

$$W_b = W_b(\boldsymbol{r}, \boldsymbol{d}_H, X^h)$$

is related to conservative body forces, with power

$$P_T^b = \frac{\partial W_b}{\partial \boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + \frac{\partial W_b}{\partial \boldsymbol{d}_H} \cdot \dot{\boldsymbol{d}}_H.$$

The field equation are derived as Euler Lagrange equation of the action functional and are

$$\begin{cases} \left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,i}}\right)_{,i} - \dfrac{\partial W_b}{\partial \boldsymbol{r}} = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\boldsymbol{r}}}, \\ \\ \left(\dfrac{\partial W}{\partial \boldsymbol{d}_{H,i}}\right)_{,i} - \dfrac{\partial W}{\partial \boldsymbol{d}_H} - \dfrac{\partial W_b}{\partial \boldsymbol{d}_H} = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\boldsymbol{d}}_H}. \end{cases} \tag{7.1.8}$$

From a physical point of view, the forces acting at the microstructural level are different from those acting at the macrolevel, and it makes sense to assume W_b to be the sum $W_b^M + W_b^m$ of a macrobody force

$$W_b^M = W_b^M(\boldsymbol{r}, \boldsymbol{X}) \tag{7.1.9}$$

and a microbody force

$$W_b^m = W_b^m(\boldsymbol{d}_H, \boldsymbol{X}). \tag{7.1.10}$$

In this case equations (7.1.8) become

$$\begin{cases} \left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,i}}\right)_{,i} - \dfrac{\partial W_b^M}{\partial \boldsymbol{r}} = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\boldsymbol{r}}} \\[2ex] \left(\dfrac{\partial W}{\partial \boldsymbol{d}_{H,i}}\right)_{,i} - \dfrac{\partial W}{\partial \boldsymbol{d}_H} - \dfrac{\partial W_b^m}{\partial \boldsymbol{d}_H} = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\boldsymbol{d}}_H}. \end{cases} \tag{7.1.11}$$

Introducing a set of forces, coupling moments, generalized forces and moments as follows

$$\begin{cases} \boldsymbol{\sigma}^i = \dfrac{\partial W}{\partial \boldsymbol{r}_{,i}} \\[2ex] \boldsymbol{\eta}^{Hi} = \dfrac{\partial W}{\partial \boldsymbol{d}_{H,i}} \\[2ex] \boldsymbol{B} = -\dfrac{\partial W_b^M}{\partial \boldsymbol{r}} \\[2ex] \boldsymbol{\tau}^H = -\dfrac{\partial W}{\partial \boldsymbol{d}_H} \\[2ex] \boldsymbol{b}^H = -\dfrac{\partial W_b^m}{\partial \boldsymbol{d}_H} \end{cases} \tag{7.1.12}$$

the field equations can be written in Eulerian form:

$$\begin{cases} \left(\boldsymbol{\sigma}^i\right)_{,i} + \boldsymbol{B} = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\boldsymbol{r}}} \\[2ex] \left(\boldsymbol{\eta}^{Hi}\right)_{,i} + \boldsymbol{\tau}^H + \boldsymbol{b}^H = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\boldsymbol{d}}_H}. \end{cases} \tag{7.1.13}$$

7.2. Microstructures in textile fabrics

Consider a set of inextensible fibers with bending stiffness forming a surface: we want to model the static behavior of textile fabrics taking into account the effects of the microundulations.

As to the macroscopic description of the deformation of the surface, the same notations and assumptions of section 2.2 hold.

To describe the microundulations due to the weaving of the warp and the weft at the microscopic scale, we exploit the notion of vectorial microstructure (in the sense of the previous section).

Letting $\psi_1 = k_1/2$, with k_1 the microcurvature of the first set of fibers, the local stretch in the direction of $\boldsymbol{a}_1$ can be written as (Figure 7.1)

$$\varepsilon_1 = 1 - \frac{\sin \psi_1}{\psi_1}.$$

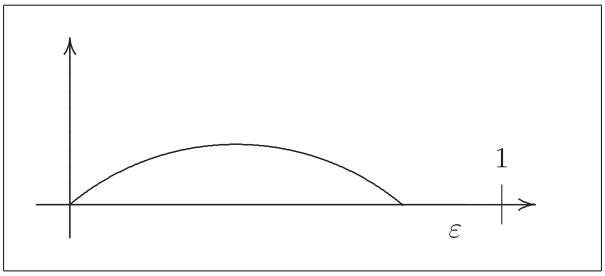

FIGURE 7.1. Forces acting on the warp thread

Hence, we are led to introduce the director fields

$$\boldsymbol{d}_1 = \varepsilon_1 \boldsymbol{a}_1, \qquad \boldsymbol{d}_2 = \varepsilon_2 \boldsymbol{a}_2, \tag{7.2.1}$$

associated to the first and second set of fibers respectively.

7.2.1. Field equations

In the general model of section 7.1.1, the strain energy density was assumed to be a function of the list

$$\boldsymbol{r}_{,h}, \boldsymbol{d}_H, \boldsymbol{d}_{H,j}, X^h,$$

but since we are interested here in the bending of the surface, the strain energy function must somehow depend on the (macroscopic) curvature of the fibers (i.e., the second derivatives of the position vector). Hence, we assume that the strain energy density has the form

$$W \quad = W(\boldsymbol{r}_{,1};\boldsymbol{r}_{,2};\boldsymbol{r}_{,11};\boldsymbol{r}_{,22};\boldsymbol{d}_1;\boldsymbol{d}_2;\boldsymbol{d}_{1,1};\boldsymbol{d}_{2,2};X^1;X^2). \tag{7.2.2}$$

We restrict to a strain energy density in additive form, such that there is no coupling between shearing, bending and microondulations. We assume that W is the sum of three components. The first component, W_0, is due to the shear stress; since this stress component responds to changes in the angle between the two families of fibers, we assume that W_0 is a function of $\boldsymbol{a}_1 \cdot \boldsymbol{a}_2$. The second component of the strain energy is associated to bending, and is assumed to be a quadratic form in the macroscopic curvatures of the fibers; the third component of the strain energy is associated to the microondulations and the local macrostretch. Precisely, we assume that (7.2.2) can be written in the explicit form

$$\begin{aligned} W \quad &= W(\sin\gamma;\mathcal{K}_1;\mathcal{K}_2;\varepsilon_1;\varepsilon_2;\varepsilon_{1,1};\varepsilon_{2,2}) = \\ &= W_0(\sin\gamma) + \frac{1}{2}\Gamma[(\mathcal{K}_1)^2 + (\mathcal{K}_2)^2] + \frac{1}{2}(A_1(\varepsilon_1)^2 + A_2(\varepsilon_2)^2) \\ &+\frac{1}{2}B_1[(\varepsilon_{1,1})^2 + (\varepsilon_1\mathcal{K}_1)^2] + \frac{1}{2}B_2[(\varepsilon_{2,2})^2 + (\varepsilon_2\mathcal{K}_2)^2]. \end{aligned} \tag{7.2.3}$$

Also, we assume that the kinetic energy is

$$T = \frac{1}{2}\rho\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}, \tag{7.2.4}$$

with ρ the body mass density. The total energy is given by

$$E = \int_{\mathcal{S}}(T - W)dA$$

with $\mathcal{S}$ the surface formed by the inextensible fibers.

For natural boundary conditions, the field equations can be derived via a variational principle as Euler-Lagrange equations of the functional

$$\mathcal{E} = \int_{t_0}^{t_1}\left[\int_{\mathcal{S}}(T - W - \lambda_1(\boldsymbol{a}_1\cdot\boldsymbol{a}_1 - 1) - \lambda_2(\boldsymbol{a}_2\cdot\boldsymbol{a}_2 - 1))dA\right]dt \tag{7.2.5}$$

with λ_i the Lagrange multipliers associated to the constrain of inextensibility of the two sets of fibers:

$$\boldsymbol{a}_1\cdot\boldsymbol{a}_1 = \boldsymbol{a}_2\cdot\boldsymbol{a}_2 = 1. \tag{7.2.6}$$

We cannot use directly the form (7.1.11) of the Euler-Lagrange equations, since $\boldsymbol{a}_i = \boldsymbol{r}_{,i}$ and the variations of the director fields $\boldsymbol{d}_i$ are not independent of the variation of $\boldsymbol{r}$. Indeed,

$$\delta \boldsymbol{d}_i = (\delta\varepsilon_i)\boldsymbol{a}_i + \varepsilon_i(\delta \boldsymbol{r}_{i,i}).$$

Hence, the microstructural equations in this case must be obtained by taking independent variations of the energy functional with respect to $\boldsymbol{r}$ and ε_1, ε_2:

$$\begin{cases} -\left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,ih}}\right)_{,ih} + \left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,i}}\right)_{,i} - (\lambda_i \boldsymbol{r}_{,i})_{,i} = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\boldsymbol{r}}} \\[2ex] \left(\dfrac{\partial W}{\partial \varepsilon_{1,1}}\right)_{,1} - \dfrac{\partial W}{\partial \varepsilon_1} = 0, \\[2ex] \left(\dfrac{\partial W}{\partial \varepsilon_{2,2}}\right)_{,2} - \dfrac{\partial W}{\partial \varepsilon_2} = 0. \end{cases} \tag{7.2.7}$$

Using the expression (7.2.3) for the strain energy functions, the field equations become:

$$\begin{cases} -\left[(\Gamma + B_1\varepsilon_1^2)\mathcal{K}_1\boldsymbol{n}_1\right]_{,11} - \left[(\Gamma + B_2\varepsilon_2^2)\mathcal{K}_2\boldsymbol{n}_2\right]_{,22} + \left(\dfrac{\mathrm{d}W_0}{\mathrm{d}\sin\gamma}\right)_{,1}\boldsymbol{a}_2 + \dfrac{\mathrm{d}W_0}{\mathrm{d}\sin\gamma}\dfrac{\partial \boldsymbol{a}_2}{\partial X^1} + \\ \quad + \left(\dfrac{\mathrm{d}W_0}{\mathrm{d}\sin\gamma}\right)_{,2}\boldsymbol{a}_1 + \dfrac{\mathrm{d}W_0}{\mathrm{d}\sin\gamma}\dfrac{\partial \boldsymbol{a}_1}{\partial X^2} + (\lambda_1\boldsymbol{a}_1)_{,1} + (\lambda_2\boldsymbol{a}_2)_{,2} = \rho\ddot{\boldsymbol{r}}, \\[2ex] B_1\varepsilon_{1,11} - (A_1 + B_1\mathcal{K}_1^2)\varepsilon_1 = 0, \\[2ex] B_2\varepsilon_{2,22} - (A_2 + B_2\mathcal{K}_2^2)\varepsilon_2 = 0. \end{cases} \tag{7.2.8}$$

Notice that, even though for our special choice of the strain energy function microundulation and shear are uncoupled, the resulting equations are indeed coupled.

To understand the role of the additional equations related to the microstructure, consider the simple case of straight fibers, i.e., $\mathcal{K}_1 = 0$ and $\mathcal{K}_2 = 0$, so that equations $(7.2.8)_2$ and $(7.2.8)_3$ reduce to

$$\begin{cases} B_1\varepsilon_{1,11} - A_1\varepsilon_1 = 0 \\ B_2\varepsilon_{2,22} - A_2\varepsilon_2 = 0. \end{cases} \tag{7.2.9}$$

The second order differential equation associated to $(7.2.9)_1$ has solution

$$\varepsilon_1(X_1) = C_1 e^{\sqrt{\frac{A_1}{B_1}}X_1} + C_2 e^{-\sqrt{\frac{A_1}{B_1}}X_1} \qquad \text{with} \qquad C_1, C_2 \in \mathbb{R}, \tag{7.2.10}$$

when A_1 and B_1 have the same sign, and

$$\varepsilon_1(X_1) = C_3 \cos\left(\frac{1}{2}\sqrt{-\frac{A_1}{B_1}}X_1\right) + C_4 \sin\left(\frac{1}{2}\sqrt{-\frac{A_1}{B_1}}X_1\right) \qquad \text{with} \qquad C_3, C_4 \in \mathbb{R}, \tag{7.2.11}$$

when A_1 and B_1 have opposite sign. Now, it is reasonable to assume that A_1 and B_1 are both positive, in order that the energy functional be lower bounded. In this case, the microundulations are given by (7.2.10), and this would indicate the presence of boundary-layer phenomena.

We now show that the first set $(7.2.8)_1$ of Euler–Lagrange equations, in the static case, with $\boldsymbol{f} = \boldsymbol{0}$ and $B_1 = 0, B_2 = 0$, is equivalent to equations (4.2.13) of Wang and Pipkin [**66**] and Luo and Steigmann [**37**] (for the case without twist), i.e.,

$$\begin{cases} \boldsymbol{F}_1 = T_1\boldsymbol{a}_1 + \dfrac{\mathrm{d}\, W_0}{\mathrm{d}\, \sin\gamma}\boldsymbol{a}_2 - \Gamma\boldsymbol{a}_{1,11} \\ \boldsymbol{F}_2 = T_2\boldsymbol{a}_2 + \dfrac{\mathrm{d}\, W_0}{\mathrm{d}\, \sin\gamma}\boldsymbol{a}_1 - \Gamma\boldsymbol{a}_{2,22} \\ \dfrac{\partial \boldsymbol{F}_1}{\partial X^1} + \dfrac{\partial \boldsymbol{F}_2}{\partial X^2} + \boldsymbol{f} = \boldsymbol{0}. \end{cases} \tag{7.2.12}$$

In fact, substituting $(7.2.12)_1$, $(7.2.12)_2$ in $(7.2.12)_3$ we obtain

$$\begin{aligned} &\frac{\partial(T_1\boldsymbol{a}_1)}{\partial X^1} + \left(\frac{\mathrm{d}W_0}{\mathrm{d}\sin\gamma}\right)_{,1}\boldsymbol{a}_2 + \frac{\mathrm{d}W_0}{\mathrm{d}\sin\gamma}\frac{\partial \boldsymbol{a}_2}{\partial X^2} - \Gamma\left(\frac{\partial \boldsymbol{a}_{1,11}}{\partial X^1}\right) + \\ &\frac{\partial(T_2\boldsymbol{a}_2)}{\partial X^2} + \left(\frac{\mathrm{d}W_0}{\mathrm{d}\sin\gamma}\right)_{,2}\boldsymbol{a}_1 + \frac{\mathrm{d}W_0}{\mathrm{d}\sin\gamma}\frac{\partial \boldsymbol{a}_1}{\partial X^2} - \Gamma\left(\frac{\partial \boldsymbol{a}_{2,22}}{\partial X^2}\right) + \boldsymbol{f} = \boldsymbol{0}, \end{aligned} \tag{7.2.13}$$

where $T_1 = \lambda_1$ and $T_2 = \lambda_2$ are reactions to the constraint of fiber inextensibility. The above equation (7.2.13) is just $(7.2.8)_1$ in the static case, with $\boldsymbol{f} = \boldsymbol{0}$ and $B_1 = 0, B_2 = 0$.

We conclude by noting that the microstructural equations $(7.2.8)_2$ and $(7.2.8)_3$, which complement the classical equations of the standard models of Wang-Pipkin and Luo-Steigmann, should be crucial to describe the effect of the microstructure on the macroscopic stretch and curvature of a textile.

APPENDIX A

The variational formulation

A.1. The Euler-Lagrange equations

For the sake of simplicity, we restrict our analysis to a single yarn in this appendix. We derive two equivalent formulations of the Euler Lagrange equations, both for a strain energy function of the form

$$W = W(\boldsymbol{r}, \boldsymbol{r}_{,\alpha}, \boldsymbol{r}_{,\alpha\alpha}, \boldsymbol{r}_{,\alpha\alpha\alpha}) \tag{A.1.1}$$

where either $\alpha = 1$ or $\alpha = 2$ (no sum on repeated indices; the index α refers to the yarn, $\alpha = 1$ for the first family and $\alpha = 2$ for the second), and when the energy is a function of the curvature and torsion of the yarn, viz.

$$W = \widetilde{W}(\mathcal{K}, \tau). \tag{A.1.2}$$

We denote by $\boldsymbol{a}$, $\boldsymbol{n}$, and $\boldsymbol{b}$ the tangent vector, the normal vector, and the binormal vector respectively for the yarn. We would like to show the equivalence of the Euler–Lagrange equations of (A.1.1) and (A.1.2).

We restrict to inextensible yarns ($||\boldsymbol{r}_{,\alpha}|| = 1$), and restrict to energy functions of the form

$$W = W(\boldsymbol{r}_{,\alpha\alpha}, \boldsymbol{r}_{,\alpha\alpha\alpha}). \tag{A.1.3}$$

We first compute the variational derivative of (A.1.3):

$$\begin{aligned}
\delta W &= \frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}} \cdot \delta \boldsymbol{r}_{,\alpha\alpha} + \frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \cdot \delta \boldsymbol{r}_{,\alpha\alpha\alpha} \\
&= \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}} \cdot \delta \boldsymbol{r}_{,\alpha} \right)_{,\alpha} - \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}} \right)_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} + \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \cdot \delta \boldsymbol{r}_{,\alpha\alpha} \right)_{,\alpha} - \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \right)_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha\alpha} \\
&= \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}} \cdot \delta \boldsymbol{r}_{,\alpha} \right)_{,\alpha} - \left(\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}} \right)_{,\alpha} \cdot \delta \boldsymbol{r} \right)_{,\alpha} + \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}} \right)_{,\alpha\alpha} \cdot \delta \boldsymbol{r} + \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \cdot \delta \boldsymbol{r}_{,\alpha\alpha} \right)_{,\alpha} \\
&\quad - \left(\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \right)_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} \right)_{,\alpha} + \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \right)_{,\alpha\alpha} \cdot \delta \boldsymbol{r}_{,\alpha}
\end{aligned}$$

which yields

$$\begin{aligned}
\delta W &= \left[\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}} \right)_{,\alpha\alpha} - \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \right)_{,\alpha\alpha\alpha} \right] \cdot \delta \boldsymbol{r} + \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}} \cdot \delta \boldsymbol{r}_{,\alpha} \right)_{,\alpha} - \left(\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}} \right)_{,\alpha} \cdot \delta \boldsymbol{r} \right)_{,\alpha} \\
&\quad + \left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \cdot \delta \boldsymbol{r}_{,\alpha\alpha} \right)_{,\alpha} - \left(\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \right)_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} \right)_{,\alpha} + \left(\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \right)_{,\alpha\alpha} \cdot \delta \boldsymbol{r} \right)_{,\alpha} . \quad \text{(A.1.4)}
\end{aligned}$$

In case the strain energy is a function of curvature and torsion, as in (A.1.2), the variational derivative is

$$\delta \widetilde{W} = \frac{\partial \widetilde{W}}{\partial \mathcal{K}}\, \delta \mathcal{K} + \frac{\partial \widetilde{W}}{\partial \tau}\, \delta \tau. \tag{A.1.5}$$

In order to compare this expression with (A.1.4), we have to express $\delta\mathcal{K}$ and $\delta\tau$ in terms of $\delta\boldsymbol{r}$. We will focus on $\dfrac{\partial \widetilde{W}}{\partial \mathcal{K}}\, \delta \mathcal{K}$, and then on $\dfrac{\partial \widetilde{W}}{\partial \tau}\, \delta \tau$.

Taking the variation of Frenet formula $(2.2.4)_1$ (i.e., $\boldsymbol{a}_{,\alpha} = \mathcal{K}\boldsymbol{n}$), we obtain

$$\delta\mathcal{K}\boldsymbol{n} + \mathcal{K}\delta\boldsymbol{n} = \delta\boldsymbol{a}_{,\alpha} \tag{A.1.6}$$

that, taking the scalar product with $\boldsymbol{n}$ and recalling that $||\boldsymbol{n}|| = 1$, reduces to:

$$\delta\mathcal{K} = \boldsymbol{n} \cdot \delta\boldsymbol{a}_{,\alpha}. \tag{A.1.7}$$

Using $(A.1.7)$, one further has:

$$\begin{aligned}
\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\delta\mathcal{K} = \frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\boldsymbol{n}\cdot\delta\boldsymbol{a}_{,\alpha} &= \left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\boldsymbol{n}\cdot\delta\boldsymbol{a}\right)_{,\alpha} - \left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\boldsymbol{n}\right)_{,\alpha}\cdot\delta\boldsymbol{a} \\
&= \left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\boldsymbol{n}\cdot\delta\boldsymbol{a}\right)_{,\alpha} - \left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\right)_{,\alpha}\boldsymbol{n}\cdot\delta\boldsymbol{a} - \frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\boldsymbol{n}_{,\alpha}\cdot\delta\boldsymbol{a}.
\end{aligned}$$

By $(2.2.4)_2$ (i.e., $\boldsymbol{n}_{,\alpha} = -\mathcal{K}\boldsymbol{a} + \tau\boldsymbol{b}$) and since $\boldsymbol{a}\cdot\delta\boldsymbol{a} = 0$,

$$\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\delta\mathcal{K} = \left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\boldsymbol{n}\cdot\delta\boldsymbol{a}\right)_{,\alpha} - \left[\left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\right)_{,\alpha}\boldsymbol{n} + \frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\tau\boldsymbol{b}\right]\cdot\delta\boldsymbol{a}.$$

Recalling that the tangent vector $\boldsymbol{a}$ is the derivative of the position vector $\boldsymbol{r}$ with respect to arc length,

$$\begin{aligned}
\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\delta\mathcal{K} &= \left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\boldsymbol{n}\cdot\delta\boldsymbol{a}\right)_{,\alpha} + \left\{\left[\left(-\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\right)_{,\alpha}\boldsymbol{n} - \frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\tau\boldsymbol{b}\right]\cdot\delta\boldsymbol{r}\right\}_{,\alpha} \\
&\quad + \left\{\left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\right)_{,\alpha}\boldsymbol{n} + \frac{\partial \widetilde{W}}{\partial \mathcal{K}}\,\tau\boldsymbol{b}\right\}_{,\alpha}\cdot\delta\boldsymbol{r} \qquad \text{(A.1.8)}
\end{aligned}$$

In a similar way, starting from Frenet formula $(2.2.4)_2$, recalling that $||\boldsymbol{b}|| = 1$ and that $\boldsymbol{b}\cdot\boldsymbol{a} = 0$, we obtain

$$\delta\tau = \boldsymbol{b}\cdot\delta\boldsymbol{n}_{,\alpha} + \boldsymbol{b}\cdot\mathcal{K}\delta\boldsymbol{a}, \qquad \text{(A.1.9)}$$

so that

$$\begin{aligned}
\frac{\partial \widetilde{W}}{\partial \tau}\,\delta\tau &= \frac{\partial \widetilde{W}}{\partial \tau}\,(\boldsymbol{b}\cdot\delta\boldsymbol{n}_{,\alpha} + \boldsymbol{b}\cdot\mathcal{K}\delta\boldsymbol{a}) \\
&= \left(\frac{\partial \widetilde{W}}{\partial \tau}\,\boldsymbol{b}\cdot\delta\boldsymbol{n}\right)_{,\alpha} - \left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\boldsymbol{b}\cdot\delta\boldsymbol{n} - \frac{\partial \widetilde{W}}{\partial \tau}\boldsymbol{b}_{,\alpha}\cdot\delta\boldsymbol{n} + \frac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b}\cdot\delta\boldsymbol{a}
\end{aligned}$$

from $(2.2.4)_3$ ($\boldsymbol{b}_{,\alpha} = -\tau\boldsymbol{n}$), we obtain $\dfrac{\partial \widetilde{W}}{\partial \tau}\boldsymbol{b}_{,\alpha}\cdot\delta\boldsymbol{n} = -\dfrac{\partial \widetilde{W}}{\partial \tau}\tau\boldsymbol{n}\cdot\delta\boldsymbol{n} = 0$, which implies that

$$\frac{\partial \widetilde{W}}{\partial \tau}\,\delta\tau = \left(\frac{\partial \widetilde{W}}{\partial \tau}\,\boldsymbol{b}\cdot\delta\boldsymbol{n}\right)_{,\alpha} - \left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\boldsymbol{b}\cdot\delta\boldsymbol{n} + \frac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b}\cdot\delta\boldsymbol{a}.$$

Taking the inner product of (A.1.6) with $\boldsymbol{b}$ we obtain the identity $\boldsymbol{b}\cdot\delta\boldsymbol{n} \;=\; \dfrac{1}{\mathcal{K}}\,\boldsymbol{b}\,\cdot\delta\boldsymbol{a}_{,\alpha}$ which, when inserted in the above equation gives:

$$
\begin{aligned}
\frac{\partial\widetilde{W}}{\partial\tau}\,\delta\tau \;&=\; -\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\,\cdot\delta\boldsymbol{a}_{,\alpha}+\frac{\partial\widetilde{W}}{\partial\tau}\mathcal{K}\boldsymbol{b}\cdot\delta\boldsymbol{a}+\left(\frac{\partial\widetilde{W}}{\partial\tau}\,\boldsymbol{b}\cdot\delta\boldsymbol{n}\right)_{,\alpha}\\
&=\; -\left(\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\,\cdot\delta\boldsymbol{a}\right)_{,\alpha}+\left(\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\right)_{,\alpha}\cdot\delta\boldsymbol{a}\\
&\quad +\frac{\partial\widetilde{W}}{\partial\tau}\mathcal{K}\boldsymbol{b}\cdot\delta\boldsymbol{a}+\left(\frac{\partial\widetilde{W}}{\partial\tau}\,\boldsymbol{b}\cdot\delta\boldsymbol{n}\right)_{,\alpha}\\
&=\; -\left[\left(\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\right)_{,\alpha}+\frac{\partial\widetilde{W}}{\partial\tau}\mathcal{K}\boldsymbol{b}\right]_{,\alpha}\cdot\delta\boldsymbol{r}+\left\{\left[\left(\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\right)_{,\alpha}+\frac{\partial\widetilde{W}}{\partial\tau}\mathcal{K}\boldsymbol{b}\right]\cdot\delta\boldsymbol{r}\right\}_{,\alpha}\\
&\quad -\left(\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\,\cdot\delta\boldsymbol{a}\right)_{,\alpha}+\left(\frac{\partial\widetilde{W}}{\partial\tau}\,\boldsymbol{b}\cdot\delta\boldsymbol{n}\right)_{,\alpha}. \qquad\qquad \text{(A.1.10)}
\end{aligned}
$$

Inserting (A.1.8) and (A.1.10) into (A.1.5), we obtain

$$
\begin{aligned}
\delta\widetilde{W} \;&=\; \left\{\left(\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\right)_{,\alpha}\boldsymbol{n}+\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\,\tau\boldsymbol{b}\right\}_{,\alpha}\cdot\delta\boldsymbol{r}-\left[\left(\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\right)_{,\alpha}+\frac{\partial\widetilde{W}}{\partial\tau}\mathcal{K}\boldsymbol{b}\right]_{,\alpha}\cdot\delta\boldsymbol{r}\\
&\quad +\left(\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\,\boldsymbol{n}\cdot\delta\boldsymbol{a}\right)_{,\alpha}+\left\{\left[\left(-\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\right)_{,\alpha}\boldsymbol{n}-\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\,\tau\boldsymbol{b}\right]\cdot\delta\boldsymbol{r}\right\}_{,\alpha}+\left(\frac{\partial\widetilde{W}}{\partial\tau}\,\boldsymbol{b}\cdot\delta\boldsymbol{n}\right)_{,\alpha}\\
&\quad +\left\{\left[\left(\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\right)_{,\alpha}+\frac{\partial\widetilde{W}}{\partial\tau}\mathcal{K}\boldsymbol{b}\right]\cdot\delta\boldsymbol{r}\right\}_{,\alpha}-\left(\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\,\cdot\delta\boldsymbol{a}\right)_{,\alpha} \qquad \text{(A.1.11)}
\end{aligned}
$$

where the second and the third line correspond to the boundary conditions, and the first line corresponds to the Euler–Lagrange equations of the energy function (A.1.3)

$$
\left[\left(\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\right)_{,\alpha}\boldsymbol{n}-\left(\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\right)_{,\alpha}+\left(\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\,\tau-\frac{\partial\widetilde{W}}{\partial\tau}\mathcal{K}\right)\boldsymbol{b}\right]_{,\alpha}=0 \qquad \text{(A.1.12)}
$$

A.2. Derivatives

In order to prove that the two relations (A.1.4) and (A.1.11) are equivalent, we need to evaluate $\dfrac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}$, $\dfrac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}$, $\dfrac{\partial \mathcal{K}}{\partial \boldsymbol{r}_{,\alpha\alpha}}$ and $\dfrac{\partial \mathcal{K}}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}$. From Frenet formula $(2.2.4)_1$, it follows that the following relation hold:

$$\boldsymbol{r}_{,\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha} = \mathcal{K}^2$$

so that

$$\frac{\partial \mathcal{K}}{\partial \boldsymbol{r}_{,\alpha\alpha}} = \frac{1}{\mathcal{K}}\, \boldsymbol{r}_{,\alpha\alpha} \tag{A.2.1}$$

or, equivalently,

$$\frac{\partial \mathcal{K}}{\partial \boldsymbol{r}_{,\alpha\alpha}} = \frac{1}{\mathcal{K}} \mathcal{K}\boldsymbol{n} = \boldsymbol{n} \tag{A.2.2}$$

and

$$\frac{\partial \mathcal{K}}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} = 0. \tag{A.2.3}$$

Now, differentiating Frenet formula $(2.2.4)_1$ $\boldsymbol{r}_{,\alpha\alpha} = \mathcal{K}\boldsymbol{n}$ and using $(2.2.4)_2$, we obtain

$$\boldsymbol{r}_{,\alpha\alpha\alpha} = \mathcal{K}_{,\alpha}\boldsymbol{n} + \mathcal{K}\boldsymbol{n}_{,\alpha} = \mathcal{K}_{,\alpha}\boldsymbol{n} + \mathcal{K}\left(-\mathcal{K}\boldsymbol{a} + \tau\boldsymbol{b}\right) = \mathcal{K}_{,\alpha}\boldsymbol{n} - \mathcal{K}^2\boldsymbol{a} + \mathcal{K}\tau\boldsymbol{b}$$

from which it follows that

$$\begin{aligned} \boldsymbol{r}_{,\alpha\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha\alpha} &= \mathcal{K}_{,\alpha}^2 + \mathcal{K}^4 + \mathcal{K}^2\tau^2 \\ \boldsymbol{r}_{,\alpha\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha} &= \mathcal{K}_{,\alpha}\mathcal{K} \end{aligned} \tag{A.2.4}$$

From $(\text{A.2.4})_2$ and recalling that $\boldsymbol{r}_{,\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha} = \mathcal{K}^2$, it follows that

$$\mathcal{K}_{,\alpha}^2 = \left(\frac{\boldsymbol{r}_{,\alpha\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha}}{\mathcal{K}}\right)^2 = \frac{\left(\boldsymbol{r}_{,\alpha\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha}\right)^2}{\boldsymbol{r}_{,\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha}}. \tag{A.2.5}$$

Further, by (A.2.5) and $(\text{A.2.4})_1$, we obtain

$$\tau^2 = \frac{\boldsymbol{r}_{,\alpha\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha\alpha} - \mathcal{K}_{\alpha}^2 - \mathcal{K}^4}{\mathcal{K}^2} = \frac{\boldsymbol{r}_{,\alpha\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha\alpha}}{\boldsymbol{r}_{,\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha}} - \frac{\left(\boldsymbol{r}_{,\alpha\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha}\right)^2}{\left(\boldsymbol{r}_{,\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha}\right)^2} - \boldsymbol{r}_{,\alpha\alpha} \cdot \boldsymbol{r}_{,\alpha\alpha} \tag{A.2.6}$$

and consequentely, after some calculation, we find the relation

$$\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}} = -\frac{1}{\tau\mathcal{K}^2}\left\{\left(2\mathcal{K}^2 + \tau^2 - \left(\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}}\right)^2\right)\boldsymbol{r}_{,\alpha\alpha} + \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}}\, \boldsymbol{r}_{,\alpha\alpha\alpha}\right\} \tag{A.2.7}$$

which corresponds to

$$\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}} = -\frac{\tau}{\mathcal{K}}\, \boldsymbol{n} - \frac{2\mathcal{K}}{\tau}\, \boldsymbol{n} + \frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\, \boldsymbol{a} - \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\, \boldsymbol{b}. \tag{A.2.8}$$

In a similar manner, starting from (A.2.6), we find the expression

$$\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} = \frac{1}{\tau \mathcal{K}^2} \left\{ \boldsymbol{r}_{,\alpha\alpha\alpha} - \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}} \boldsymbol{r}_{,\alpha\alpha} \right\} \tag{A.2.9}$$

which is equivalent to

$$\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} = -\frac{1}{\tau}\, \boldsymbol{a} + \frac{1}{\mathcal{K}}\, \boldsymbol{b}. \tag{A.2.10}$$

A.3. Comparison

For simplicity, we use the following notation for the addenda in (A.1.4) and (A.1.11):

Table 1: Notation for equation (A.1.4)

Terms	**Notation**
$\left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha\alpha} \cdot \delta\boldsymbol{r}$	a
$-\left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha\alpha\alpha} \cdot \delta\boldsymbol{r}$	b
$\left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}} \cdot \delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha}$	c
$-\left(\left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha} \cdot \delta\boldsymbol{r}\right)_{,\alpha}$	d
$\left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}} \cdot \delta\boldsymbol{r}_{,\alpha\alpha}\right)_{,\alpha}$	e

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Terms	**Notation**
$-\left(\left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha}\right)_{,\alpha}$	f
$\left(\left(\dfrac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha\alpha} \cdot \delta \boldsymbol{r}\right)_{,\alpha}$	g

Table 2: Notation for equation (A.1.11)

Terms	**Notation**
$\left\{\left(\dfrac{\partial \widetilde{W}}{\partial \mathcal{K}}\right)_{,\alpha} \boldsymbol{n} + \dfrac{\partial \widetilde{W}}{\partial \mathcal{K}} \tau \boldsymbol{b}\right\}_{,\alpha} \cdot \delta \boldsymbol{r}$	A
$-\left[\dfrac{1}{\mathcal{K}}\left(\dfrac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \boldsymbol{b}\right]_{,\alpha\alpha} \cdot \delta \boldsymbol{r}$	B
$-\left[\dfrac{\partial \widetilde{W}}{\partial \tau} \mathcal{K} \boldsymbol{b}\right]_{,\alpha} \cdot \delta \boldsymbol{r}$	A_1
$\left(\dfrac{\partial \widetilde{W}}{\partial \mathcal{K}} \boldsymbol{n} \cdot \delta \boldsymbol{a}\right)_{,\alpha}$	C
$\left\{\left[\left(-\dfrac{\partial \widetilde{W}}{\partial \mathcal{K}}\right)_{,\alpha} \boldsymbol{n} - \dfrac{\partial \widetilde{W}}{\partial \mathcal{K}} \tau \boldsymbol{b}\right] \cdot \delta \boldsymbol{r}\right\}_{,\alpha}$	D
$\left\{\left(\dfrac{1}{\mathcal{K}}\left(\dfrac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \boldsymbol{b}\right)_{,\alpha} \cdot \delta \boldsymbol{r}\right\}_{,\alpha}$	G

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Terms	**Notation**
$\left\{\dfrac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}$	D_1
$-\left(\dfrac{1}{\mathcal{K}}\left(\dfrac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\boldsymbol{b}\cdot\delta\boldsymbol{a}\right)_{,\alpha}$	F
$\left(\dfrac{\partial \widetilde{W}}{\partial \tau}\boldsymbol{b}\cdot\delta\boldsymbol{n}\right)_{,\alpha}$	E

Each term in (A.1.4) may be expressed through the derivatives of $\mathcal{K}$ and τ with respect to $\boldsymbol{r}_{,\alpha\alpha}$ and $\boldsymbol{r}_{,\alpha\alpha\alpha}$. We first focus on a, c, d (the terms containing the derivatives of W with respect to $\boldsymbol{r}_{,\alpha\alpha}$), and we prove that they correspond to the terms A, C, D, in (A.1.11) i.e. the terms related to curvature.

- <u>term c</u>:

$$\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha} = \left[\left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\frac{\partial \mathcal{K}}{\partial \boldsymbol{r}_{,\alpha\alpha}} + \frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha}$$

Using (A.2.2), the previous expression becomes

$$\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha} = \left[\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\boldsymbol{n}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} + \left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha}. \tag{A.3.1}$$

The first term on the right side of the above equation corresponds to C in equation (A.1.11). Denoting by c_1 the second, we thus have

$$\boxed{c = C + c_1.}$$

• <u>term a</u>:

$$\begin{aligned}
\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r} &= \left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\frac{\partial \mathcal{K}}{\partial \boldsymbol{r}_{,\alpha\alpha}}+\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r}\\
&= \left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\boldsymbol{n}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r}+\left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r}\\
&= \left[\left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\right)_{,\alpha}\boldsymbol{n}+\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\boldsymbol{n}_{,\alpha}\right]_{,\alpha}\cdot\delta\boldsymbol{r}+\left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r}
\end{aligned}$$

and, using the equation $\boldsymbol{n}_{,\alpha}=-\mathcal{K}\boldsymbol{a}+\tau\boldsymbol{b}$, we obtain

$$\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r}=\left[\left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\right)_{,\alpha}\boldsymbol{n}+\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\tau\boldsymbol{b}\right]_{,\alpha}\cdot\delta\boldsymbol{r}-\left[\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\mathcal{K}\boldsymbol{a}\right]_{,\alpha}\cdot\delta\boldsymbol{r}+\left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r}. \tag{A.3.2}$$

The first term on the right-hand side of the above equation corresponds to A in equation (A.1.11) so that, writing a_1 and a_2 for the second and third term respectively, it follows that

$$\boxed{a=A+a_1+a_2.}$$

Moreover, expanding the term a_2, we can recognize the term A_1 among its components. In fact, using (A.2.8):

$$\begin{aligned}
\left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r} &= \left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}+\frac{\partial \widetilde{W}}{\partial \tau}\left(\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha}\right]_{,\alpha}\cdot\delta\boldsymbol{r}\\
&= \left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right]_{,\alpha}\cdot\delta\boldsymbol{r}+\left\{\frac{\partial \widetilde{W}}{\partial \tau}\left[\left(-\frac{\tau}{\mathcal{K}}-\frac{2\mathcal{K}}{\tau}\right)\boldsymbol{n}+\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\boldsymbol{a}-\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\boldsymbol{b}\right]_{,\alpha}\right\}_{,\alpha}\cdot\delta\boldsymbol{r},
\end{aligned}$$

and from $(2.2.4)_1$, we have

$$\begin{aligned}
\left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r} &= \left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right]_{,\alpha}\cdot\delta\boldsymbol{r}+\left\{\frac{\partial \widetilde{W}}{\partial \tau}\left[-\frac{\tau}{\mathcal{K}}\boldsymbol{n}+\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\boldsymbol{a}-\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\boldsymbol{b}\right]_{,\alpha}\right\}_{,\alpha}\cdot\delta\boldsymbol{r}\\
&\quad+\left\{\frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{2\mathcal{K}}{\tau}\right)_{,\alpha}\boldsymbol{n}\right\}_{,\alpha}\cdot\delta\boldsymbol{r}+\left\{\frac{\partial \widetilde{W}}{\partial \tau}\frac{2\mathcal{K}^2}{\tau}\boldsymbol{a}\right\}_{,\alpha}\cdot\delta\boldsymbol{r}-2\left\{\frac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b}\right\}_{,\alpha}\cdot\delta\boldsymbol{r}.
\end{aligned}$$

Letting

$$
\begin{aligned}
a_3 \;=\;& \left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right]_{,\alpha}\cdot \delta \boldsymbol{r}+\left\{\frac{\partial \widetilde{W}}{\partial \tau}\left[-\frac{\tau}{\mathcal{K}}\boldsymbol{n}+\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a}-\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right]_{,\alpha}\right\}_{,\alpha}\cdot \delta\boldsymbol{r}\\
&+\left\{\frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{2\mathcal{K}}{\tau}\right)_{,\alpha}\boldsymbol{n}\right\}_{,\alpha}\cdot\delta\boldsymbol{r}+\left\{\frac{\partial \widetilde{W}}{\partial \tau}\frac{2\mathcal{K}^2}{\tau}\boldsymbol{a}\right\}_{,\alpha}\cdot\delta\boldsymbol{r}-\left\{\frac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b}\right\}_{,\alpha}\cdot\delta\boldsymbol{r}
\end{aligned}
$$

we henceforth obtain

$$
\boxed{a_2 = A_1 + a_3.}
$$

•<u>term d</u>:

$$
\begin{aligned}
-\left(\left(\frac{\partial W}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha}\cdot\delta\boldsymbol{r}\right)_{,\alpha} &= -\left[\left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\frac{\partial \mathcal{K}}{\partial \boldsymbol{r}_{,\alpha\alpha}}+\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha}\cdot\delta\boldsymbol{r}\right]_{,\alpha}\\
&= -\left[\left(\frac{\partial \widetilde{W}}{\partial \mathcal{K}}\boldsymbol{n}\right)_{,\alpha}\cdot\delta\boldsymbol{r}\right]_{,\alpha}-\left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha}\cdot\delta\boldsymbol{r}\right]_{,\alpha}
\end{aligned}
$$

Denoting by d_1 and d_2 the first and second term in the right hand side of the above equation, we may write

$$
\boxed{d = d_1 + d_2.}
$$

Expanding d_2 we can recognize the term D_1 among its components. Performing calculations similar to those done for a_3, we have:

$$
\begin{aligned}
-\left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha}\cdot\delta\boldsymbol{r}\right]_{,\alpha} &= -\left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\cdot\delta\boldsymbol{r}\right]_{,\alpha}-\left[\frac{\partial \widetilde{W}}{\partial \tau}\left(\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right)_{,\alpha}\cdot\delta\boldsymbol{r}\right]_{,\alpha}\\
&= -\left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\cdot\delta\boldsymbol{r}\right]_{,\alpha}-\left\{\frac{\partial \widetilde{W}}{\partial \tau}\left[-\frac{\tau}{\mathcal{K}}\boldsymbol{n}+\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a}-\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right]_{,\alpha}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}\\
&\quad-\left\{\frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{2\mathcal{K}}{\tau}\right)_{,\alpha}\boldsymbol{n}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}-\left\{\frac{\partial \widetilde{W}}{\partial \tau}\frac{2\mathcal{K}^2}{\tau}\boldsymbol{a}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}\\
&\quad+2\left\{\frac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b}\cdot\delta\boldsymbol{r}\right\}_{,\alpha},
\end{aligned}
$$

so that, letting

$$
\begin{aligned}
d_3 \;=\; & -\left[\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha}}\cdot\delta\boldsymbol{r}\right]_{,\alpha}-\left\{\frac{\partial\widetilde{W}}{\partial\tau}\left[-\frac{\tau}{\mathcal{K}}\boldsymbol{n}+\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a}-\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right]_{,\alpha}\cdot\delta\boldsymbol{r}\right\}_{,\alpha} \\
& -\left\{\frac{\partial\widetilde{W}}{\partial\tau}\left(-\frac{2\mathcal{K}}{\tau}\right)_{,\alpha}\boldsymbol{n}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}-\left\{\frac{\partial\widetilde{W}}{\partial\tau}\frac{2\mathcal{K}^2}{\tau}\boldsymbol{a}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}+\left\{\frac{\partial\widetilde{W}}{\partial\tau}\mathcal{K}\boldsymbol{b}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}
\end{aligned}
$$

we have

$$\boxed{d_2 = D_1 + d_3.}$$

We will now prove that

$$\boxed{a_1 + d_1 = D}.$$

Indeed,

$$-\left[\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\mathcal{K}\boldsymbol{a}\right]_{,\alpha}\cdot\delta\boldsymbol{r}-\left[\left(\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\boldsymbol{n}\right)_{,\alpha}\cdot\delta\boldsymbol{r}\right]_{,\alpha}=-\left[\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\mathcal{K}\boldsymbol{a}\cdot\delta\boldsymbol{r}\right]_{,\alpha}+\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\mathcal{K}\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha}-\left[\left(\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\boldsymbol{n}\right)_{,\alpha}\cdot\delta\boldsymbol{r}\right]_{,\alpha}.$$

Now, since $||\boldsymbol{a}|| = 1$, then $\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha} = 0$, and

$$
\begin{aligned}
-\left[\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\mathcal{K}\boldsymbol{a}\right]_{,\alpha}\cdot\delta\boldsymbol{r}-\left[\left(\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\boldsymbol{n}\right)_{,\alpha}\cdot\delta\boldsymbol{r}\right]_{,\alpha} &= -\left\{\left[\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\mathcal{K}\boldsymbol{a}+\left(\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\boldsymbol{n}\right)_{,\alpha}\right]\cdot\delta\boldsymbol{r}\right\}_{,\alpha} \\
&= -\left\{\left[\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\mathcal{K}\boldsymbol{a}+\left(\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\right)_{,\alpha}\boldsymbol{n}+\left(\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\right)(-\mathcal{K}\boldsymbol{a}+\tau\boldsymbol{b})\right]\cdot\delta\boldsymbol{r}\right\}_{,\alpha} \\
&= \left\{\left[\left(-\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\right)_{,\alpha}\boldsymbol{n}-\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\,\tau\boldsymbol{b}\right]\cdot\delta\boldsymbol{r}\right\}_{,\alpha}
\end{aligned}
$$

We now discuss the terms related to torsion. Notice that terms b, e, f, g are functions of $\frac{\partial W}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}$ which, by (A.2.3), reduces to:

$$\frac{\partial W}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}=\frac{\partial\widetilde{W}}{\partial\mathcal{K}}\frac{\partial\mathcal{K}}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}+\frac{\partial\widetilde{W}}{\partial\tau}\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}=\frac{\partial\widetilde{W}}{\partial\tau}\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}} \tag{A.3.3}$$

and we denote the terms corresponding to torsion in (A.1.4) as follows

Table 3:

b	$-\left(\dfrac{\partial \widetilde{W}}{\partial \tau}\dfrac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha\alpha\alpha}\cdot \delta \boldsymbol{r}$
e	$\left(\dfrac{\partial \widetilde{W}}{\partial \tau}\dfrac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\cdot \delta \boldsymbol{r}_{,\alpha\alpha}\right)_{,\alpha}$
f	$-\left(\left(\dfrac{\partial \widetilde{W}}{\partial \tau}\dfrac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\cdot \delta \boldsymbol{r}_{,\alpha}\right)_{,\alpha}$
g	$\left(\left(\dfrac{\partial \widetilde{W}}{\partial \tau}\dfrac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha\alpha}\cdot \delta \boldsymbol{r}\right)_{,\alpha}$

• <u>term b</u>:

$$-\left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha\alpha\alpha}\cdot \delta \boldsymbol{r} = -\left(\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha\alpha}\cdot \delta \boldsymbol{r} - \left(\frac{\partial \widetilde{W}}{\partial \tau}\left(\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\right)_{,\alpha\alpha}\cdot \delta \boldsymbol{r}.$$

which, by (A.2.10), becomes

$$-\left(\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\frac{1}{\mathcal{K}}\,\boldsymbol{b}\right)_{,\alpha\alpha}\cdot \delta \boldsymbol{r} + \left(\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha}\frac{1}{\tau}\,\boldsymbol{a}\right)_{,\alpha\alpha}\cdot \delta \boldsymbol{r} - \left(\frac{\partial \widetilde{W}}{\partial \tau}\left(\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\right)_{,\alpha\alpha}\cdot \delta \boldsymbol{r}.$$

Letting b_1 and b_2 for the second and third term in the above expression, we have

$$\boxed{b = B + b_1 + b_2.}$$

• <u>term e</u>: using (A.2.10), this term becomes

$$\left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\cdot \delta \boldsymbol{r}_{,\alpha\alpha}\right)_{,\alpha} = \left(\frac{\partial \widetilde{W}}{\partial \tau}\frac{1}{\mathcal{K}}\,\boldsymbol{b}\cdot \delta \boldsymbol{r}_{,\alpha\alpha}\right)_{,\alpha} + \left(-\frac{\partial \widetilde{W}}{\partial \tau}\frac{1}{\tau}\,\boldsymbol{a}\cdot \delta \boldsymbol{r}_{,\alpha\alpha}\right)_{,\alpha} \tag{A.3.4}$$

and writing e_1 for the second term on the right hand side of the above equation, we find

$$\boxed{e = E + e_1.}$$

• term f:

$$-\left(\left(\frac{\partial\widetilde{W}}{\partial\tau}\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha}=-\left(\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha}-\left(\frac{\partial\widetilde{W}}{\partial\tau}\left(\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha}$$

and using (A.2.10) this expression is equals to

$$\left(\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\frac{1}{\tau}\,\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha}-\left(\frac{1}{\mathcal{K}}\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\boldsymbol{b}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha}-\left(\frac{\partial\widetilde{W}}{\partial\tau}\left(\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha}$$

where the term $\left(\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\frac{1}{\tau}\,\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha}$ vanishes. If we set $f_1=-\left(\frac{\partial\widetilde{W}}{\partial\tau}\left(\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha}$, it follows that

$$\boxed{f=F+f_1.}$$

• term g:

$$\left(\left(\frac{\partial\widetilde{W}}{\partial\tau}\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r}\right)_{,\alpha}=\left\{\left[\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}\right]_{,\alpha}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}+\left\{\left[\frac{\partial\widetilde{W}}{\partial\tau}\left(\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\right]_{,\alpha}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}$$

and using (A.2.10):

$$=\left\{\left[\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\frac{1}{\mathcal{K}}\,\boldsymbol{b}\right]_{,\alpha}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}-\left\{\left[\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\frac{1}{\tau}\,\boldsymbol{a}\right]_{,\alpha}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}+\left\{\left[\frac{\partial\widetilde{W}}{\partial\tau}\left(\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\right]_{,\alpha}\cdot\delta\boldsymbol{r}\right\}_{,\alpha}$$

so that writing g_1 and g_2 for the second and third term in the above expression, we have

$$\boxed{g=G+g_1+g_2.}$$

Now, collecting the relations in the boxes

$$\begin{aligned}&a=A+a_1+a_2,\quad b=B+b_1+b_2,\quad c=C+c_1,\quad d=d_1+d_2,\quad e=E+e_1\\&f=F+f_1,\quad g=G+g_1+g_2,\quad d_2=D_1+d_3,\quad a_1+d_1=D,\quad a_2=A_1+a_3\end{aligned}\tag{A.3.5}$$

we have

$$a+b+c+d+e+f+g = A + A_1 + B + C + D_1 + E + F + G + D + a_3 + b_1 + b_2 + c_1 + d_3 + e_1 + f_1 + g_1 + g_2$$

and we only have to prove that $a_3 + b_1 + b_2 + c_1 + d_3 + e_1 + f_1 + g_1 + g_2$ vanishes.

We start by showing that $\underline{c_1 + e_1 + f_1 = 0}$, and then prove that $\underline{a_3 + b_1 + b_2 + d_3 + g_1 + g_2}$ is zero.

The relation $\boldsymbol{a} \cdot \delta\boldsymbol{r}_{,\alpha} = 0$ implies that

$$\begin{aligned}
e_1 &= \left(-\frac{\partial\widetilde{W}}{\partial\tau}\frac{1}{\tau}\,\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha\alpha}\right)_{,\alpha} = \left(-\frac{\partial\widetilde{W}}{\partial\tau}\frac{1}{\tau}\,\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha}\right)_{,\alpha\alpha} - \left[\left(-\frac{\partial\widetilde{W}}{\partial\tau}\frac{1}{\tau}\,\boldsymbol{a}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} \\
&= \left[\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\frac{1}{\tau}\,\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} + \left[\frac{\partial\widetilde{W}}{\partial\tau}\left(\frac{1}{\tau}\,\boldsymbol{a}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} = \left[\frac{\partial\widetilde{W}}{\partial\tau}\left(\frac{1}{\tau}\,\boldsymbol{a}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} \\
&= \left[\frac{\partial\widetilde{W}}{\partial\tau}\left(\frac{1}{\tau}\right)_{,\alpha}\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} + \left[\frac{\partial\widetilde{W}}{\partial\tau}\frac{1}{\tau}\,\boldsymbol{a}_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} = \left[\frac{\partial\widetilde{W}}{\partial\tau}\frac{\mathcal{K}}{\tau}\,\boldsymbol{n}\,\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha}
\end{aligned}$$

and using (A.2.10), the Frenet formulas and the identity $\boldsymbol{a} \cdot \delta\boldsymbol{r}_{,\alpha} = 0$, we find

$$\begin{aligned}
f_1 &= -\left[\frac{\partial\widetilde{W}}{\partial\tau}\left(\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} = -\left[\frac{\partial\widetilde{W}}{\partial\tau}\left(-\frac{1}{\tau}\,\boldsymbol{a} + \frac{1}{\mathcal{K}}\,\boldsymbol{b}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} \\
&= \left[\frac{\partial\widetilde{W}}{\partial\tau}\left(\frac{1}{\tau}\right)_{,\alpha}\boldsymbol{a}\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} + \left[\frac{\partial\widetilde{W}}{\partial\tau}\frac{1}{\tau}\mathcal{K}\,\boldsymbol{n}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} + \left[\frac{\partial\widetilde{W}}{\partial\tau}\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} + \left[\frac{\partial\widetilde{W}}{\partial\tau}\frac{1}{\mathcal{K}}\tau\boldsymbol{n}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} \\
&= \left[\frac{\partial\widetilde{W}}{\partial\tau}\frac{\mathcal{K}}{\tau}\,\boldsymbol{n}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} + \left[\frac{\partial\widetilde{W}}{\partial\tau}\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} + \left[\frac{\partial\widetilde{W}}{\partial\tau}\frac{\tau}{\mathcal{K}}\boldsymbol{n}\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha}
\end{aligned}$$

and recalling (A.2.8):

$$\begin{aligned}
c_1 &= \left[\left(\frac{\partial\widetilde{W}}{\partial\tau}\frac{\partial\tau}{\partial\boldsymbol{r}_{,\alpha\alpha}}\right)\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} = \left[\frac{\partial\widetilde{W}}{\partial\tau}\left(-\frac{\tau}{\mathcal{K}}\,\boldsymbol{n} - \frac{2\mathcal{K}}{\tau}\,\boldsymbol{n} + \frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a} - \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right)\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha} \\
&= \left[\frac{\partial\widetilde{W}}{\partial\tau}\left(-\frac{\tau}{\mathcal{K}}\,\boldsymbol{n} - \frac{2\mathcal{K}}{\tau}\,\boldsymbol{n} - \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right)\cdot\delta\boldsymbol{r}_{,\alpha}\right]_{,\alpha},
\end{aligned}$$

so that the sum of c_1, e_1, and f_1, is zero.

In order to prove that $\underline{a_3 + b_1 + b_2 + d_3 + g_1 + g_2}$ is equal to zero, we first focus on the terms $\underline{b_1 + b_2 + g_1 + g_2}$, and then we evaluate $\underline{a_3 + d_3}$:

$$
\begin{aligned}
g_1 + g_2 &= -\left\{\left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \frac{1}{\tau}\, \boldsymbol{a}\right]_{,\alpha} \cdot \delta \boldsymbol{r}\right\}_{,\alpha} + \left\{\left[\frac{\partial \widetilde{W}}{\partial \tau}\left(\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\right]_{,\alpha} \cdot \delta \boldsymbol{r}\right\}_{,\alpha} \\
&= -\left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \frac{1}{\tau}\, \boldsymbol{a}\right]_{,\alpha\alpha} \cdot \delta \boldsymbol{r} - \left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \frac{1}{\tau}\, \boldsymbol{a}\right]_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} + \left[\frac{\partial \widetilde{W}}{\partial \tau}\left(\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\right]_{,\alpha\alpha} \cdot \delta \boldsymbol{r} \\
&\quad + \left[\frac{\partial \widetilde{W}}{\partial \tau}\left(\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\right]_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha}
\end{aligned}
$$

and since:

$$
\begin{aligned}
b_1 + b_2 &= \left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \frac{1}{\tau}\, \boldsymbol{a}\right]_{,\alpha\alpha} \cdot \delta \boldsymbol{r} - \left[\frac{\partial \widetilde{W}}{\partial \tau}\left(\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\right]_{,\alpha\alpha} \cdot \delta \boldsymbol{r} \\
b_1 + b_2 + g_1 + g_2 &= -\left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \frac{1}{\tau}\, \boldsymbol{a}\right]_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} + \left[\frac{\partial \widetilde{W}}{\partial \tau}\left(\frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha\alpha}}\right)_{,\alpha}\right]_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} \\
&= -\left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \frac{1}{\tau}\, \boldsymbol{a}\right]_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} + \left[\frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{1}{\tau}\, \boldsymbol{a} + \frac{1}{\mathcal{K}}\, \boldsymbol{b}\right)_{,\alpha}\right]_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} \\
&= -\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha\alpha} \frac{1}{\tau}\, \boldsymbol{a} \cdot \delta \boldsymbol{r}_{,\alpha} \\
&= -\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \left(\frac{1}{\tau}\, \boldsymbol{a}\right)_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} + \left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \left(-\frac{1}{\tau}\, \boldsymbol{a} + \frac{1}{\mathcal{K}}\, \boldsymbol{b}\right)_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} \\
&\quad + \frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{1}{\tau}\, \boldsymbol{a} + \frac{1}{\mathcal{K}}\, \boldsymbol{b}\right)_{,\alpha\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} \\
&= -2\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \left(\frac{1}{\tau}\, \boldsymbol{a}\right)_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} + \left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \left(\frac{1}{\mathcal{K}}\, \boldsymbol{b}\right)_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} \\
&\quad + \frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{1}{\tau}\, \boldsymbol{a} + \frac{1}{\mathcal{K}}\, \boldsymbol{b}\right)_{,\alpha\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} \qquad \text{(A.3.6)}
\end{aligned}
$$

Now, considering the term $\underline{a_3 + d_3}$, we can write

$$
\begin{aligned}
d_3 \quad = \quad & -\left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}} \cdot \delta \boldsymbol{r}\right]_{,\alpha} - \left\{\frac{\partial \widetilde{W}}{\partial \tau}\left[-\frac{\tau}{\mathcal{K}}\boldsymbol{n} + \frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a} - \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right]_{,\alpha} \cdot \delta \boldsymbol{r}\right\}_{,\alpha} \\
& - \left\{\frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{2\mathcal{K}}{\tau}\right)_{,\alpha} \boldsymbol{n} \cdot \delta \boldsymbol{r}\right\}_{,\alpha} - \left\{\frac{\partial \widetilde{W}}{\partial \tau}\frac{2\mathcal{K}^2}{\tau}\boldsymbol{a} \cdot \delta \boldsymbol{r}\right\}_{,\alpha} + \left\{\frac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b} \cdot \delta \boldsymbol{r}\right\}_{,\alpha} \\
= \quad & -\left[\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}}\right]_{,\alpha} \cdot \delta \boldsymbol{r} - \left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}} \cdot \delta \boldsymbol{r}_{,\alpha} \\
& - \left\{\frac{\partial \widetilde{W}}{\partial \tau}\left[-\frac{\tau}{\mathcal{K}}\boldsymbol{n} + \frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a} - \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right]_{,\alpha}\right\}_{,\alpha} \cdot \delta \boldsymbol{r} \\
& - \frac{\partial \widetilde{W}}{\partial \tau}\left[-\frac{\tau}{\mathcal{K}}\boldsymbol{n} + \frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a} - \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right]_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} - \left\{\frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{2\mathcal{K}}{\tau}\right)_{,\alpha} \boldsymbol{n}\right\}_{,\alpha} \cdot \delta \boldsymbol{r} - \frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{2\mathcal{K}}{\tau}\right)_{,\alpha} \boldsymbol{n} \cdot \delta \boldsymbol{r}_{,\alpha} \\
& - \left\{\frac{\partial \widetilde{W}}{\partial \tau}\frac{2\mathcal{K}^2}{\tau}\boldsymbol{a}\right\}_{,\alpha} \cdot \delta \boldsymbol{r} - \frac{\partial \widetilde{W}}{\partial \tau}\frac{2\mathcal{K}^2}{\tau}\boldsymbol{a} \cdot \delta \boldsymbol{r}_{,\alpha} + \left\{\frac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b}\right\}_{,\alpha} \cdot \delta \boldsymbol{r} + \frac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b} \cdot \delta \boldsymbol{r}_{,\alpha}.
\end{aligned}
$$

Recalling (A.3.3) and using (A.2.8):

$$
\begin{aligned}
a_3 + d_3 \quad = \quad & -\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \frac{\partial \tau}{\partial \boldsymbol{r}_{,\alpha\alpha}} \cdot \delta \boldsymbol{r}_{,\alpha} - \frac{\partial \widetilde{W}}{\partial \tau}\left[-\frac{\tau}{\mathcal{K}}\boldsymbol{n} + \frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a} - \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right]_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} \\
& - \frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{2\mathcal{K}}{\tau}\right)_{,\alpha} \boldsymbol{n} \cdot \delta \boldsymbol{r}_{,\alpha} - \frac{\partial \widetilde{W}}{\partial \tau}\frac{2\mathcal{K}^2}{\tau}\boldsymbol{a} \cdot \delta \boldsymbol{r}_{,\alpha} + \frac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b} \cdot \delta \boldsymbol{r}_{,\alpha} \\
= \quad & -\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \left(-\frac{\tau}{\mathcal{K}}\,\boldsymbol{n} - \frac{2\mathcal{K}}{\tau}\,\boldsymbol{n} + \frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a} - \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right) \cdot \delta \boldsymbol{r}_{,\alpha} \\
& - \frac{\partial \widetilde{W}}{\partial \tau}\left[-\frac{\tau}{\mathcal{K}}\boldsymbol{n} + \frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a} - \frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right]_{,\alpha} \cdot \delta \boldsymbol{r}_{,\alpha} \\
& - \frac{\partial \widetilde{W}}{\partial \tau}\left(-\frac{2\mathcal{K}}{\tau}\right)_{,\alpha} \boldsymbol{n} \cdot \delta \boldsymbol{r}_{,\alpha} - \frac{\partial \widetilde{W}}{\partial \tau}\frac{2\mathcal{K}^2}{\tau}\boldsymbol{a} \cdot \delta \boldsymbol{r}_{,\alpha} + \frac{\partial \widetilde{W}}{\partial \tau}\mathcal{K}\boldsymbol{b} \cdot \delta \boldsymbol{r}_{,\alpha}. \qquad \text{(A.3.7)}
\end{aligned}
$$

Finally, the last step is to show that the sum of (A.3.6) and (A.3.7) is zero. We first show that the sum of the terms containing $\left(\frac{\partial \widetilde{W}}{\partial \tau}\right)_{,\alpha} \delta \boldsymbol{r}_{,\alpha}$ is zero, then we discuss the remaining terms:

$$
\begin{aligned}
&-\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\left(-\frac{\tau}{\mathcal{K}}\,\boldsymbol{n}-\frac{2\mathcal{K}}{\tau}\,\boldsymbol{n}+\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a}-\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right)\cdot\delta\boldsymbol{r}_{,\alpha}-2\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\left(\frac{1}{\tau}\,\boldsymbol{a}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\\
&+\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\left(\frac{1}{\mathcal{K}}\,\boldsymbol{b}\right)_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}=-\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\left(-\frac{\tau}{\mathcal{K}}\,\boldsymbol{n}-\frac{2\mathcal{K}}{\tau}\,\boldsymbol{n}+\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a}-\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right)\cdot\delta\boldsymbol{r}_{,\alpha}\\
&-2\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\left(\frac{1}{\tau}\right)_{,\alpha}\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha}-2\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\frac{1}{\tau}\,\mathcal{K}\boldsymbol{n}\cdot\delta\boldsymbol{r}_{,\alpha}+\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\left(\frac{1}{\mathcal{K}}\right)_{,\alpha}\boldsymbol{b}\cdot\delta\boldsymbol{r}_{,\alpha}\\
&-\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\frac{1}{\mathcal{K}}\,\tau\boldsymbol{n}\cdot\delta\boldsymbol{r}_{,\alpha}=\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\left\{-\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a}+\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}-2\left(\frac{1}{\tau}\right)_{,\alpha}\boldsymbol{a}+\left(\frac{1}{\mathcal{K}}\right)_{,\alpha}\boldsymbol{b}\right\}\cdot\delta\boldsymbol{r}_{,\alpha},
\end{aligned}
$$

which, recalling that $\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha}=0$, is equal to

$$
\left(\frac{\partial\widetilde{W}}{\partial\tau}\right)_{,\alpha}\left\{\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}+\left(\frac{1}{\mathcal{K}}\right)_{,\alpha}\boldsymbol{b}\right\}\cdot\delta\boldsymbol{r}_{,\alpha}=0.
$$

We will now prove that also the sum of the remaining terms is zero:

$$
\begin{aligned}
&-\frac{\partial\widetilde{W}}{\partial\tau}\left[-\frac{\tau}{\mathcal{K}}\boldsymbol{n}+\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a}-\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right]_{,\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}-\frac{\partial\widetilde{W}}{\partial\tau}\left(-\frac{2\mathcal{K}}{\tau}\right)_{,\alpha}\boldsymbol{n}\cdot\delta\boldsymbol{r}_{,\alpha}-\frac{\partial\widetilde{W}}{\partial\tau}\frac{2\mathcal{K}^2}{\tau}\boldsymbol{a}\cdot\delta\boldsymbol{r}_{,\alpha}\\
&+\frac{\partial\widetilde{W}}{\partial\tau}\mathcal{K}\boldsymbol{b}\cdot\delta\boldsymbol{r}_{,\alpha}+\frac{\partial\widetilde{W}}{\partial\tau}\left(-\frac{1}{\tau}\,\boldsymbol{a}+\frac{1}{\mathcal{K}}\,\boldsymbol{b}\right)_{,\alpha\alpha}\cdot\delta\boldsymbol{r}_{,\alpha}\\
=\;&\frac{\partial\widetilde{W}}{\partial\tau}\left\{\left[\frac{\tau}{\mathcal{K}}\boldsymbol{n}\right]_{,\alpha}-\left[\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a}\right]_{,\alpha}+\left[\frac{\mathcal{K}_{,\alpha}}{\mathcal{K}^2}\,\boldsymbol{b}\right]_{,\alpha}+\left(\frac{2\mathcal{K}}{\tau}\right)_{,\alpha}\boldsymbol{n}+\mathcal{K}\boldsymbol{b}\right\}\cdot\delta\boldsymbol{r}_{,\alpha}\\
&+\frac{\partial\widetilde{W}}{\partial\tau}\left\{-\left[\left(\frac{1}{\tau}\right)_{,\alpha}\boldsymbol{a}\right]_{,\alpha}-\left[\frac{1}{\tau}\mathcal{K}\boldsymbol{n}\right]_{,\alpha}+\left[\left(\frac{1}{\mathcal{K}}\right)_{,\alpha}\boldsymbol{b}\right]_{,\alpha}-\left[\frac{1}{\mathcal{K}}\tau\boldsymbol{n}\right]_{,\alpha}\right\}\cdot\delta\boldsymbol{r}_{,\alpha}\\
=\;&\frac{\partial\widetilde{W}}{\partial\tau}\left\{-\left[\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\,\boldsymbol{a}\right]_{,\alpha}+\left(\frac{2\mathcal{K}}{\tau}\right)_{,\alpha}\boldsymbol{n}+\mathcal{K}\boldsymbol{b}-\left[\left(\frac{1}{\tau}\right)_{,\alpha}\boldsymbol{a}\right]_{,\alpha}-\left[\frac{\mathcal{K}}{\tau}\boldsymbol{n}\right]_{,\alpha}\right\}\cdot\delta\boldsymbol{r}_{,\alpha}\\
=\;&\frac{\partial\widetilde{W}}{\partial\tau}\left\{-\left[\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\right]_{,\alpha}\boldsymbol{a}-\left[\frac{\mathcal{K}_{,\alpha}}{\tau\mathcal{K}}\right]\mathcal{K}\boldsymbol{n}+2\left(\frac{\mathcal{K}}{\tau}\right)_{,\alpha}\boldsymbol{n}+\mathcal{K}\boldsymbol{b}-\left(\frac{1}{\tau}\right)_{,\alpha\alpha}\boldsymbol{a}-\left(\frac{1}{\tau}\right)_{,\alpha}\mathcal{K}\boldsymbol{n}\right\}\cdot\delta\boldsymbol{r}_{,\alpha}\\
&+\frac{\partial\widetilde{W}}{\partial\tau}\left\{-\left[\frac{\mathcal{K}}{\tau}\right]_{,\alpha}\boldsymbol{n}-\left[\frac{\mathcal{K}}{\tau}\right]\boldsymbol{n}_{,\alpha}\right\}\cdot\delta\boldsymbol{r}_{,\alpha}=
\end{aligned}
$$

and using Frenet formulas and the identity $\left(\frac{\mathcal{K}}{\tau}\right)_{,\alpha} \boldsymbol{n} = \left(\frac{1}{\tau}\right)_{,\alpha} \mathcal{K}\boldsymbol{n} + \frac{\mathcal{K}_{,\alpha}}{\tau}\boldsymbol{n}$, we finally find that the above expression is equal to

$$\frac{\partial \widetilde{W}}{\partial \tau} \left\{ \mathcal{K}\boldsymbol{b} + \left(\frac{\mathcal{K}}{\tau}\right) \mathcal{K}\boldsymbol{a} - \left(\frac{\mathcal{K}}{\tau}\right) \tau \boldsymbol{b} \right\} \cdot \delta \boldsymbol{r}_{,\alpha} = 0,$$

and the thesis is proved.

APPENDIX B

The structure of fabrics

We present here a short glossary containing the relevant terms related to textile fabrics.

Fiber: a long and narrow hair-like component of plant or animal tissue and, by extension, the smallest linear component (natural or artificial), used to create a yarn.

Yarn: the general term of any assemblage of fibers that has been put together in a continuous strand suitable for weaving or knitting.

Thread: a type of yarn used for hand-sewing or machine-sewing.

The techniques used to create textiles can produce different structures, such as knitted materials or woven textiles.

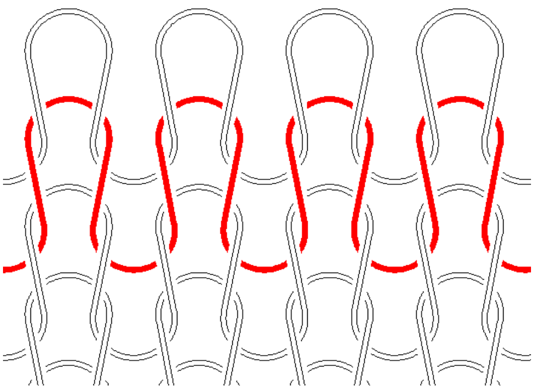

FIGURE B.1. Knitted fabric

Knitting technique: a technique using a single element or yarn in which a loop is drawn through a previous loop at the edge of a fabric.

Weaving technique: a technique used to interlace weft and warp yarns.

Warp: On a loom, the warp is the set of elements stretched in place before the weft is introduced during the weaving process. In a finished fabric with two or more sets of elements, the warp is the longitudinal set of fibers.

Weft: On a loom, the weft is inserted over and under the warp during the weaving process. In a finished fabric with two or more sets of elements, the warp is the transverse set.

The way the warp and weft threads are crossed with each other in the weaving process is called weave. Most common types of weave are: the plain weave and the twill weave.

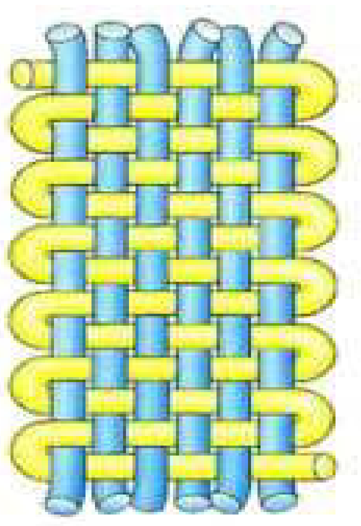

FIGURE B.2. Plain weave

Plain weave: The simplest possible interlacing of warp and weft elements in which each weft element passes alternately over and under successive warp elements (over-one, under-one), and each reverses the procedure of the one before it.

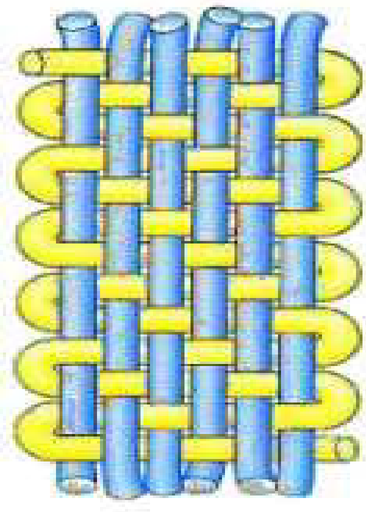

FIGURE B.3. Twill weave

Twill weave: It is made by interlacing warp and weft elements in which each weft element passes over one warp element and then under two or more warp elements. This way of interlacing threads creates a pattern of diagonal parallel ribs.

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VU LES RAPPORTS ETABLIS PAR :
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à soutenir devant un jury de l'INSTITUT NATIONAL POLYTECHNIQUE DE LORRAINE, une thèse intitulée :

"Mechanical models for 2D fibber networks and textiles"

en vue de l'obtention du titre de :

DOCTEUR DE L'INSTITUT NATIONAL POLYTECHNIQUE DE LORRAINE

Spécialité : « **Mécanique et énergétique** »

Fait à Vandoeuvre, le 18 février 2008
Le Président de l'I.N.P.L.,
F. LAURENT

INSTITUT NATIONAL POLYTECHNIQUE DE LORRAINE - NANCY

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